An automata model for Borel-MSO

Antonio Casares · RPTU Kaiserslautern

Joint work with: *Mikołaj Bojańczyk, Sven Manthe and Paweł Parys*

Séminaire Move, 28 *November* 2025

Decidability results in logic

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$(S, <)$ ordered set (*or any structure*)

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Infinite word over the unary alphabet {●}

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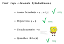

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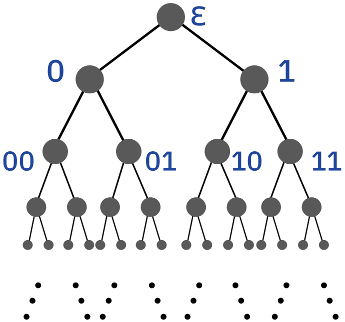

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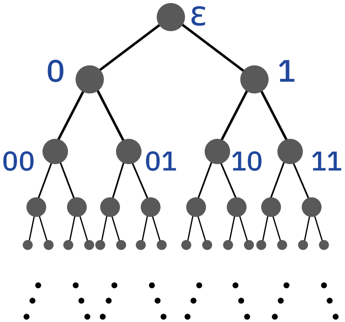

$(2^{<\omega}, \text{left}, \text{right})$

$\{0,1\}^*$

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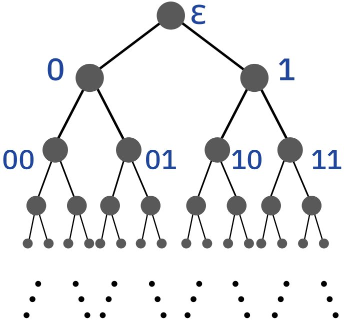

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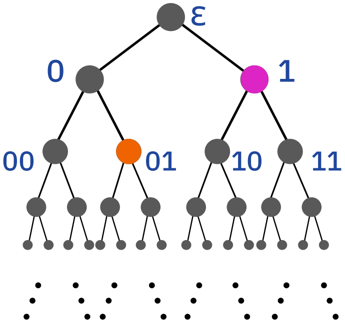

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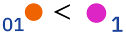

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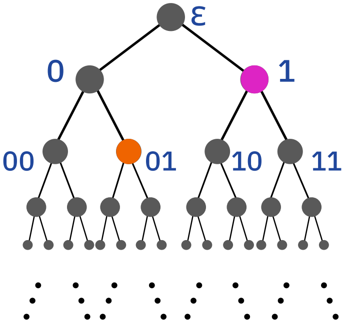

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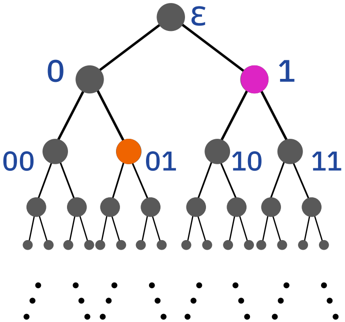

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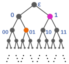

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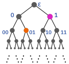

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$\mathrm{MSO}(2^{<\omega}, \text{left}, \text{right}) \longleftrightarrow$ Automata over infinite trees

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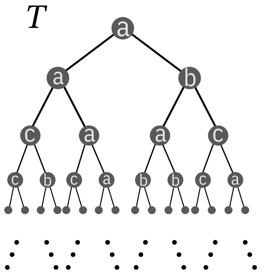

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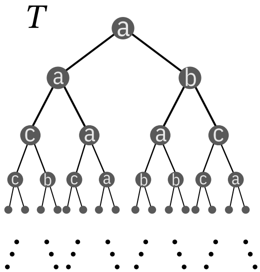
States: $Q = \{1, 2, 3, ..., n\}$



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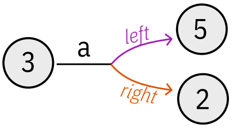

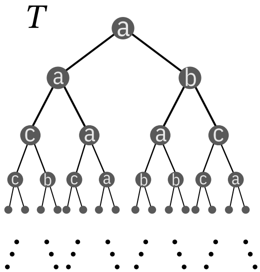

Input: A Σ-labelled tree

Output: A $[1,n]$-labelled tree

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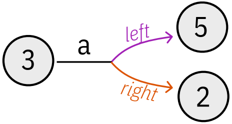

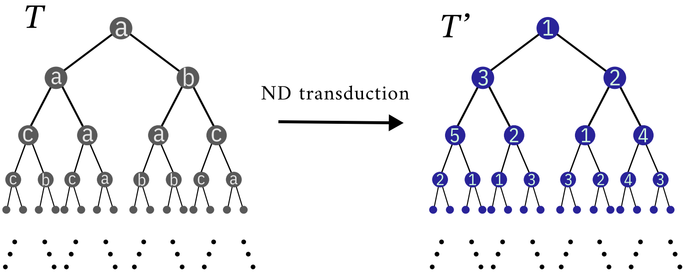

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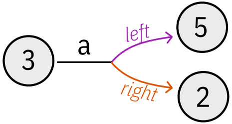

