An automata model for Borel-MSO

Antonio Casares · RPTU Kaiserslautern

Joint work with: Mikołaj Bojańczyk, Sven Manthe and Paweł Parys

Séminaire Plume, 24 November 2025

Who am I?

Math Bachelor, Valladolid

PhD, Bordeaux
(with N. Fijalkow & I. Walukiewicz)

PostDoc, Kaiserslautern
(with A. Lin)

CNRS, LIP ?

Master LMFI, Paris

PostDoc, Warsaw
(with M. Bojańczyk)

Courses in
- Model theory
- Descriptive complexity

ω-automata

Games on graphs

Logic

Transducers

Graph databases

Arithmetic theories

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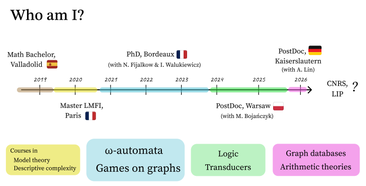

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Infinite word over the unary alphabet $\{\bullet\}$

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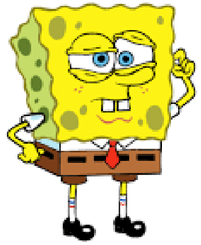

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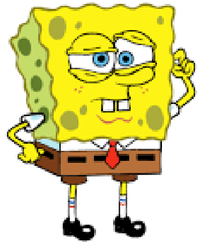

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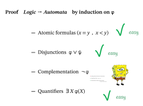

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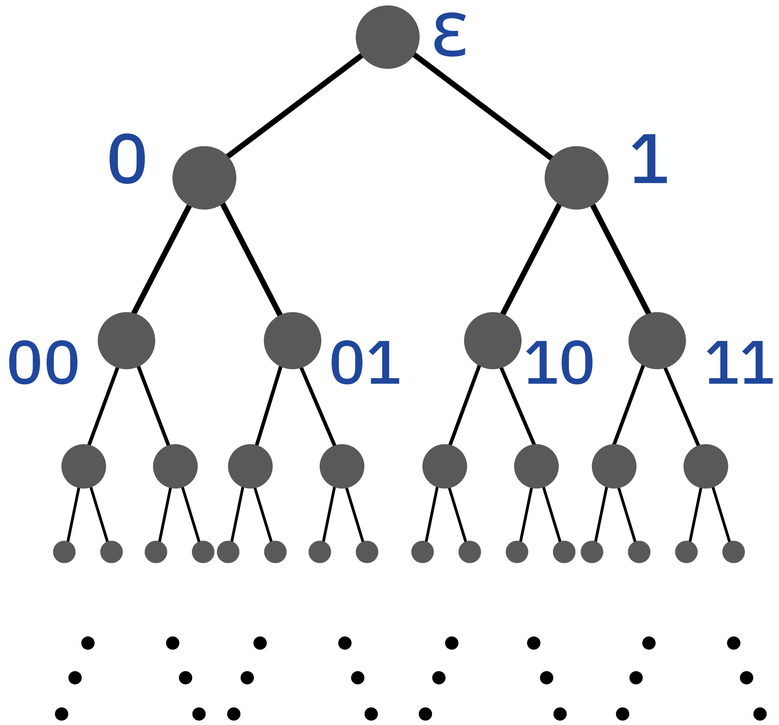

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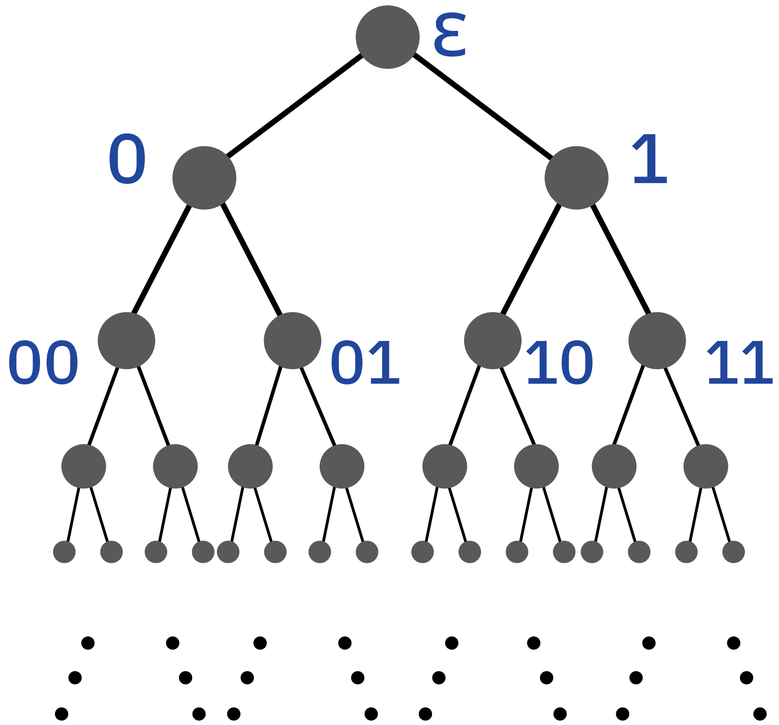

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$\{0,1\}^*$

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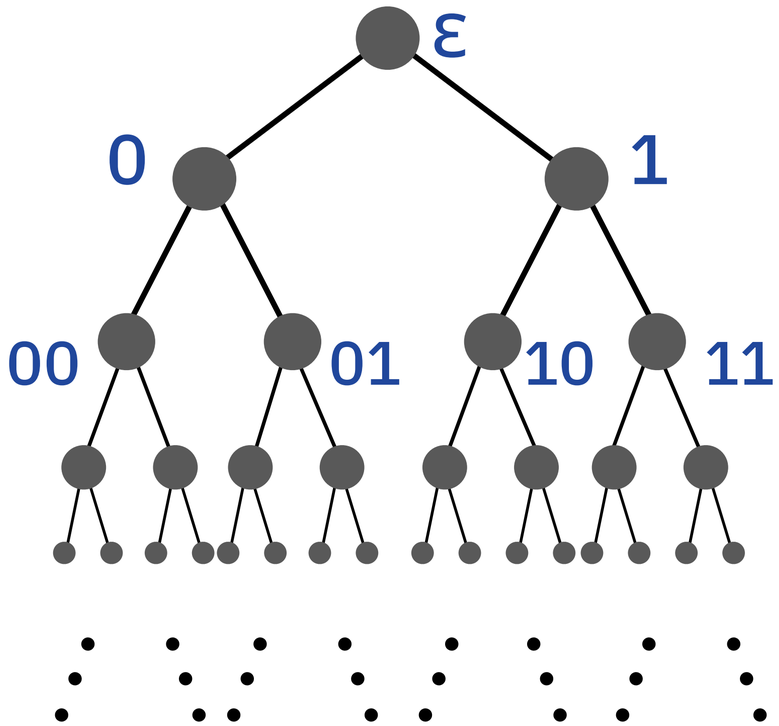

