INFINITE LEXICOGRAPHIC PRODUCTS OF POSITIONAL OBJECTIVES

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Abstract. This paper contributes to the study of positional determinacy of infinite duration games played on potentially infinite graphs. Recently, [Ohlmann, TheoretiCS 2023] established that positionality of prefix-independent objective is preserved by finite lexicographic products. We propose two different notions of infinite lexicographic products indexed by arbitrary ordinals, and extend Ohlmann's result by proving that they preserve positionality. In the context of one-player positionality, this extends positional determinacy results of [Grädel and Walukiewicz, Logical Methods in Computer Science 2006] to edge-labelled games and arbitrarily many priorities for both Max-Parity and Min-Parity. Applying these results, we obtain positional languages that are complete for $\mathbf{\Delta}^0_3$ and $\mathbf{\Sigma}^0_3$ as well as new insight about closure under unions and neutral letters.

1. Introduction

1.1. Context: Positionality in games on graphs.

We consider infinite duration games played on directed graphs whose edges are coloured with labels from a set of colours C, with a specified objective $W \subseteq C^\omega$. Both the game graph and the set of colours may be infinite. The two players, Eve and Adam, take turns in moving a token along the edges of the graph. If the sequence of colours appearing on the produced path belongs to W, then Eve wins, otherwise Adam wins. If the objective W is Borel, then the game is determined, meaning one of the two player has a winning strategy [Mar75].

This paper is part of a long line of research aiming at understanding which Borel objectives are *positional*. A positional strategy depends only on the current vertex of the game and not on the whole history of the play so far. An objective is positional for Eve (just positional[1] in the following) if whenever Eve has a winning strategy in a game with this objective then she has a positional one.

[1] In some parts of the literature, these are called half-positional or memoryless for Eve.

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Recently, Ohlmann [Ohl23] introduced universal graphs to the study of positional objectives, and proved that an objective[2] W is positional if and only if it admits monotone well-ordered universal graphs (see Section 2 for formal definitions). Universal graphs, which have already proven key to obtain results about positionality [BCRV24, CO24], are the central object of study in this work.

Closure properties and lexicographic products. Some of the questions surrounding positional objectives concern their closure properties, with two major open problems in the area focusing on this aspect:

- Kopczyński's Conjecture [Kop08, Conjecture 7.1]: Are positional objectives closed under finite and countable unions? This question has been answered positively for finite unions of ω-regular languages [CO24] and negatively for positionality over finite graphs [Koz24].
- Neutral Letter Conjecture [Ohl23]: Are positional objectives closed under the addition of a neutral letter, that is, a letter whose addition or removal from a word w does not change whether w belongs to W? This conjecture has important consequences for the completeness of the characterisation of positionality via universal graphs (see [Ohl23] or Section 2 for details).

One of the few known closure properties of positional objectives is given by *finite lexicographic products*, obtained as a corollary of the characterisation based on universal graphs [Ohl23]. The lexicographic product of a sequence of objectives $(W_i \subseteq C_i^\omega)_{i<k}$ is their hierarchical combination: a word w belongs to the product if $\pi_i(w) \in W_i$, where i is the largest index such that w contains infinitely many colours from C_i, and $\pi_i(w)$ is the subword obtained by restricting w to these colours. This hierarchical combination of objectives naturally appears due to the alternation of quantifiers of some logics, such as the fixpoint operators in modal μ-calculus.

A paradigmatic example of such hierarchical construction is given by the parity objective

$$\mathrm{Parity}_d = \{w \in \{0,1,\ldots,d\}^\omega \mid \limsup w \text{ is even}\},$$

which enjoys a special status: it is one of the first objective shown to be positional over arbitrary game graphs [EJ91, Mos91], a result which is central in modern proofs of Rabin's Theorem on the decidability of the logic S2S [Rab69, GTW02], as well as in the algorithmic study of infinite duration games [FBBD$^+$]. It holds that the parity objective can be obtained as a finite lexicographic product of trivial objectives [Ohl23] (see also Section 2), giving an alternative positionality proof and highlighting the fundamental role of lexicographic products in the theory of positionality.

From finite to infinite products. A natural goal is to extend the previous ideas to infinite sequences of objectives. The simplest example of such a construction is the min-parity objective over ω, defined by

$$\mathrm{MinParity}_\omega = \{w \in \omega^\omega \mid \liminf w \text{ is finite and even}\}.$$

$\mathrm{MinParity}_\omega$ was first studied by Grädel and Walukiewicz [GW06], who established its bi-positionality, that is, positionality for both the objective and its complement. This result was proved for vertex-labelled game graphs. Here, the distinction between vertex-labels and edge-labels is crucial; in fact, it is easy to see (see Figure 1) that $\mathrm{MinParity}_\omega$ is not positional

[2]All objectives in this paper are prefix-independent and admit a neutral letter, as explained in Section 2.

for the opponent when edge-labels are considered.[3] Grädel and Walukiewicz [GW06] also observed that bi-positionality does not hold when considering $\mathrm{MaxParity}_\omega$, or when considering $\mathrm{MinParity}_\alpha$ for $\alpha > \omega$. However, failure of bi-positionality in these cases is due to phenomena akin to Figure 1: playing an increasing sequence of priorities requires memory, and therefore Adam requires memory. Positionality over edge-labelled graphs of all these objectives is neither proved nor disproved in their work.

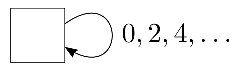

Figure 1: An edge-labelled game controlled by Adam where he requires non-positional strategies to ensure that $\mathrm{MinParity}_\omega$ is not met.

Another motivation for the study of infinite lexicographic products is the development of tools to propose new, complex, positional objectives. Indeed, an important obstacle to advance in Kopczyński's and the Neutral Letter conjectures is the lack of such tools. In fact, as we will see, some candidate objectives to disprove the Neutral Letter Conjecture can be described in this framework, namely $\mathrm{MinParity}_\omega$ and the ω-Büchi objective, defined by

$$\omega\text{-Büchi} = \{w \in \omega^\omega \mid \exists i, |w|_i \text{ is infinite}\}.$$

Positionality for both of them was not known prior to our work, and they are drastically altered when adding a neutral letter.

1.2. **Contributions.** We provide two ways of defining lexicographic products of families of objectives indexed by ordinals, namely, the max-lexicographic product and the min-lexicographic product. We show that these operations preserve positionality. The proof relies on providing adequate constructions of well-ordered monotone universal graphs for the two lexicographic products. We now discuss some consequences of these results.

Topological complexity. We obtain positionality of

$$\mathrm{MaxParity}_\alpha = \{w \in \alpha^\omega \mid \limsup w \text{ is odd}^4\},$$

for any ordinal α. It is known that Parity_d objectives are Wadge-complete for finite levels of the difference hierarchy [Skr13]. The objectives $\mathrm{MaxParity}_\alpha$ remain in $\mathbf{\Delta}^0_3$ (and we believe that they are Wadge-complete for the corresponding level in the difference hierarchy over $\mathbf{\Pi}^0_2$). On the other hand, min-lexicographic products of trivial objectives can go beyond $\mathbf{\Delta}^0_3$. This is the case of the ω-Büchi objective, which is Wadge-complete for $\mathbf{\Sigma}^0_3$. As far as we are aware, this is the first known positional objective in this class. This gives a first step into the possibility of exploring positionality beyond $\mathbf{\Delta}^0_3$.

[3] It is easy to encode vertex-labels into edge-labels, and therefore if an objective is edge-labelled positional then it is vertex-labelled positional; but the converse is not true.

[4] An ordinal is odd if it rewrites as $\beta + n$, with β either 0 or a limit ordinal and $n < \omega$ odd. The use of odd ordinals is crucial in this definition, the reason being that limit ordinals are even, and should be rejected for positionality.

Closure under addition of neutral letters for some objectives. Any objective admitting a well-ordered monotone universal graph satisfies that it is not only positional, but its extension with a neutral letter is too [Ohl23] (conversely, the restriction of a positional objective to a subset of colours always remains positional). Therefore, all the positionality results that we establish hold for both the objectives and their extensions with neutral letters. This is in particular the case for ω-Büchi and $\mathrm{MinParity}_\alpha$ (for any ordinal α), suggesting that adding neutral letters may preserve positionality in general, as up to date these were the best potential candidates to disprove this conjecture.

Locally finite memory. Casares and Ohlmann [CO22] recently proposed to study objectives W with locally finite memory, meaning that in any game with objective W, if Eve has a winning strategy then she has a winning strategy that only uses finitely many memory states for each game vertex. They proved that objectives admitting well-monotone universal graphs which are well-partial-orders (wpo) have locally finite memory, and that this class (which broadly generalises positional objectives or finite memory objectives) is closed under finite intersections [CO22, Corollary 32]. Our construction can also be applied to well-monotone graphs which are wpo's, which proves that this class of objectives is also closed under infinite (min and max) lexicographic products.

Structure of the paper. We first recall the necessary definitions, including finite lexicographic products, in Section 2. Then we present max-lexicographic products in Section 3 and (the more complex) min-lexicographic products in Section 4.

2. Preliminaries and finite lexicographic products

Graphs. In this paper, graphs are directed, edge-coloured, typically infinite, and may have sinks. Formally, a *C-graph* G, where C is an arbitrary set of colours, is given by a set of vertices $V(G)$ and a set of edges $E(G) \subseteq V(G) \times C \times V(G)$. We will usually denote edges as $v \xrightarrow{c} v'$. A *path* in a graph G is a sequence of edges in $E(G)$ with matching endpoints,

$$v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} v_2 \xrightarrow{c_2} \dots .$$

A path can be finite (even empty) or infinite. We say that it is a path *from* v_0, and, if it is finite and contains i edges, *towards* v_i. The finite or infinite word $c_0c_1\dots$ is called the *label* of the path. When there is a path from v towards v', we say that v' is *reachable* from v in G.

A *morphism* between two C-graphs G and H is a map $\phi : V(G) \to V(H)$ such that for all edge $v \xrightarrow{c} v' \in E(G)$, it holds that $\phi(v) \xrightarrow{c} \phi(v')$ is an edge in $G(H)$. We write $G \to H$ if we just want to state the existence of such a morphism. Note that ϕ need not be injective or surjective. Morphisms compose into morphisms. A *subgraph* G' of G is obtained from G by removing vertices and edges of G; note that in that case $G' \to G$. If $R \subseteq V(G)$, the subgraph of G obtained by removing all vertices in $V(G) \setminus R$ and keeping all edges between vertices in R is called the *restriction of G to R*. Given a vertex $v \in V(G)$, we let $G[v]$ denote the restriction of G to vertices reachable from v. The size of a graph G is the cardinal $|V(G)|$. If $v \xrightarrow{c} v' \in E(G)$ then we say that v is a c-predecessor of v' and v' is a c-successor of v. An edge $v \xrightarrow{c} v$ is called a loop around v.