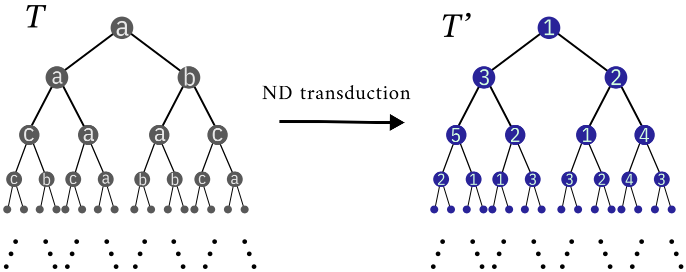

Acceptance: Parity condition

States: $Q = \{1, 2, 3, \ldots, n\}$

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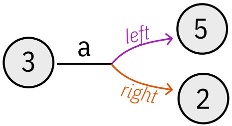

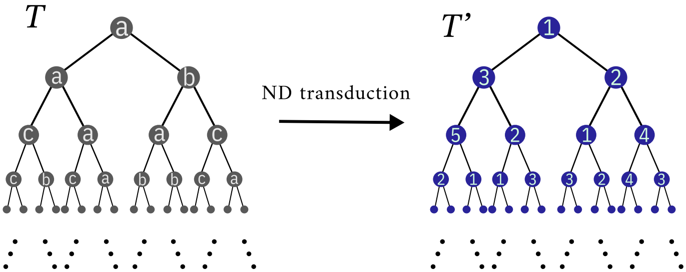

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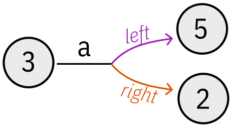
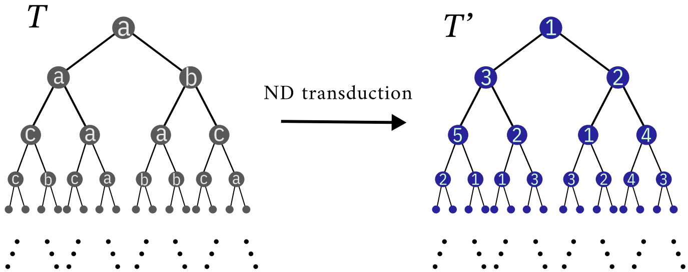
A branch π in T' is *good* if $\limsup_{i \in \pi} i$ is even

States: $Q = \{1, 2, 3, \ldots, n\}$

Transitions:



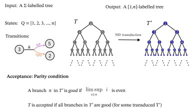


Acceptance: Parity condition

A branch π in T' is *good* if $\limsup_{i \in \pi} i$ is even

T is *accepted* if all branches in T' are good (for some transduced T')

MSO($2^{<\omega}$, left, right) $\longleftrightarrow$ Automata over infinite trees



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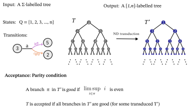

- For each sentence φ there is an automaton $\mathcal{A}_\varphi$ such that

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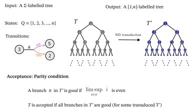

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- Emptiness of automata over infinite trees is decidable

(*Solving a parity game*)

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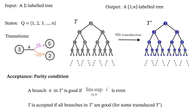

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Decidability

Proof *Logic* → *Automata* **by induction on** φ

- Atomic formulas
- Disjunctions $\varphi \lor \psi$
- Complementation $\neg \varphi$
- Quantifiers $\exists X\, \varphi(X)$

Proof ***Logic*** $\rightarrow$ ***Automata*** **by induction on φ**

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$\neg\varphi$

- Challenge: parity tree-automata do not determinize

$\neg \varphi$

- Challenge: parity tree-automata do not determinize

- Original proof by Rabin: extremely complex

TREES, AUTOMATA, AND GAMES

Yuri Gurevich and Leo Harrington

ABSTRACT. In 1969 Rabin introduced tree automata and proved one of the deepest decidability results. If you worked on decision problems you did most probably use Rabin's result. But did you make your way through Rabin's cumbersome proof with its induction on countable ordinals? Building on ideas

$\neg\,\varphi$

- Challenge: parity tree-automata do not determinize
- Original proof by Rabin: extremely complex

- Modern presentation: Use games!

We have a tree automaton

Objective: Build an automaton for the complement language

Rejection game over a given Σ-tree

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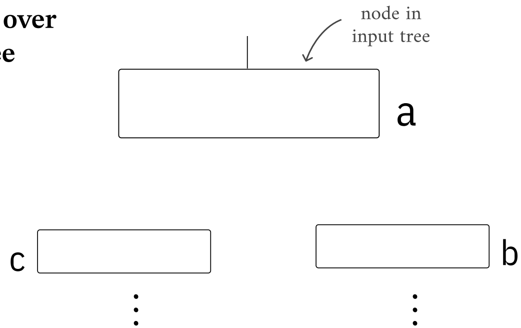

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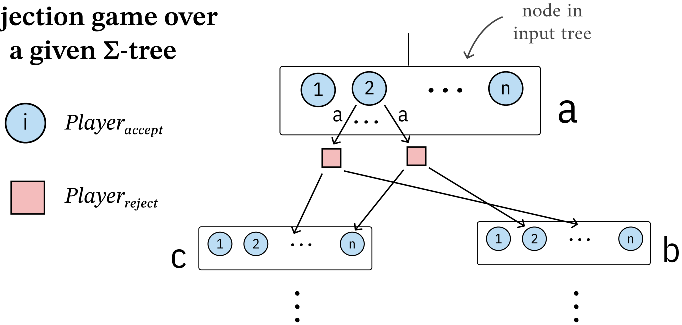