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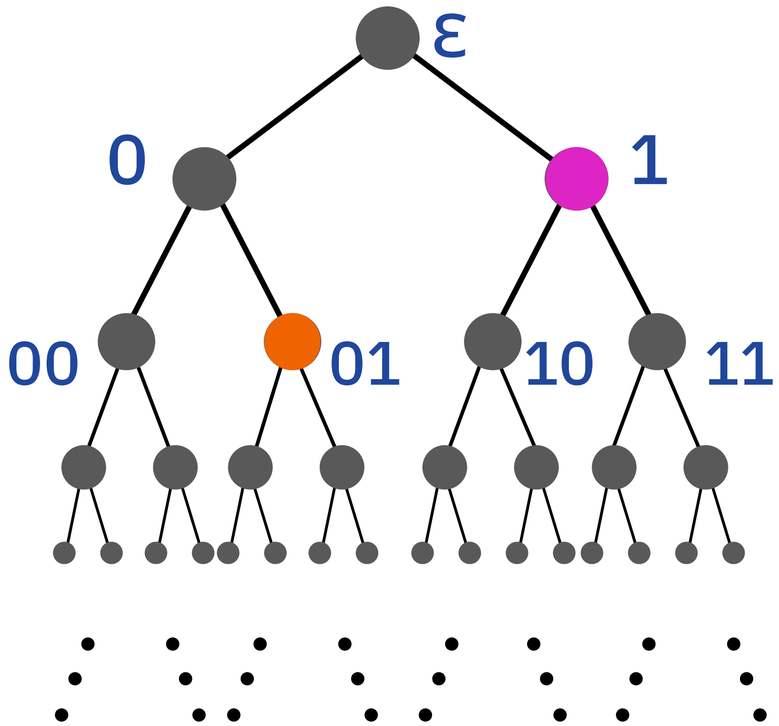

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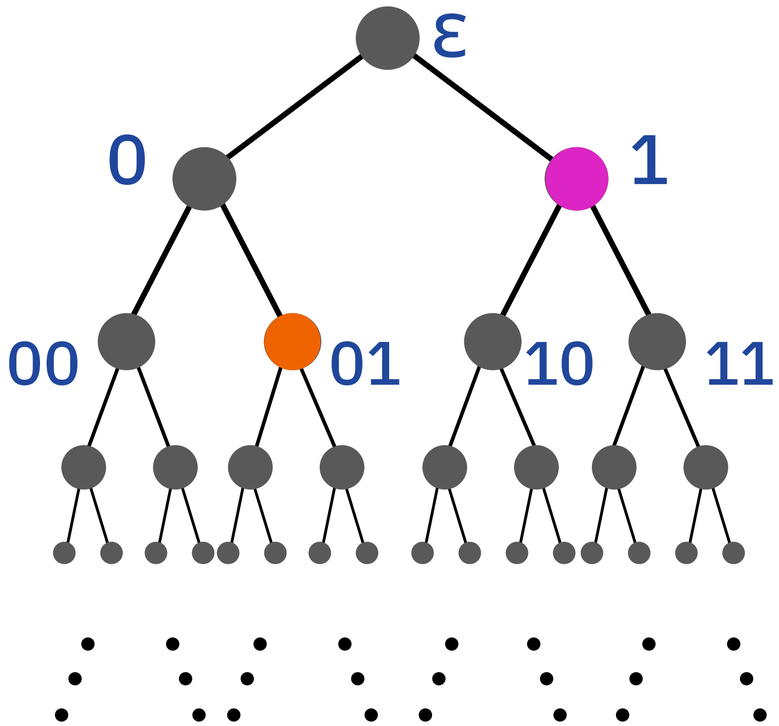

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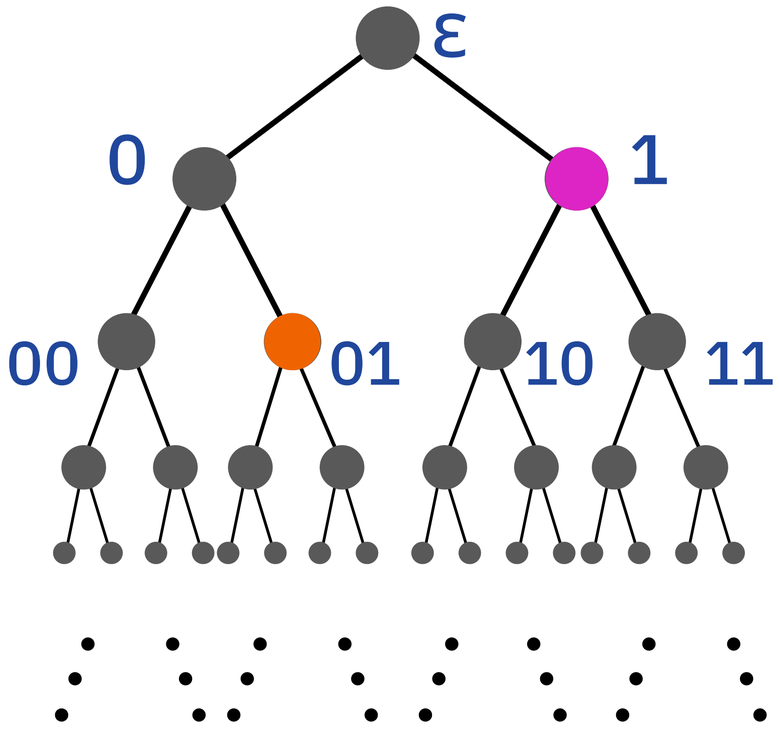

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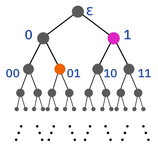

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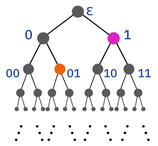

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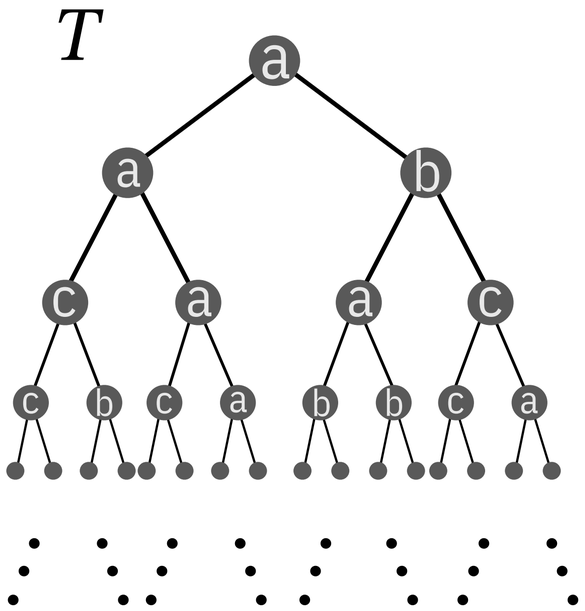