Ordered graphs, monotonicity and directed sums. We will often consider *ordered graphs*, which are pairs $(G, \geq)$ where $\geq$ is a (partial) order over $V(G)$. By a slight abuse of notation, we sometimes omit $\geq$ from the notation of an ordered graph. We will pay special attention to graphs in which the order satisfies some of the following properties:

- is total,
- is well-founded (any non-empty subset has a minimal element),
- is a well-order (total and well-founded),
- is a well-partial order (is well-founded and contains no infinite antichain).

An ordered C-graph $(G, \geq)$ is said to be *monotone* if for all $u, v, u', v' \in V(G)$ and $c \in C$ we have

$$u \geq v \xrightarrow{c} v' \geq u' \text{ in } G \quad \implies \quad u \xrightarrow{c} u'.$$

In proofs, it is sometimes convenient to break monotonicity into left-monotonicity ($u \geq v \xrightarrow{c} v' \implies u \xrightarrow{c} v'$) and right-monotonicity ($v \xrightarrow{c} v' \geq u' \implies v \xrightarrow{c} u'$); it is a direct check that monotonicity is equivalent to their conjunction.

Given a family of (ordered) C-graphs $(G_\mu)_{\mu<\alpha}$, where α is an arbitrary ordinal, we define their *directed sum* $\overleftarrow{\sum}_{\mu<\alpha} G_\mu$ to be the disjoint union of the G_μ's with added edges from each G_μ to all the graphs before it in the sequence. Formally, $G = \overleftarrow{\sum}_{\mu<\alpha} G_\mu$ is a graph with vertices $V(G) = \bigsqcup_{\mu<\alpha} V(G_\mu) \times \{\mu\}$, and edges

$$(v, \mu) \xrightarrow{c} (v', \mu') \in E(G) \quad \text{ if } \mu > \mu', \text{ or } [\mu = \mu' \text{ and } v \xrightarrow{c} v' \in E(G_\mu)].$$

Note that for all μ, it holds that $G_\mu \to G$. If the G_μ's are ordered, then so is their sum, by the order

$$(v, \mu) \geq (v', \mu') \quad \text{ if } \mu > \mu' \text{ or } [\mu = \mu' \text{ and } v \geq v' \text{ in } G_\mu].$$

Observe that for any property X among being totally ordered, well-founded, or monotone, if the G_μ's have property X then so does their directed sum. By a slight abuse, when the $V(G_\mu)$'s are disjoint sets, we define for convenience the sum over $\bigsqcup_{\mu<\alpha} V(G_\mu)$ instead of $\bigsqcup_{\mu<\alpha} V(G_\mu) \times \{\mu\}$. In the case where the G_μ's are all equal to some (ordered) graph G, we denote their directed sum by $G \overleftarrow{\otimes} \alpha$.

Objectives and universality. A *C-objective* is a language[5] of infinite words $W \subseteq C^\omega$. In this paper, we will always only consider prefix-independent objectives, meaning those such that $cW = W$ for all $c \in C$ (equivalently, membership of a word in W is not affected by addition or removal of a finite prefix).

We say that a C-graph G *satisfies an objective* W if the label of any of its infinite paths belongs to W. In particular, a graph without infinite paths satisfies any objective.

Given a cardinal κ, we say that a C-graph U is *κ-universal for W* if

- U satisfies W; and
- every graph G of size $< \kappa$ and satisfying W admits a morphism to U, i.e. $G \to U$.

Next lemma indicates that, for prefix-independent objectives, it is in fact sufficient to find (monotone, well-ordered) universal graphs with weaker requirements.

Lemma 2.1 ([Ohl23, Lemma 4.5])**.** *Let W be a prefix-independent C-objective, κ a cardinal and U be a C-graph such that:*

[5]Formally, an objective is a pair (C, W), where C is non-empty and $W \subseteq C^\omega$. For simplicity, we just write objectives as W, as this does not create confusion or ambiguity.

- *U satisfies W; and*
- *for all graphs G which satisfy W and have size $< \kappa$, there is a vertex $v \in V(G)$ such that $G[v] \to U$.*

Then $U \overset{\leftarrow}{\otimes} \kappa$ is κ-universal for W.

Following [Ohl23], we sometimes say that a graph U as above is almost (κ, W)-universal.

Proof. Let U be such a graph and let G be a graph $< \kappa$ satisfying W; we should prove that $G \to U \overset{\leftarrow}{\otimes} \kappa$. By hypothesis, there is a vertex v_0 such that $G[v_0] \to U$.

Now let λ be any ordinal and assume constructed vertices v_μ for $\mu < \lambda$. Then we let $G_\lambda = G \setminus \bigcup_{\mu<\lambda} G[v_\mu]$ be the restriction of G to vertices which are not reachable from any of the v_μ's. Since $|G_\lambda| < \kappa$, there is v_λ such that $G_\lambda[v_\lambda] \to U$.

Now note that G is the disjoint union of the $G_\lambda[v_\lambda]$'s, and moreover G_λ is empty if $\lambda \geq \kappa$. Moreover, any edge in G is either part of some $G_\lambda[v_\lambda]$, or goes from $G_\lambda[v_\lambda]$ to $G_{\lambda'}[v_{\lambda'}]$ for some $\lambda > \lambda'$. We conclude that $G \to U \overset{\leftarrow}{\otimes} \kappa$ by mapping $G_\lambda[v_\lambda]$ in the λ-th copy of U for each λ. □

Universal graphs for the study of positionality and memory in games. We introduce definitions of games and positionality for completeness. However, in all the paper we will study positionality through the lenses of universal graphs (by using Theorem 2.2), and will not directly use the game-based definition of positionality.

A *W-game* is given by a sinkless $C \cup \{\varepsilon\}$-graph, together with a (prefix-independent) C-objective W and a partition of the vertices into those controlled by one player, called Eve, and her adversary, called Adam. Players play by moving a token in the graph for an infinite amount of time; the player controlling the current vertex choses which edge to take. The result of a play is an infinite path in the game graph. Who wins the play is determined by the projection of the labels on C: Eve wins if this projection is finite or belongs to W, otherwise Adam is the winner. This definition makes ε a neutral letter. A strategy (for Eve) is a function assigning to each finite path ending in a vertex controlled by Eve the next edge she should take. Such a strategy is winning from a vertex v if all infinite paths from v following the strategy are winning.

A strategy is *positional* if it can be described by a function from the set of Eve's vertices to edges; the strategy always points to the same outgoing edge, independently of the past of the play. An objective W is *positional*[6] if for every W-game, Eve has a positional strategy σ such that if she has a winning strategy from a vertex v, she wins from v using strategy σ.

Theorem 2.2 ([Ohl23, Theorem 3.1])**.** *A prefix-independent objective W is positional if and only if for every cardinal κ there exists a well-ordered monotone κ-universal graph for W.*

In the following, we will use the term "positional objective" as a synonym of an objective admitting well-ordered monotone κ-universal graphs for all κ. More generally, we say that a prefix-independent objective W has wpo-monotone graphs if for every cardinal κ, there exists a well-partially ordered monotone κ-universal graph for W. Such objectives are interesting because they have locally finite memory, are closed under finite intersections, and generalise ω-regular objectives, as shown in [CO22].

[6]It is not known whether the presence of a neutral colour ε affects positionality. Theorem 2.2 concerns positionality in the presence of a neutral colour due to the way we have defined games here.

Trivial objectives. For a non-empty set of colours C, we call $\mathrm{TW}_C = C^\omega$ the trivially winning objective, and $\mathrm{TL}_C = \emptyset \subseteq C^\omega$ the trivially losing objective over C. We will write TW_c and TL_c if $c \in C$ is a colour. These objectives are positional: it is easy to see that the single vertex C-graph $\overset{C}{\circlearrowleft}\!\bullet$ with all possible loops is κ-universal for TW_C for all κ. For TL_C, the graph of the order relation for cardinal κ is κ-universal. This graph, that we denote $\bullet \overset{\leftarrow C}{\otimes} \kappa$, has as set of nodes all ordinals $< \kappa$ and contains an edge $\lambda \xrightarrow{c} \lambda'$ for every $c \in C$ and ordinals $\lambda > \lambda'$.

Finite lexicographic products of objectives. Let C_0 and C_1 be two disjoint sets of colours, and let $C = C_0 \cup C_1$. Given an infinite word $w \in C^\omega$ and $i \in \{0,1\}$, we let $\pi_i(w)$ denote the (finite or infinite) word obtained by restricting w to letters in C_i.

We then define the *max-lexicographic product* of two prefix-independent objectives $W_0 \subseteq C_0^\omega$ and $W_1 \subseteq C_1^\omega$ by

$$\begin{aligned} W_0 \rtimes W_1 = \{w \in C^\omega \mid {} & [\pi_1(w) \text{ is infinite and belongs to } W_1] \\ & \text{or } [\pi_1(w) \text{ is finite and } \pi_0(w) \in W_0]\}. \end{aligned}$$

Note that $W_0 \rtimes W_1$ is prefix-independent. This operation is associative, and the min-lexicographic product of p conditions is

$$W_0 \rtimes \cdots \rtimes W_p = \{w \in C^\omega \mid \pi_\ell(w) \in W_\ell, \text{ where } \ell \text{ is maximal such that } \pi_\ell(w) \text{ is infinite}\}.$$

Clearly, this operation is not commutative. We write $W_0 \ltimes W_1$ to denote $W_1 \rtimes W_0$; we call it the *min-lexicographic product* of the objectives, for which more importance is given to W_0. The difference will be important once we study infinite products. We define infinite max-lexicographic products in the next section, and later consider (infinite) min-lexicographic products in Section 4.

In the rest of this section we discuss an associated operation of *max-lexicographic product of two ordered graphs* over disjoint sets colours. Given an ordered C_0-graph $(G_0, \geq_0)$ and an ordered C_1-graph $(G_1, \geq_1)$, where $C_0 \cap C_1 = \emptyset$, we define $(G_0 \rtimes G_1, \geq)$ to be the ordered $C_0 \cup C_1$-graph with vertices $V(G_0 \ltimes G_1) = V(G_0) \times V(G_1)$ ordered by

$$(v_0, v_1) \geq (v_0', v_1') \quad \Leftrightarrow \quad v_1 >_1 v_1' \text{ or } [v_1 = v_1' \text{ and } (v_0 \geq v_0')].$$

and whose edges are

$$\begin{aligned} E(G_0 \ltimes G_1) = {} & \{(v_0, v_1) \xrightarrow{c_1} (v_0, v_1') \mid c_1 \in C_1 \text{ and } v_1 \xrightarrow{c_1} v_1' \in E(G_1)\} \cup \\ & \{(v_0, v_1) \xrightarrow{c_0} (v_0', v_1') \mid c_0 \in C_0 \text{ and } [v_1 >_1 v_1' \text{ or} \\ & \qquad (v_1 = v_1' \text{ and } v_0 \xrightarrow{c_0} v_0' \in E(G_0))]\}. \end{aligned}$$

Once again, it is immediate to check that, if G_0 and G_1 are well-ordered, monotone or well-partially ordered, then so is their lexicographic product.

