Chapter 3

Positionality and Memory

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In this chapter, we study the subject of *strategy complexity* for games, focusing on the notion of positionality. We have already shown positionality of some objectives, such as `Reach` or `Parity`. The objective of this chapter is twofold. First, we provide tools to show positionality of objectives, either over finite or over infinite arenas; in particular, we establish the positionality of many objectives from other chapters. Second, we aim to provide a comprehensive study of the subject of positionality in itself, establishing both necessary and sufficient conditions for an arbitrary objective to be (bi-)positional.

The chapter is organised as follows.

- Section 3.1 focuses on sufficient conditions for (bi-)positionality over finite arenas. One of the main results presented is a *one-to-two-player lift*: in order to verify whether an objective is bi-positional over finite arenas, it suffices to check whether both players can play optimally in a positional way in finite one-player arenas. As a consequence of the results from this section, we establish the positionality over finite arenas of many of the classical objectives defined in **??**, such as `Parity`, `Rabin`, `MeanPayoff`, `Energy`, and `TotalPayoff`.
- In Section 3.2, we provide a characterisation of bi-positional qualitative objectives over infinite arenas; in the case of prefix independent objectives, these exactly correspond to parity objectives. As a corollary, we obtain a one-to-two-player lift over infinite arenas. The techniques presented in this section focus on necessary conditions for positionality.
- In Section 3.3, we discuss several natural definitions of positionality in the case of quantitative objectives. We show that they yield equivalent notions over finite arenas, and show how to reduce their study to the case of qualitative objectives.
- In Section 3.4, we focus on the study of positionality over infinite arenas, and present a characterisation of it via *monotone universal graphs*. As a result, we es-

tablish the positionality over infinite arenas of objectives such as $\mathtt{MeanPayoff}^{-}_{>0}$ and $\mathtt{DiscountedPayoff}_{\lambda}$. In Section 3.4.5, we discuss Kopczyński's conjecture about the closure under union of prefix independent positional objectives, and we present in Section 3.4.6 results about positionality for the class of ω-regular objectives.

- In Section 3.5, we introduce several models of memories for games and provide examples separating the different notions. We discuss how the results from the previous sections can be generalised to strategies with memory.

The first four sections of this chapter can be read independently, and all give various kinds of tools to better approach positionality in different contexts.

3.1 Fundamental positionality results over finite arenas

We aim here to understand some common underlying properties of objectives that make them (bi-)positional over finite arenas. Results from this section are meant to be widely applicable and easy to use. We will focus on two results, which are sufficient to prove (bi-)positionality of most standard qualitative and quantitative objectives, such as the ones defined in **??** that satisfy this property.

The first result we present is a sufficient (but not necessary) property for uniform positionality over finite arenas of prefix independent objectives. The second one is a characterisation of uniform bi-positionality over finite arenas, reducing the problem to the simpler study of *one-player* arenas.

3.1.1 Positionality of submixing objectives

We introduce one property of objectives which, along with prefix independence, entails uniform positionality.

Definition 1 (Submixing objective)**.** *An objective $\Phi \colon C^{\omega} \to \mathbb{R} \cup \{\pm\infty\}$ is submixing if for all sequences of non-empty finite words $\rho_1^0, \rho_1^1, \ldots \in C^+$ and $\rho_2^0, \rho_2^1, \ldots \in C^+$, we have*

$$\Phi(\rho_1^0 \rho_2^0 \rho_1^1 \rho_2^1 \ldots) \leq \max\left\{\Phi(\rho_1^0 \rho_1^1 \ldots),\ \Phi(\rho_2^0 \rho_2^1 \ldots)\right\}.$$

The submixing property means that two infinite words cannot be combined into a better infinite word (where "better" is with respect to the objective and to player Max) by intertwining them. The submixing property can hold for an objective Φ but not its opposite $-\Phi$. It can be rephrased as follows for qualitative objectives $\Omega \subseteq C^{\omega}$: if $\rho_1^0 \rho_1^1 \ldots \notin \Omega$ and $\rho_2^0 \rho_2^1 \ldots \notin \Omega$, then $\rho_1^0 \rho_2^0 \rho_1^1 \rho_2^1 \ldots \notin \Omega$. Observe that it corresponds to the property of Rabin objectives highlighted in (**??**) (**??**).

In the upcoming result, we also assume determinacy of the objective, which is a reasonable requirement thanks to **??**.

Theorem 1 (Prefix independent and submixing imply uniform positionality)**.** *Every determined, prefix independent, and submixing objective is uniformly positional over finite arenas.*

Proof. Let $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ be a determined, prefix independent, and submixing quantitative objective. We show the result by induction over the following quantity of finite arenas: the total outdegree of vertices controlled by Max, minus the number of vertices controlled by Max. For an arena $\mathscr{A} = (G, V_{\mathrm{Max}}, V_{\mathrm{Min}})$, this is the quantity $n(\mathscr{A}) = \left(\sum_{v \in V_{\mathrm{Max}}} d(v)\right) - |V_{\mathrm{Max}}|$, where $d(v)$ is the outdegree of vertex v. Our base case occurs when $n(\mathscr{A})$ is 0. Since we assume that every vertex has an outgoing edge, the base case corresponds to the situation in which each vertex of Max has exactly one outgoing edge. In this case, Max has only one strategy, which is positional and optimal.

We move on to the inductive step. Let $\mathscr{G} = (\mathscr{A}, \Phi(\mathfrak{c}))$ be a game on a finite arena $\mathscr{A}$ such that $n(\mathscr{A}) > 0$. Hence, there is an element $v \in V_{\mathrm{Max}}$ with outdegree at least 2; we pick such a v. We partition the outgoing edges of v in two non-empty subsets which we call E_1^v and E_2^v. We define two games $\mathscr{G}_1$ and $\mathscr{G}_2$: the game $\mathscr{G}_1$ is obtained from $\mathscr{G}$ by removing the edges from E_2^v, and symmetrically for $\mathscr{G}_2$. For $i \in \{1,2\}$, if $\mathscr{A}_i$ is the arena on which $\mathscr{G}_i$ is played, observe that $n(\mathscr{A}_i) < n(\mathscr{A})$.

W.l.o.g., we assume that $\mathrm{val}^{\mathscr{G}_1}(v) \geq \mathrm{val}^{\mathscr{G}_2}(v)$. By induction hypothesis, we take σ_1 a positional optimal strategy of Max in $\mathscr{G}_1$. Observe that $\mathrm{val}^{\mathscr{G}_1,\sigma_1}(v) = \mathrm{val}^{\mathscr{G}_1}(v)$ (by optimality of σ_1) and that $\mathrm{val}^{\mathscr{G}_1}(v) \leq \mathrm{val}^{\mathscr{G}}(v)$ (as every strategy of Max in $\mathscr{G}_1$ can be seen as a strategy in $\mathscr{G}$, and Min has no additional power in $\mathscr{G}$).[1] We show that we even have $\mathrm{val}^{\mathscr{G}_1}(v) = \mathrm{val}^{\mathscr{G}}(v)$; to do so, it is left to show that $\mathrm{val}^{\mathscr{G}}(v) \leq \mathrm{val}^{\mathscr{G}_1}(v)$. This implies that the positional strategy σ_1, seen as a strategy on $\mathscr{G}$, is also optimal from v in $\mathscr{G}$.

For some $\varepsilon > 0$, let τ_1 and τ_2 be strategies of Min ensuring a payoff of at most $\mathrm{val}^{\mathscr{G}_1}(v) + \varepsilon$ from v respectively in $\mathscr{G}_1$ and $\mathscr{G}_2$ (τ_1 and τ_2 exist as we assumed that Φ is determined, and that $\mathrm{val}^{\mathscr{G}_2}(v) \leq \mathrm{val}^{\mathscr{G}_1}(v)$). We build a strategy τ of Min in $\mathscr{G}$ (defined only from v) based on τ_1 and τ_2. It has two modes, 1 and 2. The strategy simulates τ_1 from the mode 1 and τ_2 from the mode 2. Whenever v is visited, the mode is updated by observing which outgoing edge is taken by Max: if the outgoing edge is in E_1^v, the new mode is 1; otherwise, it is 2. When simulating τ_1, we ignore the parts of the play using mode 2, so removing them yields a play consistent with τ_1. The same goes for τ_2.

Consider a play π from v consistent with τ. It can be decomposed following which mode the play is in. Play π looks like

$$\begin{array}{lccccc}
\text{mode 1} & \overbrace{v\dots v}^{\pi_1^0} & & \overbrace{v\dots v}^{\pi_1^1} & & \cdots \\
\text{mode 2} & & \underbrace{v\dots v}_{\pi_2^0} & & \underbrace{v\dots v}_{\pi_2^1} & \cdots
\end{array}$$

where $\pi_1 = \pi_1^0 \pi_1^1 \dots$ is consistent with τ_1 and $\pi_2 = \pi_2^0 \pi_2^1 \dots$ is consistent with τ_2 (in the graphical representation above, we assumed that Max starts with an edge in E_1^v, but this has no bearing on the proof).

There are two cases: either exactly one of π_1 and π_2 is finite, or π_1 and π_2 are both infinite. The latter case happens if and only if v is visited infinitely often along π and edges from both E_1^v and E_2^v are taken infinitely often.

[1] For convenience, we will sometimes consider the same positional strategy (here, σ_1) in different games over the same state space. We therefore extend notation val and write e.g. $\mathrm{val}^{\mathscr{G}_1,\sigma_1}$ to avoid any ambiguity on the game being considered.

In the former case, if (w.l.o.g.) π_1 is finite, then we have

$$\begin{aligned}\Phi(\pi) &= \Phi(\pi_2) && \text{by prefix independence of } \Phi\\ &\leq \mathrm{val}^{\mathcal{G}_1}(v)+\varepsilon && \text{as } \pi_2 \text{ is consistent with } \tau_2.\end{aligned}$$

If π_1 and π_2 are both infinite, then we have

$$\begin{aligned}\Phi(\pi) &\leq \max\{\Phi(\pi_1),\Phi(\pi_2)\} && \text{as } \Phi \text{ is submixing}\\ &\leq \mathrm{val}^{\mathcal{G}_1}(v)+\varepsilon && \text{as } \pi_1 \text{ (resp. } \pi_2\text{) is consistent with } \tau_1 \text{ (resp. } \tau_2\text{)}.\end{aligned}$$

Therefore, $\mathrm{val}^{\mathcal{G}}(v) \leq \mathrm{val}^{\mathcal{G}_1}(v)+\varepsilon$ for all $\varepsilon > 0$, so $\mathrm{val}^{\mathcal{G}}(v) \leq \mathrm{val}^{\mathcal{G}_1}(v)$. This ends the proof that σ_1 is optimal from v in $\mathcal{G}$.

We now show that σ_1 is optimal from any other vertex as well. Let $v_0 \in V$ be an initial vertex. We again have that $\mathrm{val}^{\mathcal{G}_1}(v_0) \leq \mathrm{val}^{\mathcal{G}}(v_0)$ by construction; to get the equality, we show that $\mathrm{val}^{\mathcal{G}}(v_0) \leq \mathrm{val}^{\mathcal{G}_1}(v_0)$. As $\mathrm{val}^{\mathcal{G}_1}(v_0) = \mathrm{val}^{\mathcal{G}_1,\sigma_1}(v_0) = \mathrm{val}^{\mathcal{G},\sigma_1}(v_0)$, this will show that σ_1 is also optimal from v_0 in $\mathcal{G}$.

Let σ be any strategy of Max in $\mathcal{G}$. We modify σ into another strategy $f(\sigma)$ in the following way: on every finite play π, $f(\sigma)$ plays as σ if v is not (yet) visited along π, and plays as σ_1 if v was already visited. We have that $\mathrm{val}^{\mathcal{G},\sigma}(v_0) \leq \mathrm{val}^{\mathcal{G},f(\sigma)}(v_0)$. Indeed, any path consistent with $f(\sigma)$ that does not go through v is also consistent with σ, and switching to σ_1 when in v guarantees a value at least as high (by optimality of σ_1 from v, and using prefix independence to ignore the path from v_0 to v). Observe that $f(\sigma)$ can be seen as a strategy in $\mathcal{G}_1$, as no edge of E_2^v is ever taken. Hence,

$$\mathrm{val}^{\mathcal{G}}(v_0) = \sup_\sigma \mathrm{val}^{\mathcal{G},\sigma}(v_0) \leq \sup_\sigma \mathrm{val}^{\mathcal{G},f(\sigma)}(v_0) = \sup_\sigma \mathrm{val}^{\mathcal{G}_1,f(\sigma)}(v_0) \leq \mathrm{val}^{\mathcal{G}_1}(v_0).$$

This ends the proof that the positional strategy σ_1 is optimal in $\mathcal{G}$. □

We now give a few examples of objectives to which this result provides an easy proof of positionality.

Theorem 2. *The* `Buchi`, `CoBuchi`, `Parity`, `Rabin`, `MeanPayoff`$^+$, *and* `Energy`$_{\geq +\infty}$ *objectives are all prefix independent and submixing. In particular, thanks to Theorem 1, they are uniformly positional over finite arenas.*

Partial proof. The fact that the Rabin objective is prefix independent and submixing was already discussed as part of the proof of **??**.

We now give arguments for the `MeanPayoff`$^+$ objective. Prefix independence of the `MeanPayoff`$^+$ objective will be discussed again in **??**; we focus here on showing that it is submixing. Let $\rho_1^0, \rho_1^1, \ldots$ and $\rho_2^0, \rho_2^1, \ldots$ be sequences of words in $\mathbb{Z}^+$. Let $\rho_1 = \rho_1^0\rho_1^1\ldots$, $\rho_2 = \rho_2^0\rho_2^1\ldots$, and $\rho = \rho_1^0\rho_2^0\rho_1^1\rho_2^1\ldots$. We rename each colour sequentially in all three sequences as $\rho = c^0c^1\ldots$, $\rho_1 = c_1^0c_1^1\ldots$, and $\rho_2 = c_2^0c_2^1\ldots$, where all c^i, c_1^i, and c_2^i are in $\mathbb{Z}$. For $k \in \mathbb{N}$ and $j \in \{1,2\}$, let n_j^k be the number of indices originating from ρ_j among the first k colours of ρ. Observe that $n_1^k + n_2^k = k$ for all $k \in \mathbb{N}$.

For $k \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{1}{k}\sum_{i=0}^{k-1} c^i &= \frac{1}{k}\sum_{i=0}^{n_1^k} c_1^i + \frac{1}{k}\sum_{i=0}^{n_2^k} c_2^i \\
&= \frac{n_1^k}{k}\left(\frac{1}{n_1^k}\sum_{i=0}^{n_1^k} c_1^i\right) + \frac{n_2^k}{k}\left(\frac{1}{n_2^k}\sum_{i=0}^{n_2^k} c_2^i\right) \\
&\leq \max\left\{\frac{1}{n_1^k}\sum_{i=0}^{n_1^k} c_1^i, \frac{1}{n_2^k}\sum_{i=0}^{n_2^k} c_2^i\right\},
\end{aligned}$$

where the last inequality holds because the previous line is a convex combination of the two values. Taking the $\limsup$ as $k \to \infty$ of the first and the last expression, we obtain $\texttt{MeanPayoff}^+(\rho) \leq \max\{\texttt{MeanPayoff}^+(\rho_1), \texttt{MeanPayoff}^+(\rho_2)\}$. $\square$

Note that `Buchi`, `CoBuchi`, `Parity`, and `Rabin` objectives are even positional over *infinite* arenas; this was shown using more specific techniques in **??**. However, positionality over infinite arenas does not hold for the $\texttt{MeanPayoff}^+$ objective (see [Put05, Example 8.10.2]). In particular, there are prefix independent submixing objectives that are not positional over *infinite* arenas. There is therefore no hope of extending Theorem 1 to infinite arenas as is.

Perhaps surprisingly, the dual mean payoff objective $\texttt{MeanPayoff}^-$ is not submixing (even if it is uniformly positional over finite arenas, as we will see later in Theorem 4). To see it, take a sequence in $\{0,1\}^\omega$ whose mean payoff oscillates between values close to 0 and 1: a way to build such a sequence is to have long stretches with only 0's or 1's, the lengths of which increase sufficiently fast. Its $\texttt{MeanPayoff}^-$ is 0. Now if you take the "opposite" sequence (inverting 1's and 0's), you get a sequence that also has a $\texttt{MeanPayoff}^-$ of 0, but that can be intertwined with the first into the sequence $(10)^\omega$, which has a better $\texttt{MeanPayoff}^-$ of $\frac{1}{2}$. Hence, $\texttt{MeanPayoff}^-$ is not submixing. In the above proof for $\texttt{MeanPayoff}^+$, the argument that would fail for $\texttt{MeanPayoff}^-$ is the application of $\liminf$ over max (unlike for $\limsup$, it does not hold that $\liminf_k \max\{a_k, b_k\} = \max\{\liminf_k a_k, \liminf_k b_k\}$ in general).

3.1.2 Reduction of bi-positionality to one-player arenas

We show a second useful tool to establish the existence of positional optimal strategies. This time, the result is about uniform *bi-positionality* (i.e., positionality of both an objective Φ and its inverse $-\Phi$). It reduces uniform bi-positionality over finite arenas to uniform bi-positionality over finite *one-player* arenas of both players. A *one-player arena* is simply an arena in which the same player controls all vertices. There are two kinds: an arena $(G, V_{\text{Max}}, V_{\text{Min}})$ is a one-player arena of Max if $V_{\text{Min}} = \emptyset$, and is a one-player arena of Min if $V_{\text{Max}} = \emptyset$.