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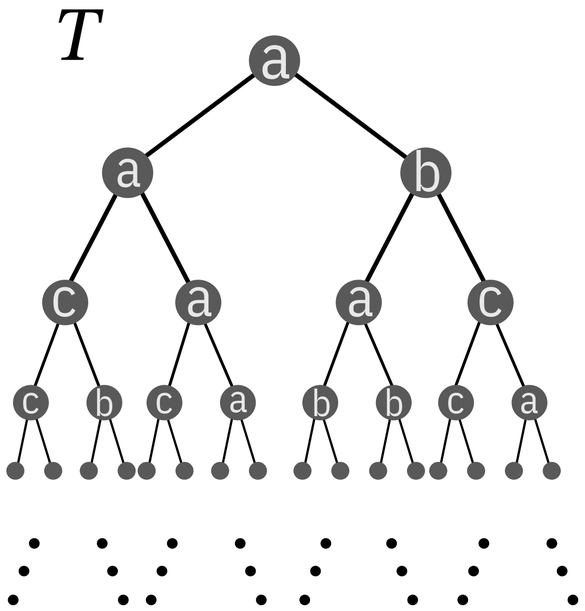

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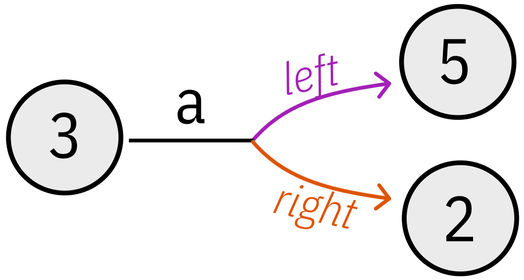
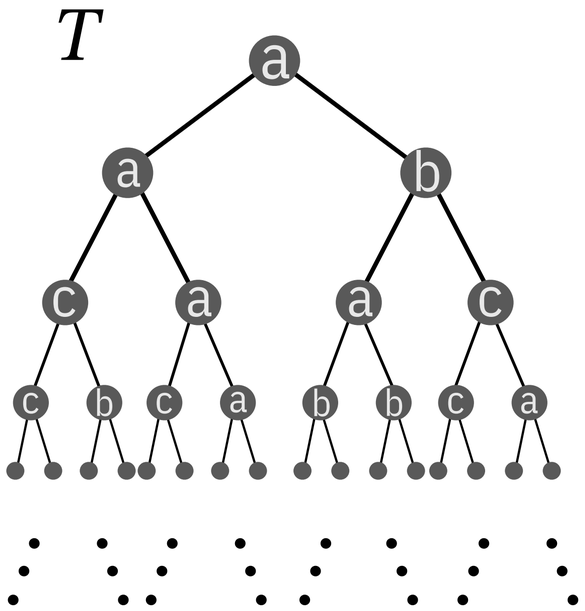
Transitions:





Input: A Σ-labelled tree

Output: A $[1,n]$-labelled tree

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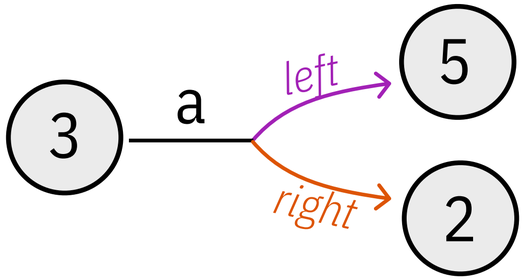

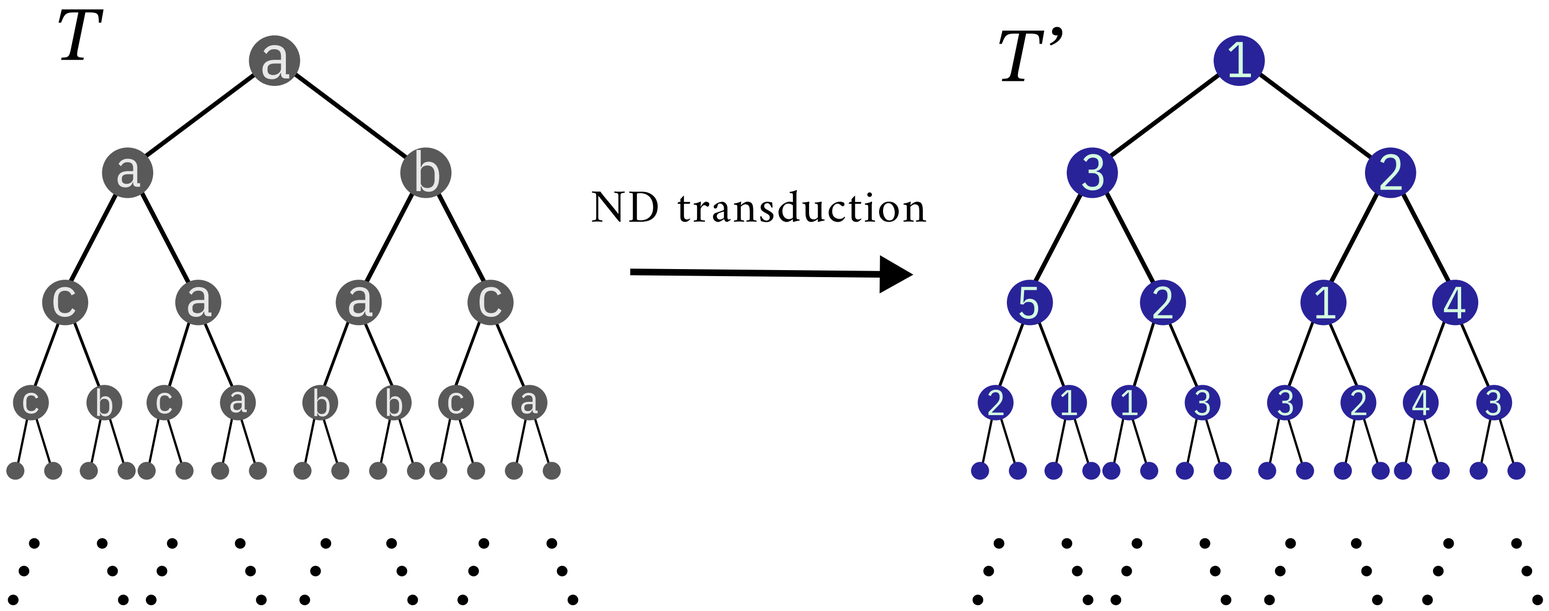

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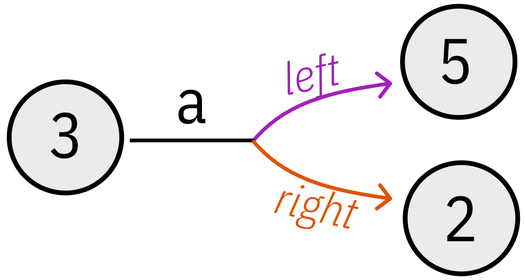