Ohlmann[7] related finite lexicographic products of positional objectives with lexicographic products of their universal graphs as follows.

Theorem 2.3 ([Ohl23, Theorem 5.2]). *Let $W_0 \subseteq C_0^\omega$, $W_1 \subseteq C_1^\omega$ be prefix-independent objectives with $C_0 \cap C_1 = \emptyset$. Let κ be a cardinal, and assume that the graphs U_0 and U_1 are κ-universal for W_0 and W_1, respectively. Then $U_0 \rtimes U_1$ is κ-universal for $W_0 \rtimes W_1$.*

[7]Formally, it was only proved for totally ordered graphs in [Ohl23], but the proof for non-totally ordered graphs, presented in [CO22] for completeness, is the same.

As a direct consequence, we get the following closure properties.

Corollary 2.4. *Prefix-independent positional objectives, as well as prefix-independent objectives having wpo-monotone graphs, are closed under finite lexicographic products.*

As an important example, the parity condition can be defined as the lexicographic product

$$\mathrm{Parity}_d = \mathrm{TW}_0 \rtimes \mathrm{TL}_1 \rtimes \mathrm{TW}_2 \rtimes \cdots \rtimes \mathrm{TL}_{d-1} \rtimes \mathrm{TW}_d,$$

where d is an even integer. Then, by Theorem 2.3 and κ-universality of $\overset{c}{\circlearrowleft\!\bullet}$ and $\bullet \overset{\overleftarrow{c}}{\otimes} \kappa$ for TW_c and TL_c, respectively we get that the graph

$$\overset{0}{\circlearrowleft\!\bullet} \rtimes (\bullet \overset{\overleftarrow{1}}{\otimes} \kappa) \rtimes \overset{2}{\circlearrowleft\!\bullet} \rtimes \cdots \rtimes (\bullet \overset{\overleftarrow{d-1}}{\otimes} \kappa) \rtimes \overset{d}{\circlearrowleft\!\bullet}$$

is κ-universal for Parity_d. A closer examination reveals that this graph corresponds to Walukiewicz's signatures [Wal96], or to Emerson and Jutla's positionality proof [EJ91] (we also refer the reader to [Ohl21, Chapter 5] for discussions around this construction).

The purpose of this paper is to introduce extensions of finite lexicographic products to infinite families of objectives, indexed by ordinals, and then to give corresponding constructions over universal graphs in order to generalize Theorem 2.3 and obtain closure properties. As we will see, in the infinite case max-lexicographic products and min-lexicographic products behave quite differently. We treat them separately in Sections 3 and 4.

3. Infinite max-lexicographic products

Setting. Fix a countable ordinal α. We fix a family of pairwise disjoint sets of colours $(C_\lambda)_{\lambda<\alpha}$, a family of prefix-independent objectives $(W_\lambda)_{\lambda<\alpha}$ with $W_\lambda \subseteq C_\lambda^\omega$. We define $C = \bigcup_{\lambda<\alpha} C_\lambda$ and $C_{<\lambda}, C_{\leq\lambda}, C_{>\lambda}, C_{\geq\lambda}$ as expected.

For a word $w \in C^\omega$, and an ordinal $\lambda < \alpha$, we let $\pi_\lambda(w) \in C_\lambda^* \cup C_\lambda^\omega$ denote the (finite or infinite) restriction of w to colours in C_λ. For a (finite or infinite) word $w = c_0 c_1 \cdots \in C^* \cup C^\omega$, we also let $\mathrm{ind}(w) = \lambda_0 \lambda_1 \cdots \in \alpha^* \cup \alpha^\omega$ denote the (finite or infinite) word of ordinals such that for all i we have $w_i \in C_\lambda$. Given $\Lambda = \lambda_0 \lambda_1 \cdots \in \alpha^\omega$, recall that

$$\limsup \Lambda = \min_{i<\omega} \sup\{\lambda_i, \lambda_{i+1}, \dots\},$$

We note that $\limsup \Lambda$ is always defined and $\leq \alpha$, as it is a min of a set of ordinals $\leq \alpha$.

We define the max-lexicographic product of the family $(W_\lambda)_{\lambda<\alpha}$ to be

$$\prod_{\lambda<\alpha}^{\text{max-lex}} W_\lambda = \{w \in C^\omega \mid \pi_\lambda(w) \in W_\lambda \text{ where } \lambda = \limsup \mathrm{ind}(w)\}.$$

Note that for w to be in the product, it should be that in particular $\pi_\lambda(w)$ is an infinite word, where $\lambda = \limsup \mathrm{ind}(w)$, which means that the limsup of the indices is seen infinitely often.

Our main result in this section is the following.

Theorem 3.1. *Prefix-independent positional objectives, as well as prefix-independent objectives having wpo-monotone graphs, are closed under countable max-lexicographic products.*

Colour-increasing unions. We start by establishing the following weakening of Kopczyński's conjecture, which will be the key lemma in the proof of Theorem 3.1 and may be of independent interest.

Theorem 3.2. *Let $(C_\lambda)_{\lambda<\alpha}$ be a family of sets colours satisfying $C_\lambda \subseteq C_{\lambda'}$ for $\lambda < \lambda' < \alpha$, where α is countable. Let $(W_\lambda)_{\lambda<\alpha}$ be a family of prefix-independent positional objectives (resp. prefix-independent objectives having wpo-monotone graphs) over the respective sets of colours such that for each $\lambda < \lambda'$ it holds that $C_\lambda^\omega \cap W_{\lambda'} = W_\lambda$. Then the union W of the W_λ's is positional (resp. has wpo-monotone graphs).*

We will say that a family of objectives as above is colour-increasing. The proof is a simple application of Lemma 2.1.

Proof. Let κ be a cardinal and let $U_0, U_1, \dots$ be well-ordered (resp. wpo) monotone κ-universal graphs for the respective objectives. Let $U = \overleftarrow{\sum}_{\lambda<\alpha} U_\lambda$; we claim that U is almost (κ, W)-universal and therefore $U \overleftarrow{\otimes} \kappa$ is (κ, W)-universal. First, observe that U indeed satisfies W: this follows from prefix-independence and the fact that each U_λ satisfies $W_\lambda \subseteq W$.

Now, consider a graph G of size $< \kappa$ satisfying W. We should prove that for some v, $G[v] \to U$. We claim that there exists $v \in V(G)$ such that all colours appearing on paths from v belong to C_λ for some λ. Assume by contridiction that this fails. Then, by an easy induction we obtain a path visiting edges with colours in C_{λ_0}, $C_{\lambda_1}, \dots$ where we choose $\lambda_0, \lambda_1, \dots,$ to be a cofinal sequence of α; such a path cannot satisfy any W_λ and therefore it does not satisfy W.

We conclude that for some v and some λ, it holds that $G[v]$ satisfies $W \cap C_\lambda^\omega = W_i$. Therefore $G[v] \to U_i$ which concludes since $U_i \to U$. □

Proof of Theorem 3.1. For $\alpha' \leq \alpha$, we let $W_{<\alpha'}$ denote the max-lexicographic product of the family $(W_\lambda)_{\lambda<\alpha'}$. To prove the Theorem 3.1, we proceed by induction over α'. There are two cases, corresponding to α' being a successor or a limit. First, we prove that for successor ordinals, our definition behaves just like finite lexicographic products.

Lemma 3.3. *For any $\alpha' < \alpha$, we have*

$$W_{<\alpha'+1} = W_{<\alpha'} \rtimes W_{\alpha'}.$$

Proof. Let $w \in C^\omega_{<\alpha'+1}$.

- First assume that $\pi_{\alpha'}(w)$ is infinite. Then $\limsup \operatorname{ind}(w) = \alpha'$ and we have
$$w \in W_{<\alpha'+1} \Leftrightarrow \pi_{\alpha'}(w) \in W_{\alpha'} \Leftrightarrow w \in W_{<\alpha'} \rtimes W_{\alpha'} \ .$$
- Otherwise, $\pi_{\alpha'}(w)$ is finite, and we let w' denote a suffix of w with $\pi_{\alpha'}(w') = \epsilon$. Then we have
$$w \in W_{<\alpha'+1} \Leftrightarrow w' \in W_{<\alpha'+1} \Leftrightarrow w' \in W_{<\alpha'} \Leftrightarrow w \in W_{<\alpha'} \rtimes W_{\alpha'}.$$ □

On the other hand, for limit ordinals, our definition resembles a union.

Lemma 3.4. *For any limit ordinal $\alpha' \leq \alpha$, we have*

$$W_{<\alpha'} = \bigcup_{\lambda<\alpha'} W_{<\lambda}.$$

Proof. It is a direct check that for $\lambda < \alpha'$ we have $W_{<\alpha'} \cap C^{\omega}_{<\lambda} = W_{<\lambda}$, and thus the right-to-left inclusion holds. Conversely, let $w \in W_{<\alpha'}$. Then $\lambda = \limsup \operatorname{ind}(w)$ is $\leq \alpha'$ and $\pi_\lambda(w)$ is infinite, so $\lambda < \alpha'$. Thus $w \in W_\lambda \subseteq W_{<\lambda+1}$. $\square$

Together, Lemmas 3.3 and 3.4 give an alternative inductive definition of the max-lexicographic product. Now note that for any $\alpha' < \alpha$, the above union is colour-increasing: $(C_{<\lambda})_{\lambda<\alpha'}$ is an increasing sequence of sets of colours, $W_{<\lambda} \subseteq C^{\omega}_{<\lambda}$ and for any $\lambda < \lambda'$ we have $C_{<\lambda} \cap W_{<\lambda'} = W_\lambda$. Thus Theorem 3.1 holds by induction on α: the successor case follows from Lemma 3.3 and Theorem 2.3 and the limit case follows from Lemma 3.4 and Theorem 3.2.

Max-parity. We now discuss the important case of the Max-Parity languages. Let $C_\lambda = \{\lambda\}$ for $\lambda < \alpha$ and

$$W_\lambda = \begin{cases} \mathrm{TL}_\lambda \text{ if } \lambda \text{ is even,}^{8} \\ \mathrm{TW}_\lambda \text{ otherwise.} \end{cases}$$

We define the max-parity objective $\mathrm{MaxParity}_\alpha$ as the lexicographic product of the W_λ's. Equivalently, it can be written as:

$$\mathrm{MaxParity}_\alpha = \{w \in \alpha^\omega \mid \limsup w \text{ is odd}\}.$$

Note that if $\lambda = \limsup w$ is an odd ordinal, it is necessarily non-limit, and therefore $\pi_\lambda(w)$ is infinite (this justifies our choice of odd priorities to be winning, rather than the more standard even ones).