Theorem 3 (One-to-two-player lift over finite arenas)**.** *Every objective that is uniformly bi-positional over finite one-player arenas of both players is uniformly bi-positional over all finite arenas.*

We emphasise that the hypothesis of this result actually makes a requirement about both players: it requires that in all finite one-player arenas of both Eve/Max and Adam/Min, there exists a positional optimal strategy. Having this hypothesis about both players is necessary to obtain a general result: there exist objectives for which Max (but not Min!) has positional optimal strategies in her finite one-player arenas, but for which memory is required in some finite two-player arenas (see, e.g., the example in [Kop06, Proposition 2]). Note that unlike for Theorem 1, the proof of Theorem 3 will not use the determinacy of the objective.

The point of Theorem 3 is that one-player arenas are usually easier to reason with: they are essentially graphs, in which no quantification over strategies of the opponent must be made. A positional strategy from a given vertex in a finite one-player arena always induces an ultimately periodic play $\pi_1\pi_2^\omega$ (sometimes called a "lasso") in which π_1 is a path and π_2 is a cycle, and both are *simple* in the graph-theoretic sense (they do not go through the same vertex twice). Proving that an objective is bi-positional over finite one-player arenas therefore reduces to showing that whenever there is an arbitrarily complex winning play for a player, there is also a winning play that is a "simple lasso" as described above. Be careful that the hypothesis requires *uniform* bi-positionality: uniformity requires no additional argument for prefix independent objectives thanks to **??**, but needs to be shown in general.

The proof of Theorem 3 we will present shares many similarities with the one of Theorem 1: it also proceeds by induction on the number of outgoing edges of arenas, and the construction of a strategy of the opponent will be reminiscent of the previous proof. However, it is more involved; one reason is that, in a given arena, we will need to show the existence of a positional optimal strategy for both players (not only for Max). To this end, we first prove a sufficient condition for the optimality of two strategies.

Let $\mathscr{G} = (\mathscr{A}, f)$ be a quantitative game and v be a vertex of $\mathscr{A}$. We say that a strategy σ of Max is a *best response* to a strategy τ of Min from v if $\mathrm{val}^\tau(v) = f(\pi^v_{\sigma,\tau})$. This means that σ attains the supremum in the expression $\sup_{\sigma'} f(\pi^v_{\sigma',\tau})$. We can define symmetrically a best response of Min to a strategy σ of Max. A *pair of best responses* from a vertex v is a pair of strategies (σ,τ) such that σ is a best response of Max to τ from v and τ is a best response of Min to σ from v. We prove two properties of pairs of best responses: (i) strategies that are part of such a pair are optimal, and (ii) if strategies σ of Max and τ of Min are each part of a pair of best responses, then (σ,τ) is also a pair of best responses. This notion of *best response* will be reused in a more general (non-zero-sum) setting in **??**, but the following two lemmas need the zero-sum assumption.

Lemma 1. *Let (σ,τ) be a pair of best responses from a vertex v. Then, σ and τ are both optimal from v.*

Proof. We prove that σ is optimal from v; the proof for τ is symmetric. Let σ' be any strategy of Max. We show that $\mathrm{val}^{\sigma}(v) \geq \mathrm{val}^{\sigma'}(v)$. Indeed, we have

$$\begin{aligned}
\mathrm{val}^{\sigma}(v) &= \inf_{\tau'} f(\pi^{v}_{\sigma,\tau'}) \\
&= f(\pi^{v}_{\sigma,\tau}) && \text{as } \tau \text{ is a best response to } \sigma \text{ from } v \\
&\geq f(\pi^{v}_{\sigma',\tau}) && \text{as } \sigma \text{ is a best response to } \tau \text{ from } v \\
&\geq \inf_{\tau'} f(\pi^{v}_{\sigma',\tau'}) \\
&= \mathrm{val}^{\sigma'}(v).
\end{aligned}$$

This shows the optimality of σ from v. □

Lemma 2. *Let (σ, τ') and (σ', τ) be two pairs of best responses from a vertex v. Then, (σ, τ) is also a pair of best responses from v. In other words, the set of pairs of best responses is a Cartesian product.*

Proof. We prove that σ is a best response to τ from v; the other direction is symmetric. Let σ'' be any strategy of Max. We show that $f(\pi^{v}_{\sigma'',\tau}) \leq f(\pi^{v}_{\sigma,\tau})$. Indeed, we have

$$\begin{aligned}
f(\pi^{v}_{\sigma'',\tau}) &\leq f(\pi^{v}_{\sigma',\tau}) && \text{as } \sigma' \text{ is a best response to } \tau \\
&\leq f(\pi^{v}_{\sigma',\tau'}) && \text{as } \tau \text{ is a best response to } \sigma' \\
&\leq f(\pi^{v}_{\sigma,\tau'}) && \text{as } \sigma \text{ is a best response to } \tau' \\
&\leq f(\pi^{v}_{\sigma,\tau}) && \text{as } \tau' \text{ is a best response to } \sigma.
\end{aligned}$$

This shows that σ is a best response to τ from v. □

We can now prove Theorem 3.

Proof of Theorem 3. Let $\Phi\colon C^{\omega} \to \mathbb{R} \cup \{\pm\infty\}$ be a quantitative objective that is uniformly bi-positional over finite one-player arenas of both players. Our proof shows that for every arena, there is a pair of positional strategies (σ, τ) that is a pair of best responses from every vertex. In particular, we obtain that σ and τ are positional optimal strategies using Lemma 1, which is what we want to show.

Let $\mathscr{G} = (\mathscr{A}, \Phi(\mathfrak{c}))$ be a game played on the finite arena $\mathscr{A} = (G, V_{\mathrm{Max}}, V_{\mathrm{Min}})$, where $V = V_{\mathrm{Max}} \cup V_{\mathrm{Min}}$. We proceed by induction on the following quantity of arenas: $n'(\mathscr{A}) = (\sum_{v \in V} d(v)) - |V|$, where $d(v)$ is the outdegree of vertex v. As every vertex has at least one outgoing edge, the equality $n'(\mathscr{A}) = 0$ means that every vertex has exactly one outgoing edge. In this case, both players have only one strategy (which is positional), so this pair of positional strategies is a pair of best responses. This constitutes the base case of the induction.

We now assume that $n'(\mathscr{A}) > 0$. We need to show the existence of a pair of positional strategies (σ, τ) that is a pair of best responses. We will focus on player Max and prove the following claim: if there is a vertex $v \in V_{\mathrm{Max}}$ with at least two outgoing edges, then there exists a pair of best responses (σ, τ') such that σ (but not necessarily τ') is positional. A symmetric argument would show that if there is a vertex $v \in V_{\mathrm{Min}}$ with at least two outgoing edges, then there is a pair of best responses (σ', τ) with τ positional. We show that this suffices to conclude. There are two cases to consider.

- If there exists a vertex with two outgoing edges both in V_{Max} and in V_{Min}, then there is a pair of best responses (σ, τ') with σ positional and there is a pair of best responses (σ', τ) with τ positional. We conclude using Lemma 2 that (σ, τ) is a pair of best responses where both σ and τ are positional.

- If V_{Min} contains no such vertex, as $n'(\mathscr{A}) > 0$, then V_{Max} contains one. So there is a pair of best responses (σ, τ') such that σ is positional. Observe that when V_{Min} contains no such vertex, Min has a single strategy which is positional, so τ' is necessarily positional in the above pair. The case "V_{Max} contains no such vertex" is symmetric.

The rest of the proof is devoted to the existence of a pair of best responses (σ, τ') such that σ is positional when there is a vertex in V_{Max} with at least two outgoing edges.

We pick such a $v \in V_{\text{Max}}$ with at least two outgoing edges. We partition the outgoing edges of v in two non-empty subsets which we call E_1^v and E_2^v. We define two games $\mathscr{G}_1$ and $\mathscr{G}_2$: the game $\mathscr{G}_1$ is obtained from $\mathscr{G}$ by removing the edges from E_2^v, and symmetrically for $\mathscr{G}_2$. For $i \in \{1,2\}$, setting $\mathscr{A}_i$ as the arena on which $\mathscr{G}_i$ is played, observe that $n'(\mathscr{A}_i) < n'(\mathscr{A})$.

Using the induction hypothesis, we obtain the existence of two pairs of positional best responses: (σ_1, τ_1) in $\mathscr{G}_1$ and (σ_2, τ_2) in $\mathscr{G}_2$. We will show that either σ_1 or σ_2, seen as strategies on $\mathscr{G}$, is optimal in $\mathscr{G}$. To do so, we construct a *one-player* arena $\widehat{\mathscr{A}}$ of Max using $\mathscr{A}$, τ_1, and τ_2 to which we will apply our hypothesis.

We define $\widehat{\mathscr{A}}$ as follows: we make two copies of $\mathscr{A}$ (one labelled 1 and the other labelled 2), which we merge on vertex v. This means that every vertex u of $\mathscr{A}$ has two copies called u_1 and u_2, except for v which has a single (unlabelled) copy. Edges in E_1^v are directed towards their original target in copy 1, while edges in E_2^v are directed towards copy 2. We give the control of all vertices to Max, but we only keep a single outgoing edge of vertices previously controlled by Min: in copy 1, we follow strategy τ_1, and in copy 2, we follow strategy τ_2. We illustrate this construction in Figure 3.1.

Arena $\widehat{\mathscr{A}}$ is a one-player arena of Max, and therefore (by hypothesis) Max has a positional optimal strategy $\widehat{\sigma}$ in $\widehat{\mathscr{A}}$. We are first interested in the side (1 or 2) that $\widehat{\sigma}$ picks in v. We assume w.l.o.g. that $\widehat{\sigma}$ picks an edge toward copy 1 (i.e., that $\widehat{\sigma}(v) \in E_1^v$). From this, we show that the positional strategy σ_1 is actually part of a pair of best responses in game $\mathscr{G}$.

We build a (non-necessarily positional) strategy τ of Min on $\mathscr{G}$ such that (σ_1, τ) is a pair of best responses from every vertex. The construction is very similar to the one in Theorem 1, although we need to be more careful with the initialisation. Strategy τ has two modes, 1 and 2. The initial one is chosen as 1 (this is not arbitrary: recall that side 1 was preferred by Max in arena $\widehat{\mathscr{A}}$ through strategy $\widehat{\sigma}$). Whenever v is visited, the mode is updated: if the outgoing edge is in E_1^v, the new mode is 1; otherwise, it is 2. The strategy uses edges prescribed by τ_1 from the mode 1 and by τ_2 from the mode 2. Note that we rely on the positionality of τ_1 and τ_2 for this definition: a finite play may use edges both in E_2^v and E_1^v, which do not exist in $\mathscr{G}_1$ and $\mathscr{G}_2$ respectively, but τ_1 and τ_2 ignore the past and only look at the current vertex. It is left to show that (σ_1, τ) is a pair of best responses in $\mathscr{G}$ from every vertex. Let v_0 be any vertex of $\mathscr{A}$.

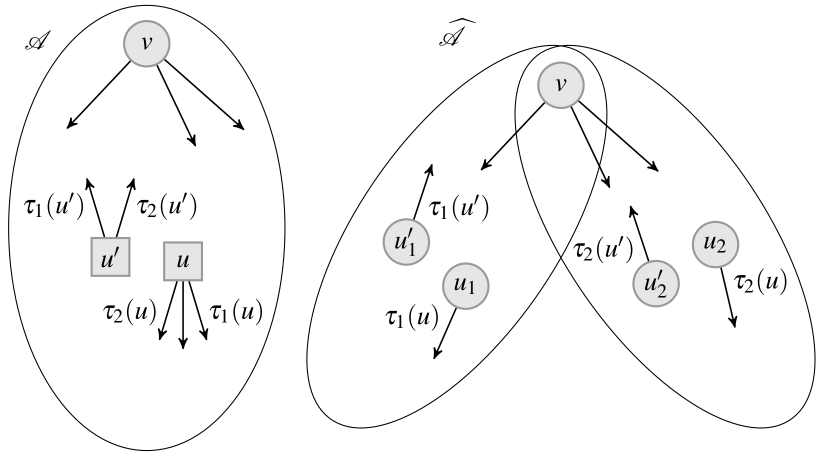

Figure 3.1: Construction of $\widehat{\mathscr{A}}$ (right) from $\mathscr{A}$ (left) in the proof of Theorem 3. We assume that E_1^v contains the leftward edge from v, and that E_2^v contains the two rightward edges. We depict the transformation applied to two vertices of Min u and u': two copies of each vertex are created, only keeping the outgoing edge given respectively by τ_1 and τ_2.

We first show that τ is a best response to σ_1 from v_0. Let τ' be any strategy of Min in $\mathscr{G}$. We have that

$$\begin{aligned}
\Phi(\pi^{\mathscr{G},v_0}_{\sigma_1,\tau'}) &= \Phi(\pi^{\mathscr{G}_1,v_0}_{\sigma_1,\tau'}) && \text{as } \sigma_1 \text{ only chooses edges in } E_1^v \text{ from } v \\
&\geq \Phi(\pi^{\mathscr{G}_1,v_0}_{\sigma_1,\tau_1}) && \text{as } \tau_1 \text{ is a best response to } \sigma_1 \text{ in } \mathscr{G}_1 \\
&= \Phi(\pi^{\mathscr{G},v_0}_{\sigma_1,\tau}) && \text{as } \tau \text{ only uses mode 1 against } \sigma_1 \text{ in } \mathscr{G}.
\end{aligned}$$

This shows that τ is a best response to σ_1 from v_0.

We now show that σ_1 is a best response to τ in $\mathscr{G}$ from v_0. Let σ' be any strategy of Max in $\mathscr{G}$. We denote $\widehat{\sigma}'$ for the strategy corresponding to σ' in $\widehat{\mathscr{G}}$: it plays as σ' would, ignoring in which copy of the arena it currently is. For convenience, we also denote $\widehat{\tau}$ the only "empty" strategy of Min in arena $\widehat{\mathscr{A}}$ (recall that Min controls no vertex in $\widehat{\mathscr{A}}$). We have

$$\begin{aligned}
\Phi(\pi^{\mathscr{G},v_0}_{\sigma',\tau}) &= \Phi(\pi^{\widehat{\mathscr{G}},(v_0)_1}_{\widehat{\sigma'},\widehat{\tau}}) && \text{by construction of } \widehat{\mathscr{A}} \text{ and } \tau \\
&\leq \Phi(\pi^{\widehat{\mathscr{G}},(v_0)_1}_{\widehat{\sigma},\widehat{\tau}}) && \text{by optimality of } \widehat{\sigma} \text{ in } \widehat{\mathscr{G}} \\
&= \Phi(\pi^{\mathscr{G}_1,v_0}_{\widehat{\sigma},\tau_1}) && \text{as } \widehat{\sigma} \text{ always stays in copy 1 from } (v_0)_1 \text{ in } \widehat{\mathscr{G}} \\
&\leq \Phi(\pi^{\mathscr{G}_1,v_0}_{\sigma_1,\tau_1}) && \text{as } \sigma_1 \text{ is a best response to } \tau_1 \text{ is } \mathscr{G}_1 \\
&= \Phi(\pi^{\mathscr{G},v_0}_{\sigma_1,\tau}) && \text{as } \tau \text{ only resorts to } \tau_1 \text{ in } \mathscr{G} \text{ against } \sigma_1.
\end{aligned}$$

This shows that σ_1 is a best response to τ in $\mathscr{G}$, ending the proof. □

Theorem 3 is a characterisation, as the other implication is immediate (the finite one-player arenas form a subset of the finite arenas). Therefore, uniform bi-positionality over finite arenas of all objectives satisfying this property can be established using this theorem. It also implies that if an objective is *not* uniformly bi-positional, this is witnessed by some one-player arena for at least one of the two players. We give some examples of objectives whose uniform bi-positionality had not been established yet and for which the use of Theorem 3 greatly simplifies the proof.

Theorem 4. *The* $\mathtt{Reach}$, $\mathtt{ShortestPath}$, $\mathtt{Energy}$, $\mathtt{MeanPayoff}^{-}$, $\mathtt{MeanPayoff}^{+}$, *and* $\mathtt{TotalPayoff}$ *objectives are uniformly bi-positional over finite arenas.*

Partial proof. We give arguments for the $\mathtt{MeanPayoff}^{-}$ objective. We show that in all finite one-player arenas of Max, Max has a positional optimal strategy. Symmetric arguments will provide the same result for one-player arenas of Min, and we can conclude using Theorem 3 that $\mathtt{MeanPayoff}^{-}$ is uniformly bi-positional over finite arenas.

Let $\mathscr{A}$ be a finite one-player arena of Max (so Max controls all vertices of $\mathscr{A}$). For each cycle of $\mathscr{A}$, we define its *mean payoff* as the average of its colours. Let v_0 be an initial vertex. We consider all cycles reachable from v_0. The main observation is that the maximal mean payoff among all these cycles is attained at a *simple* cycle. Indeed, the mean payoff of an arbitrary cycle can always be expressed as a convex combination of mean payoffs of the simple cycles it contains. In particular, the mean payoff of an arbitrary cycle is less than or equal to the mean payoff of some simple cycle it contains. Moreover, in a finite arena, there are only finitely many simple cycles, so the maximum is attained. Based on this observation, here is therefore a positional optimal strategy from v_0: reach such a maximal simple cycle in a positional way (by prefix independence, the exact path does not matter) and then loop around it forever. No other play from v_0 can obtain a larger mean payoff (to see it, decompose any other play into a prefix and infinitely many cycles).

Formally, we have now shown that from every vertex of $\mathscr{A}$, there is a positional optimal strategy. We must still prove uniformity, i.e., the existence of a single positional strategy that is optimal from every vertex. In this particular case, as $\mathtt{MeanPayoff}^{-}$ is prefix independent, we can simply apply **??**.

Observe that this proof also works with no change for $\mathtt{MeanPayoff}^{+}$. □

There is one previously defined objective that is uniformly bi-positional over finite arenas but for which we have not yet given a proof of this fact: the discounted payoff. Resorting to Theorem 3 to establish its bi-positionality is possible, but proving its positionality over one-player arenas is still arguably difficult. Later in the book, two very different (and more direct) proofs will be given: the first uses *universal graphs* (Section 3.4.4); another, relying on Banach fixed point theorem, will be given in **??**.