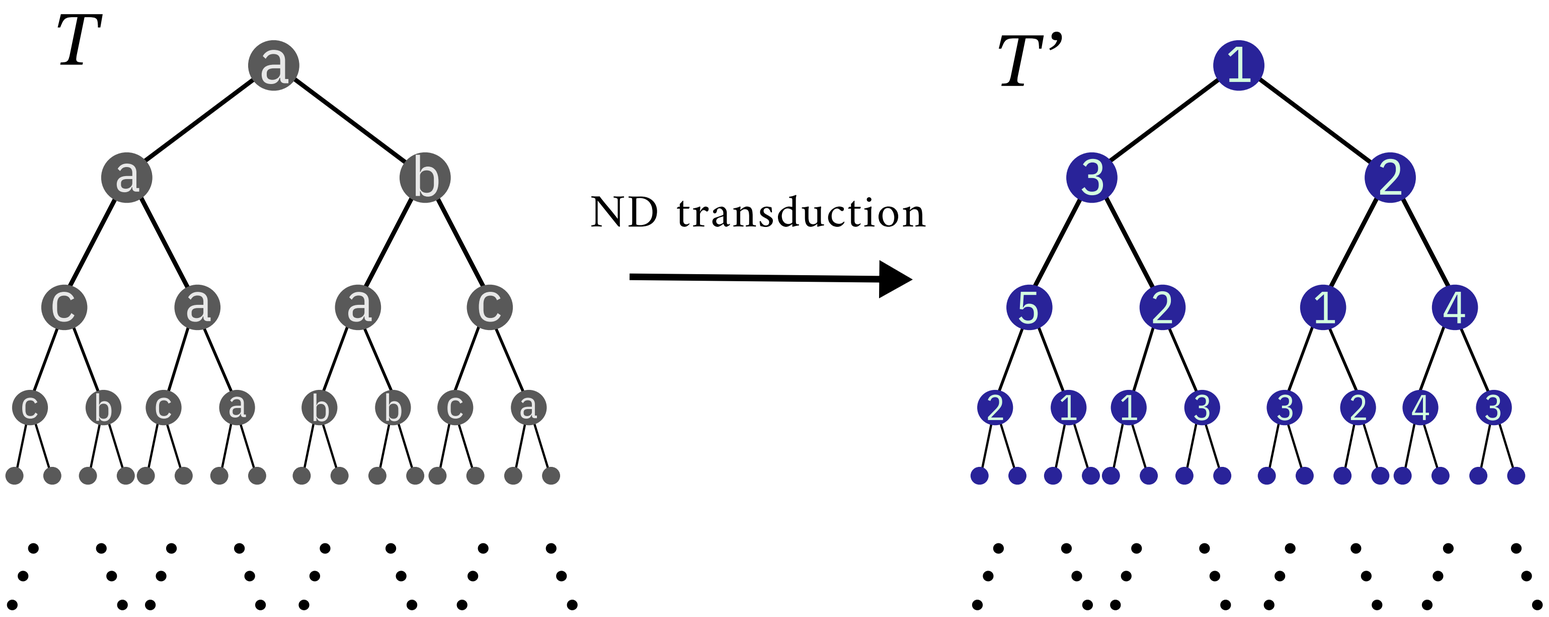

Acceptance: Parity condition

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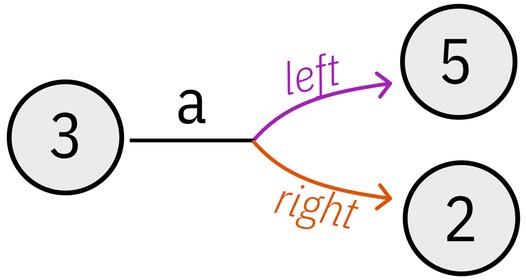

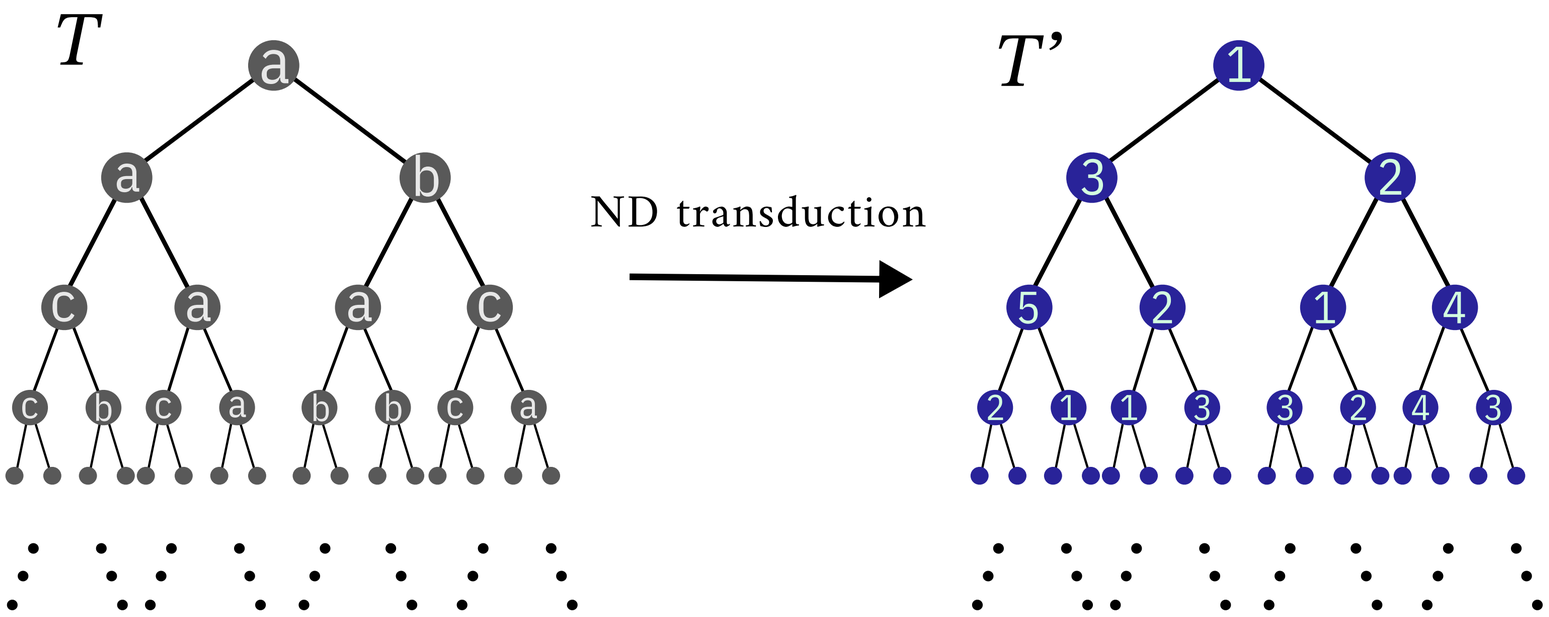

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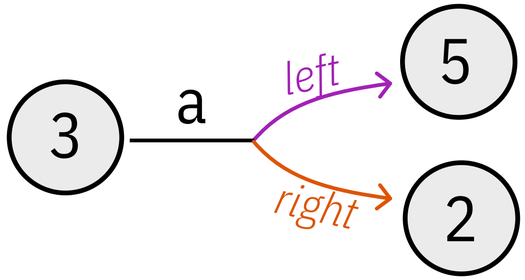