Corollary 3.5. *For every countable ordinal α, MaxParity$_\alpha$ is positional.*

The universal graph obtained by unravelling the above proof, using the graphs $\overset{c}{\circlearrowleft}\!\bullet$ and $\bullet \overset{\leftarrow c}{\otimes} \kappa$ as starting blocks for the trivially winning and trivially losing objectives, provides a natural generalisation of Walukiewicz's signatures [Wal96] to ordinal priorities. We provide an explicit construction of such a graph in Appendix A.

4. Infinite min-lexicographic products

We introduce infinite min-lexicographic products of a sequence of winning objectives. Intuitively, the objectives at the beginning of the sequence have priority over those that appear later. The sequence can be indexed by any ordinal. We show that if winning conditions in the product are positional then the min-lexicographic product objective is positional too (Theorem 4.6). For this we provide an adequate construction of an universal graph from universal graphs for the components.

As we shall see, min-lexicographic products turn out to be more complex than max-lexicographic ones, for different aspects listed below.

- Finding the natural definition of infinite min-lexicographic products indexed over ordinals $> \omega$ is not obvious (see also Remark 4.2 below for more explanations).
- Topologically, min-lexicographic products generally lie beyond Δ_3 (for instance, the product of ω-many trivially winning conditions is in fact Σ_3-complete).

[8] The parity of an ordinal α is the parity of the unique $n < \omega$ such that α rewrites as $\alpha' + n$ for α' either 0 or a limit ordinal.

- Constructions (and universality proofs) establishing their positionality turn out to be substantially more involved.

4.1. Definitions and statement of the result.

Setting. In this section, we fix a cardinal $\kappa \geq 2$, an ordinal α, a family of pairwise disjoint sets of colours $(C_\lambda)_{\lambda<\alpha}$, and a family of prefix-independent objectives $(W_\lambda)_{\lambda<\alpha}$ with $W_\lambda \subseteq C_\lambda^\omega$ for all λ. We assume that each W_λ has a κ-universal well-founded monotone graph $(U_\lambda, \geq_\lambda)_{\lambda<\alpha}$. We will use $C = \bigcup_{\lambda<\alpha} C_\lambda$, as well as $C_{<\lambda}, C_{\leq\lambda}, C_{>\lambda}, C_{\geq\lambda}$ defined as expected. For a word $w \in C^\omega$, and an ordinal $\lambda < \alpha$, we let $\pi_\lambda(w) \in C_\lambda^* \cup C_\lambda^\omega$ denote the (finite or infinite) projection of w to colours in C_λ. Likewise, we let $\pi_{<\lambda}(w)$ denote the projection of w to colours in $C_{<\lambda}$.

Min-lexicographic products. We say that a word w is λ-*supported*[9] if $\pi_\lambda(w)$ is infinite and $\pi_{<\lambda}(w)$ is finite. A word is *supported* if it is λ-supported for some λ. In other words, a word is λ-supported if (i) after a finite prefix, λ is the smallest index of colours that appears, and (ii) a colour from C_λ appears infinitely often. In particular, λ is uniquely determined by w. For example, if $\alpha = \omega + 1$ and $C_\lambda = \{\lambda\}$, then the word $0\omega 1\omega 2\omega \cdots \in C^\omega$ is not supported, and the word $12131415\ldots$ is 1-supported.

We define the *min-lexicographic product* of $(W_\lambda)_{\lambda<\alpha}$ to be

$$\prod_{\lambda<\alpha}^{\text{min-lex}} W_\lambda = \{w \in C^\omega \mid w \text{ is } \lambda\text{-supported and } \pi_\lambda(w) \in W_\lambda\}.$$

Note that for $\alpha < \omega$, every word is supported, and thus in this case our definition coincides with finite min-lexicographic products.

Lemma 4.1. *The min-lexicographic product is associative. Formally, let $(\mu_i)_{i<\beta}$ be a strictly increasing sequence of ordinals that is cofinal in α, that is, $\mu_i < \alpha$ and for all $\lambda < \alpha$ there is i such that $\mu_i > \lambda$. Then*

$$\prod_{\lambda<\alpha}^{\text{min-lex}} W_\lambda = \prod_{i<\beta}^{\text{min-lex}} \left(\prod_{\mu_i \leq \lambda < \mu_{i+1}}^{\text{min-lex}} W_\lambda \right).$$

Proof. Let W be the objective on the left of the equality and $\widetilde{W}$ the one on the right. Assume that $w \in W$. Then, w is λ-supported for some $\lambda < \alpha$ (for the α-partition of the set of colours) and $\pi_\lambda(w) \in W_\lambda$. Let i be the unique ordinal such that $\mu_i \leq \lambda < \mu_{i+1}$. Then, w is i-supported (for the β-partition of the set of colours), and $\pi_i(w)$ is, in turn, λ-supported and $\pi_\lambda(\pi_i(w)) = \pi_l(w) \in W_\lambda$, so $\pi_i(w) \in \prod_{\mu_i \leq \lambda < \mu_{i+1}}^{\text{min-lex}} W_\lambda$ and $w \in \widetilde{W}$.

Conversely, assume $w \notin W$. If w is supported, we conclude using the same argument as above. Assume w is not supported (for the α-partition of the set of colours). Then, for all $i < \beta$, $\pi_i(w)$ is not λ-supported for any $\mu_i \leq \lambda < \mu_{i+1}$, so $\pi_i(w) \notin \prod_{\mu_i \leq \lambda < \mu_{i+1}}^{\text{min-lex}}$, so $w \notin \widetilde{W}$. $\square$

[9] Formally, being λ-supported depends on the sequence $(C_\lambda)_{\lambda<\alpha}$. We do not explicitly include this dependence in the notation for simplicity.

Remark 4.2. Another possible definition of min-lexicographic product could be

$$W' = \{w \in C^\omega \mid \lambda_0 = \mathrm{mininf}(w) \text{ is defined and } \pi_{\lambda_0}(w) \in W_{\lambda_0}\},$$

where $\mathrm{mininf}(w)$ is the minimal $\lambda < \alpha$ such that there are infinitely many occurrences of colours from C_λ in w. The two definitions coincide for $\alpha \leq \omega$, but they are different for $\alpha = \omega + 1$. Indeed, take $C_\lambda = \{\lambda\}$, $W_i = \mathrm{TL}_i$ for $i < \omega$ and $W_\omega = \mathrm{TW}_{\{\omega\}}$ (we write $\mathrm{TW}_{\{\omega\}}$ instead of TW_ω to avoid any ambiguities here). Observe that $\mathrm{mininf}(0\omega1\omega2\omega\dots) = \omega$ while this word is not supported. So this word is not in the min-lexicographic product, but it is in W', showing that the two definitions are different.

However, this modified definition has several disadvantages, that already appear for the example above. Firstly, the modified operation is not associative. Indeed, the product of the W_i's for $i < \omega$ is exactly TL_ω, the trivially losing objective over ω (for both definitions). Hence, $w \notin \mathrm{TL}_\omega \ltimes \mathrm{TW}_{\{\omega\}}$, so $\mathrm{TL}_\omega \ltimes \mathrm{TW}_{\{\omega\}} \neq W'$.

Moreover, W' is even not positional: to win in the game from Figure 2, Eve cannot use a positional strategy. So the modified definition does not preserve positionality. As we will show, our definition does preserve positionality.

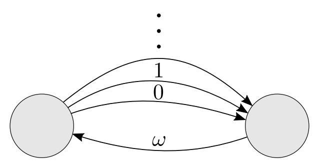

Figure 2: A game in which Eve requires memory to ensure objective W', for instance by playing a path labelled $0\omega1\omega\dots$.

Main result. We can now state the main result of the section: the closure of positional objectives under infinite min-lexicographic products.

Theorem 4.3. *Prefix-independent positional objectives, as well as prefix-independent objectives having wpo-monotone graphs, are closed under arbitrary min-lexicographic products.*

Before moving on to the proof, which spans the rest of the section, we present an application.

Application: ω-Büchi. For $\alpha = \omega$, $C_i = \{i\}$ and $W_i = \mathrm{TW}_i$ for $i < \alpha$, the min-lexicographic product yields

$$\omega\text{-Büchi} = \{w \in \omega^\omega \mid \exists i, |w|_i \text{ is infinite}\},$$

which one can see as an infinite union of Büchi objectives. Grädel and Walukiewicz [GW06] proved bi-positionality of ω-Büchi over vertex-labelled game graphs. Theorem 4.3 implies positionality,[10] over edge-labelled game graphs; it is easy to see that positionality for the opponent fails for edge-labelled graphs.

[10] We recall here that in this paper, positionality means "positionality in the presence of a neutral letter". For this objective, the fact that adding a neutral letter retains positionality is non-trivial.

4.2. **Universal graph: Construction.** In the rest of the section, we let $W = \prod_{\lambda<\alpha}^{\text{min-lex}} W_\lambda$. To show positionality of W, we define for every ordinal β the *power graph* U^β, using the universal graphs $(U_\lambda, \geq_\lambda)_{\lambda<\alpha}$. We show that U^β is κ-universal for W if β is chosen large enough (Theorem 4.6). In all the section β is an arbitrary but fixed ordinal.

For each λ, we consider the ordered graph $U_\lambda^\top$ obtained from U_λ by adding a fresh maximal vertex $\top_\lambda$ with no incoming edge and all possible outgoing edges except towards itself; formally, $E(U_\lambda^\top) = E(U_\lambda) \cup (\{\top_\lambda\} \times C_\lambda \times V(U_\lambda))$. Note that $U_\lambda^\top$ is well-founded, monotone, and κ-universal for W_λ.

Vertices of U^β. The vertices of U^β are the pairs (f, S), where $f : \alpha \to \beta$ is a non-increasing function and $S : \alpha \to \bigcup_{\lambda<\alpha} V(U_\lambda^\top)$ is such that $S(\lambda) \in V(U_\lambda^\top)$ for all $\lambda < \alpha$. Moreover, the two functions are linked by the condition: For all $\lambda < \alpha$,

$$S(\lambda) \neq \top_\lambda \quad \implies \quad f(\lambda) > f(\lambda + 1). \tag{4.1}$$

This condition implies that there may be only finitely many λ's for which $S(\lambda) \neq \top_\lambda$.