3.2 Bi-positionality over infinite arenas for qualitative objectives

In this section, we prove that the only prefix independent qualitative objectives that are bi-positionally determined over infinite arenas are, in some sense, parity objectives. Bi-positionality of the parity objective (over finite arenas) was derived in Theorem 2 (a proof for infinite arenas follows from Section 3.4.3). We therefore focus on showing the necessity of the condition for bi-positionality. We note that, by **??**, we already know that parity objectives are the only bi-positional ones among Muller objectives.

In Section 3.2.2, we generalise the characterisation of bi-positionality to objectives that are not necessarily prefix independent. As a result, we obtain a one-to-two-player lift over infinite arenas (Corollary 2), analogous to the one presented in the previous section.

3.2.1 Prefix independent qualitative objectives

We say that a qualitative objective $\Omega \subseteq C^\omega$ (over a possibly infinite set of colours) is *equivalent to a parity objective* if there exists $d \in \mathbb{N}$ and a mapping $\phi \colon C \to [1,d]$ such that, for all $\rho \in C^\omega$:

$$\rho \in \Omega \iff \limsup \phi(\rho_i) \text{ is even.}$$

The main theorem of this section is the following.

Theorem 5. *Every prefix independent qualitative objective that is bi-positional over one-player infinite arenas of both players is equivalent to a parity objective.*

We note that parity objectives are uniformly bi-positionally determined over infinite arenas. Therefore, the previous theorem yields a characterisation of bi-positionality for prefix independent objectives:

Corollary 1. *Let Ω be a prefix independent qualitative objective. The following are equivalent:*

- *Ω is bi-positional over infinite one-player arenas.*
- *Ω is equivalent to a parity objective.*
- *Ω is uniformly bi-positionally determined over infinite arenas.*

The rest of the subsection is devoted to the proof of Theorem 5. We will use the following remark repeatedly.

Remark 1. *If Ω is prefix independent, then $(uv)^\omega \in \Omega$ if and only if $(vu)^\omega \in \Omega$, for all $u, v \in C^+$.*

Lemma 3. *Let $\Omega \subseteq C^\omega$ be a prefix independent objective that is bi-positional over infinite one-player arenas, and let $w_1, w_2, \ldots \in C^+$ be a sequence of words such that $w_i^\omega \in \Omega$ (resp. $w_i^\omega \notin \Omega$) for all i. Then $w_1 w_2 w_3 \ldots \in \Omega$ (resp. $w_1 w_2 w_3 \ldots \notin \Omega$).*

Proof. We show the contrapositive. Assume that $w_1w_2w_3\ldots \notin \Omega$, and consider the Adam-game in which Adam controls one vertex v, from which there are self-loops labelled w_i for each i. Adam can win by producing the word $w_1w_2\ldots \notin \Omega$, so, by bi-positionality, he has a positional strategy which always picks a same self-loop labelled w_i. Therefore, $w_i^\omega \notin \Omega$. □

Lemma 4. *Let $\Omega \subseteq C^\omega$ be a prefix independent objective that is bi-positional over infinite one-player arenas. Let $A,B \subseteq C$ such that for all $a \in A$ and $b \in B$, $(ab)^\omega \in \Omega$ (resp. $(ab)^\omega \notin \Omega$). Then, $(AB^*)^\omega \subseteq \Omega$ (resp. $(AB^*)^\omega \notin \Omega$).*

Proof. We show that for all $a \in A$, $(aB^*)^\omega \subseteq \Omega$, and conclude using Lemma 3.

First, we prove that for all $u \in B^*$ there is $n \in \mathbb{N}$ such that $(a^n u)^\omega \in \Omega$ by induction on the length of u. If $|u| = 1$, this follows by hypothesis for $n = 1$. Let $u = u'b$, $b \in B$. By induction hypothesis, there is n such that $(a^n u')^\omega \in \Omega$. Then, by Lemma 3, $((a^n u')(ba))^\omega \in \Omega$, and so $(a^{n+1}u)^\omega \in \Omega$.

We now show that there is a fixed k such that $(a^k u)^\omega \in \Omega$ for all $u \in B^*$. On the contrary, assume that for each k there is $u_k \in B^*$ such that $(a^k u_k)^\omega \notin \Omega$. Let $k_1, k_2, \ldots \in \mathbb{N}$ be a sequence such that $(a^{k_{i+1}} u_{k_i})^\omega \in \Omega$ (which exists by the previous claim), and consider the game in which Eve controls a vertex with self-loops labelled $a^k u_k$, for all k. She can win this game by producing the sequence $a^{k_1}(u_1 a^{k_2})(u_2 a^{k_2})\ldots$, which belongs to Ω by Lemma 3 and Remark 1. However, she cannot win positionally, a contradiction.

Finally, we prove that the minimal k such that $(a^k u)^\omega \in \Omega$ for all $u \in B^*$ must be $k = 1$. If $k > 1$, there are words $u_1, u_2 \in B^*$ such that $(a^{k-1}u_1)^\omega \notin \Omega$ and $(a^{k-1}u_2)^\omega \notin \Omega$. This implies that $a^{k-1}u_1u_2a^{k-1} \notin \Omega$, and so $(a^{k-2}(a^k u_1 u_2))^\omega \notin \Omega$, a contradiction. □

We can now prove Theorem 5.

Proof of Theorem 5. Let

$$C_{\text{Even}} = \{c \in C \mid c^\omega \in \Omega\} \text{ and } C_{\text{Odd}} = C \setminus C_{\text{Even}}.$$

For $c \in C_{\text{Even}}$, we define

$$\mathsf{smallerOdd}(c) = \{x \in C_{\text{Odd}} \mid (cx)^\omega \in \Omega\}.$$

We define the preorder over C_{Even} given by $c \preceq c'$ if $\mathsf{smallerOdd}(c) \subseteq \mathsf{smallerOdd}(c')$.

We claim that the preorder $\preceq$ is total over C_{Even}. Assume by contradiction that there are two incomparable elements $c, c' \in C_{\text{Even}}$. Then, there are colours $x, x' \in C_{\text{Odd}}$ such that

$$\begin{aligned} (cx)^\omega \in \Omega, \quad (cx')^\omega \notin \Omega, \\ (c'x)^\omega \notin \Omega, \quad (c'x')^\omega \in \Omega. \end{aligned}$$

Consider the game in which Eve controls a vertex v from which there are self-loops labelled $c'x$ and cx'. We show that she can win this game by producing the sequence $\rho = (c'xcx')^\omega = c'((xc)(x'c'))^\omega$. Indeed, by Lemma 3, $(xcx'c')^\omega \in \Omega$, so by prefix independence, $\rho \in \Omega$. However, she cannot win positionally, a contradiction.

Secondly, we prove that $\preceq$ admits finitely many equivalent classes. On the contrary, assume that it contains either an infinite strictly increasing sequence or a strictly decreasing one. We assume w.l.o.g. that we are in the former case, so there is a sequence $c_1 \prec c_2 \prec \dots$ of elements of C_{Even}. By definition, there are $x_1, x_2, \dots \in C_{\mathrm{Odd}}$ such that $(c_i x_i)^\omega \in \Omega$ and $(c_i x_{i+1})^\omega \notin \Omega$, for all i. We consider the game in which Eve controls a vertex with self-loops labelled $c_i x_{i+1}$, for $i \geq 1$. She can win by producing the sequence $(c_1 x_2)(c_2 x_3)\dots = c_1(x_2 c_2)(x_3 c_3)\dots$, which belongs to Ω by Lemma 3 and prefix independence. However, Eve cannot win this game positionally, a contradiction.

We conclude that C_{Even} admits a partition into finitely many subsets $C_1, \dots, C_d$ such that $C_1 \prec C_2 \prec \dots \prec C_d$. We define the sought mapping $\phi : C \to [1, 2d]$ as follows:

$$\begin{array}{lll} \text{For } c \in C_{\mathrm{Even}}, & \phi(c) = 2i & \text{if } c \in C_i, \\ \text{For } c \in C_{\mathrm{Odd}}, & \phi(c) = 2i-1 & \text{if } c \in \mathsf{smallerOdd}(C_i) \setminus \mathsf{smallerOdd}(C_{i-1}), \end{array}$$

where we let $\mathsf{smallerOdd}(C_0) = \emptyset$.

Finally, we prove that for $\rho \in C^\omega$, $\rho \in \Omega$ if and only if $i_{\max} = \limsup \phi(\rho_i)$ is even, as wanted. We prove this in the case where $i_{\max}$ is even; the odd case is analogous.

We let $A = \{c \in C \mid \phi(c) = i_{\max}\}$ and $B = \{c \in C \mid \phi(c) < i_{\max}\}$. We note that $A \subseteq C_{\mathrm{Even}}$, as $i_{\max}$ is assumed even. We claim that for all $a \in A$ and $b \in B$ we have that $(ab)^\omega \in \Omega$. Indeed, if $b \in C_{\mathrm{Even}}$, then $(ab)^\omega \in \Omega$ by Lemma 3. If $b \in C_{\mathrm{Odd}}$, then $b \in \mathsf{smallerOdd}(a)$, and $(ab)^\omega \in \Omega$ by definition. We deduce by Lemma 4 that $(AB^*)^\omega \subseteq \Omega$. Since $i_{\max}$ is the maximal index appearing infinitely often in ρ, ρ contains a suffix in $(AB^*)^\omega$, so it belongs to Ω.

This concludes the proof that, for all $\rho \in C^\omega$:

$$\rho \in \Omega \iff \limsup \phi(\rho_i) \text{ is even.} \qquad \square$$

Remark 2. *Theorem 5 relies on the fact that games are edge-labelled; there are objectives that are bi-positional over state-labelled infinite arenas that are not equivalent to a parity objective. One such example is the Muller objective "see both colours* 1 *and* 2 *infinitely often".*

3.2.2 General qualitative objectives

We now state a characterisation of bi-positionality over infinite arenas without assuming prefix independence, generalising the previous result. We only provide a partial proof of it. The main ingredient in this characterisation consists in the *residuals of an objective*, as introduced here.

Residuals. Let $\Omega \subseteq C^\omega$ be a qualitative objective. We define the *residual of* Ω with respect to a finite word $u \in C^*$ as:

$$u^{-1}\Omega = \{w \in C^\omega \mid uw \in \Omega\}.$$

We denote by $\mathrm{Res}(\Omega)$ the set of residuals of Ω. We usually consider the order on the residuals given by the inclusion relation (which is a *partial* order in general).

Remark 3. *An objective Ω is prefix independent if and only if* $\mathrm{Res}(\Omega)$ *is a singleton.*

Remark 4. *If Ω is ω-regular, then* $\mathrm{Res}(\Omega)$ *is finite. Unlike for regular languages of finite words, there are non-ω-regular languages with finitely many residuals. For instance, there are prefix independent objectives that are not ω-regular, such as* $\texttt{MeanPayoff}^+$.

We can associate to any objective Ω its *automaton of residuals*, which is the deterministic automaton $\mathbf{R}_\Omega = (Q, q_0, \Delta)$ given by:

- $Q = \mathrm{Res}(\Omega)$,
- $q_0 = \varepsilon^{-1}\Omega$,
- $\Delta = \{(u^{-1}\Omega, a, (ua)^{-1}\Omega) \mid u \in C^*, a \in C\}$.

This is not formally an automaton as defined in **??**: it is defined without an acceptance condition and it may have infinitely many states.

We say that Ω can be *recognised by a parity automaton on top of its automaton of residuals* if there is some $d \in \mathbb{N}$ and some colouring $\mathfrak{c}\colon \Delta \to [1,d]$ such that the automaton $\mathbf{R}_\Omega$ with the parity condition induced by $\mathfrak{c}$ recognises Ω.

We introduce now a property about the residuals of a language, called progress consistency, that is necessary for positionality.

Progress consistency. We say that an objective $\Omega \subseteq C^\omega$ is *progress consistent* if for all $u, w \in C^*$

$$u^{-1}\Omega \subsetneq (uw)^{-1}\Omega \implies uw^\omega \in \Omega.$$

Intuitively, $u^{-1}\Omega \subsetneq (uw)^{-1}\Omega$ means that when producing w after the word u we "make a progress towards Ω", in the sense that there are now strictly more winning continuations than after u. The objective Ω is progress consistent if, in such a situation, repeating w forever after u yields a winning word.

Example 1. *Let $C = \{a, b\}$ and consider the objective*

$$\Omega = \texttt{Reach}(aa) = \{w \in C^\omega \mid w \text{ contains the factor } aa\}.$$

The objective Ω has three residuals: $\varepsilon^{-1}\Omega \subsetneq a^{-1}\Omega \subsetneq (aa)^{-1}\Omega$. We claim that is not progress consistent. Indeed, it suffices to take $u = \varepsilon$ and $w = ba$. We have that $\varepsilon^{-1}\Omega \subsetneq (ba)^{-1}\Omega$, but $(ba)^\omega \notin \Omega$. As we will see (Lemma 6), progress consistency is a necessary condition for positionality. Non-positionality of Ω can be shown by considering a game with a vertex controlled by Eve with self-loops labelled by ba and by ab.

The characterisation. We can now state the main characterisation theorem.

Theorem 6. *Let $\Omega \subseteq C^\omega$ be a qualitative objective that is uniformly bi-positional over infinite one-player arenas of both players. Then,*

- $\mathrm{Res}(\Omega)$ *is finite and totally ordered by inclusion,*

- *both Ω and $C^\omega \setminus \Omega$ are progress consistent, and*
- *the objective Ω can be recognised by a parity automaton on top of its automaton of residuals (in particular, Ω is ω-regular).*

Furthermore, in this case, Ω is uniformly bi-positionally determined over all arenas.

We note that if Ω is prefix independent, then the first two conditions in the previous theorem are trivially satisfied. In this case, the third condition amounts to saying that Ω should be recognised by a parity automaton with a single state. This is equivalent to the fact that Ω is equivalent to a parity objective; we recover therefore Theorem 5.

Corollary 2 (One-to-two-player lift over infinite arenas)**.** *Every objective that is uniformly bi-positional over infinite one-player arenas of both players is uniformly positionally determined over all infinite arenas.*

We discuss the proof of necessity of the first two items of the characterisation. These proofs exemplify some common techniques for proving non-positionality. For a proof of necessity of the third item, we refer to [BRV23]. Sufficiency of the conditions can be established using monotone universal graphs, introduced in Section 3.4; we refer to [**?**] for a proof.

Necessity of well-order of residuals and progress consistency.

Lemma 5. *If $\Omega \subseteq C^\omega$ is uniformly positional over infinite one-player arenas of Eve, then* $\mathrm{Res}(\Omega)$ *is well-ordered by inclusion (that is, well-founded and totally ordered).*

Proof. We show the contrapositive. Assume first that the inclusion on $\mathrm{Res}(\Omega)$ is not a total order. Then, there are two residuals $u_1^{-1}\Omega$ and $u_2^{-1}\Omega$ that are incomparable, that is, there are words $w_1, w_2 \in C^\omega$ such that

$$\begin{aligned} u_1 w_1 \in \Omega, &\quad u_1 w_2 \notin \Omega, \\ u_2 w_1 \notin \Omega, &\quad u_2 w_2 \in \Omega. \end{aligned}$$

It suffices to consider the (infinite sized) game in Figure 3.2. Eve can win from v_1 and v_2; however, no positional strategy ensures that she will win from both these vertices.

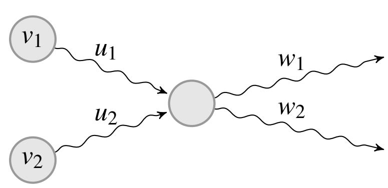

Figure 3.2: Necessity of total order of residuals: game in which Eve can win, but not positionally when $u_i w_i \in \Omega$ and $u_i w_j \notin \Omega$ for $i \neq j$. Squiggly arrows are used as a shorthand for possibly multiple edges (they are labelled by an element of C^*).

Now, assume that $\mathrm{Res}(\Omega)$ is not well-founded, and let $u_1^{-1}\Omega \supsetneq u_2^{-1}\Omega \supsetneq \dots$ be an infinite strictly decreasing chain of residuals. Then, there are words $w_1, w_2, \dots \in C^\omega$ such that $u_i w_i \in \Omega$, but $u_i w_j \notin \Omega$ for all $j > i$. To conclude, consider the game from Figure 3.3. Eve can win from all vertices v_i, but no positional strategy guarantees a win from all of them. $\square$

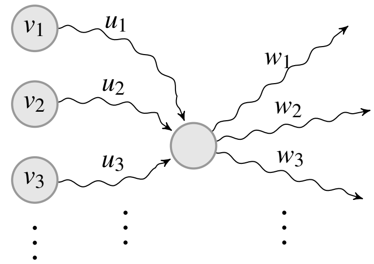

Figure 3.3: Necessity of well-foundedness of residuals: game in which Eve can win from every vertex v_i, but no positional strategy ensures a win from all those vertices.

Corollary 3. *If $\Omega \subseteq C^\omega$ is uniformly bi-positional over infinite one-player arenas (of both players), then* $\mathrm{Res}(\Omega)$ *is finite and totally ordered by inclusion.*

Proof. By the previous lemma, both $\mathrm{Res}(\Omega)$ and $\mathrm{Res}(C^\omega \setminus \Omega) = \{C^\omega \setminus R \mid R \in \mathrm{Res}(\Omega)\}$ are well-ordered, so $\mathrm{Res}(\Omega)$ must be finite. $\square$

Lemma 6. *If $\Omega \subseteq C^\omega$ is positional over infinite one-player arenas of Eve, then it is progress consistent.*

Proof. We prove the contrapositive. Assume that Ω is not progress consistent. Then, there are words $u, w \in C^*$ such that $u^{-1}\Omega \subsetneq (uw)^{-1}\Omega$ but $uw^\omega \notin \Omega$. The strict inclusion of the residuals implies that there is $w' \in C^\omega$ such that $uw' \notin \Omega$ and $uww' \in \Omega$. Consider the game in Figure 3.4. Eve can win from v_0 by producing the output uww', but she cannot win positionally. $\square$

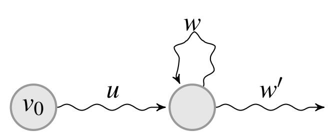

Figure 3.4: Necessity of progress consistency: Game in which Eve can win from v_0, but not positionally.