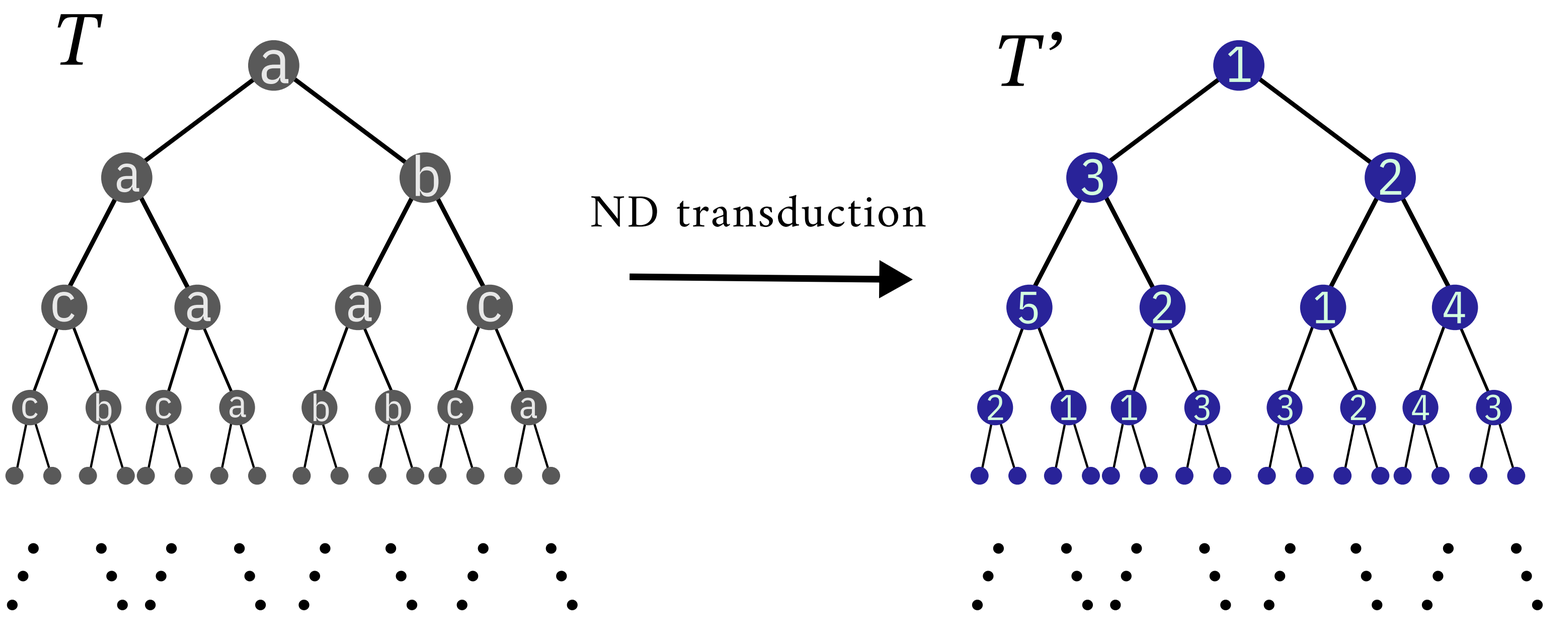

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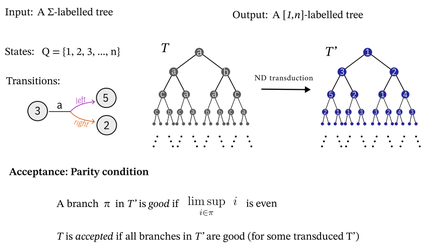
T is *accepted* if all branches in *T'* are good (for some transduced T')

$\mathrm{MSO}(2^{<\omega}, \text{left}, \text{right}) \longleftrightarrow$ Automata over infinite trees



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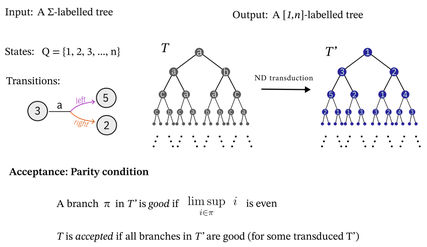

- For each sentence φ there is an automaton $\mathcal{A}_\varphi$ such that

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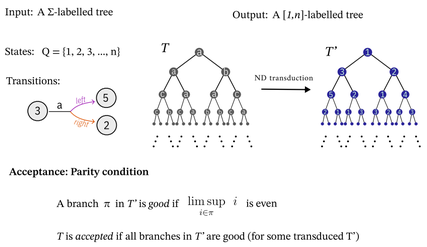

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(*Solving a parity game*)

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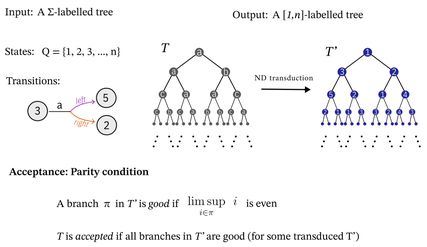

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Decidability

Proof ***Logic*** $\rightarrow$ ***Automata*** **by induction on** φ

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TREES, AUTOMATA, AND GAMES

Yuri Gurevich and Leo Harrington

ABSTRACT. In 1969 Rabin introduced tree automata and proved one of the deepest decidability results. If you worked on decision problems you did most probably use Rabin's result. But did you make your way through Rabin's cumbersome proof with its induction on countable ordinals? Building on ideas

$\neg\varphi$

- Challenge: parity tree-automata do not determinize
- Original proof by Rabin: extremely complex

- Modern presentation: Use games!