Order over U^β. A vertex (f, S) is a pair of sequences. Vertices are ordered by lexicographic order over the interleaving of these two sequences: $f(0), S(0), f(1), S(1) \ldots$ where lesser coordinates matter the most. To define this formally we introduce a piece of notation. Given $(f, S) \in V(U^\beta)$ and $\lambda \leq \alpha$, we let $(f, S)_{<\lambda}$ be obtained by restricting the domains of f and S to λ. We let $(f, S) > (f', S')$ if and only if

$$\exists \lambda < \alpha, \quad (f, S)_{<\lambda} = (f', S')_{<\lambda} \text{ and } [f(\lambda) > f'(\lambda) \text{ or } (f(\lambda) = f(\lambda) \text{ and } S(\lambda) >_\lambda S'(\lambda))].$$

Clearly, the above order is total assuming each $\geq_\lambda$ is.

It is also convenient to define $(f, S)_{\lessgtr\lambda}$ that is obtained from (f, S) by restricting f to $\lambda + 1$ and S to λ (equivalently, $(f, S)_{\lessgtr\lambda}$ is obtained by extending the map from $(f, S)_{<\lambda}$ by $\lambda \mapsto f(\lambda)$).

Using this notation, we get that $(f, S) > (f', S')$ if and only if there exists $\lambda < \alpha$ such that

$$[(f, S)_{<\lambda} = (f', S')_{<\lambda} \text{ and } f(\lambda) > f'(\lambda)] \quad \text{or} \quad [(f, S)_{\lessgtr\lambda} = (f', S')_{\lessgtr\lambda} \text{ and } S(\lambda) >_\lambda S'(\lambda)].$$

Edges of U^β. For a colour $c_\lambda \in C_\lambda$ and vertices $(f, S), (f', S') \in V(U^\beta)$, we let $(f, S) \xrightarrow{c_\lambda} (f', S') \in E(U^\beta)$ if and only if

$$(f, S)_{\lessgtr\lambda} > (f', S')_{\lessgtr\lambda} \quad \text{or} \quad [(f, S)_{\lessgtr\lambda} = (f', S')_{\lessgtr\lambda} \text{ and } S(\lambda) \xrightarrow{c_\lambda} S'(\lambda) \in E(U_\lambda^\top)].$$

This definition ensures a property we will often use in proofs:

$$\text{if } (f, S) \xrightarrow{c_\lambda} (f', S') \text{ and } c_\lambda \in C_\lambda \text{ then } (f, S)_{\lessgtr\lambda} \geq (f', S')_{\lessgtr\lambda}. \tag{4.2}$$

meaning that a transition on a colour c_λ does not increase the part of the state before coordinate λ, nor the f component of the coordinate λ.

4.3. **Universal graph: Monotonicity and compositionality.**

Monotonicity and satisfiability of W. We show that the power graph U^β is monotone and satisfies W.

Lemma 4.4. *The graph U^β defined above is:*

1. *well-founded,*
2. *monotone,*
3. *satisfies W,*
4. *is a wqo if all $(U_\lambda, \geq_\lambda)$ are wqo's.*

Proof. We prove the four items in order.

1. Towards a contradiction, consider an infinite decreasing sequence $(f^i, S^i)_{i\in\omega}$ of vertices of U^β. Let λ_0 be the minimal $\lambda < \alpha$ such that $(f^i(\lambda), S^i(\lambda))$ is not constant. Since $(f^i, S^i)_{<\lambda_0}$ is constant, it must be that $(f^i(\lambda_0), S^i(\lambda_0))_{i\in\omega}$ is non-increasing. Now since $\beta \times V(U^\top_{\lambda_0})$ is well-founded, the above sequence is ultimately constant. Let i_0 be the last index of strict decrease: $(f^{i_0}(\lambda_0), S^{i_0}(\lambda_0)) > (f^{i_0+1}(\lambda_0), S^{i_0+1}(\lambda_0)) = (f^i(\lambda_0), S^i(\lambda_0))$, for all $i > i_0$.

   We show that $f^{i_0}(\lambda_0) > f^{i_0+1}(\lambda_0 + 1)$. By the definition of order we have two cases. If $f^{i_0}(\lambda_0) > f^{i_0+1}(\lambda_0)$ then the property holds as $f^{i_0+1}(\lambda_0) \geq f^{i_0+1}(\lambda_0 + 1)$. The second case is when $f^{i_0}(\lambda_0) = f^{i_0+1}(\lambda_0)$ and $S^{i_0}(\lambda_0) > S^{i_0+1}(\lambda_0)$. But then $S^{i_0+1}(\lambda_0) \neq \top$ and therefore $f^{i_0+1}(\lambda_0) > f(i_0 + 1)(\lambda_0 + 1)$ by the condition (4.1) on vertices of U^β. So the property holds in this case too.

   Now, repeating the same argument on the suffix $(f^i, S^i)_{i>i_0}$ we find $\lambda_1 > \lambda_0$ and $i_1 > i_0$ such that $f^{i_0}(\lambda_0) > f^{i_0+1}(\lambda_0 + 1) \geq f^{i_1}(\lambda_1) > f^{i_1+1}(\lambda_1 + 1)$. Iterating this construction, we obtain an infinite decreasing sequence of ordinals: a contradiction.
2. Let $(f, S), (f', S'), (f'', S'')$ be vertices of U^β and let $c_\lambda \in C_\lambda$. We consider only the left monotonicity, right monotonicity being similar. Assume $(f, S) \xrightarrow{c_\lambda} (f', S') > (f'', S'')$. Using (4.2), we have the following chain of non-strict inequalities
$$(f, S)_{\overline{<}\lambda} \geq (f', S')_{\overline{<}\lambda} \geq (f'', S'')_{\overline{<}\lambda}$$
   and conclude that $(f, S) \xrightarrow{c_\lambda} (f'', S'')$ if any of them is strict. Otherwise, the above are equalities, so the definition of transitions and order gives us $S(\lambda) \xrightarrow{c_\lambda} S'(\lambda) \geq S''(\lambda)$ in $U^\top_\lambda$. By monotonicity of $U^\top_\lambda$ we have $S(\lambda) \xrightarrow{c_\lambda} S''(\lambda)$ giving us the desired $(f, S) \xrightarrow{c_\lambda} (f'', S'')$.
3. Consider an infinite path $(f^0, S^0) \xrightarrow{c^0_{\lambda^0}} (f^1, S^1) \xrightarrow{c^1_{\lambda^1}} \dots$ in U^β where for all i, $c^i_{\lambda^i} \in C_{\lambda^i}$. Let $w = c^0_{\lambda^0} c^1_{\lambda^1} \dots$. The aim is to prove that $w \in W$. Let λ_0 be minimal among the λ^i's and distinguish two cases.
   - If λ_0 appears infinitely often among the λ^i's, then w is λ_0-supported, so we must prove that $\pi_{\lambda_0}(w) \in W_\lambda$. Since all λ^i's are $\geq \lambda_0$, property (4.2) gives $(f^i, S^i)_{\overline{<}\lambda_0} \geq (f^{i+1}, S^{i+1})_{\overline{<}\lambda_0}$ for all i. Therefore, thanks to well-foundedness, $(f^i, S^i)_{\overline{<}\lambda_0}$ is eventually constant, say starting from index i_0. Consider any $i \geq i_0$. If $\lambda^i = \lambda_0$ we must have $S^i(\lambda_0) \xrightarrow{c^i_{\lambda_0}} S^{i+1}(\lambda_0)$. Otherwise, $\lambda^i > \lambda_0$, so we have both $(f^i, S^i)_{\overline{<}\lambda^i} \geq (f^{i+1}, S^{i+1})_{\overline{<}\lambda^i}$ and $(f^i, S^i)_{\overline{<}\lambda_0} = (f^{i+1}, S^{i+1})_{\overline{<}\lambda_0}$, which implies that $S^i(\lambda_0) \geq_{\lambda_0} S^{i+1}(\lambda_0)$. Therefore for $i \geq i_0$, we have $S^i(\lambda_0) \xrightarrow{c^i_{\lambda^i}} S^{i_0+1}(\lambda_0) \in E(U^\top_\lambda)$ if $\lambda^i = \lambda$, and $S^i(\lambda_0) \geq S^{i+1}(\lambda_0)$ otherwise. Thanks to monotonicity of transitions in $U^\top_{\lambda_0}$ we conclude that $\pi_{\lambda_0}(w_{\geq i_0})$ labels a path of $U^\top_{\lambda_0}$. By universality of $U^\top_{\lambda_0}$, this path satisfies W_{λ_0}. By prefix-independence $\pi_{\lambda_0}(w)$ also satisfies W_{λ_0}.

- Assume now that λ_0 appears only finitely often among the λ^i's. Let i_0 be the maximal i such that $\lambda^i = \lambda_0$. We show that

$$\begin{aligned} \text{either} \quad & (f^0, S^0)_{\lessdot\lambda_0} > (f^{i_0+1}, S^{i_0+1})_{\lessdot\lambda_0} \quad \text{or} \\ & (f^0, S^0)_{\lessdot\lambda_0} = (f^{i_0+1}, S^{i_0+1})_{\lessdot\lambda_0} \text{ and } f^{i_0}(\lambda_0) > f^{i_0+1}(\lambda_0+1) \end{aligned}$$

  Thanks to (4.2), for all i we have $(f^i, S^i)_{\lessdot\lambda_0} \geq (f^{i+1}, S^{i+1})_{\lessdot\lambda_0}$. If $(f^0, S^0)_{\lessdot\lambda_0} > (f^{i_0+1}, S^{i_0+1})_{\lessdot\lambda_0}$ we are done. Otherwise, we must have $(f^{i_0}, S^{i_0})_{\lessdot\lambda_0} = (f^{i_0+1}, S^{i_0+1})_{\lessdot\lambda_0}$ and $S^{i_0}(\lambda_0) \xrightarrow{c^{i_0}_{\lambda^{i_0}}} S^{i_0+1}(\lambda_0) \in E(U^\top_{\lambda_0})$. Thus $S^{i_0+1}(\lambda_0) \neq \top_{\lambda_0}$ hence $f^{i_0}(\lambda_0) > f^{i_0+1}(\lambda_0+1)$ by the condition (4.1) on vertices.
  In the next step we let $\lambda_1 > \lambda_0$ be the minimum λ^i for $i > i_0$, and if it appears only finitely often, define i_1 to be maximal such that $\lambda^{i_1} = \lambda_1$. Just like above, we obtain:

$$\begin{aligned} \text{either} \quad & (f^{i_0+1}, S^{i_0+1})_{\lessdot\lambda_1} > (f^{i_1+1}, S^{i_1+1})_{\lessdot\lambda_1} \quad \text{or} \\ & (f^{i_0+1}, S^{i_0+1})_{\lessdot\lambda_1} = (f^{i_1+1}, S^{i_1+1})_{\lessdot\lambda_1} \text{ and } f^{i_1}(\lambda_1) > f^{i_1+1}(\lambda_1+1) \end{aligned}$$

  Observe that if the second case occurs, and we have $f^{i_0}(\lambda_0) > f^{i_0+1}(\lambda_0+1)$ then we can combine these inequalities to $f^{i_0}(\lambda_0) > f^{i_0+1}(\lambda_0+1) \geq f^{i_0+1}(\lambda_1) = f^{i_1}(\lambda_1)$ (here the second inequality is monotonicity of f as $\lambda_1 \geq \lambda_0 + 1$).
  To finish we observe that this process cannot continue forever. Indeed, the first case cannot occur infinitely often due to well-foundedness proved in the first item of the lemma. If eventually only the second case occurs then this also leads to a contradiction as we can combine the inequalities we have observed above to obtain an infinite strictly decreasing chain $f^{i_0}(\lambda_0) > f^{i_1}(\lambda_1) > f^{i_2}(\lambda_2) > \dots$.