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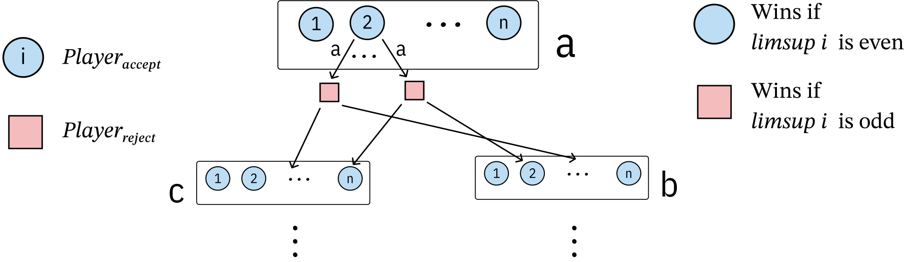

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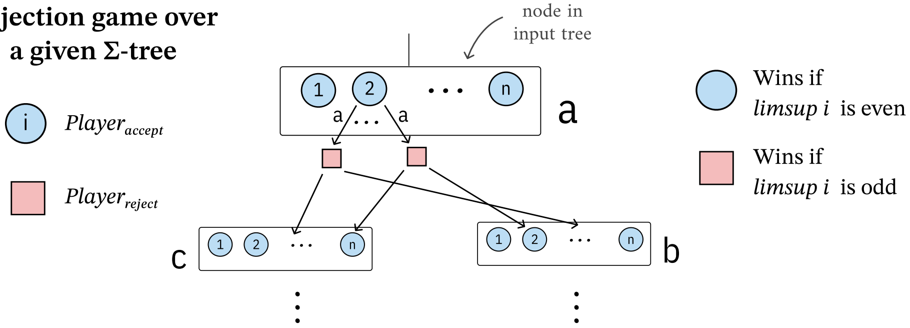
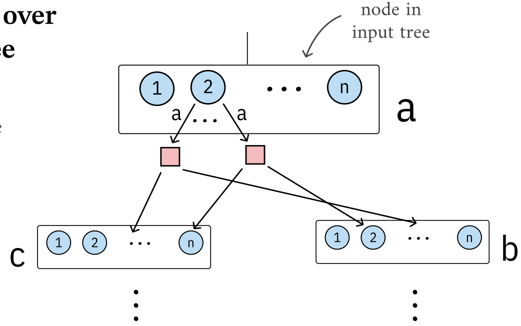


T rejected by the automaton iff P_{reject} has a winning strategy

Rejection game over a given Σ-tree



i *Player*$_{accept}$

Player$_{reject}$

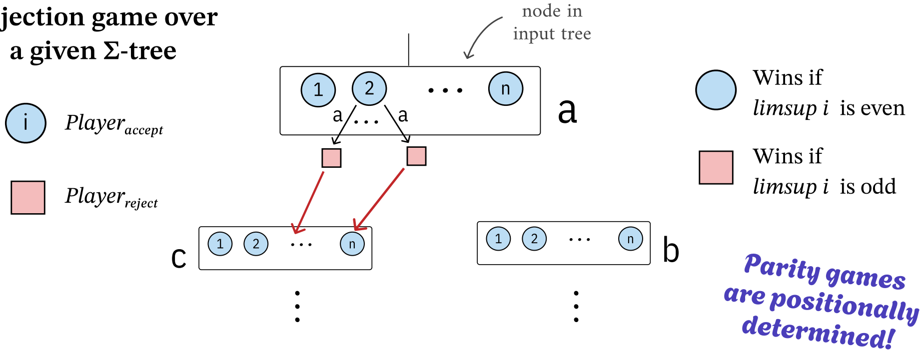
Wins if *limsup i* is even

Wins if *limsup i* is odd

Parity games are positionally determined!

T rejected by the automaton iff P$_{reject}$ *has a winning strategy*

Rejection game over a given Σ-tree



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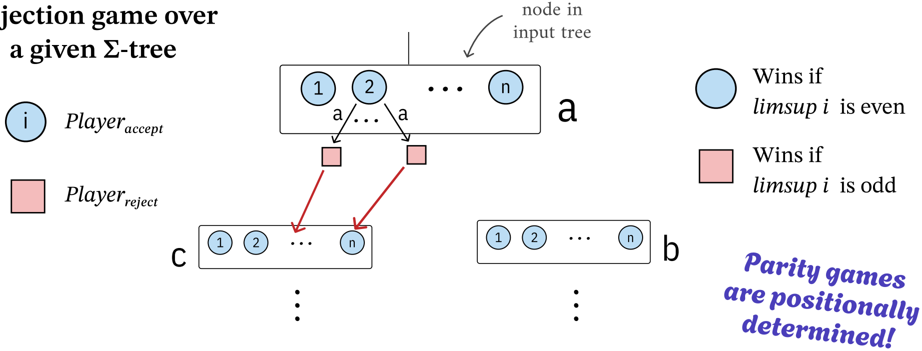

T rejected by the automaton iff P_{reject} *has a* ***positional*** *winning strategy*

$$\textit{Positional strategy} \; = \; \begin{array}{l}\textit{Labelling of the tree by functions} \\ \sigma : \mathrm{TransAut} \to \{\textit{left}, \textit{right}\}\end{array}$$