3.3 Positionality for quantitative objectives

In the previous section, we studied positionality of qualitative objectives. We can wonder whether neat characterisations such as the one presented in Theorem 5 can be obtained for quantitative objectives. In this section, we will see in Proposition 2 that the study of positionality of quantitative objectives can be reduced to the study of qualitative ones.

However, in the case of quantitative objectives, various natural definitions of positionality are possible. We start by presenting the three main such definitions and studying their relations. We show that these three notions coincide over finite arenas and provide examples separating them over infinite arenas. The study of positionality of each of them reduces to qualitative objectives in a slightly different way.

Definition 2. *Let $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ be a quantitative objective. We say that it is*

- positional for Max *if for every game $\mathcal{G}$ with objective Φ, every Max strategy σ, and every vertex v, Max has a positional strategy σ' such that*
$$val^{\sigma}_{\mathrm{Max}}(v) \leq val^{\sigma'}_{\mathrm{Max}}(v),$$
- limit-optimal positional for Max *if for every game $\mathcal{G}$ with objective Φ and every vertex v,*
$$val^{\mathcal{G}}_{\mathrm{Max}}(v) = \sup_{\sigma \text{ positional}} val^{\sigma}_{\mathrm{Max}}(v),$$
- positional and admitting optimal strategies for Max *if for every game $\mathcal{G}$ with objective Φ and every vertex v, Max has a positional strategy σ from v such that*
$$val^{\sigma}_{\mathrm{Max}}(v) = val^{\mathcal{G}}_{\mathrm{Max}}(v), \text{ (i.e. } \sigma \text{ is optimal from } v).$$

Remark 5. *An objective can be limit-optimal positional but not positional if the following three conditions occur in some game:*

- *Max has an optimal strategy (strategy that reaches $val^{\mathcal{G}}_{\mathrm{Max}}(v)$),*
- *Max does not have a positional optimal strategy, and*
- *Max has positional ε-optimal strategies for all $\varepsilon > 0$.*

Proposition 1. *If Φ is positional, then Φ is limit-optimal positional. Moreover, over finite arenas, if Φ is limit-optimal positional then Φ is positional and admits optimal strategies.*

Proof. (Positionality $\Longrightarrow$ limit-optimal positionality). Let $\varepsilon > 0$. We aim to prove that Max has a positional strategy ensuring a value at least $\mathrm{val}^{\mathcal{G}}_{\mathrm{Max}}(v) - \varepsilon$. By definition, Max has a (not necessarily positional) strategy σ ensuring a value greater than $\mathrm{val}^{\mathcal{G}}_{\mathrm{Max}}(v) - \varepsilon$. As Φ is assumed positional, there exists a positional strategy σ' such that
$$\mathrm{val}^{\mathcal{G}}_{\mathrm{Max}}(v) - \varepsilon \leq \mathrm{val}^{\sigma}_{\mathrm{Max}}(v) \leq \mathrm{val}^{\sigma'}_{\mathrm{Max}}(v).$$

(Over finite arenas, limit-optimal positionality $\Longrightarrow$ positionality and existence of optimal strategies). We note that over a finite arena there are finitely many positional strategies. Therefore, if for every $\varepsilon > 0$ Max has a positional strategy ensuring a value greater than $\mathrm{val}^{\mathcal{G}}_{\mathrm{Max}}(v) - \varepsilon$, one of these strategies must ensure the value $\mathrm{val}^{\mathcal{G}}_{\mathrm{Max}}$. $\square$

Reduction to the qualitative case

We now show that we can reduce the study of the positionality of quantitative objectives to the positionality of qualitative objectives. We introduce the following notation for this purpose.

For a quantitative objective $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ and $x \in \mathbb{R} \cup \{\pm\infty\}$, we let

$$\Phi_{\geq x} = \{\rho \in C^\omega \mid \Phi(w) \geq x\} \quad \text{and} \quad \Phi_{>x} = \{\rho \in C^\omega \mid \Phi(w) > x\}.$$

We also define $\Phi_{<x} = C^\omega \setminus \Phi_{\geq x}$ and $\Phi_{\leq x} = C^\omega \setminus \Phi_{>x}$.

Proposition 2. *Let $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ be a quantitative objective. The following statements can be instantiated over any class of arenas.*

1. *Φ is positional if and only if for all $x \in \mathbb{R} \cup \{\pm\infty\}$, the qualitative objective $\Phi_{\geq x}$ is positional.*

2. *Φ is limit-optimal positional if and only if for all $x \in \mathbb{R} \cup \{\pm\infty\}$, the qualitative objective $\Phi_{>x}$ is positional.*

3. *Φ is positional and admits optimal strategies if and only if for all $x \in \mathbb{R} \cup \{\pm\infty\}$, the qualitative objective $\Phi_{\geq x}$ is positional and $\Phi(C^\omega)$ does not contain any infinite strictly increasing sequence.*

Proof. In the following, $\mathcal{G}$ stands for a game labelled with colours in C, which can be viewed as a game using objective Φ, $\Phi_{\geq x}$, or $\Phi_{>x}$.

1. Assume that Φ is positional, and suppose that Eve has a winning strategy σ for the objective $\Phi_{\geq x}$ in $\mathcal{G}$ from a vertex v. By positionality of Φ, there is positional strategy σ' for Max such that $x \leq \mathrm{val}^{\sigma}_{\mathrm{Max}}(v) \leq \mathrm{val}^{\sigma'}_{\mathrm{Max}}(v)$. The strategy σ' is therefore a positional strategy winning for $\Phi_{\geq x}$ from v. The uniform case is analogous.

   Conversely, assume that $\Phi_{\geq x}$ is positional for all $x \in \mathbb{R} \cup \{\pm\infty\}$. Let σ be a strategy for Max from v and let $x = \mathrm{val}^{\sigma}_{\mathrm{Max}}(v)$. Clearly, σ is a winning strategy from v for the objective $\Phi_{\geq x}$. By positionality of $\Phi_{\geq x}$, Max has a positional strategy σ' such that $x = \mathrm{val}^{\sigma}_{\mathrm{Max}}(v) \leq \mathrm{val}^{\sigma'}_{\mathrm{Max}}(v)$, as wanted.

2. Let σ be a winning strategy for $\Phi_{>x}$ from a vertex v. By taking $\varepsilon < \mathrm{val}^{\sigma}_{\mathrm{Max}}(v) - x$ and applying limit-optimal positionality, we conclude that Max has a positional strategy ensuring a value strictly greater than x, which is therefore winning for $\Phi_{>x}$.

   Conversely, let $\varepsilon > 0$ and let σ be an strategy ensuring a value strictly greater than $x = \mathrm{val}^{\sigma}_{\mathrm{Max}}(v) - \varepsilon$. Then, by positionality of $\Phi_{>x}$, Max has a positional strategy ensuring a value strictly greater than x.

3. Assume that Φ is positional and admits optimal strategies. The fact that $\Phi_{\geq x}$ is positional for all x is implied by the first item. Suppose by contradiction that there is a sequence of infinite words $w_1, w_2, \ldots \in C^\omega$ whose image by Φ is $x_1 < x_2 < \ldots$. Consider the infinite game in which Max controls a vertex v from which

she can choose to take a path labelled w_i, for every i. No strategy in this game ensures $\mathrm{val}^{\mathcal{G}}_{\mathrm{Max}}(v)$, so Φ does not admit optimal strategies.

We show the converse. Consider a sequence of strategies ensuring values that tend to $x = \mathrm{val}^{\mathcal{G}}_{\mathrm{Max}}(v)$. Since $\Phi(C^\omega)$ does not contain infinite strictly increasing sequences of values, such a sequence eventually stabilises at x, in particular, there is a strategy with value x. We conclude by positionality of $\Phi_{\geq x}$. □

Separation over infinite arenas

We provide two examples showing that the three notions under study are not equivalent over infinite arenas.

Example 2. *Let $C = \mathbb{Z}$ and consider the quantitative objective:*

$$\texttt{First}(\rho_1\rho_2\rho_3\dots) = \rho_1.$$

Clearly, $\texttt{First}(C^\omega) = \mathbb{Z}$ *contains infinite strictly increasing sequences, therefore, by Proposition 2, it does not admit optimal strategies (for any of the players). By the same proposition, we obtain that* $\texttt{First}$ *is bi-positional, as the qualitative objectives* $\texttt{First}_{\geq x} = \{\rho \in C^\omega \mid \rho_1 \geq x\}$ *and* $\texttt{First}_{\leq x}$ *are trivially positional.*

Example 3. *Consider the limsup objective over $C = \mathbb{Z}$:*

$$\texttt{LimSup}(\rho) = \limsup \rho.$$

We claim that $\texttt{LimSup}$ *is limit-optimal bi-positional but not positional.*

For each $x \in \mathbb{R}$, $\texttt{LimSup}_{>x}$ *is a Büchi objective (Eve wins if numbers strictly larger than x are produced infinitely often), so it is positional for her. Also,* $\texttt{LimSup}_{>\infty}$ *is the trivial empty objective, and* $\texttt{LimSup}_{>-\infty}$ *can be seen as a countable union of Büchi objectives (ρ is winning if there is one negative number seen infinitely often, or if positive numbers are seen infinitely often). This latter objective will be shown to be positional in Section 3.4.3. Therefore,* $\texttt{LimSup}$ *is limit-optimal positional for Max. From the point of view of Min,* $\texttt{LimSup}_{<x}$ *is a coBüchi objective for $x \in \mathbb{R}$, and* $\texttt{LimSup}_{<\infty} = \{\rho \in C^\omega \mid \rho \text{ is bounded}\}$ *can be shown to be positional using Proposition 4, so* $\texttt{LimSup}$ *is also limit-optimal positional for Min. We conclude that* $\texttt{LimSup}$ *is limit-optimal bi-positional.*

However, $\texttt{LimSup}$ *is not positional, as* $\texttt{LimSup}_{\geq\infty}$ *is not positional.*

3.4 Positionality over infinite arenas

This section is concerned with understanding positional (qualitative or quantitative) objectives: those for which, on arbitrary arenas, if Eve wins, then she can do so with a positional strategy. Just like for bi-positionality, it turns out that uniform positionality appears to be better behaved than the non-uniform counterpart. In a nutshell, results about positionality are sparser and often more difficult than for bi-positionality. No general simple characterisations akin to Theorems 3 or 6 are available.

Most of the section is devoted to the relationship between positionality over arbitrary arenas and so-called monotone universal graphs. These are combinatorial objects which allow to reason about positional objectives. Section 3.4.1 introduces monotone universal graphs, then Section 3.4.2 states and proves the characterisation. Section 3.4.3 illustrates the technique with a number of examples, and then Section 3.4.4 shows how it adapts to finite degree arenas.

An important conjecture about positionality, called Kopczyński's conjecture, states that prefix independent positional objectives are closed under unions; Section 3.4.5 discusses results around this conjecture. We end with a discussion on positionality of objectives which are ω-regular in Section 3.4.6.

Throughout Section 3.4, it is more convenient, for technical reasons that will appear below, to take the point of view of player Min. Therefore, by default, "positionality" means "positionality for Min". We will establish general results about quantitative objectives, and freely allow ourselves to apply these to qualitative objectives Ω, by identifying them to their indicator function.

3.4.1 Universal graphs

We now introduce the required vocabulary for talking about universal graphs.

Graphs and morphisms. We consider *C-graphs*, which are (potentially infinite) directed graphs whose edges are labelled by colours in C, and which exclude sinks: all vertices have some outgoing edge. Edges of C-graphs are denoted by $v \xrightarrow{c} v'$. We think of graphs as being arenas controlled by the opponent Max, therefore, given a quantitative objective $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$, the *value* $\mathrm{val}(v)$ of a vertex v in a graph is defined to be the supremum value of $\Phi(\pi)$, where π ranges over infinite paths from v.

A *morphism* from a C-graph G to a C-graph H is a map h from vertices of G to vertices of H such that for all edges $v \xrightarrow{c} v'$ in G, it holds that $h(v) \xrightarrow{c} h(v')$ is an edge in H. Note that h needs not be injective; for example any graph admits a morphism towards the 1-vertex graph with all possible self-loops. We write $G \xrightarrow{h} H$ when h is a morphism from G to H, and $G \to H$ when there is a morphism from G to H. Note that when $G \xrightarrow{h} H$, paths in G are mapped to paths in H with the same labels, and in particular, the value of a vertex in G is not larger than the value of its image in H.

Universal graphs. We say that a morphism $G \xrightarrow{h} H$ is Φ*-preserving*, where $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ is a quantitative objective, if for every vertex v in G, the value of $h(v)$ in H is in fact equal to the value of v in G. When Φ corresponds to a qualitative objective Ω, note that a morphism is Φ-preserving if and only if vertices satisfying Ω in G are mapped to vertices satisfying Ω in H.

Given a quantitative objective Φ and a cardinal κ (which is usually taken infinite in this section), we say that a graph U is (κ, Φ)*-universal* if all graphs of cardinality $< \kappa$ admit a Φ-preserving morphism towards U.

Remark 6. *We often work with prefix independent objectives Ω. In this case, graphs have no edges from vertices satisfying Ω to vertices which do not. As a consequence,* *one may simply focus on vertices satisfying Ω, and alter the definition as follows: a graph U is (κ, Ω)-universal if it satisfies Ω, and all graphs of cardinality $< \kappa$ which satisfy Ω have a morphism towards U. When considering prefix independent objectives, for instance for* `CoBuchi` *and* `Buchi` *below, we use this definition.*

To prove universality results, we often use the notion of rank of a well-founded tree which is recalled now. A tree T is *well-founded* if it does not admit any infinite branch. The *rank* of a vertex in a well-founded tree is defined to be 0 for leaves, and one plus the supremum rank of its successor for non-leaves; the rank of a tree is the rank of its root. Intuitively, the rank of a tree is an ordinal quantifying its depth. We give a few examples in Figure 3.5.

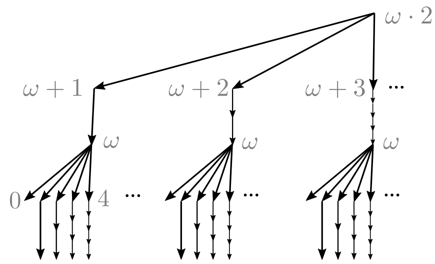

Figure 3.5: A well-founded tree of rank $\omega \cdot 2$. The ranks of a few vertices are displayed.

We now give some examples of universal graphs for standard objectives.

Example: CoBüchi. Let $C = \{2,3\}$ and Ω be the `CoBuchi` objective, meaning the set of words where 3 has at most finitely many occurrences. Fix a cardinal κ. Consider the graph U over κ given by $u \xrightarrow{3} u' \iff u > u'$ and $u \xrightarrow{2} u' \iff u \geq u'$ (see Figure 3.6).

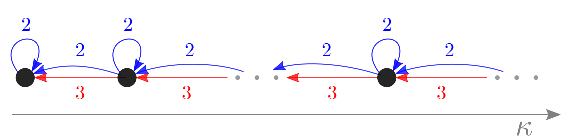

Figure 3.6: The graph U for the coBüchi objective. Transitive edges are excluded for readability.

Let us prove that U is (κ, Ω)-universal. First, we should show that U satisfies the `CoBuchi` objective Ω; this is clear since paths in U are non-increasing and follow a strict decrease when a 3 is seen.

Next, we should prove that any graph G of cardinality $< \kappa$ which satisfies `CoBuchi` has a morphism towards U. Take such a graph G and fix a vertex v in G. Since G satisfies `CoBuchi`, paths from v visit only finitely many 3-edges; stated differently, the tree obtained by unfolding G at v and contracting 2-edges is well-founded. We assign to v the rank $h(v)$ of this tree. Intuitively, $h(v)$ measures the maximal number of 3-edges that appear on paths from v. Since $G < \kappa$ we have $h(v) < \kappa$, so $h(v)$ is a vertex in U.

By definition, h satisfies that whenever $v \xrightarrow{3} v'$ is an edge in G, then $h(v) > h(v')$, and whenever $v \xrightarrow{2} v'$ is an edge in G, then $h(v) \geq h(v')$. Stated differently, h defines a morphism from G to U, as required.

Example: Büchi. Let $C = \{1,2\}$ and Ω be the `Buchi` objective, meaning the set of words where 2 has infinitely many occurrences. Fix a cardinal κ. Consider the graph U over κ given by $u \xrightarrow{2} u'$ for every pair of vertices u, u' and $u \xrightarrow{1} u' \iff u > u'$ (see Figure 3.7).

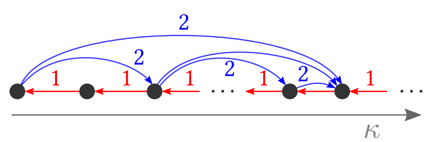

Figure 3.7: The graph U for the Büchi objective. Most edges are excluded for readability.

To prove that U is (κ, Ω)-universal, we start by showing that it satisfies the `Buchi` objective Ω: all paths have infinitely many 2's. Indeed, sequences of 1's force a strict decrease and are therefore finite.

Next, take a graph G of cardinality $< \kappa$ which satisfies Ω. To any vertex v, we associate an ordinal $h(v)$ which counts the number of 1's before the next 2; formally $h(v)$ is the rank of the well-founded tree obtained from removing 2-edges and then unfolding from v. It satisfies $h(v) > h(v')$ whenever $v \xrightarrow{1} v'$ is an edge, thus it is indeed a morphism from G to U.