4. Well-foundedness was established in the first item, so we should show that antichains in U^β are finite. Consider a non-empty antichain $A \subseteq V(U^\beta)$. Towards a contradiction suppose A is infinite. Let λ_0 be the smallest among λ's such that there is a difference among elements of A on position λ, namely, there are $(f, S), (f', S') \in A$ with $S(\lambda) \neq S'(\lambda)$. Observe that the smallest difference cannot appear between f and f' components, as all elements of A are incomparable. Consider the set $\{S(\lambda_0) : (f, S) \in A\}$. It must be an antichain, because A is an antichain and all elements of A are the same up to λ_0. Hence, this set is finite because all antichains in $(U_\lambda, \geq_\lambda)$ are finite. Since it is an antichain and has more than one element, $\top$ is not in this set. As we have assumed that A is infinite there must be $(f_0, S_0) \in A$ for which the set $A_{(f_0,S_0),\lambda_0} = \{(f, S) \in A : (f, S)_{<\lambda_0+1} = (f_0, S_0)_{<\lambda_0+1}\}$ is infinite. Observe that $S_0(\lambda_0) \neq \top$.
   We can repeat the reasoning starting from $A_{(f_0,S_0),\lambda_0}$ instead of A. This gives us $\lambda_1 > \lambda_0$ and (f_1, S_1). Continuing like this we obtain an infinite sequence $A_{(f_i,S_i),\lambda_i}$ such that: $S_i(\lambda_i) \neq \top$ and $(f_i, S_i)_{\leq\lambda_i+1} = (f_{i+1}, S_{i+1})_{\leq\lambda_i+1}$. This gives us $f_0(\lambda_0) = f_1(\lambda_0) > f_1(\lambda_1) = f_2(\lambda_1) > f_2(\lambda_2)$ the strict inequalities following from $S_i(\lambda_j) \neq \top$ for $i \geq j$. A contradiction, as the ordinals are well-founded. □

Compositionality properties. For every $\lambda < \alpha$ we can define the graph $U^\beta_{<\lambda}$ in the same way as U^β, but considering the sequence up to λ instead of up to α. We can also define the graph $U^\beta_{\geq\lambda}$ by considering the subsequence starting from λ. In this second case, it will be convenient to assume the vertices of $U^\beta_{\geq\lambda}$ are of the form $f : [\lambda, \alpha) \to \beta$ and $S : [\lambda, \alpha) \to \bigcup_{\lambda \leq \lambda' < \alpha} V(U^\top_{\lambda'})$.

The lemma below proves a useful compositionality property; in some sense it states that our construction extends finite lexicographic products.

Lemma 4.5. *For all ordinals β, β' and for $\lambda < \alpha$ it holds that $U^{\beta}_{<\lambda} \ltimes U^{\beta'}_{\geq\lambda} \to U^{\beta+\beta'}$.*

Proof. For $v = ((f,S),(f',S')) \in V(U^{\beta}_{\lambda} \ltimes U^{\beta'}_{[\lambda,\alpha)})$, we define $\phi(v) = (g,R)$ by

$$g(\lambda') = \begin{cases} \beta' + f(\lambda') & \text{if } \lambda' < \lambda \\ f'(\lambda') & \text{otherwise} \end{cases} \qquad \text{and} \qquad R(\lambda') = \begin{cases} S(\lambda') & \text{if } \lambda' < \lambda \\ S'(\lambda') & \text{otherwise.} \end{cases}$$

It is direct to check that $(g,R) \in V(U^{\beta+\beta'})$; in particular g is non-increasing since both f and f' are and values of f' are $< \beta'$. To show that ϕ defines a morphism from $U^{\beta}_{\lambda} \ltimes U^{\beta'}_{[\lambda,\alpha)}$ to $U^{\beta+\beta'}$, we pick an edge $((f_0,S_0),(f'_0,S'_0)) \xrightarrow{c_{\lambda'}} ((f_1,S_1),(f'_1,S'_1))$ with $c_{\lambda'} \in C_{\lambda'}$. By the definition of $\ltimes$ this edge comes from one of the three cases.

- If $\lambda' < \lambda$, then $(f_0,S_0) \xrightarrow{c_{\lambda'}} (f_1,S_1) \in E(U^{\beta}_{\lambda})$, that is,

  $$(f_0,S_0)_{\eqslantless\lambda'} > (f_1,S_1)_{\eqslantless\lambda'} \text{ or } [(f_0,S_0)_{\eqslantless\lambda'} = (f_1,S_1)_{\eqslantless\lambda'} \text{ and } S_0(\lambda') \xrightarrow{c_{\lambda'}} S_1(\lambda') \in E(U^{\top}_{\lambda'})].$$

  We have $(g_0,R_0)_{\eqslantless\lambda'} = (f_0+\beta',S_0)_{\eqslantless\lambda'}$, and likewise $(g_1,R_1)_{\eqslantless\lambda'} = (f_1+\beta',S_1)_{\eqslantless\lambda'}$, so the result follows.
- If $\lambda' \geq \lambda$ and $(f_0,S_0) > (f_1,S_1)$. Then we have $(g_0,R_0)_{\leq\lambda} > (g_1,R_1)_{\leq\lambda}$ which implies $(g_0,R_0)_{\eqslantless\lambda'} > (g_1,R_1)_{\eqslantless\lambda'}$, thus $(g_0,R_0) \xrightarrow{c_{\lambda'}} (g_1,R_1)$.
- Otherwise, $\lambda' \geq \lambda$, $(f_0,S_0) = (f_1,S_1)$ and $(f'_0,S'_0) \xrightarrow{c_{\lambda'}} (f'_1,S'_1) \in E(U^{\beta'}_{[\lambda,\alpha)})$, which rewrites as

  $$(f'_0,S'_0)_{\eqslantless\lambda'} > (f'_1,S'_1)_{\eqslantless\lambda'} \text{ or } [(f'_0,S'_0)_{\eqslantless\lambda'} = (f'_1,S'_1)_{\eqslantless\lambda'} \text{ and } S'_0(\lambda') \xrightarrow{c_{\lambda'}} S'_1(\lambda') \in E(U^{\top}_{\lambda'})].$$

  (In the line above, notation $(f'_0,S'_0)_{\eqslantless\lambda}$ refers to maps $[\lambda,\alpha] \to \beta'$ and $[\lambda,\alpha) \to \bigcup_{\lambda\leq\lambda'\leq\alpha} V(U^{\top}_{\lambda'})$.) Then since $(g_0,R_0)_{<\lambda} = (f_0+\beta',S_0) = (f_1+\beta',S_1) = (g_1,R_1)_{<\lambda}$, it follows that

  $$(g_0,R_0)_{\eqslantless\lambda'} > (g_1,R'_1)_{\eqslantless\lambda'} \text{ or } [(g_0,R_0)_{\eqslantless\lambda'} = (g_1,R_1)_{\eqslantless\lambda'} \text{ and } R_0(\lambda') \xrightarrow{c_{\lambda'}} R_1(\lambda') \in E(U^{\top}_{\lambda'})],$$

  the wanted result. □

4.4. **Universal graph: Universality.**

We are now ready to prove our main result.

Theorem 4.6. *Suppose $(C_\lambda)_{\lambda<\alpha}$ is a sequence of pairwise disjoint sets of colours, and $(W_\lambda)_{\lambda<\alpha}$ is a sequence of prefix-independent objectives with $W_\lambda \subseteq C^{\omega}_{\lambda}$ for all λ. Let κ be some cardinal and assume that for every $\lambda < \alpha$ there is a κ-universal graph $(U_\lambda, \geq_\lambda)$ for W_λ. Then there is β such that the power graph $(U^{\beta}, \geq)$ is κ-universal for the min-lexicographic product of $(W_\lambda)_{\lambda<\alpha}$.*

We say that a C-graph G *can be mapped* if for some ordinal β it holds that $G \to U^{\beta}$; otherwise we say that G cannot be mapped. Since U^{β} satisfies W (Lemma 4.4), any graph that can be mapped satisfies W. Our goal is to prove the converse: graphs of size $< \kappa$ that satisfy W can be mapped. This implies Theorem 4.6, by taking β large enough so that any graph smaller than κ satisfying W can be mapped into U^{β}.

Our first step is to show that if every graph in a sequence can be mapped then the directed sum of the sequence can be mapped.

Lemma 4.7. *Let $(G_\mu)_{\mu<M}$ be a family of graphs such that for all $\mu < M$, G_μ can be mapped. Then $\overleftarrow{\sum}_{\mu<M} G_\mu$ can be mapped.*

Proof. For each $\mu < M$, let β_μ be such that $G_\mu \to U^{\beta_\mu}$. Let $\varphi_\mu(v) = (f^v_\mu, S^v_\mu)$ be a morphism $\varphi_\mu : G_\mu \to U^{\beta_\mu}$. Recall that $f^v_\mu : \alpha \to \beta_\mu$ and $S^v_\mu : \alpha \to \bigcup_{\lambda<\alpha} V(U^\top_\lambda)$. Let $G = \overleftarrow{\sum}_{\mu<M} G_\mu$ and let $\beta = \sum_{\mu<M} \beta_\mu$. We define a map $\psi : V(G) \to V(U^\beta)$ by

$$\varphi(v) = (f^v_\mu + \sum_{\mu'<\mu} \beta'_\mu, S^v_\mu), \qquad \text{if } v \in V(G_\mu) \text{ and } \varphi_\mu(v) = (f^v_\mu, S^v_\mu)$$

It is direct to check that $\varphi(v)$ is an element of $V(U^\beta)$. In particular for the condition (4.1) we check that if $S^v_\mu(\lambda) \neq \top$ then $(f^v_\mu + \sum_{\mu'<\mu} \beta'_\mu)(\lambda) > (f^v_\mu + \sum_{\mu'<\mu} \beta'_\mu)(\lambda + 1)$. This follows directly from the fact that f^v_μ satisfies this condition.