Positional strategy = *Labelling of the tree by functions* $\sigma : \mathrm{TransAut} \to \{\mathit{left}, \mathit{right}\}$

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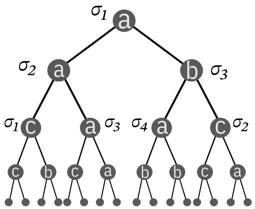

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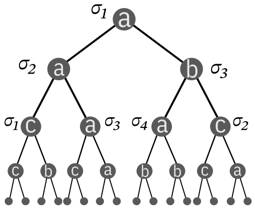

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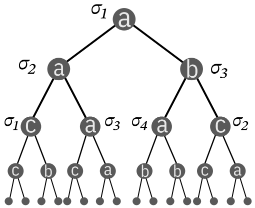

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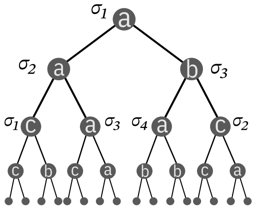

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$\downarrow$ *projection in the Σ - component*

(P_{reject} *guesses a positional strategy*)

ND automaton for $L(\mathcal{A})^c$

$\neg \varphi$

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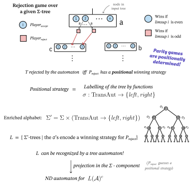

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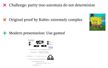

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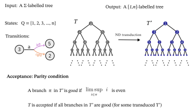

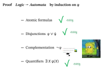

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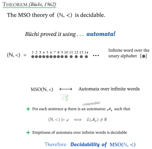
Decidability

Some classic results

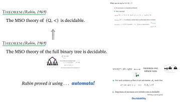
Büchi, 1962

MSO theory of $(\mathbb{N}, <)$



Rabin, 1969

MSO theory of $(\mathbb{Q}, <)$ and the full binary tree



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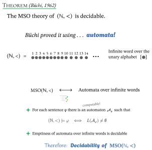
THEOREM *(Büchi, 1962)*

The MSO theory of $(\mathbb{N}, <)$ is decidable.

Büchi proved it using . . . ***automata!***

$(\mathbb{N}, <) \; = $ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 … Infinite word over the unary alphabet {●}

$\mathrm{MSO}(\mathbb{N}, <) \longleftrightarrow$ Automata over infinite words

+ For each sentence φ there is an automaton $\mathcal{A}_\varphi$ such that (computable!)

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Therefore: ***Decidability of*** $\mathrm{MSO}(\mathbb{N}, <)$

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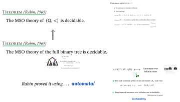
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$\Uparrow$

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Shelah, 1975

MSO theory of $(\mathbb{R}, <)$

Our methods are model-theoretic, and we do not use automaton theory.

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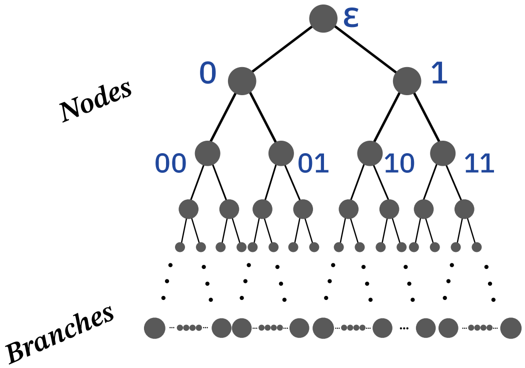
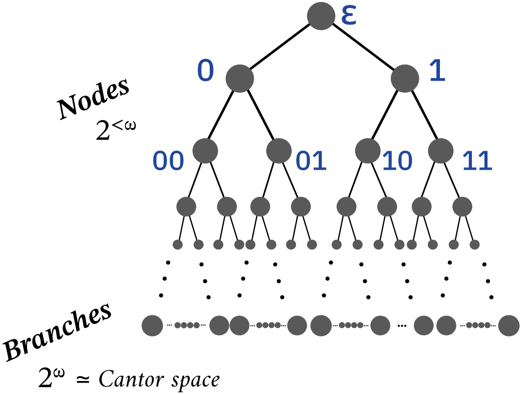
The MSO theory of the ***full binary tree with branches*** is ***undecidable***.

full binary tree with branches

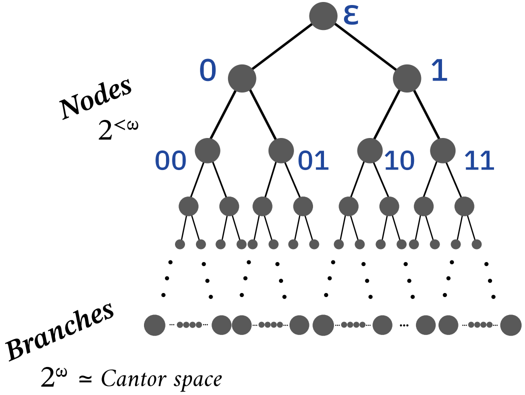


full binary tree with branches



$(2^{\leq\omega}, <, \sqsubseteq)$

full binary tree with branches



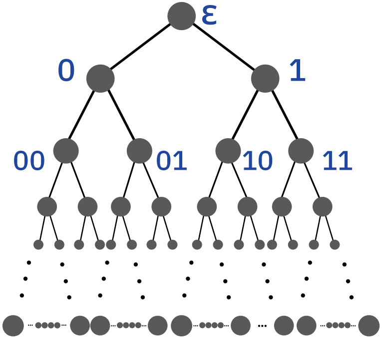
$(2^{\leq\omega}, <, \sqsubseteq)$

lexicographic order

prefix relation

(allows to say node ∈ branch)

full binary tree with branches



Nodes $2^{<\omega}$

Branches

$2^{\omega} \simeq$ *Cantor space*

$$(2^{\leq\omega}, <, \sqsubseteq)$$

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(*allows to say node* $\in$ *branch*)