We have a tree automaton

Objective: Build an automaton for the complement language

Rejection game over a given Σ-tree

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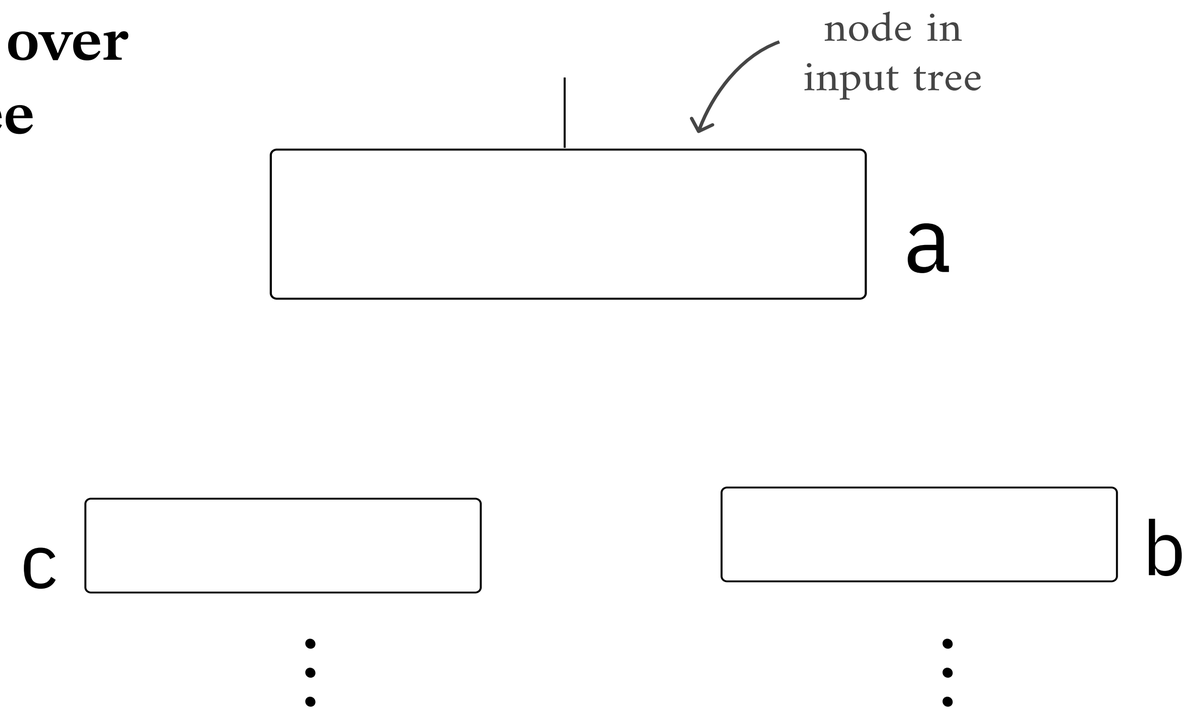

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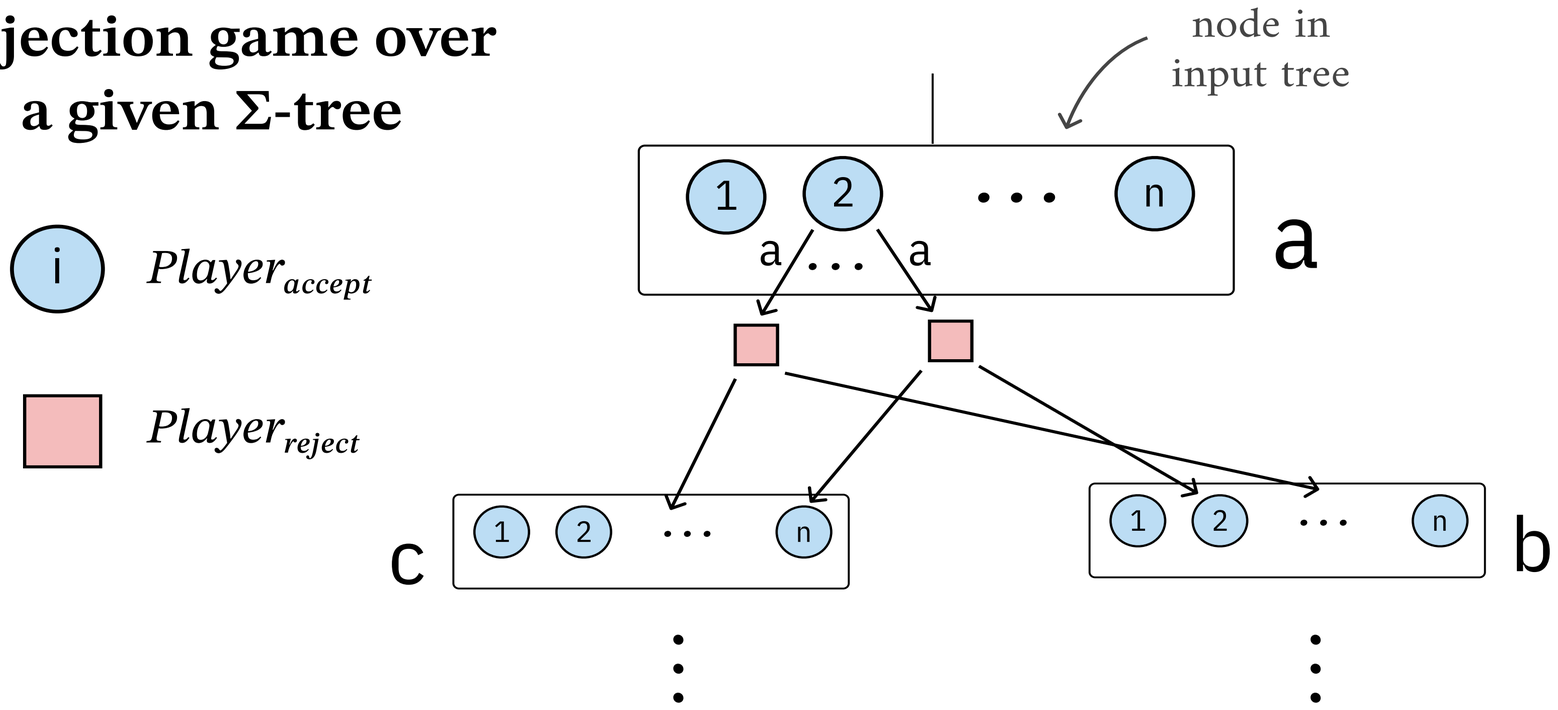