To show that φ defines a morphism $G \to U^\beta$, we take an edge $v \xrightarrow{c_\lambda} v' \in E(G)$ with $c \in C_\lambda$; by definition of $G = \overleftarrow{\sum}_{\mu<M} G_\mu$ there are two cases.

- The first case is when $v \xrightarrow{c_\lambda} v' \in E(G_\mu)$ for some $\mu < M$. Since ϕ_μ is a morphism $G_\mu \to U^{\beta_\mu}$, we have $\phi_\mu(v) \xrightarrow{c_\lambda} \phi_\mu(v')$, which rewrites as

$$\begin{aligned} &(f^v_\mu, S^v_\mu)_{\overline{<}\lambda} > (f^{v'}_\mu, S^{v'}_\mu)_{\overline{<}\lambda}, \quad \text{or} \\ &(f^v_\mu, S^v_\mu)_{\overline{<}\lambda} = (f^{v'}_\mu, S^{v'}_\mu)_{\overline{<}\lambda}, \quad \text{and} \quad [S^v_\mu]_\lambda \xrightarrow{c_\lambda} [S^{v'}_\mu]_\lambda \in E(U^\top_\lambda). \end{aligned}$$

  Since f^v and $f^{v'}$ are obtained by shifting respectively f^v_μ and $f^{v'}_\mu$ by $\sum_{\mu'<\mu} \beta_{\mu'}$, and $S^v = S^v_\mu$ and $S^{v'} = S^{v'}_\mu$, we get that $(f^v, S^v) \xrightarrow{c_\lambda} (f^{v'}, S^{v'})$, as required.
- Otherwise, $v \in V(G_\mu)$ and $v' \in V(G_{\mu'})$ for $\mu > \mu'$. Then

$$f^v(0) = f^v_\mu(0) + \sum_{\mu''<\mu} \beta_{\mu''} \;>\; f^{v'}_{\mu'}(0) + \sum_{\mu''<\mu'} \beta_{\mu''} = f^{v'}(0),$$

  where the inequality holds since $\beta_{\mu'} > f^{v'}_{\mu'}(0)$. Therefore, $(f^v, S^v) \xrightarrow{c_\lambda} (f^{v'}, S^{v'})$. □

We now prove that any $C_{\geq\lambda}$-graph that can be mapped into U^β can be also mapped into $U^{\beta'}_{\geq\lambda}$, for some bigger β'.

Lemma 4.8. *Let $\lambda < \alpha$ and let G be a $C_{\geq\lambda}$-graph which can be mapped. Then there is an ordinal β' such that $G \to U^{\beta'}_{\geq\lambda}$.*

Proof. Let $\phi : G \to U^\beta$. For each $(f', S') \in V(U^\beta_{<\lambda})$, we let $G_{(f',S')}$ be the restriction of G to vertices in

$$\phi^{-1}\{(f, S) \in V(U^\beta) \mid (f, S)_{<\lambda} = (f', S')\}.$$

(It may be that some of the $(G_{(f',S')})$ are empty; this is not an issue in the proof below.) We now make two claims which are proved below.

Claim 4.9. *For each (f', S'), it holds that $G_{(f',S')} \to U^\beta_{\geq\lambda}$.*

Claim 4.10. *It holds that $G \to \overleftarrow{\sum}_{(f',S')} G_{(f',S')}$, where the (f', S')'s are ordered as in $V(U^\beta_{<\lambda})$.*

Putting the two claims together with Lemma 4.7 yields the desired result. For Claim 4.9, it suffices to consider the restriction of ϕ to $G_{(f',S')}$ (restrictions of morphisms are morphisms). For Claim 4.10, is suffices to recall property (4.2) saying that for every edge $(f_0, S_0) \xrightarrow{c} (f_1, S_1) \in E(U^\beta)$ such that $c \in C_{\geq\lambda}$, we have $(f_0, S_0)_{<\lambda} \geq (f_1, S_1)_{<\lambda}$. □

We now prove a crucial ingredient to the proof of Theorem 4.6. Recall that $G[v]$ is the restriction of G to vertices reachable from v.

Lemma 4.11. *If G satisfies W and cannot be mapped then there is $v \in V(G)$ such that $G[v]$ cannot be mapped. Moreover, v can be picked so that it has a predecessor in G.*

Proof of Lemma 4.11. Assume for contradiction that for all $v \in V[G]$, $G[v]$ can be mapped. Take a well-ordering $(v_\mu)_{\mu<M}$ of all vertices of G, and define G_μ to be

$$G_\mu = G[v_\mu] - \bigcup_{\mu'<\mu} V(G[v_{\mu'}]).$$

Then for each $\mu < M$, it holds that $G_\mu \to G[v_\mu]$ therefore G_μ can be mapped. Hence, by Lemma 4.7, $\overleftarrow{\sum}_{\mu<M} G_\mu$ can be mapped. But $G \to \overleftarrow{\sum}_{\mu<M} G_\mu$, so G can also be mapped: a contradiction proving that there is $v \in V(G)$ such that $G[v]$ cannot be mapped.

We now show that v can be taken to have a predecessor, so assume that the v constructed above does not have a predecessor (in particular, there is no loop around v). We show that for some successor v' of v, $G[v']$ cannot be mapped. Observe that if v is reachable from some of its successors v' then $G[v] = G[v']$. So we can take v' in this case. Otherwise, we adapt the argument from the previous paragraph. Assume that for all successors v' of v, $G[v']$ can be mapped, well-order them into $(v'_\mu)_{\mu<M'}$, and let

$$G'_\mu = G[v'_\mu] - \bigcup_{\mu'<\mu} V(G[v'_{\mu'}]).$$

Then the G'_μ's can be mapped. Let $G_{M'}$ be the restriction of G to $\{v\}$. Since v is not reachable from any of its successors, there is no loop around v. So $G_{M'}$ is edgeless, and therefore it can be mapped. Now observe that

$$G[v] \to \overleftarrow{\sum}_{\mu<M'} G'_\mu \overleftarrow{+} G_{M'},$$

which can be mapped thanks to Lemma 4.7. A contradiction showing that this case is impossible. □

We are now ready to present an inductive proof of Theorem 4.6.

Proof of Theorem 4.6. The proof goes by induction over the ordinal α therefore we assume the result known for ordinals $< \alpha$:

For any $\lambda < \alpha$ and for any $C_{<\lambda}$-graph G satisfying W, there is β such that $G \to U^\beta_{<\lambda}$.

Let G be a C-graph satisfying W and assume towards contradiction that G cannot be mapped.

Claim 4.12. *For any $\lambda < \alpha$, the restriction $G_{\geq\lambda}$ of G to edges with colours in $C_{\geq\lambda}$ cannot be mapped.*

Proof. Suppose for a contradiction that there is λ such that $G_{\geq\lambda}$ can be mapped. We construct a $C_{<\lambda}$-graph G' that can be mapped and use $G' \ltimes G_{\geq\lambda}$ to show that G can be mapped.

The $C_{<\lambda}$-graph G' has the same vertices as G, $V(G') = V(G)$. The edges have colours $c \in C_{<\lambda}$, and are given by:

$$v \xrightarrow{c} v' \in E(G') \quad \text{when} \quad \exists u, u' \in V(G), \quad v \overset{C^*_{\geq\lambda}}{\rightsquigarrow} u \xrightarrow{c} u' \overset{C^*_{\geq\lambda}}{\rightsquigarrow} v \text{ in } G,$$

where the notation $v \overset{C^*_{\geq\lambda}}{\rightsquigarrow} u$ means that there is a path in G from v to u using only edges with colours in $C_{\geq\lambda}$ (stated differently, a path in $G_{\geq\lambda}$). Note that these paths may be empty, and in particular, edges of G with colour in $C_{<\lambda}$ also belong to G'.

Let us prove that G' satisfies W. Consider an infinite path $\pi' = v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \dots$ in G'; it is labelled by the word $w = c_0c_1 \dots \in C^\omega_{<\lambda}$. Then there is a path of the form

$$\pi = v_0 \rightsquigarrow u_0 \xrightarrow{c_0} u'_0 \rightsquigarrow v_1 \rightsquigarrow u_1 \xrightarrow{c_1} u'_1 \rightsquigarrow \dots,$$

in G, where paths $v_i \rightsquigarrow u_i$ and $u'_i \overset{v}{\rightsquigarrow}_{i+1} v_{i+1}$ are labelled by colours in $C_{\geq\lambda}$. Since G satisfies W, the label w of π belongs to W, in particular it is λ'-supported for some λ', and since w has infinitely-many occurrences of letters from $C_{<\lambda}$, it must be that $\lambda' < \lambda$. Thus w' is also λ'-supported and $\pi_{\lambda'}(w') = \pi_{\lambda'}(w) \in W_{\lambda'}$ and thus $w' \in W$. Therefore, G' satisfies W hence we obtain by induction that $G' \to U^{\beta'}_{<\lambda}$ for some β'.

Since $G' \to U^{\beta'}_{<\lambda}$ we can find a minimal morphism $\phi' : G' \to U^{\beta'}_{<\lambda}$. This means, it is a morphism not pointwise bigger than any other morphism $G' \to U^{\beta'}_{<\lambda}$. Such a morphism has a property that for any pair (v, v') of vertices, if for all colour c, all c-successors of v' are also c-successors of v, then $\phi'(v) \geq \phi'(v')$ (otherwise we could obtain a smaller morphism by mapping v' to $\phi(v)$).

We now show that $G \to U^{\beta'}_{<\lambda} \ltimes G^{\geq\lambda}$. Consider the map ϕ between these graphs given by $\phi(v) = (\phi'(v), v)$, which we show to be a morphism. Take an edge $v \xrightarrow{c} v' \in E(G)$.

- If $c \in C_{<\lambda}$ then $v \xrightarrow{c} v' \in E(G')$ thus $\phi'(v) \xrightarrow{c} \phi'(v') \in E(U^{\beta'}_{<\lambda})$ which implies the result.
- Otherwise, $c \in C_{\geq\lambda}$. Then in G', $out(v) \supseteq out(v')$. By the above-mentioned property of minimal morphisms, this implies that $\phi'(v) \geq \phi'(v')$. Together with the fact that $v \xrightarrow{c} v' \in E(G^{\geq\lambda})$, this implies that $v \xrightarrow{c} v' \in E(G' \ltimes G^{\geq\lambda})$, as required.