$(\mathbb{R}, <) \simeq 2^{\omega} \setminus$ Eventually constant sequences

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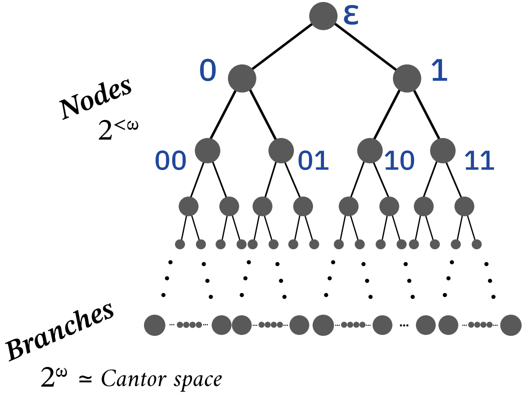

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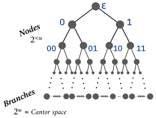

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Conjecture 1

The MSO theory of $(\mathbb{R}, <)$ is decidable if the set quantification is restricted to Borel sets.

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$$\Uparrow$$

Conjecture 2

The MSO theory of $(2^{\leq\omega}, <, \sqsubseteq)$ is decidable if the set quantification is restricted to Borel subsets of branches.

(not so)

Borel ≡ Topologically simple

(not so)

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$\mathrm{MSO}_{\mathrm{Borel}}$: $\exists X \varphi(X)$ ≡ There exists X that is Borel and φ(X)

(not so)

Borel $\equiv$ Topologically simple

$\mathrm{MSO}_{\mathrm{Borel}}$:

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$\forall X \varphi(X)$ $\equiv$ For every Borel X, φ(X)

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$\exists X \varphi(X) \equiv$ There exists X that is Borel and $\varphi(X)$

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$\mathrm{MSO}_{\mathrm{Borel}}$ theory of $(\mathbb{R}, <)$ $\longleftrightarrow$ First order theory of (Borel subsets of $\mathbb{R}$, $\subseteq$, singleton , $<$)

Why $\mathrm{MSO}_{\mathrm{Borel}}$?

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- Borel sets are very natural

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- Sentences in $\mathrm{MSO}_{\mathrm{Borel}}$ have a different (interesting) meaning!!

Perfect Set Property

Every Borel set is either countable, or contains a perfect set.

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non-empty closed without isolated points (ex: $[a,b]$)

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non-empty closed without isolated points (*ex*: [*a*,*b*])

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False in MSO (under AC)

True in $\mathrm{MSO}_{\mathrm{Borel}}$

True in MSO (under Ax. Determinacy)

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PERFECT SET PROPERTY

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You can also define Cantor sets, meager sets...

In $\mathrm{MSO}_{\mathrm{Borel}}$, $\varphi(X) = X$ is countable!!

over $2^{\leq\omega}$

$\varphi_{\mathrm{det}} =$ Every game is determined

False in MSO (under AC)

True in $\mathrm{MSO}_{\mathrm{Borel}}$

True in MSO (under Ax. Determinacy)

Conjecture 1

The MSO theory of $(\mathbb{R}, <)$ is decidable if the set quantification is restricted to Borel sets.

$$\Uparrow$$

Conjecture 2

The MSO theory of $(2^{\leq\omega}, <, \sqsubseteq)$ is decidable if the set quantification is restricted to Borel subsets of branches.

Borel $\equiv$ Topologically simple (not so)

$\mathrm{MSO}_{\mathrm{Borel}}$: $\exists X \varphi(X)$ $\equiv$ There exists X that is Borel and φ(X)

$\forall X \varphi(X)$ $\equiv$ For every Borel X, φ(X)

$\mathrm{MSO}_{\mathrm{Borel}}$ theory of $(\mathbb{R}, <)$ $\longleftrightarrow$ First order theory of (Borel subsets of $\mathbb{R}$, $\subseteq$, singleton, $<$)

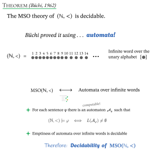
Why $\mathrm{MSO}_{\mathrm{Borel}}$?

- Borel sets are very natural
- The undecidability proof uses non-Borel, complex sets
- Sentences in $\mathrm{MSO}_{\mathrm{Borel}}$ have a different (interesting) meaning!!

Some classic results

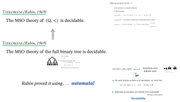
Büchi, 1962

MSO theory of $(\mathbb{N}, <)$



Rabin, 1969

MSO theory of $(\mathbb{Q}, <)$ and the full binary tree



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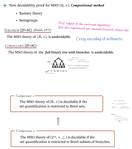

Some ~~classic~~ brand new results!