Monotone graphs. We consider *ordered graphs*, which are simply graphs together with a linear order on their vertices. We say that an ordered graph is *monotone* if

$$u \geq u' \xrightarrow{c} v \geq v' \implies u \xrightarrow{c} v'.$$

Note that the universal graphs for `CoBuchi` and `Buchi` above (Figures 3.6 and 3.7) are monotone. Observe also that in a monotone graph, if $u \leq u'$ then the value of u (with respect to some quantitative objective) is not greater than the value of u'.

Neutral letters for quantitative objectives. Neutral letters are defined (and exemplified) in the preliminaries, for qualitative objectives. For a quantitative objective Φ, a letter ε is neutral if for all $w \in C^\omega$,

$$\Phi(w) = \begin{cases} \Phi(w') & \text{if the word } w' \text{ obtained from } u \text{ by removing } \varepsilon\text{'s is infinite} \\ -\infty & \text{otherwise.} \end{cases}$$

3.4.2 Characterisation of positionality

In Figures 3.6 and 3.7, the proposed universal graphs provide a way to measure the quality of a vertex in a strategy: in Figure 3.6 a vertex v is of good quality if paths from v visit few 3-edges, and in Figure 3.7, a vertex v is of good quality if paths from v visit a 2-edge as quickly as possible. Monotonicity of the universal graph formalises the intuition that this way of measuring the quality is sound. To derive positionality by measuring strategies as above, we also require well-foundedness of the universal graph (any set of vertices admit a minimum), which formalises the requirement that for any vertex in the game (which corresponds to a set of vertices in the strategy), there is a best (i.e. minimal) occurrence of this vertex in the strategy.

This intuition leads to the main result of this section.

Theorem 7. *Let $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ be a quantitative objective. If for all κ, there exists a (κ, Φ)-universal graph which is monotone and well-ordered, then Φ is uniformly positional (over arbitrary arenas). The converse holds assuming Φ admits a neutral letter.*

Remark 7. *In the statement above, κ ranges over arbitrary cardinals. Monotone universal graphs with respect to finite cardinals have also been studied, because they give rise to value iteration algorithms for solving the corresponding games. For more details, see [**?**].*

Proof of the direct implication (from structure to positionality). Consider a game $\mathcal{G}$, and let κ be a cardinal strictly greater than $|\mathrm{Paths}(G)|$. Our aim is to build a positional strategy which is uniformly optimal for Min in $\mathcal{G}$. Take a well-ordered monotone (κ, Φ)-universal graph U.

To a Min strategy $\tau\colon \mathrm{Paths}(G) \to E$ from v_0 naturally corresponds a graph G_τ over $\mathrm{Paths}(G, v_0)$ with edges

$$\pi \xrightarrow{c} \pi' \iff \pi' = \pi \xrightarrow{c} v' \text{ and if } v \in V_{\mathrm{Eve}} \text{ then } \sigma(\pi) = v \xrightarrow{c} v'.$$

For each strategy τ let h_τ be a Φ-preserving morphism from G_τ to U.

Define h to be the map assigning to each vertex v in G the minimal value of all possible $h_\tau(\pi)$, where τ ranges over Min strategies and π over paths ending in v. Then $h(v)$ is a well-defined vertex of U thanks to well-orderedness. For each vertex v, we additionally fix a choice of a strategy $\tau(v)$ and a path $\pi(v)$ ending in v where the minimum is met: $h_{\tau(v)}(\pi(v)) = h(v)$. Then we consider the positional strategy τ_{pos} defined by assigning to $v \in V_{\mathrm{Min}}$ the edge $\tau(v)(\pi(v))$.

We now show that h defines a morphism from the graph $\mathcal{G}[\tau_{\mathrm{pos}}]$ of τ_{pos} to U. Indeed, if $v \xrightarrow{c} v'$ is an edge in G, then $\pi(v) \xrightarrow{c} \pi(v)$ is an edge in $G_{\tau(v)}$, and therefore

$$h_{\tau(v)}(\pi(v)) \xrightarrow{c} h_{\tau(v)}(\pi(v) \xrightarrow{c} v')$$

is an edge in U. Now the left term is $h(v)$ by definition, and the right term is $\geq h(v')$ in U since $\pi(v)e$ is a path ending in v'. We conclude that $h(v) \xrightarrow{c} h(v')$ by monotonicity of U.

Since h is a morphism, the value of v in $\mathscr{G}[\tau_{\text{pos}}]$ is not larger than the value of $h(v)$ in U. Now, for every $\varepsilon > 0$, there is a strategy τ which is ε-optimal from v, which means that the value of the empty path v in G_τ is at most the value of v in $\mathscr{G}$ plus ε. Since the morphism h_τ is Φ-preserving, the same is true for the value of $h_\tau(v)$ in U (here, v stands for the empty path at v). Now since $h(v)$ is smaller than $h(v)$ in U, it follows that the value of v in $\mathscr{G}[\tau_{\text{pos}}]$ is not larger than the value of v in $\mathscr{G}$ plus ε. We conclude by letting ε go to zero.

Proof of the converse implication (from positionality to structure). We now prove the converse implication, so assume Φ is a uniformly positional quantitative objective which admits a neutral letter ε. The crucial result is the following.

Lemma 7. *Let Φ be a quantitative objective admitting a neutral letter ε and let H be a graph. There exists a well-ordered monotone graph H' with a Φ-preserving morphism $H \to H'$.*

It not hard to see that the lemma implies the theorem as follows. Apply the lemma to H being be the disjoint union of all graphs of cardinality $< \kappa$, up to isomorphism. Then the obtained graph H' is well-ordered, monotone, and (κ, f)-universal by composition of Φ-preserving morphisms.

Proof of Lemma 7. We build a game $\mathscr{G}$ from the graph H as follows. The vertices of $\mathscr{G}$ are those of H, which belong to Max, together with non-empty subsets of H, which are given to Min. Edges within the copy of H are just like in H, and we add edges of the form $v \xrightarrow{\varepsilon} S$ and $S \xrightarrow{\varepsilon} v$ whenever $v \in S$ (see Figure 3.8).

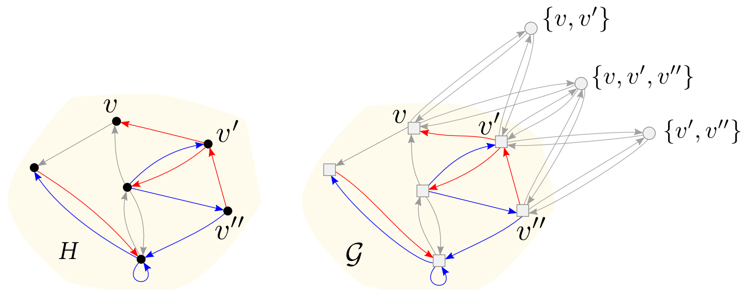

Figure 3.8: An example depicting the game in the proof of Lemma 7. Gray edges are neutral. We represent only 3 of the $2^6 - 1$ Min vertices.

First, we claim that for any vertex v of H, its value in $\mathscr{G}$ is just as in H. Indeed, the value is not smaller in $\mathscr{G}$ since Max can just follow any path while remaining in the copy of H. However, it is not larger since Min can choose to play back to vertex v, whenever the play reaches a subset S by an edge $v \xrightarrow{\varepsilon} S$; this has the neutral effect of adding two ε's.

Now thanks to uniform positionality, this means that Min has a positional strategy τ ensuring the optimal value from each vertex. Note that τ corresponds to a choice function over vertices of H: it assigns a chosen vertex to any non-empty set of vertices. We let H_1 be the graph obtained from H by adding every edge of the form $v \xrightarrow{\varepsilon} v'$, whenever $v \in S$ and $\tau(S) = S \xrightarrow{\varepsilon} v'$.

We claim that values in H_1 are just the same as in H. Clearly, values are not smaller in H_1 since we just added edges, creating more paths. Conversely, any path π_1 in H_1 can be converted to a path π in H, where the label of π is obtained from the label of π_1 by replacing some occurrences of ε^2 by ε; this does not affect the value of Φ. Note that in H_1, the relation $\xrightarrow{\varepsilon}$ has the property that for every non-empty subset S of vertices, there is $v' \in S$ (namely, the one such that $\tau(S) = S \xrightarrow{\varepsilon} v'$) such that for each $v \in S$, it holds that $v \xrightarrow{\varepsilon} v'$; this property is already close to being a well-order, it simply lacks transitivity and anti-symmetry.

We let H_2 be obtained from H_1 by adding every edge of the form $v \xrightarrow{c} v'$, whenever $v \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon}\xrightarrow{c}\xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} v'$ occurs in H_1. Again, it is easy to see that thanks to neutrality, values in H_2 are the same as in H_1. Now over H_2, the relation $\xrightarrow{\varepsilon}$ is transitive and moreover it satisfies the monotonicity requirement: $v \xrightarrow{\varepsilon}\xrightarrow{c}\xrightarrow{\varepsilon} v'$ implies $v \xrightarrow{c} v'$.

There remains to guarantee anti-symmetry, which poses no issue. Indeed, equivalent vertices $v \xrightarrow{\varepsilon} v' \xrightarrow{\varepsilon} v$ in H_2 have the same incoming and outgoing edges; therefore the quotient $H' = H_2 \;/ \xrightarrow{\varepsilon}$ is well-defined, well-ordered and monotone. Now we have $H \to H_1 \to H_2 \to H'$ which concludes the proof of the lemma, and the theorem. $\square$

About neutral letters. The proof of the converse in Theorem 7 used a neutral letter. A quantitative objective Φ over C admits a unique extension to an objective Φ^ε over $C \cup \{\varepsilon\}$ for which ε is a neutral letter, obtained by setting $\Phi^\varepsilon(u') = \Phi(u)$ if u' is obtained from an infinite word u by adding (arbitrarily many) occurrences of ε, and $\Phi^\varepsilon(u\varepsilon^\omega) = -\infty$ for finite words u.

It follows from Theorem 7 that existence of well-ordered monotone (κ, Φ)-universal graphs is equivalent to uniform positionality of Φ^ε. However, it is not known whether uniform positionalityof Φ entails uniform positionality of Φ^ε.

3.4.3 Examples

We now give a few examples to illustrate how to use Theorem 7 to establish positionality.

Parity. We consider the parity objective over $C = \{1, 2, \dots, d\}$, where d is even, which generalises both the `CoBuchi` and `Buchi` examples from Section 3.4.1. Fix a cardinal κ. We write elements of $\kappa^{d/2}$ as tuples indexed by odd numbers $d-1, d-3, \dots, 3, 1$, and order them lexicographically. Consider the graph U over $\kappa^{d/2}$ given by edges

$$(u_{d-1}, \dots, u_1) \xrightarrow{p} (u'_{d-1}, \dots, u'_1) \iff \begin{cases} (u_{d-1}, \dots, u_{p+1}) \geq (u'_{d-1}, \dots, u'_{p+1}) & \text{if } p \text{ is even} \\ (u_{d-1}, \dots, u_p) > (u'_{d-1}, \dots, u'_p) & \text{if } p \text{ is odd.} \end{cases}$$

We claim that U is $(\kappa, \texttt{Parity})$-universal.

First, we should prove that U satisfies `Parity`. Take an infinite path in U and assume towards contradiction that the maximal priority seen infinitely often is an odd number k. Then $(u_{d-1},\dots,u_k)$ only decreases along the path, with infinitely many strict decreases; this is not possible.

Second, we should prove that any graph of cardinality $< \kappa$ which satisfies `Parity` has a morphism into U, so take such a graph G. To each vertex v of G, we associate a tuple $h(v) = (h_{d-1}(v),\dots,h_3(v),h_1(v))$, where $h_k(v)$ is an ordinal counting the number of occurrences of the odd priority k before a greater priority over paths from v in G. Formally, $h_k(v)$ is defined to be the rank of the well-founded tree obtained from G by contracting edges with priority $< k$, cutting edges with priority $> k$ and unfolding at v. It is a direct check that h defines a morphism from G to U.

Thanks to Theorem 7, this gives uniform positionality of `Parity`. In fact, this proof coincides with the original proof of Emerson and Jutla (see the bibliographic references at the end of the chapter), rephrased in the language of universal graphs.

Energy. Recall the quantitative objective

$$\texttt{Energy}^+(\rho) = \sup_k \sum_{i=0}^{k-1} \rho_i$$

over $C = \mathbb{Z}$ (recall also that we take the point of view of Min). Consider the graph U over $\mathbb{N} \cup \{\infty\}$ with an edge $u \xrightarrow{t} u'$ if and only if $u \geq u' + t$. Note that vertex ∞ in U has all possible outgoing edges (including self-loops), and no incoming edges besides self-loops. In particular, it has value ∞.

We claim that in fact, for all $u \in \mathbb{N} \cup \{\infty\}$, vertex u has value u in U. This is already argued above for $u = \infty$, so let $u \in \mathbb{N}$. Then paths from u are of the form $u = u_0 \xrightarrow{t_0} u_1 \xrightarrow{t_1} \dots$, where the u_i's belong to $\mathbb{N}$, and for each i, $u_i \geq u_{i+1} + t_i$. This gives for any k that $\sum_{i=0}^{k-1} t_i = u_0 - u_k \leq u_0 = u$, therefore the value of u is smaller than u. Conversely, the path $u \xrightarrow{u} 0 \xrightarrow{0} 0 \xrightarrow{0} \dots$ has value u and therefore the value of u is u.

Next, we prove that any graph has an $\texttt{Energy}^+$-preserving morphism towards U, and therefore U is $(\kappa, \texttt{Energy}^+)$-universal for all cardinals κ. Indeed, take a graph G, and map each vertex v to its value. Then consider an edge $e = v \xrightarrow{t} v'$, and a path π' from v' witnessing the value u' of v'. Then $e\pi'$ is a path from v with value $\geq u' + t$. Therefore, the value u of v satisfies $u \geq u' + t$, so the map above is indeed a morphism; it is $\texttt{Energy}^+$-preserving by definition.

Finitely many letters occur infinitely often. In practice, for proving (or disproving) positionality of a given objective Ω, it is often not too difficult to construct well-ordered monotone universal graphs by trial and error. We give a detailed example on how such a method can help to disprove positionality of a given objective.

Consider the objective $\Omega = \{w \in \mathbb{N}^\omega \mid \texttt{Inf}(w) \text{ is finite}\}$, where $\texttt{Inf}(w) \subseteq \mathbb{N}$ is the set of letters occurring infinitely often in w. We aim for a well-ordered monotone universal graph for Ω (for some large infinite cardinal). First, notice that the graph G_n with a single vertex g_n and self-loops labelled by all integers $\leq n$ satisfies the objective.

Further, the graph G comprised of the union of the G_n's with all possible edges pointing from g_i to g_j when $i > j$ (see Figure 3.9) also satisfies Ω.

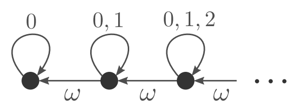

Figure 3.9: A first candidate for a universal graph.

This gives us a first candidate for a universal graph: could it be that G is universal? Assume that it is the case. Then in each (small) graph H satisfying Ω, there is a vertex h_0 which is mapped to a minimal position g_n in G; observe that any path from h_0 should only be comprised of edges with label $\leq n$. So is it the case that all graphs satisfying Ω admit a vertex from which all reachable edges are $\leq n$?

No, for instance the graph P comprised of only the path $p_0 \xrightarrow{0} p_1 \xrightarrow{1} p_2 \xrightarrow{2} \dots$ satisfies Ω but does not admit such a vertex. Therefore, $P \not\to G$ and G is not universal, so we should find a way of integrating P to our construction. A closer look at P reveals that it seems difficult to endow it with a well-order so as to make it monotone: the only reasonable possible well-order over P would be $p_0 < p_1 < \dots$, and making P monotone would require adding edges $p_i \xrightarrow{i} p_0$, and therefore including a bad path $p_0 \xrightarrow{0} p_1 \xrightarrow{1} p_0 \xrightarrow{0} p_1 \xrightarrow{1} p_2 \xrightarrow{2} p_0 \xrightarrow{0} \dots$. By applying a (simplified variant of) the gadget in the proof of Lemma 7, one obtains the game from Figure 3.10, which disproves Ω's positionality.

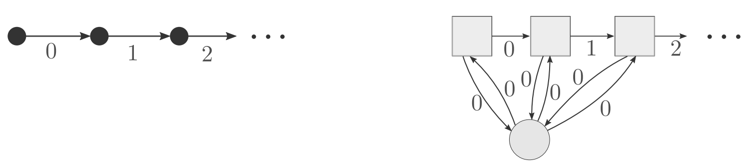

Figure 3.10: On the left the graph P, on the right a game disproving positionality of Ω.

ω-Büchi. We end with a more involved construction of a universal graph. Consider the objective $\Omega = \{w \in \mathbb{N}^\omega \mid \mathtt{Inf}(w) \neq \emptyset\}$, which is a countable union of Büchi objectives. Fix an ordinal α, and consider elements of ω^α, which are finitely supported α-sequences of natural numbers ordered with least-important coordinate first. Let U_α be the graph over ω^α given by edges

$$u \xrightarrow{n} u' \iff u > u' \text{ or } \exists \lambda, [u_\lambda > n \text{ and } u_{\geq\lambda} \geq u'_{\geq\lambda}],$$

where $s' = s_{\geq\lambda}$ is the sequence given by $s'_\beta = s_{\beta+\lambda}$. It is not difficult to check that U is monotone. We claim that U is (κ, Ω)-universal.

Claim 1. *The graph U satisfies Ω.*

To prove the claim, take an infinite path $\pi = u^0 \xrightarrow{n_0} u^1 \xrightarrow{n_1} \dots$ in U, and assume for contradiction that $w = n_0 n_1 \dots \notin \Omega$. Let λ_0 be maximal such that $k_0 = u^0_{\lambda_0}$ is nonzero. Since $w \notin \Omega$, there is some suffix of the path which does not contain any edge $\leq k_0$. Then observe that on this suffix, $u^i_{\geq \lambda_0}$ can not increase when i grows, so it converges to some fixed value; let i_0 be large enough so that $u^{i_0}_{\geq \lambda_0}$ is constant.