Thus $G \to U^{\beta'}_{<\lambda} \ltimes G^{\geq\lambda}$. Now if $G^{\geq\lambda}$ could be mapped, then by Lemma 4.8 we get $G \to U^{\beta}_{\geq\lambda}$, therefore it follows from Lemma 4.5 that G can be mapped, a contradition. □

Let $G_0 = G$ and let v_0 be such that $G_0[v_0]$ cannot be mapped, obtained from Lemma 4.11 (here, the fact that v_0 has a predecessor in G_0 is not used). We will construct a decreasing sequence of subgraphs $G_0, G_1, \dots$ of G and vertices $v_0, v_1, \dots$ with non-empty paths π_i from v_i to v_{i+1} in G_i, with the property that for all i, all edges in G_{i+1} (and therefore also in subsequent graphs) have colours in $C^{>\lambda_i}$, where λ_i is the maximal colour of an edge in $\pi_{<i} = \pi_0 \dots \pi_{i-1}$. This implies the desired contradiction as the label of π is not supported, and thus does not satisfy W. The crucial invariant in the construction is that the $G_i[v_i]$'s cannot be mapped.

Assume constructed the path up to v_i (see also Figure 3), and let λ_i be as above (or $\lambda_i = 0$ if $i = 0$). Since $G_i[v_i]$ cannot be mapped, Claim 4.12 says that $G_i[v_i]^{\geq\lambda_i+1}$ cannot be mapped. So we let $G_{i+1} = G_i[v_i]^{\geq\lambda_i+1}$, and then apply Lemma 4.11 to G_{i+1} to obtain

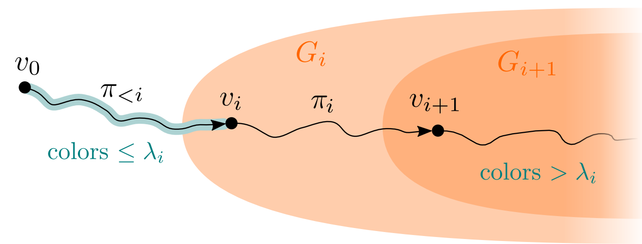

Figure 3: Constructing a path violating W.

$v_{i+1} \in V(G_{i+1})$ such that $G_{i+1}[v_{i+1}]$ cannot be mapped and v_{i+1} has a predecessor u_{i+1} in G_{i+1}. Since G_{i+1} is a subgraph of $G_i[v_i]$, there is a path π_i in G_i from v_i to v_{i+1}, which we can take to go through u_i. This ensures the path is non-empty. $\square$

Our main result, Theorem 4.3, follows from Theorem 4.6 and Lemma 4.4.

5. Conclusions

In this work, we have introduced two positionality-preserving operations of objectives generalising lexicographic products to arbitrary ordinals: max- and min-lexicographic products. These two operations extend our understanding of positionality in two orthogonal manners.

Max-lexicographic products yield a natural generalisation of the Parity languages. An interesting open problem is to prove that these languages are complete for infinite levels of the difference hierarchy over $\mathbf{\Pi}^0_2$, which would provide a path forward the systematic study of positional objectives within $\mathbf{\Delta}^0_3$, the natural topological generalisation of ω-regular objectives.

Min-lexicographic products, on the other hand, easily go beyond $\mathbf{\Delta}^0_3$. They provide a tool to show positionality of objectives in $\mathbf{\Sigma}^0_3$ (as, for instance, ω-Büchi), the higher level in the Borel hierarchy in which positional objectives have been found.[11] An interesting question is whether there are positional objectives in all the levels of the Borel hierarchy.

Furthermore, we have proved a special case of Kopczyński's conjecture, namely, closure of positionality under colour-increasing unions of objectives (Theorem 3.2). The lexicographic product of a family of objectives provides a sort of underapproximation to their union. Whether the positionality of lexicographic products can help to resolve the general case of Kopczyński's conjecture is an exciting open problem.

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Appendix A. Signatures for parity games with infinitely many priorities

Recall the Max-parity objective

$$\mathrm{MaxParity}_\alpha = \{w \in \alpha^\omega \mid \limsup w \text{ is odd}\},$$

which is the max-lexicographic product of the objectives

$$W_\lambda = \begin{cases} \mathrm{TL}_\lambda \text{ if } \lambda \text{ is even,} \\ \mathrm{TW}_\lambda \text{ otherwise.} \end{cases}$$

Fix a cardinal κ. Let $U_{<\alpha}$ be given by $V(U_{<\alpha}) = \kappa^{\alpha_{\text{even}}}$ (with ordinal exponentiation),[12] where α_{even} denotes the set of even ordinals $< \alpha$, and

$$E(U_{<\alpha}) = \{v \xrightarrow{\lambda} v' \mid [\lambda \text{ even and } v_{\geq\lambda} > v'_{\geq\lambda}] \text{ or } [\lambda \text{ odd, } v_{\geq\lambda+1} > 0 \text{ and } v_{\geq\lambda+1} \geq v'_{\geq\lambda+1}]\}.$$

It is a direct check that U_α is well-ordered and monotone, when ordered lexicographically. This appendix is devoted to the proof of the following theorem.

Theorem A.1. *The graph $U_{<\alpha} \overset{\leftarrow\alpha}{\otimes} \kappa$ is κ-universal for MaxParity$_\alpha$.*

Proof. We proceed by induction over α, call $P(\alpha)$ the assertion "$U_{<\alpha}$ is almost $(\kappa, \text{MaxParity}_\alpha)$-universal." From there, Lemma 2.1 concludes.

Zero case. The graph U_0 has a single vertex with no edge; therefore it satisfies $\text{MaxParity}_0 = \varnothing$. Now, the only graphs satisfying MaxParity_0 are graphs with no infinite paths, and such graphs have sinks; in other words, in any graph G satisfying MaxParity_0, there is a vertex v such that $G[v] \mapsto U_0$.

Even successor case. Assume α is even and $P(\alpha)$ holds. We aim to prove $P(\alpha+1)$. Let $U_\alpha = \bullet \overset{\leftarrow\alpha}{\otimes} \kappa$, so that U_α is (κ, W_α)-universal. Let us prove that

$$U_{<\alpha+1} = U_{<\alpha} \rtimes U_\alpha.$$

Since α is even, it is an easy check that the two vertex sets coincide with $\kappa^{\alpha_{\text{even}}+1}$. Now $v \xrightarrow{\lambda} v' \in E(U_{<\alpha+1})$ if and only if $v_\alpha > v'_\alpha$ or $[v_\alpha = v'_\alpha$ and $v_{<\alpha} \xrightarrow{\lambda} v'_{<\alpha}]$ if and only if $v \xrightarrow{\lambda} v' \in E(U_{<\alpha} \rtimes U_\alpha)$. Now $\text{MaxParity}_{\alpha+1} = \text{MaxParity}_\alpha \rtimes W_\alpha$, therefore we conclude by Theorem 2.3.

Odd successor case. Assume α is odd and $P(\alpha)$ holds. We aim to prove $P(\alpha+1)$. Let $U_\alpha = \overset{\alpha}{\circlearrowleft}\!\bullet$ so that U_α is (κ, W_α)-universal. Let $U'_{<\alpha+1} = U_{<\alpha} \rtimes U_\alpha$; again thanks to Theorem 2.3 we know that $U'_{<\alpha+1}$ is κ-almost universal for $\text{MaxParity}_{\alpha+1}$. Now observe that $V(U'_{<\alpha+1}) = V(U_{<\alpha}) = \kappa^{\alpha_{\text{even}}} = \kappa^{(\alpha+1)_{\text{even}}} = V(U_{\alpha+1})$, and

$$E(U'_\alpha) = \{v \xrightarrow{\lambda} v' \mid [\lambda \text{ even and } v_{\geq\lambda} > v'_{\geq\lambda}] \text{ or } [\lambda \text{ odd and } v_{\geq\lambda+1} \geq v'_{\geq\lambda+1}]\}.$$

Therefore the identity over $\kappa^{\alpha_{\text{even}}}$ defines a morphism $U_{<\alpha+1} \to U'_{<\alpha+1}$ and in particular, $U_{<\alpha+1}$ satisfies MaxParity_α. Conversely, the map assigning v' to v where $v'_\alpha = v_\alpha + 1$ and $v'_\lambda = v_\lambda$ for $\lambda < \alpha$ defines a morphism $U'_{<\alpha+1} \to U_{<\alpha+1}$, which concludes.

Limit case. Assume α is a limit and $P(\lambda)$ holds for all $\lambda < \alpha$; we aim to prove $P(\alpha)$. We first prove that $U_{<\alpha}$ satisfies MaxParity_α. Take an infinite path $v^0 \xrightarrow{\lambda_0} v^1 \xrightarrow{\lambda_1} \dots$ in $U_{<\alpha}$, and assume towards a contradiction that $\lambda = \limsup(\lambda_0\lambda_1\dots)$ is even. There are two cases.

- If $\lambda < \alpha$. Let i_0 be large enough so that all λ_i's are $\leq \lambda$ for $i \geq i_0$. Then for $i \geq i_0$ we have $v^i_{>\lambda} \geq v^{i+1}_{>\lambda}$, so by well-foundedness there is i_1 so that $v^i_{>\lambda}$ is the same for all $i \geq i_1$. Then for $i \geq i_1$ we have $v^i_{\leq\lambda} \xrightarrow{\lambda_i} v^{i+1}_{\leq\lambda}$ therefore $v^{i_1}_{\leq\lambda} \xrightarrow{\lambda_{i_1}} v^{i_1+1}_{\leq\lambda} \xrightarrow{\lambda_{i_1+1}} \dots$ defines a path in $U_{<\lambda+1}$. But by induction, $U_{<\lambda+1}$ satisfies $\text{MaxParity}_{\lambda+1} \subseteq \text{MaxParity}_\alpha$, so we conclude thanks to prefix-independence.

[12]Stated differently, $v \in V(U_{<\alpha})$ is given by $(v_\lambda)_{\lambda\leq\alpha, \lambda \text{ even}}$ such that $v_\lambda < \kappa$ and finitely many of the v_λ's are nonzero.

- If $\lambda = \alpha$. Let μ be the maximal element of the support of v^0; in particular, $v^0_{\geq \mu+2} = 0$. By induction we get that for all i, $v^i_{\geq \mu+2} = 0$ and $\lambda_i < \mu + 2$. But then $\lambda = \limsup_i \lambda_i < \mu + 2 < \alpha$, a contradiction.

We now let G be a graph $< \kappa$ satisfying $\mathrm{MaxParity}_\alpha$ and aim to prove that there is $v \in V(G)$ such that $G[v] \to U_{<\alpha}$. Note that for any $\lambda < \alpha$, we have $U_{<\lambda} \to U_{<\alpha}$ and therefore it suffices to find $v \in V(G)$ such that all priorities in $G[v]$ are $< \lambda$. Assume that there is no such v: for any v and any $\lambda < \alpha$ there is a path in G towards an edge with priority $> \lambda$. Then (just as in the proof of Theorem 3.2) we construct a path whose $\limsup$ is α which is a limit (and therefore even), violating the fact that G satisfies $\mathrm{MaxParity}_\alpha$. $\square$