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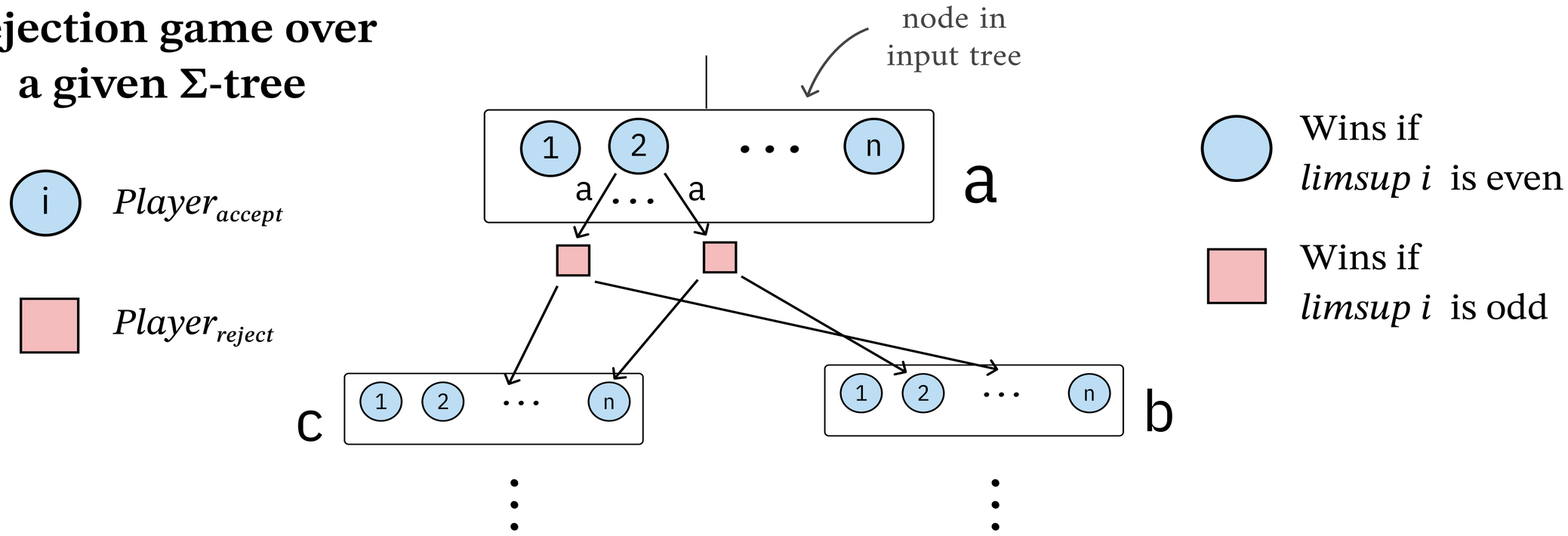

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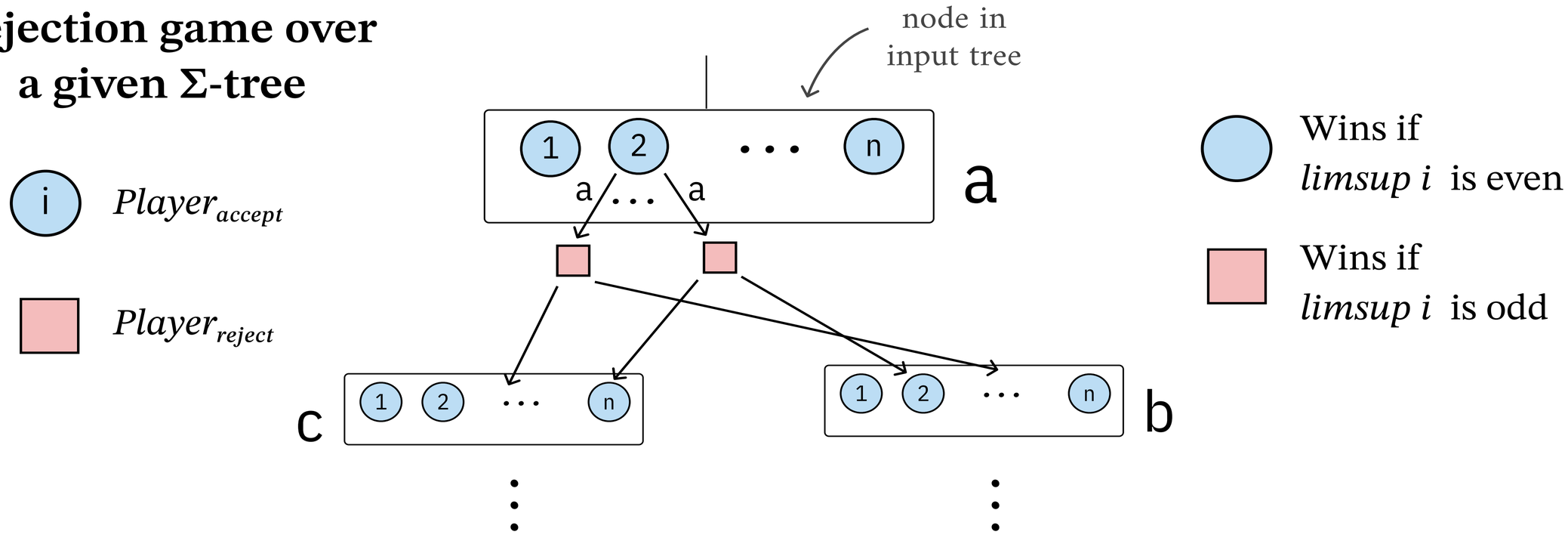

T rejected by the automaton iff P_{reject} *has a winning strategy*

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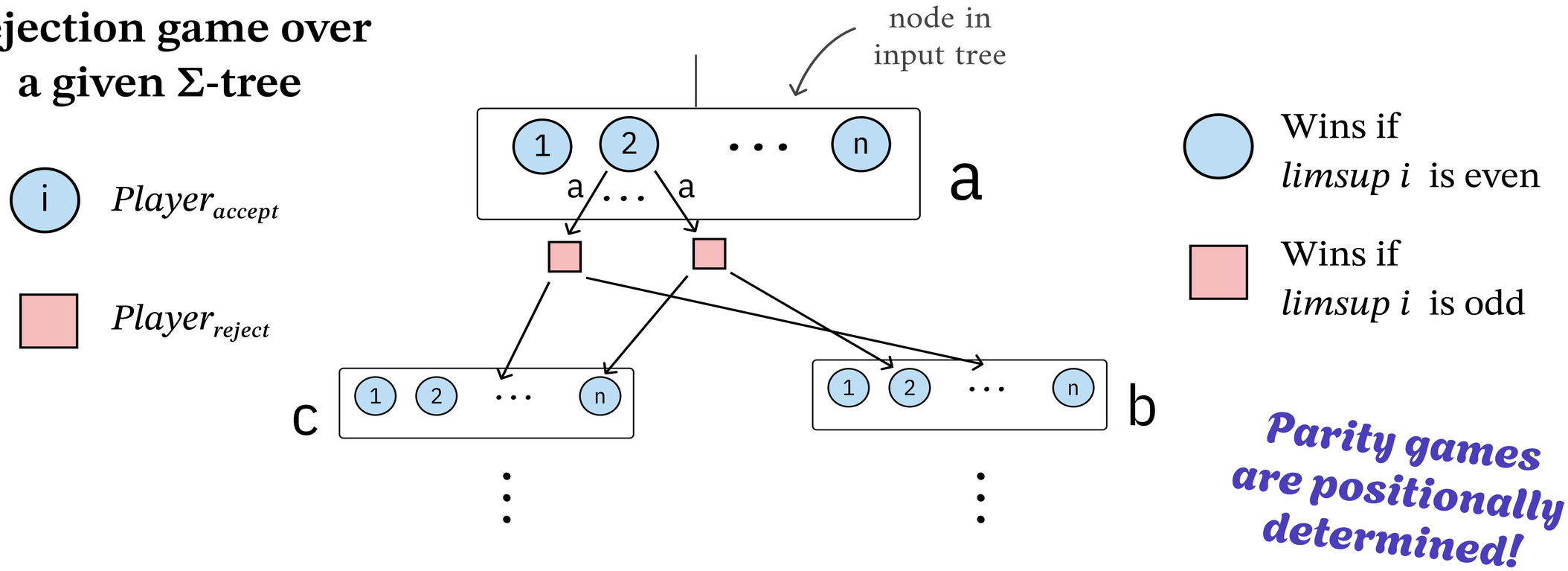