Then iterate this reasoning by taking λ_1 to be the maximal coordinate $< \lambda_0$ such that $k_1 = u^{i_0}(\lambda_1)$ is nonzero, restricting to a suffix which does not contain edges $\leq k_1$ and so on. This way we build an infinite sequence $\lambda_0 > \lambda_1 > \dots$, obtaining a contradiction that proves the claim.

Towards proving that small graphs satisfying Ω have a morphism into G_α for some α, define ranks of graphs as the smallest ordinal measure satisfying that

1. the rank of edgeless graphs is 0;
2. if $G \to G'$ then $\mathrm{rk}(G) \leq \mathrm{rk}(G')$;
3. for any n, the graph G' obtained from G by adding all possible edges with weight $< n$ has rank $\mathrm{rk}(G') \leq \mathrm{rk}(G) + 1$;
4. the graph $G_0 + G_1 + \dots$ obtained from an arbitrary ordinal sequence $G_0, G_1, \dots$ by adding to their disjoint union all edges from G_λ to G_β for $\lambda > \beta$, has rank $\mathrm{rk}(G_0 + G_1 + \dots) \leq \sum_\lambda G_\lambda$.

We let $\mathrm{rk}(G) = \infty$ if it cannot be bounded by the above rules.

Claim 2. *If* $\mathrm{rk}(G) = \infty$ *then* G *does not satisfy* Ω*.*

The claim is proved as follows. Given a graph G and a vertex v, let G^v denote the restriction of G to vertices reachable from v. Let $G = G_0$ be a graph with $\mathrm{rk}(G) = \infty$. First we prove that for some v we have $\mathrm{rk}(G^v) = \infty$. Indeed, otherwise, well-order the vertices $v_0, v_1, \dots$ and note that $G \to G^{v_0} + G^{v_1} + \dots$ therefore $\mathrm{rk}(G)$ should be defined thanks to rule 4, a contradiction.

Thus, pick a vertex v_0 such that $\mathrm{rk}(G^{v_0}) = \infty$. Then let G_1 be obtained from G^{v_0} be removing all 0-edges, note that thanks to rule 3, it holds that $\mathrm{rk}(G_1) = \infty$. Further, let v_1 be such that $\mathrm{rk}(G_1^{v_1}) = \infty$. Iterating this process, we construct by induction a path $v_0 \xrightarrow{w_0} v_1 \xrightarrow{w_1} \dots$ such that w_i contains only edges $\geq i$. Therefore G does not satisfy Ω.

We continue with the following claim.

Claim 3. *If* $\mathrm{rk}(G) = \alpha$ *then* $G \to U_\alpha$*.*

To prove the claim, we proceed by induction on the definition of the rank. There are four cases.

1. If G is edgeless, then G maps into the edgeless graph with a single vertex U_0.
2. If $G \to G'$ and, by induction, $G' \to U_\alpha$ then $G \to U_\alpha$.
3. If G is obtained from G' by adding all edges of weight $< n$ and $G' \xrightarrow{h'} U_\alpha$. Then define $h \colon G \to \omega^{\alpha+1}$ by setting $h(v) = (h'(v), n)$, meaning that we append an extra element $u_\alpha = n$ to each sequence $u = h(v) = (u_\lambda)_{\lambda < \alpha}$. It is a direct check that h' defines a morphism $G \to U_{\alpha+1}$.

4. If $G = G_0 + G_1 + \dots$ with $\alpha = \mathrm{rk}(G) = \mathrm{rk}(G_0) + \mathrm{rk}(G_1) + \cdots = \alpha_0 + \alpha_1 + \dots$ and for all λ, $G_\lambda \xrightarrow{h_\lambda} U_\lambda$. Without loss of generality we assume that the $h_\lambda(v)$'s are always nonzero.

   Then we define $u = h(v)$, when $v \in G_\lambda$ and $u' = h_\lambda(v)$, by setting $u_\lambda = u'_\beta$ if $\lambda = \sum_{\lambda' < \lambda} \alpha_{\lambda'} + \beta$ and $u_\lambda = 0$ otherwise. If $v \xrightarrow{n} v'$ is an edge in G, then either $v \in G_\lambda$ and $v' \in G_{\lambda'}$ with $\lambda > \lambda'$ in which case $h(v) > h(v')$, or $v, v' \in G_\lambda$ for some λ, in which case $h_\lambda(v) \xrightarrow{n} h_\lambda(v')$ in U_{α_λ}. In both cases, we obtain that $h(v) \xrightarrow{n} h(v')$ is an edge in U_α and thus $G \xrightarrow{h} U_\alpha$.

Finally, an easy induction yields $|\mathrm{rk}(G)| \leq |G|$, therefore it follows from the two claims above that U_κ is (κ, Ω)-universal.

3.4.4 Positionality over finite degree arenas

One may wonder what happens if the well-foundedness hypothesis is dropped in Theorem 7. It turns out that this provides a characterisation of objectives which are uniformly positional over arenas in which vertices owned by Min (resp. Eve) have finitely many outgoing edges. We say that such arenas have *finite Min-degree* (resp. *Eve-degree*).

Theorem 8. *Let $\Phi \colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ be an objective. If for all κ, there exists a monotone (κ, Φ)-universal graph, then Φ is uniformly positional over arenas with finite Min-degree. The converse holds assuming Φ admits a neutral letter.*

Proof sketch. We start just as in the proof of Theorem 7: take a game $\mathscr{G}$ and a monotone universal graph U for some large enough cardinal so that G_τ has a Φ-preserving morphism h_τ towards U for any strategy τ.

Consider a Min vertex v. Since v has only finitely-many outgoing edges, there exists an outgoing edge $\sigma(v)$ from v such that in any strategy τ and for any path π ending in v, there exists a strategy τ' and a path π' ending in v such that $\tau'(\pi') = \sigma(v)$ and $h_{\tau'}(\pi') \leq h_\tau(\pi)$. Then it is relatively straightforward to check that σ is a positional optimal strategy.

For the converse, we mimic the proof of Theorem 7 except that in the game $\mathscr{G}$, we keep only Min vertices corresponding to finite sets (we could also keep only the pairs). In particular, Min vertices have finite degree. Then the same proof provides a totally ordered (but not necessarily well-founded) universal graph. □

We give two natural examples of objectives which are positional on finite Min-degree arenas but not on arbitrary ones.

Example: Discounted payoff. Let $\lambda \in (0,1)$ and consider the discounted payoff objective over $C = [-N, N] \subseteq \mathbb{R}$ given by

$$\texttt{DiscountedPayoff}_\lambda(\rho) = \lim_k \sum_{i=0}^{k-1} \rho_i \lambda^i \in [-S, S],$$

where $S = N(1-\lambda)^{-1}$. It is not positional over arbitrary arenas; to see this, simply consider the game with a single vertex belonging to Eve, and all loops with weights > 0 (in fact, this game does not even admit optimal non-positional strategies). However, when restricted to finite degree arenas, positional optimal strategies exist. The standard proof, which is presented in **??**, uses Banach's fixed-point theorem (and applies to arenas with finite Eve-degree), but this can also be seen as a non-well-founded monotone universal graph.

Let U be the graph over $[-S,S] \subseteq \mathbb{R}$ given by $u \xrightarrow{t} u' \iff t \leq u - \lambda u'$. Clearly U is monotone (but not well-founded). It is an easy check that for any graph G, the map assigning its value to each vertex defines a (value-preserving) morphism $G \to U$ and therefore U is $(\kappa, \mathtt{DiscountedPayoff})$-universal for any κ.

Example: ω-coBüchi. Consider the objective $\Omega = \{w \in \mathbb{N}^\omega \mid \mathtt{Inf}(w) = \emptyset\}$, which is dual to the ω-Büchi objective studied in Section 3.4.3. To see that Ω is not positional on arbitrary arenas, simply consider the arena with a single Eve-vertex and all possible loops. We now prove that it is positional on arenas with finite Eve-degree via Theorem 8.

Fix a cardinal κ and consider the set κ^ω of maps $u \colon \omega \to \kappa$. Given such a u and $i \in \omega$, we let $u_{<i}$ (resp. $u_{\leq i}$) denote the restriction of u to $[0,i)$ (resp. $[0,i]$). We order κ^ω lexicographically:

$$u > u' \iff \exists i < \omega, [u_{<i} = u'_{<i} \text{ and } u(i) \geq u'(i)].$$

We raise the reader's attention on the fact that the above is not the standard ordinal exponentiation because maps are not assumed finitely supported; in fact this order is not well-founded, for instance $(1,1,1,\dots) > (0,1,1,\dots) > (0,0,1,\dots) > \dots$.

Let U be the graph over κ^ω given by $u \xrightarrow{n} u' \iff u_{\leq n} > u'_{\leq n}$; clearly U is monotone. Observe that U satisfies Ω: if an infinite path $u^0 \xrightarrow{n_0} u^1 \xrightarrow{n_1} \dots$ does not satisfy Ω, then for the smallest n which repeats infinitely often we have $u^0_{<n} \geq u^1_{<n} \geq$ with infinitely many strict inequalities, which is a contradiction.

There remains to prove that any graph $< \kappa$ satisfying Ω has a morphism towards U. Let G be such a graph. Then for each i, we let $h(v)(i)$ be the ordinal counting the number of occurrences of i-edges from v (formally, it is the rank of the well-founded tree obtained by unfolding at v and contracting $\neq i$-edges). Then it is a direct check that h defines a morphism $G \to U$.

3.4.5 Kopczyński's conjecture

Kopczyński conjectured that prefix independent positional objectives are closed under (finite or countable) unions, and gave an example showing that uncountable unions of prefix independent positional objectives may fail to be positional. Note that the prefix independence assumption is required. For instance, the qualitative objectives $\{a^\omega\}$ and $\{b^\omega\}$ are positional but their union is not. Recall also that uniform positionality is granted for prefix independent positional objectives by **??**.

We now discuss some results around this conjecture.

The conjecture fails over finite arenas. We first present a proof that the conjecture fails over finite arenas. This requires the notion of ordered groups which we recall now.

An ordered group is a group G together with a linear order $\leq$ such that left-multiplication is order-preserving: for all group elements h, g, g' it holds that $g \leq g' \implies hg \leq hg'$. From an ordered group $(G, \leq)$ we define an objective $\text{Decreasing}_{(G,\leq)}$ over $C = G$, comprised of all words admitting a decreasing sequence of prefixes; formally

$$\text{Decreasing}_{(G,\leq)} = \{w \in G^\omega \mid (\Pi_{i=0}^{k-1} w_i)_{k\in\mathbb{N}} \text{ has an infinite decreasing subsequence}\}.$$

One may prove, using Theorem 3, that for any ordered group G, the objective Decreasing_G is positional over finite arenas.

The free group with two generators F_2 is defined to be the set of finite words over alphabet $\{a, a^{-1}, b, b^{-1}\}$, up to addition or removal of infixes of the form xx^{-1} or x^-1x for $x \in \{a, b\}$. The group multiplication is concatenation. The following proposition uses the non-trivial fact that free groups admit orderings.

Proposition 3. *Let $(F_2, \leq)$ be the ordered free group with two generators. The objective*

$$\Omega = \text{Decreasing}_{(F_2,\leq)} \cup \text{Decreasing}_{(F_2,\geq)}$$

is not positional over finite arenas.

Proof. Observe that Ω is comprised of all words $w \in F_2^\omega$ such that prefixes of w take infinitely many different values when multiplied in F_2. Indeed, clearly words whose prefixes take finitely many values do not belong to Ω; conversely, a word with infinitely many prefixes has either a strictly growing or a strictly decreasing subsequence by Ramsey's theorem. Consider the arena from Figure 3.11.

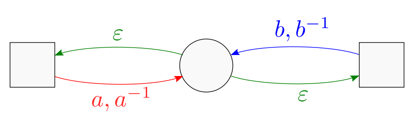

Figure 3.11: The arena for the proof of Proposition 3.

Clearly, Eve wins by alternating both sides, as prefixes of the resulting path admit no cancellations, and therefore take infinitely many values. However, she cannot win positionally: if she commits to, say, playing to the left, then Adam can simply alternate between a and a^{-1}. $\square$

The conjecture holds for countable unions of Σ_2^0 objectives. An objective $\Omega \subseteq C^\omega$ belongs to Σ_2^0 if and only if it can be written as $\Omega = \text{Finite}(L)$ for some language $L \subseteq C^*$ of finite words, where

$$\text{Finite}(L) = \{w \in C^\omega \mid \text{ finitely many prefixes of } w \text{ belong to } L\}.$$

We have the following positive result.

Proposition 4. *Let $\Omega_0, \Omega_1, \ldots \subseteq C^\omega$ be a sequence of (countably many) prefix independent positional objectives in Σ_2^0 admitting neutral letters. Then their union is positional.*

Proof. Let Ω be the union of $\Omega_0, \Omega_1, \dots$. Fix a cardinal κ, we aim to build a monotone well-ordered (κ, Ω)-universal graph. For each i, let $L_i \subseteq C^*$ be such that $\Omega_i = \mathrm{Finite}(L_i)$, and let U_i be a monotone well-ordered (κ, Ω)-universal graph. Then let $\hat{U}$ be obtained from the disjoint copy of the U_i's by adding all edges from U_j to U_i whenever $j > i$. Clearly, $\hat{U}$ is monotone and well-founded. Moreover, any infinite path in $\hat{U}$ ultimately remains in some U_i, therefore $\hat{U}$ satisfies Ω by prefix independence.

We define U to be obtained by setting κ disjoint copies of $\hat{U}$ next to one another, and adding all edges pointing from each copy to (strictly) smaller ones. Again, U is monotone and clearly satisfies Ω. There remains to prove that any graph $< \kappa$ satisfying Ω has a morphism towards U. Take such a graph G, and let G^v denote the restriction of G to vertices reachable from some vertex v.

We first prove that for some vertex v in G, there is a morphism $G^v \to U_i$ for some i. Indeed, assume that this is not the case, and pick a vertex v_0. Since G^{v_0} does not have a morphism towards U_0, and U_0 is (κ, Ω_0)-universal, it must be that G^{v_0} does not satisfy Ω_0: there is a path from v_0 with infinitely many prefixes in L_0. Let $\pi_0 \colon v_0 \xrightarrow{w_0} v_1$ be a path from v_0 such that $w_0 \in L_0$.

Likewise, G^{v_1} does not satisfy Ω_1, and thus there is a path from v_1 with label $w_1' \notin \Omega_1$, which implies by prefix independence that $w_0 w_1' \notin \Omega_1$, and therefore there is a finite path $\pi_1 \colon v_1 \xrightarrow{w_1} v_2$ such that $w_0 w_1 \in L_1$. We continue this construction in a round-robin fashion (for instance, $010120123\dots$) to construct a path from v_0 whose label w has infinitely many prefixes in each L_i. This contradicts the fact that G satisfies Ω.

Therefore there is some vertex v_0 such that $G^{v_0} \to U_i$ for some i, and therefore $G^{v_0} \xrightarrow{h_0} \hat{U}$. Now we map G to the first copy of $\hat{U}$ in U, and conclude by transfinite induction[2]. □

As a consequence, we may easily establish positionality of mean payoff objectives over arbitrary arenas.

Corollary 4. *The qualitative objective* $\texttt{MeanPayoff}^+_{<0} = \{\rho \in \mathbb{Z}^\omega \mid \limsup_k \frac{1}{k} \sum_{i=0}^{k-1} \rho_i < 0\}$ *is uniformly positional over arbitrary arenas.*

We raise the reader's attention on the fact that the choice of lim sup as well as the strict equality are required for the result to hold.

Proof. For $n > 0$, consider the objective

$$\texttt{TiltedEnergy}_n = \{\rho \in \mathbb{Z}^\omega \mid \exists N, \forall k, \sum_{i=0}^{k-1} (\rho_i + 1/n) < N\}.$$

For each n, renaming letters using the map $w \mapsto nw - 1$ embeds $\texttt{TiltedEnergy}_n$ into $\texttt{Energy}^+_{<\infty}$ and therefore these objectives are positional. Now it suffices to observe that these objectives belong to Σ^0_2, are prefix independent, and that $\texttt{MeanPayoff}^+_{<0}$ is their countable union. □

Another class of objectives where Kopczyński's conjecture is known to hold is ω-regular objectives; more discussion below.

[2] In [Ohl23], the property satisfied by $\hat{U}$ is called *almost universality*. One can always construct a universal graph from an almost universal graph by transfinite induction [Ohl23, Lemma 4.5].

3.4.6 Positionality of ω-regular objectives

Recall that an objective is ω-regular if it can be recognised by a deterministic parity automaton. In this section, we consider parity automata with ε-transitions; these are denoted by $q \xrightarrow{\varepsilon:x} q'$, where $x \in \mathbb{N}$ is some priority.

We say that an automaton with priorities in $\{0,\dots,d+1\}$, where d is even, is ε-*complete* if

- the relations $\xrightarrow{\varepsilon:1}, \xrightarrow{\varepsilon:3}, \dots, \xrightarrow{\varepsilon:d+1}$ all define total preorders, each refining the previous one; and
- for each even $x \in \{0,2,\dots,d\}$, $q \xrightarrow{\varepsilon:x} q'$ holds if and only if $q' \xrightarrow{\varepsilon:x+1} q$ does not hold. Stated differently, the relation $\xrightarrow{\varepsilon:x}$ is the strict preorder corresponding to the (non-strict) preorder $\xrightarrow{\varepsilon:x+1}$.