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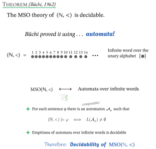

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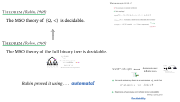

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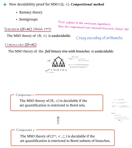

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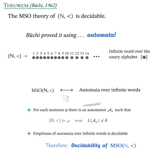

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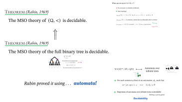

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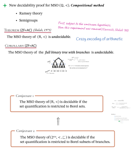

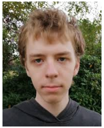

Manthe, 2024

MSO$_{\text{Borel}}$ theory of $(\mathbb{R}, <)$

THEOREM *(Manthe, 2024 , under review)*

The $\mathrm{MSO}_{\mathrm{Borel}}$ theory of $(\mathbb{R}, <)$ is decidable.

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Proof using Shelah's compositional method
$+$
Baire property of Borel sets

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Theorem *(Manthe, 2024)*

The Boolean combinations of Σ_2-sets form an elementary substructure of the Borel sets.

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✗ Highly involved

✗ Does not generalize to the binary tree with branches

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MSO theory of $(\mathbb{N}, <)$

THEOREM *(Büchi, 1962)*

The MSO theory of $(\mathbb{N}, <)$ is decidable.

Büchi proved it using . . . ***automata!***

$(\mathbb{N}, <) \;=\;$ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ● ● ● ● ● ● ● ● ● ● ● ● ● ● … Infinite word over the unary alphabet {●}

$\mathrm{MSO}(\mathbb{N}, <) \longleftrightarrow$ Automata over infinite words

+ For each sentence φ there is an automaton $\mathcal{A}_\varphi$ such that *computable!*

$$(\mathbb{N}, <) \models \varphi \iff L(\mathcal{A}_\varphi) \neq \emptyset$$

+ Emptiness of automata over infinite words is decidable

Therefore: ***Decidability of*** $\mathrm{MSO}(\mathbb{N}, <)$

Rabin, 1969

MSO theory of $(\mathbb{Q}, <)$ and the full binary tree

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Decidability

Shelah, 1975

MSO theory of $(\mathbb{R}, <)$

+ New decidability proof for MSO $(\mathbb{Q}, <)$. ***Compositional method***
  - Ramsey theory
  - Semigroups

THEOREM (ZF+AC) *(Shelah, 1975)*

The MSO theory of $(\mathbb{R}, <)$ is *undecidable.*

First, subject to the continuum hypothesis, then this requirement was removed (Gurevich, Shelah '88)

Crazy encoding of arithmetic

COROLLARY (ZF+AC)

The MSO theory of the *full binary tree with branches* is *undecidable.*

Conjecture 1

The MSO theory of $(\mathbb{R}, <)$ is decidable if the set quantification is restricted to Borel sets.

$\Uparrow$

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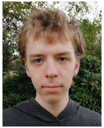

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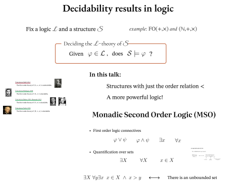
An automata model for Borel-MSO

Antonio Casares · RPTU Kaiserslautern

Joint work with: Mikołaj Bojańczyk, Sven Manthe and Paweł Parys



Séminaire Move, 28 November 2025

Goal

Show decidability of $\mathrm{MSO}_{\mathrm{Borel}}$ with automata methods.

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- Simpler proof (for automata-oriented people)

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Show decidability of $\mathrm{MSO}_{\mathrm{Borel}}$ with automata methods.

- Simpler proof (for automata-oriented people)
- Get decidability of $\mathrm{MSO}_{\mathrm{Borel}}$ over the full binary tree

Contributions

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- Introduction of Borel-tree automata

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- Introduction of prioritized Borel games

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- Introduction of prioritized Borel games

THEOREM *(Bojańczyk, C., Manthe, Parys)*

If prioritized Borel-games are finite-memory determined, then

$$\mathrm{MSO}_{\mathrm{Borel}}(2^{\leq\omega}) \equiv \mathrm{MSO}_{\mathrm{Bool}(\Sigma_2)}(2^{\leq\omega}) \equiv \text{Borel-tree-automata}$$

and this theory is decidable.

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Prioritized Borel-games are finite-memory determined.

Proof $\mathrm{MSO}_{\boldsymbol{Borel}} \to$ **Automata** **by induction on $\boldsymbol{\varphi}$**

- Atomic formulas

- Disjunctions $\varphi \vee \psi$

- Complementation $\neg \varphi$

- Quantifiers $\exists X \, \varphi(X)$

Proof $\mathrm{MSO}_{\mathit{Borel}} \to$ **Automata** **by induction on φ**

- Atomic formulas ✓ easy
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Proof $\mathrm{MSO}_{Borel} \to$ **Automata** **by induction on φ**

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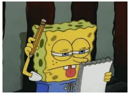
- Complementation $\neg\varphi$  Uses finite-memory of prioritized Borel games
- Quantifiers $\exists X\, \varphi(X)$ ✓ easy

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Prioritized Borel-games are finite-memory determined.