The decreasing sequence of total preorders is better visualised as a tree of height $d/2$ whose leaves correspond to states of the automaton; we refer to Figure 3.12 below for an example. On an intuitive level, $q \xrightarrow{\varepsilon:x} q'$ means that "q is much better than q'", since one may, at any point, move from q to q' and be rewarded with an even priority x on the way. On the contrary, $q' \xrightarrow{\varepsilon:x+1} q$ means that "q' is not much worse than q", since one may at any point move from q' to q at the cost of reading an odd priority $x+1$. In other words, in a ε-complete automaton, one may say that q and q' are comparable for priority x.

An automaton is ε-completable if one may add ε-transitions so as to make it ε-complete without augmenting the language.

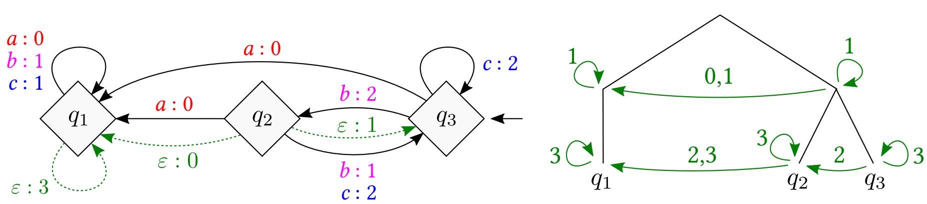

Figure 3.12: On the left, an automaton recognising the set of words with infinitely many a's, or with no occurrence of a and finitely many occurrences of the factor bb. All ε-transitions depicted on the right-hand side may be added to the automaton (like the three dashed ones) without augmenting the language. Therefore the automaton is ε-completable.

Positionality is well-understood for ω-regular objectives thanks to the following characterisation.

Theorem 9. *Let Ω be an ω-regular objective. The following are equivalent:*

- *Ω is positional over finite arenas of Eve;*
- *Ω is recognised by a deterministic ε-completable automaton;*

- *Ω is positional over arbitrary (including infinite) arenas.*

For instance, the language defined in Figure 3.12 is positional. Note that Theorem 9 implies finite-to-infinite and one-to-two-player lifts for ω-regular objectives. Moreover, positionality of an ω-regular language given by a deterministic parity automaton is decidable in polynomial time. Last, a stronger variant of Kopczyński's conjecture in known to hold for ω-regular objectives, as a consequence of Theorem 9.

Proposition 5. *Let Ω_1 and Ω_2 be ω-regular positional objectives, and assume that Ω_1 is prefix independent. Then $\Omega_1 \cup \Omega_2$ is positional.*

This proposition is incomparable with Proposition 4, since ω-regular objectives span a strict subclass of Δ^0_3 which does not contain Σ^0_2.

3.5 Generalisations to memory requirements

In this chapter, we have so far focused on general results for bi-positionality or positionality over finite or infinite arenas. In this last section, we show that many of these results can be generalised to results about finite memory (cf. **??**).

We will mostly skip proofs, as simpler arguments for the positional cases already highlight multiple of the technical challenges. References to the complete proofs are found in the bibliographic references at the end of the chapter.

Before addressing these results, we spend some time in Section 3.5.1 discussing the definition of memory models and some natural varieties of it that arise.

3.5.1 Memory models

Let X be a set (which will usually be either the set of edges of a game, or the set of colours labelling those edges). An *X-memory structure $\mathcal{M}$* is a deterministic automaton over X without an acceptance condition. Formally, an X-memory structure is a tuple $\mathcal{M} = (M, m_0, \delta)$, where M is a set of memory states, $m_0 \in M$ is the initial state and $\delta \colon M \times X \to M$ is the update function, which is extended to $\delta^* \colon X^* \to M$ by $\delta^*(\varepsilon) = m_0$ and $\delta^*(\rho x) = \delta(\delta^*(\rho), x)$. The size of a memory structure is its number of states.

Different types of memory

General memory. Let $\mathcal{G}$ be a game. A *general memory structure* for $\mathcal{G}$ is an E-memory structure $\mathcal{M} = (M, m_0, \delta)$, where E is the set of edges of the arena of $\mathcal{G}$. Let $\sigma \colon \text{Paths} \to E$ be a strategy for Eve/Max. We say that σ can be *defined based on $\mathcal{M}$* if there is a function

$$\hat{\sigma} \colon V_{\text{Eve}} \times M \to E$$

such that $\sigma(\pi) = \hat{\sigma}(\text{last}(\pi), \delta^*(\pi))$. In this case, we use expressions such as "σ uses $|M|$ memory states".

Let $\Phi \colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ be an objective. We define the *general memory requirements* of Φ, written $\text{mem}_{\text{gen}}(\Phi)$, as the minimal $m \in \mathbb{N} \cup \{\infty\}$ such that for every game $\mathcal{G}$ using Φ as an objective, Max has an optimal strategy using m memory states. We note that this induces a definition of memory requirements for qualitative objectives. Also, observe that we adopt a *uniform* version of the memory: the strategy using m memory states is optimal from any given vertex. As in the case of positionality, we may talk about the *memory requirements of* Φ *over* a class of arenas (such as finite or without uncoloured edges). The *general memory requirements over finite arenas* of Φ are denoted $\mathrm{mem}^{\mathrm{fin}}_{\mathrm{gen}}(\Phi)$. Also, one may define the *limit-optimal general memory requirements* of a quantitative objective Φ, as the natural generalisation of Definition 2.

We note that the definition of memory requirements is an asymmetric one; we take the point of view of Max/Eve and focus on her optimal strategies.

Chromatic memory. A chromatic memory is a memory structure that can only observe the information about the colours produced in a game, and not about the particular edges that are taken.

Formally, let $\mathcal{G}$ be a game and $\mathfrak{c}\colon E \to C$ a colouring of its edges. A *chromatic memory structure* for $\mathcal{G}$ (with respect to C) is a C-memory structure. The update function $\delta\colon M \times C \to M$ of such a C-memory structure induces a function $\hat{\delta}\colon M \times E \to M$ by $\hat{\delta}(m,e) = \delta(m, \mathfrak{c}(e))$. Therefore, a chromatic memory structure $\mathcal{M}$ for $\mathcal{G}$ induces a general memory structure $\hat{\mathcal{M}}$. We say that a strategy σ in $\mathcal{G}$ can be defined based on a chromatic memory $\mathcal{M}$ if it can be defined based on the general memory $\hat{\mathcal{M}}$.

We define the *chromatic memory requirements* of an objective $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$, noted $\mathrm{mem}_{\mathrm{chrm}}(\Phi)$, as the minimal $m \in \mathbb{N} \cup \{\infty\}$ such that for every game $\mathcal{G}$ using Φ as an objective, Max has an optimal strategy based on a chromatic memory with m states. The *general memory requirements over finite arenas* of Φ, defined analogously, are denoted $\mathrm{mem}^{\mathrm{fin}}_{\mathrm{chrm}}(\Phi)$.

Arena-independent memory. Let $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ be an objective. A C-memory structure $\mathcal{M}$ is an *arena-independent memory* for Φ if for all game $\mathcal{G}$ using Φ as objective, Max has an optimal strategy based on $\mathcal{M}$.

We write $\mathrm{mem}_{\mathrm{arInd}}(\Phi)$ for the minimal size of an arena-independent memory for Φ (we let $\mathrm{mem}_{\mathrm{arInd}}(\Phi) = \infty$ if such an arena-independent memory does not exist). Similarly, we let $\mathrm{mem}^{\mathrm{fin}}_{\mathrm{arInd}}(\Phi)$ be the minimal size of an arena-independent memory that Max can use to define optimal strategies in finite arenas.

We say that $\mathcal{M}$ *suffices for* Φ *for a player over a class of arenas* if for all arenas from this class, this player has strategies based on $\mathcal{M}$ that are optimal for Φ from every vertex. This definition includes the uniformity of the strategies.

Comparison between the models

We have presented three different models of memory: general, chromatic, and arena-independent, by successively adding further restrictions. It is therefore clear that, for any objective Φ:

$$\mathrm{mem}_{\mathrm{gen}}(\Phi) \leq \mathrm{mem}_{\mathrm{chrm}}(\Phi) \leq \mathrm{mem}_{\mathrm{arInd}}(\Phi).$$

We show that the second inequality is in fact an equality: we can assume that a single chromatic memory structure of minimal size suffices to play optimally in all games. However, the first inequality might be strict and arbitrary large, already for Muller objectives.

Proposition 6. *Let* $\Phi\colon C^\omega \to \mathbb{R}\cup\{\pm\infty\}$ *be an objective. Then,* $\mathrm{mem}_{\mathrm{chrm}}(\Phi) = \mathrm{mem}_{\mathrm{arInd}}(\Phi)$. *Moreover, if* C *is finite,* $\mathrm{mem}^{\mathrm{fin}}_{\mathrm{chrm}}(\Phi) = \mathrm{mem}^{\mathrm{fin}}_{\mathrm{arInd}}(\Phi)$.

Proof. It is clear that $\mathrm{mem}_{\mathrm{chrm}}(\Phi) \le \mathrm{mem}_{\mathrm{arInd}}(\Phi)$ (resp. $\mathrm{mem}^{\mathrm{fin}}_{\mathrm{chrm}}(\Phi) \le \mathrm{mem}^{\mathrm{fin}}_{\mathrm{arInd}}(\Phi)$). We show that this inequality cannot be strict. Let $k = \mathrm{mem}_{\mathrm{chrm}}(\Phi)$, and assume by contradiction that $k < \mathrm{mem}_{\mathrm{arInd}}(\Phi)$ (resp. $k < \mathrm{mem}^{\mathrm{fin}}_{\mathrm{arInd}}(\Phi)$). Let $\mathrm{Memories}_{\le k}$ be the set of all C-memory structures with at most k states. We remark that if C is finite, so is the set $\mathrm{Memories}_{\le k}$. As $k < \mathrm{mem}_{\mathrm{arInd}}(\Phi)$, for each memory structure $\mathscr{M}$ of size less or equal k, there is a game $\mathscr{G}_{\mathscr{M}}$ for which Max has no optimal strategy based on $\mathscr{M}$. Let $\mathscr{G}'$ be the disjoint union of all the games $\mathscr{G}_{\mathscr{M}}$, for $\mathscr{M} \in \mathrm{Memories}_{\le m}$ (we note that, if C is finite, this is a finite arena). As $k = \mathrm{mem}_{\mathrm{chrm}}(\Phi)$, there is a chromatic memory $\mathscr{M}'$ with no more than k states defining an optimal strategy in $\mathscr{G}'$. In particular, $\mathscr{M}'$ defines an optimal strategy in $\mathscr{G}_{\mathscr{M}'}$, a contradiction. $\square$

Proposition 7. *There is a Muller objective* $\Omega \subseteq C^\omega$ *such that* $\mathrm{mem}_{\mathrm{gen}}(\Omega) < \mathrm{mem}_{\mathrm{chrm}}(\Omega)$.

Proof. Let $C_n = \{1,\dots,n\}$ be a set of n colours, and consider the family of subsets

$$\mathscr{F}_n = \{A \subseteq C \mid |A| = 2\}\,.$$

The Muller objective associated to $\mathscr{F}_n$ is

$$\mathtt{Muller}(\mathscr{F}_n) = \{\rho \in C^\omega \mid \rho \text{ visits exactly two different colours infinitely often}\}\,.$$

Using the characterisation from **??**, it is an easy exercise to check that the general memory requirements of $\mathtt{Muller}(\mathscr{F}_n)$ are exactly 2, for $n \ge 2$.

We claim that the chromatic memory requirements of $\mathtt{Muller}(\mathscr{F}_n)$ are at least n. We will show that this objective admits no arena independent memory of size strictly less than n, and conclude by Proposition 6.

Assume by contradiction that $\mathscr{M} = (M, m_0, \delta)$ is an arena-independent memory for $\mathtt{Muller}(\mathscr{F}_n)$ and $|M| < n$. For each $c \in C_n$, let m_c be a state of $\mathscr{M}$ in a cycle labelled c^+. Since $|M| < n$, there must be two different colours $a, b \in C_n$ such that $m_a = m_b$. Let $k_1, k_2 > 0$ such that $\delta^*(m_a, a^{k_1}) = m_a$ and $\delta^*(m_a, b^{k_2}) = m_a$. Let $u \in C^*$ such that $\delta^*(m_0, u) = m_a$. Consider the game $\mathscr{G}$ in Figure 3.13. It is clear that Eve wins $\mathscr{G}$, but not with a strategy based on $\mathscr{M}$, as the memory structure would always be at the state m_a whenever we are in vertex v, so it would always take the same edge from there. $\square$

In **??** was presented a characterisation of the general memory requirements of Muller objectives over partially-coloured arenas. We show that this last hypothesis is necessary, as the memory requirements over partially or fully-coloured arenas might differ.

Proposition 8. *There is a Muller objective* $\Omega \subseteq C^\omega$ *such that the general memory requirements of* Ω *over fully-coloured arenas are strictly smaller than its general memory requirements over partially-coloured arenas.*

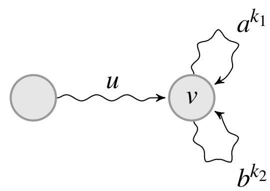

Figure 3.13: Game won by Eve by alternating the loops a^{k_1} and b^{k_2}.

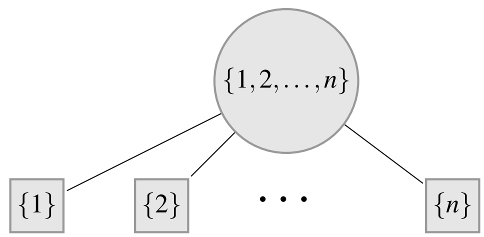

Figure 3.14: The Zielonka tree for $\mathscr{F}_n = \{A \subseteq C \mid |A| > 1\}$.

Proof. Let $C_n = \{1,\ldots,n\}$ be a set of n colours, and consider the family of subsets

$$\mathscr{F}_n = \{A \subseteq C \mid |A| > 1\}.$$

The Muller objective $\texttt{Muller}(\mathscr{F}_n)$ consists therefore in the set of sequences that contain at least two different colours infinitely often. Its Zielonka tree is depicted in Figure 3.14. The characterisation from **??** implies that the general memory requirements of $\texttt{Muller}(\mathscr{F}_n)$ over partially-coloured arenas are n.

We claim that the general memory requirements of $\texttt{Muller}(\mathscr{F}_n)$ over fully-coloured arenas are 2.

Let $\mathscr{G}$ be fully-coloured game using the objective $\texttt{Muller}(\mathscr{F}_n)$. By prefix independence we can restrict our study to Eve's winning region, so we assume without loss of generality that she wins from every vertex in $\mathscr{G}$. We associate to each vertex $v \in V_{\text{Eve}}$ a colour $c(v) \in C_n$ such that there is an outgoing transition from v coloured with $c(v)$ (such a colour exists by the fully-coloured assumption). We obtain in this way a partition $V_{\text{Eve}} = V_1 \uplus \cdots \uplus V_n$, with $v \in V_{c(v)}$. We fix a mapping $\sigma_0 : V_{\text{Eve}} \to E$ picking one outgoing edge labelled with $c(v)$ from v. For each $x \in C_n$, we consider the reachability objective $\Omega_{\neg x} =$ "reach a colour different from x". Since Eve can ensure to reach two different colours in $\mathscr{G}$ from any vertex, she can in particular ensure the objective $\Omega_{\neg x}$. We fix a positional strategy σ_x ensuring to see some colour $y \neq x$ and coinciding with σ_0 outside V_x.

We define a general memory structure for $\mathscr{G}$ with two states m_0, m_1 as follows: the state m_0 is used to remember that we have to see the colour x corresponding to the component V_x that we are in. As soon as we arrive to a vertex controlled by Eve, we use the next transition to accomplish this and we can change to state m_1 in $\mathscr{M}$. The state m_1 serves to follow the positional strategy σ_x reaching one colour different from x. We change to state m_0 if we arrive to some state in V_{Eve} not in V_x (this ensures that we will see one colour different from x), or if Eve produces a colour different from x staying in the component V_x. If she does not produce this colour and we do not go to a vertex in $V_{\text{Eve}} \setminus V_x$, that means that (since σ_x ensures that we will see a colour different from x), Adam will take some transition coloured with some colour different from x. □

3.5.2 One-to-two-player lift with memory

Recall Theorem 3: if an objective is uniformly bi-positional over finite *one-player* arenas, then it is uniformly bi-positional over finite arenas. This actually also holds if we replace "positional strategies" in this statement by "strategies using memory $\mathscr{M}$" where $\mathscr{M}$ is a fixed chromatic memory structure.

Theorem 10 (One-to-two-player lift with memory)**.** *Let Φ be an objective and $\mathscr{M}$ be a chromatic memory structure. If $\mathscr{M}$ suffices for Φ for both players over finite one-player arenas, then $\mathscr{M}$ suffices for Φ for both players over finite arenas.*

It may seem surprising that we consider the same chromatic memory structure for both players; for a given objective, memory requirements of each player may vary wildly (see for instance Rabin and Streett games). However, even if memory requirements are different in one-player games, this result still gives a useful information: if chromatic memory structure $\mathscr{M}_1$ (resp. $\mathscr{M}_2$) suffices to play optimally in the finite one-player arenas of Max (resp. Min), then the direct product $\mathscr{M}_1 \times \mathscr{M}_2$ also suffices for both players in their one-player arenas (any strategy based on $\mathscr{M}_1$ or $\mathscr{M}_2$ can also be defined based on $\mathscr{M}_1 \times \mathscr{M}_2$). Hence, we can conclude that $\mathscr{M}_1 \times \mathscr{M}_2$ suffices to play optimally for both players in all finite arenas. Under these specific hypotheses, it is unknown whether $\mathscr{M}_1$ (resp. $\mathscr{M}_2$) always suffices for Max (resp. Min) in all finite arenas. In other words, it is unknown whether the product with the memory structure for the one-player arenas of the opponent is necessary in the two-player arenas of a player.

Example 4. *Let $C = \{\perp, \top\} \times \mathbb{Z}$. We consider a quantitative objective $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ that is in some way a lexicographic combination of a reachability and a mean payoff, with the priority given to the reachability objective along the first component:*

$$\Phi((a_1,z_1)(a_2,z_2)\dots) = \begin{cases} -\infty & \text{if } a_i = \perp \text{ for all } i \geq 1, \\ \texttt{MeanPayoff}^+(z_1 z_2 \dots) & \text{otherwise}. \end{cases}$$

We want to study the memory requirements of this objective using Theorem 10. We show that the memory structure $\mathscr{M}$ with two states remembering whether $\top$ has already been seen suffices for both players over finite arenas. Memory structure $\mathscr{M}$ is depicted in Figure 3.15 along with two examples of one-player arenas (one of each player) in which $\mathscr{M}$ suffices to play optimally, but memoryless strategies would not suffice.

Over her finite one-player arenas, Max can always play optimally with $\mathscr{M}$. From any given state, if $\top$ was not seen yet, Max aims for the occurrence of $\top$ that then allows for the largest possible mean payoff, which can be obtained with a memoryless strategy. If $\top$ was already seen, Max simply aims for maximising the mean payoff.

Over his finite one-player arenas, Min can also play optimally with $\mathscr{M}$: if $\top$ was not seen yet, Min plays in a memoryless way for $\texttt{Safety}(\top)$ if possible and, if not or if $\top$ was already seen, Min aims for minimising the mean payoff.

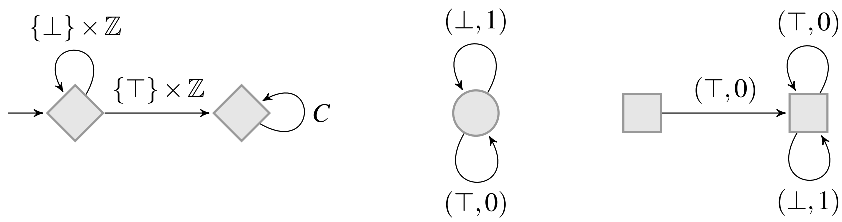

Figure 3.15: Memory structure $\mathscr{M}$ used in Example 4 (left). Arena in which Max needs $\mathscr{M}$ to play optimally for Φ (center). Arena in which Min needs $\mathscr{M}$ to play optimally in a uniform way for Φ (right). Note that memoryless strategies suffice to play optimally in a non-uniform way over finite one-player arenas of Min in general.

By Theorem 10, memory structure $\mathscr{M}$ suffices to play optimally over finite arenas for both players.

In Theorem 10, we considered the hypothesis of a fixed memory structure for both players, which is stronger than the existence of optimal finite memory strategies for both players (for which the memory structure and its size may depend on the arena). If we consider the weaker assumption with finite memory strategies, such a general one-to-two-player lift does not hold [BRO$^+$22, Section 3.4].

3.5.3 From memory to automata

We have discussed in Section 3.2 the central place of the parity objectives: they are bi-positional over arbitrary arenas (**??**), they are the only such prefix independent qualitative objectives (Theorem 5), and, more generally, any bi-positional objective can be recognised by a parity automaton built on top of its automaton of residuals (Theorem 6). In this section, we show that we can express a more general connection between the parity automata and, not only positionality, but memory requirements.

As a consequence of **??** (**??**) and the bi-positionality of parity objectives, we obtain that a deterministic parity automaton recognising an objective Ω can be used (as a memory structure) to define optimal strategies for both players in any game with objective Ω. In particular, any ω-regular objective is finite memory determined.

Proposition 9. *Let $\mathbf{A} = (Q, q_0, \delta, A)$ be a deterministic parity automaton. Let $\mathscr{M}_{\mathbf{A}} = (Q, q_0, \delta)$ be the chromatic memory structure underlying $\mathbf{A}$. Memory structure $\mathscr{M}_{\mathbf{A}}$ suffices for objective $L(\mathbf{A})$ for both players over all arenas.*

We claim that it is possible to go in the other direction. For a given objective, from a chromatic memory structure $\mathscr{M}$ that suffices for both players, it is possible to recover (with a small blow-up) a deterministic parity automaton recognising the objective. This result makes use of the *automaton of residuals* $\mathbf{R}_\Omega$ of an objective Ω, defined in Section 3.2.2.

Theorem 11. *Let Ω be a qualitative objective and $\mathscr{M}$ be a chromatic memory structure. If $\mathscr{M}$ suffices for Ω for both players over all arenas, then (i) Ω has finitely* *many residuals and* (ii) Ω *is recognised by a deterministic parity automaton on top of* $\mathbf{R}_\Omega \times \mathscr{M}$.

A corollary of this result is a characterisation of the ω-regular languages through a strategic perspective.

Corollary 5. *A qualitative objective* Ω *is* ω*-regular if and only if some chromatic memory structure suffices for* Ω *for both players over all arenas.*

Proof. Any ω-regular objective is recognised by a deterministic parity automaton. Such an automaton, as discussed in Proposition 9, can be used as a memory structure to play optimally in all arenas for both players.

Conversely, if $\mathscr{M}$ suffices for Ω for both players over all arenas, then Ω can be recognised by a deterministic parity automaton by Theorem 11. Hence, Ω is ω-regular. $\square$

3.5.4 Universal graphs and memory

In this section, we illustrate how monotone universal graphs (defined in Section 3.4) can be extended to prove upper bounds on the memory of given objectives.

We now consider partially ordered C-graphs, which are C-graphs together with a partial order on their vertices. Such a graph is monotone if

$$u \geq u' \xrightarrow{c} v \geq v' \implies u \xrightarrow{c} v',$$

just as in Section 3.4. The width of a partially ordered C-graph is the least upper bound on the size of its antichains.

Next, we should adapt the definition of universality. Given a quantitative objective Φ, we say that a tree-to-graph morphism $h\colon T \to G$ preserves the value of the root if the value of $h(t_0)$ in G, where t_0 is the root of T, is not greater than the value of t_0 in T. Given a cardinal κ, we say that a graph U is (κ,Φ)-universal for trees if all trees of cardinality $< \kappa$ admit a morphism towards U which preserves the value of the root.

Remark 8. *For a well-ordered (i.e. well-founded and totally ordered) monotone graph, as in Theorem 7, and for an infinite cardinal* κ*, the notions of being* (κ,Φ)*-universal and* (κ,Φ)*-universal for trees coincide.*

Remark 9. *We often work with prefix independent qualitative objectives* Ω*. In this case, the definitions can be simplified as follows: a graph* U *is* (κ,Ω)*-universal for trees if it satisfies* Ω *and all trees of cardinality* $< \kappa$ *which satisfy* Ω *have a morphism towards* U*.*

Monotone universal graphs characterise a variant of memory called ε*-memory*, for which the player is not allowed to update the memory state when an uncoloured edge ε is read. We may now state the characterisation.

Theorem 12. *Let* $\Phi\colon C^\omega \to \mathbb{R} \cup \{\pm\infty\}$ *be a quantitative objective and* $m \in \mathbb{N}$*. If for all* κ*, there exists a* (κ,Φ)*-universal graph which is monotone, well-founded and of width* $\leq m$ *(resp. which is monotone and of width* $\leq m$*), then* Φ *requires* $\leq m$ *states of* ε*-memory over arbitrary arenas (resp. over arenas with finite Min-degree). The converse holds assuming* Φ *is qualitative and admits a neutral letter.*

The proof is an extension of that of Theorem 7 to the setting of memory.

Example 5. *Consider the Muller objective $\Omega = \{w \in \{a,b\}^\omega \mid \mathtt{Inf}(w) = \{a,b\}\}$. It is easy to see that Ω requires at least two states of memory, for instance over the one-vertex game controlled by Eve with the two possible self-loops. Fix a cardinal κ, and consider the graph U over $\{a,b\} \times \kappa$ given by edges*

$$(x,\lambda) \xrightarrow{y} (x',\lambda') \iff [x = y = x' \text{ and } \lambda > \lambda'] \text{ or } x \neq y = x',$$

and the order $(x,\lambda) \geq (x',\lambda') \iff [x = x' \text{ and } \lambda \geq \lambda']$ (see Figure 3.16).

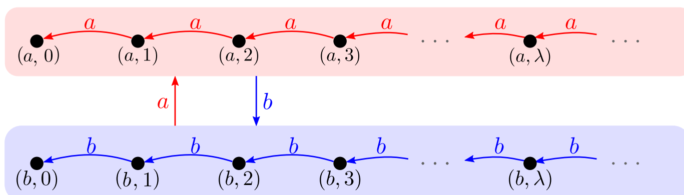

Figure 3.16: The universal graph U in Example 5

Clearly U is monotone and has width 2. One can check that U satisfies Ω, and a proof that U is (κ,Ω)-universal for trees follows the same lines as for the Büchi objective (see Section 3.4.1). We conclude that Ω requires two states of memory (and even of ε-memory).

As an application of Theorem 12 we may characterise the memory requirements of *closed* objectives.

An objective $\Omega \subseteq C^\omega$ is *closed* if it can be written as

$$\Omega = \mathtt{Safety}(S) = \{w \in C^\omega \mid w' \notin S \text{ for any prefix } w' \text{ of } w\},$$

for some language $S \subseteq C^*$.

Theorem 13. *Let Ω be a closed objective, and consider its set of residuals* $\mathrm{Res}(\Omega)$ *ordered by inclusion.*

1. *If* $\mathrm{Res}(\Omega)$ *has antichains of size m, then Ω requires $\geq m$ states of memory, already over finite degree arenas.*

2. *If antichains of* $\mathrm{Res}(\Omega)$ *have size $\leq m$, then Ω requires $\leq m$ states of memory (even of ε-memory) on arenas of finite Eve-degree.*

3. *If moreover,* $\mathrm{Res}(\Omega)$ *is well-founded, then the previous statement extends to arbitrary arenas.*

Proof. 1. Take an antichain $\{u_1^{-1}\Omega,\dots,u_m^{-1}\Omega\}$ and for each $i \neq j$, fix a finite word $w_{i,j} \in u_i^{-1}\Omega \setminus u_j^{-1}\Omega$. Consider the game with $m+1$ Adam vertices $v_0,v_1,\dots,v_m$ and one Eve vertex e, with all transitions $v_0 \xrightarrow{u_i} e$, $e \xrightarrow{\varepsilon} v_i$ and $v_i \xrightarrow{w_{i,j}}$ (additional

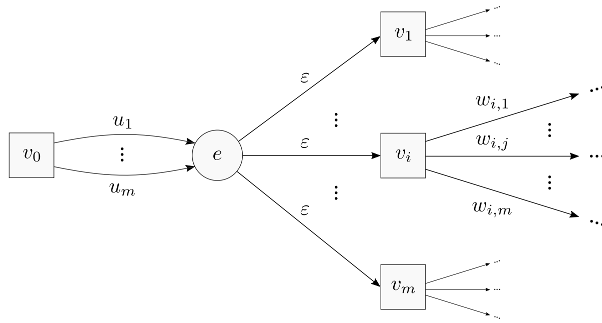

Figure 3.17: The game for the proof of the first item.

vertices within these disjoint finite and infinite paths are added adequately). See Figure 3.17.

Then Eve wins with m states of memory by choosing the edge $e \xrightarrow{\varepsilon} v_i$ whenever Adam picked $v_0 \xrightarrow{u_i} e$. However, if less than m memory states are used by a strategy σ, then by pigeonhole, there are $i \neq j$ and k such that $v_0 \xrightarrow{u_i} e \xrightarrow{\varepsilon} v_k$ and $v_1 \xrightarrow{u_j} e \xrightarrow{\varepsilon} v_k$ are consistent with σ, and thus, assuming without loss of generality that $k \neq i$, Adam wins by playing $v_0 \xrightarrow{u_i} e \xrightarrow{\varepsilon} v_k \xrightarrow{w_{k,i}}$.

2. Consider the partially ordered graph U over $\mathrm{Res}(\Omega) \setminus \{\emptyset\} \cup \{\top\}$ (ordered by inclusion and with maximal element $\top$) with all possible outgoing edges from $\top$, and edges $[u] \xrightarrow{c} [v] \iff v^{-1}\Omega \subseteq (uc)^{-1}\Omega$. Then U is monotone, and has width $\leq m$. It is an easy check that vertex $[u]$ satisfies the objective $u^{-1}\Omega$ in U; in particular, $[\varepsilon]$ satisfies Ω.

   Therefore, thanks to Theorem 12, it suffices to prove that any tree T has a morphism towards U which preserves the value of the root. Take a tree T and let t_0 denote the root. If t_0 does not satisfy Ω, then preserving the value of the root is trivial, so simply map all T to $\top$.

   So assume that t_0 satisfies Ω, and define let $h(t) = [u]$ where u labels the unique path from t_0 to t. Then h is a morphism, and $h(t_0) = [\varepsilon]$ which satisfies Ω, so h preserves the value of the root.

3. Follows directly from the previous item, and Theorem 12. $\square$

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Arguably, the first result related to positionality is Shapley's proof of existence of stationary strategies in stochastic discounted games [Sha53] (we refer to **??** for more details on stochastic games). The relevance of the question of strategy complexity was highlighted by Rabin in 1958, who showed that there are games in which one player can force a victory, but for which no computable winning strategy exists [**?**]. In 1969, Büchi and Landweber proved that ω-regular winning objectives are finite-memory determined over finite games [BL69]. In 1979, Ehrenfeucht and Mycielski provided the first proof of bi-positionality of mean-payoff games [EM79]. Interest in strategy complexity increased due to the quest for a simplified proof of Rabin's theorem [**?**]: In 1982, Gurevich and Harrington (and independently Büchi [Büc77, **?**]) explicitly described memory structures for Muller games over infinite tree-shaped arenas; in 1991 Emerson and Jutla proved bi-positionality of parity games [EJ91] (the same result was obtained independently by Mostowski [Mos91]), and in 1994, Klarlund proved the positionality of Rabin objectives [**?**].

From the late 1990s, a series of results focused on understanding more generally the class of objectives that are (bi-)positional. In 1998, Zielonka showed that the class of positional (resp. bi-positional) Muller objectives is exactly the class of Rabin objectives (resp. parity objectives), both over finite and infinite games [Zie98]. Extending this result, Dziembowski, Jurdziński, and Walukiewicz characterised the precise general memory requirements of Muller objectives [DJW97]. In 2005, Gimbert and Zielonka characterised objectives that are positional over finite arenas by means of two properties called *monotony* and *selectivity* [GZ05]. As a consequence, they obtained the one-to-two-player lift stated in Theorem 3. The proof we presented here is closer to the one in [**?**]; it is conceptually simpler (avoids going through monotony and selectivity) and, as is shown in that paper, can be generalised to a result about stochastic games. In 2006, Colcombet and Niwiński proved that parity objectives are the only prefix-independent positional objectives over infinite arenas [CN06] (corresponding to Theorem 5). The generalisation of this result to non-prefix independent objectives (Theorem 6) was first stated by Casares and Ohlmann [**?**]; the necessity of recognisability by the automaton of residuals for positionality (and more generally, for finite memory determinacy) was proven by Bouyer, Randour and Vandenhove [BRV23].

The first thorough study of positionality was carried out in Kopczyński's PhD thesis [Kop08] (where it is called *half-positionality*). He presents several sufficient conditions for positionality of prefix-independent objectives, including the submixing property, stated here in Theorem 1 (submixing qualitative objectives are therein called *concave*). The sufficient condition of Theorem 1 has been weakened in [**?**] for qualitative objectives, mostly relaxing prefix independence to a property allowing finite prefixes to influence the outcome of the game, but in a totally ordered fashion. Theorem 1 has been generalised to MDPs [Gim07] (also extending it for the first time to quantitative objectives) and then to stochastic games [**?**]: if an objective is prefix independent and submixing, it is even positional (in *pure* strategies) over finite *stochastic* arenas. One of the central topics in Kopczyński's thesis [Kop06] is the question of whether prefix independent positional objectives are closed under union, discussed in Section 3.4.5. The counterexample for finite arenas we presented is due to Kozachinskiy [**?**]. Ohlmann and Skrzypczak proved the conjecture for Σ_2^0 objectives [**?**], and Casares and Ohlmann proved it for ω-regular languages [**?**].

Universal graphs first appeared as a generalisation of universal trees [CF19, CFGO22], a tool to describe and analyse the quasi-polynomial algorithms for parity games (we refer to **??** for a definition of universal trees and to the references of **??** for further discussions). The theory of universal graphs was developed by Ohlmann in his PhD Thesis [Ohl21, Ohl23], where he proved the characterisation of positionality stated here in Theorem 7. This result was lifted to characterise memory requirements of objectives in 2023, as presented in Section 3.5.4 [**?**]. In 2024, an effective characterisation of positionality for ω-regular languages (Theorem 9) was given by Casares and Ohlmann [**?**].

We mention two other tools for positionality that were omitted from this chapter. First, for prefix independent objectives, the work of Aminof and Rubin [AR17] gives a sufficient condition for uniform positional determinacy that focuses on the properties of winning cycles. Second, the positionality of the discounted payoff and generalisations of it was discussed at length in [**?**], which gives three different proofs of its positionality.

The notion of chromatic memory was introduced by Kopczyński [Kop06], who showed that the chromatic and arena-independent memory requirements coincide for all objectives (stated in Proposition 6). He left open the question on whether these moreover coincide with the general memory requirements. The separation between general and chromatic memory requirements (Proposition 7, also Proposition 8) is due to Casares [**?**]. The extension of the one-to-two-player lift to constant chromatic memory (Theorem 10) was shown in [BRO$^+$22], with an extension to stochastic games in [**?**]. The link between chromatic memory structures and deterministic parity automata (Theorem 11) was shown in [BRV23].

Concerning positionality of quantitative objectives, as far as we are aware, the literature focuses on the notion defined here as "positional and admitting optimal strategies". The distinction between positionality and limit-positionality from Section 3.3 are original.

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