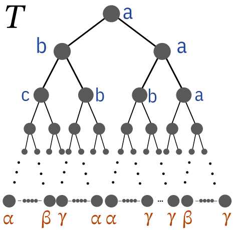
Proof in some subcases

Borel tree automata

Input:

Σ = Labels of nodes

Σ' = Labels of branches



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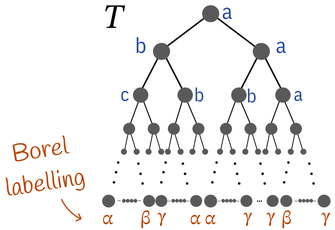

Borel tree automata

Input: Σ = Labels of nodes

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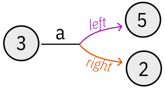
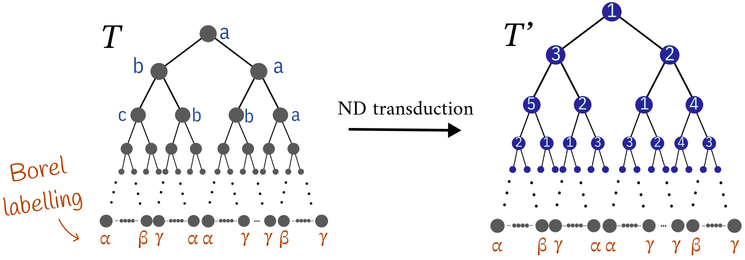
Output: A $[1,n]$-labelled tree.
Same branch labeling

States: $Q = \{1, 2, 3, ..., n\}$

Transitions:





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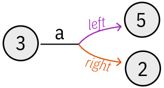

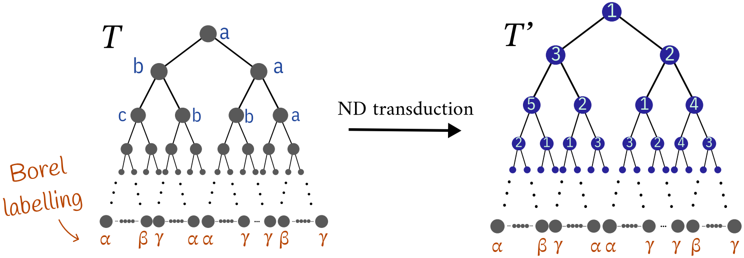

Acceptance:

For each $\alpha \in \Sigma'$

$W_\alpha \subseteq [1,n]$

Borel set

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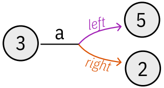

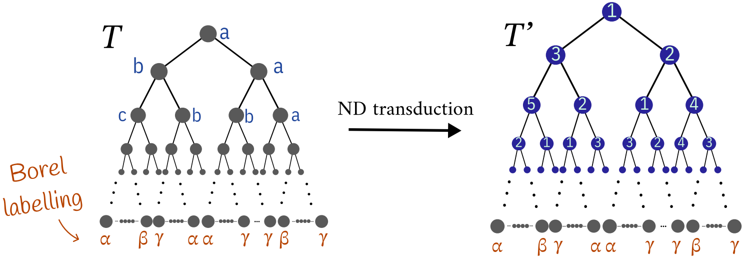

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T is *accepted* if for every branch π in T':

$$\limsup_{i \in \pi} \; i \in W_{l(\pi)}$$

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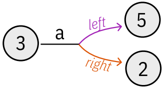

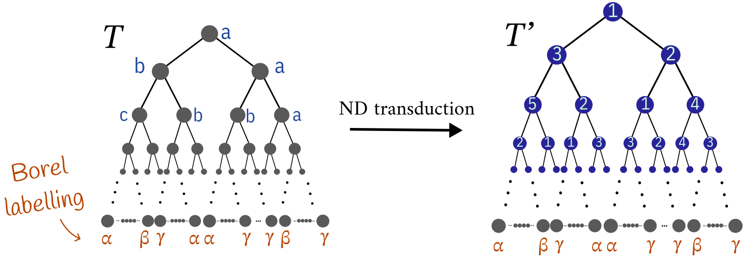

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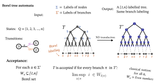
classical notion:
for all α,
W_α = *Even numbers*

Contributions



- Introduction of Borel-tree automata
- Introduction of prioritized Borel games

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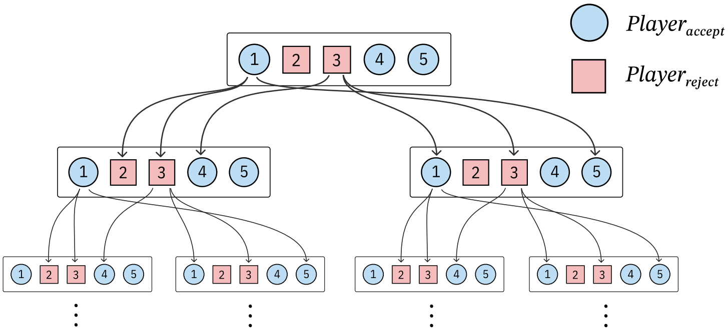
Prioritized Borel-games are finite-memory determined.

Proof in some subcases

Prioritized Borel games

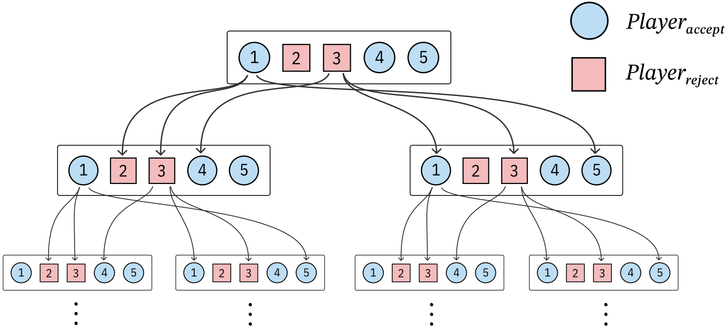


Prioritized Borel games

Winning condition

For each $i \in [1,n]$

W_i a Borel subset of branches



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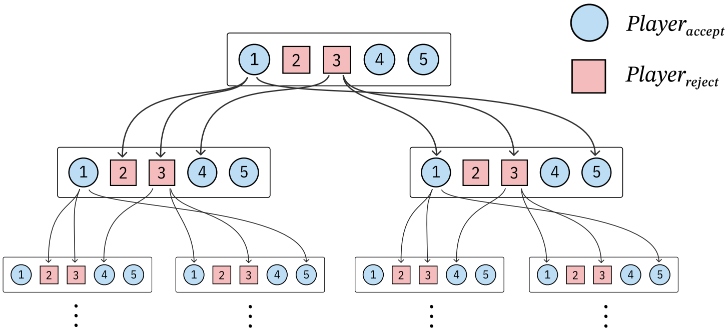
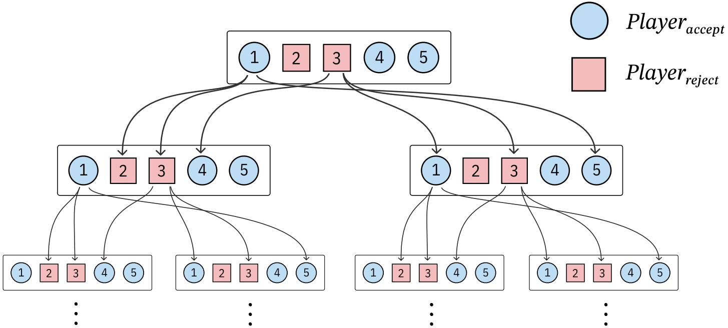


A play produces: $1342342342\ldots \in [1,n]^\omega$

and

a branch π

Prioritized Borel games



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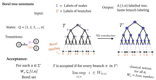
and
a branch π

Wins if $\pi \in W_i$

Contributions



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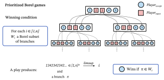

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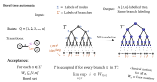
Prioritized Borel-games are finite-memory determined.

- *Proof in some subcases*

Contributions



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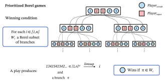

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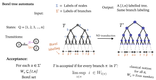
There exists a function *f(n)* such that the winner in prioritized Borel games with *n* internal nodes can win with a strategy given by an automaton of size ≤ *f(n)*.

Contributions



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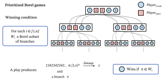

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Thanks for your attention!

An automata model for Borel-MSO

Antonio Casares · RPTU Kaiserslautern

Joint work with: Mikołaj Bojańczyk, Sven Manthe and Paweł Parys

Goal

Show decidability of $\mathrm{MSO}_{\mathrm{Borel}}$ with automata methods.

- Simpler proof (for automata-oriented people)
- Get decidability of $\mathrm{MSO}_{\mathrm{Borel}}$ over the full binary tree

Contributions

- Introduction of Borel-tree automata
- Introduction of prioritized Borel games

THEOREM *(Bojańczyk, C., Manthe, Parys)*

If prioritized Borel-games are finite-memory determined, then $\mathrm{MSO}_{\mathrm{Borel}}(2^{\leq\omega}) \equiv \mathrm{MSO}_{\mathrm{Bool}(\Sigma_2)}(2^{\leq\omega}) \equiv$ Borel-tree-automata and this theory is decidable.

CONJECTURE

Proof in some subcases

There exists a function $f(n)$ such that the winner in prioritized Borel games with n internal nodes can win with a strategy given by an automaton of size $\leq f(n)$.

Thanks for your attention!

Decidability results in logic

Fix a logic $\mathcal{L}$ and a structure $\mathcal{S}$ — *example:* FO(+,×) *and* (ℕ,+,×)

Deciding the $\mathcal{L}$-theory of $\mathcal{S}$: Given $\varphi \in \mathcal{L}$, does $\mathcal{S} \models \varphi$?

In this talk:

- Structures with just the order relation $<$
- A more powerful logic!

Monadic Second Order Logic (MSO)

- First order logic connectives: $\varphi \vee \psi$ $\quad \varphi \wedge \psi$ $\quad \exists x$ $\quad \forall x$
- Quantification over sets: $\exists X$ $\quad \forall X$ $\quad x \in X$

$\exists X\ \forall y \exists x\ x \in X \wedge x > y \quad \longleftrightarrow \quad$ There is an unbounded set

Some ~~classic~~ brand new results!

Büchi, 1962

MSO theory of (ℕ, <)

Rabin, 1969

MSO theory of (ℚ, <) and the full binary tree

Shelah, 1975

MSO theory of (ℝ, <)

Manthe, 2024

$\mathrm{MSO}_{\mathrm{Borel}}$ theory of (ℝ, <)

Séminaire Move, 28 November 2025