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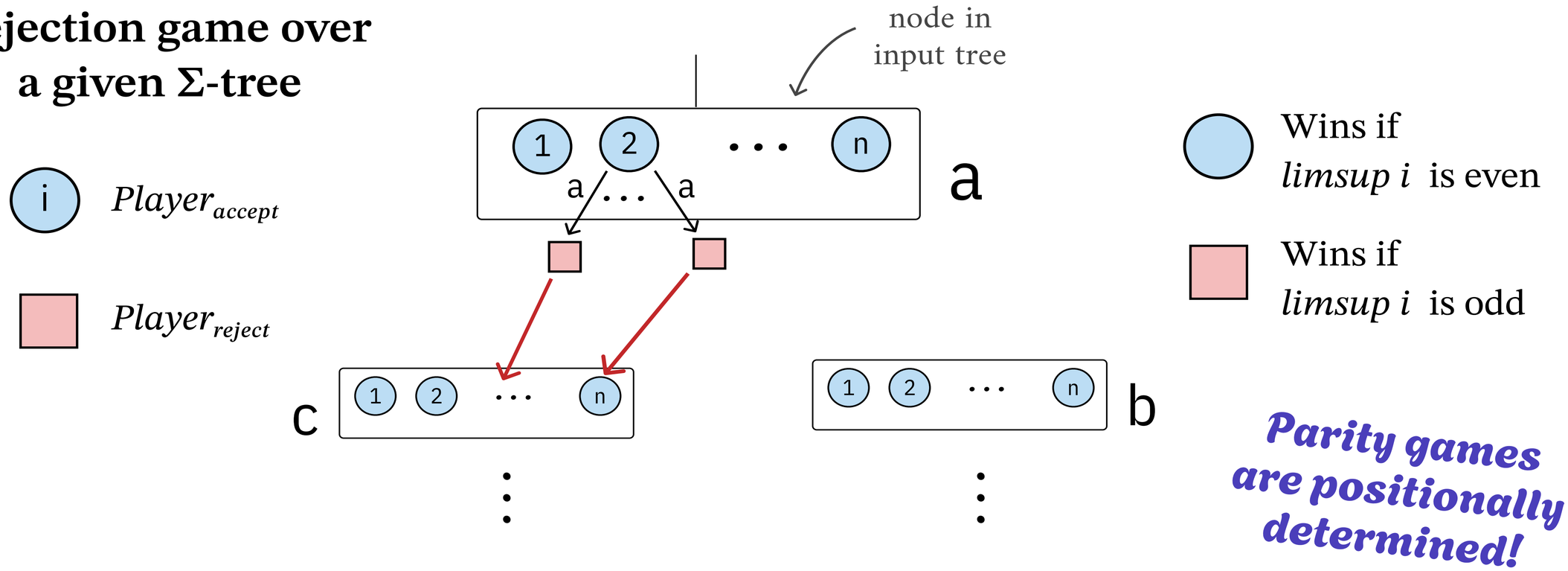

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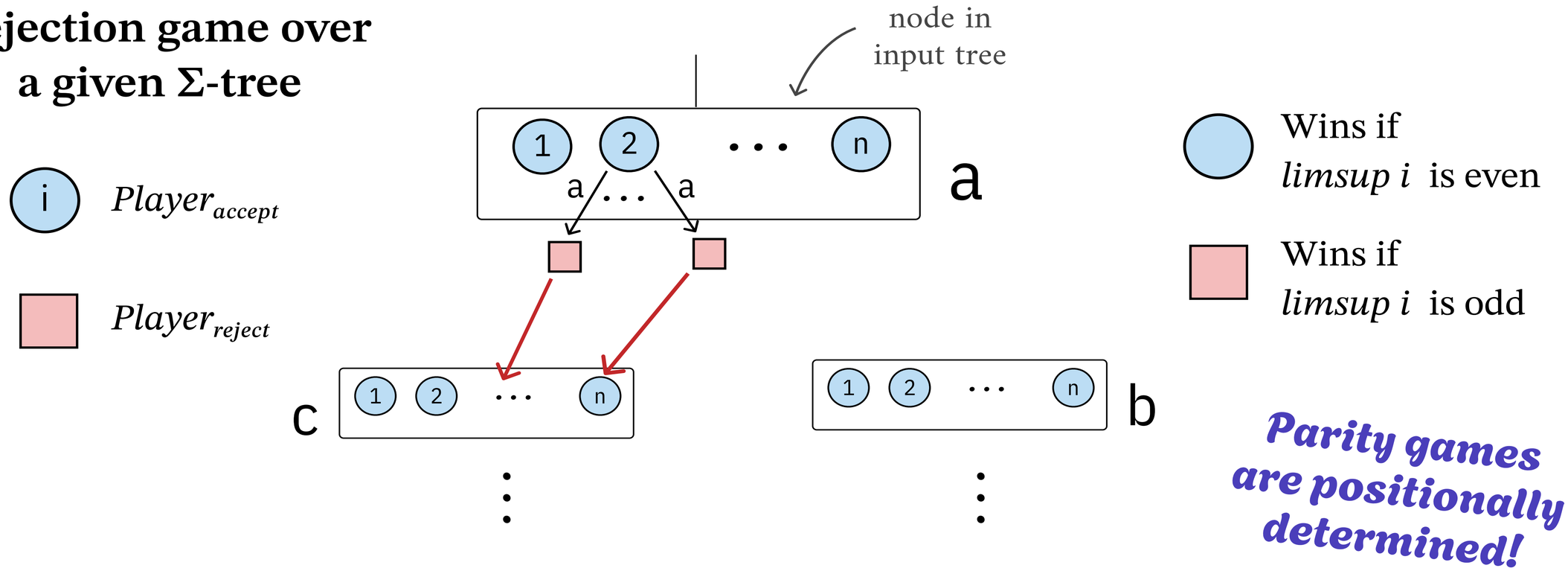

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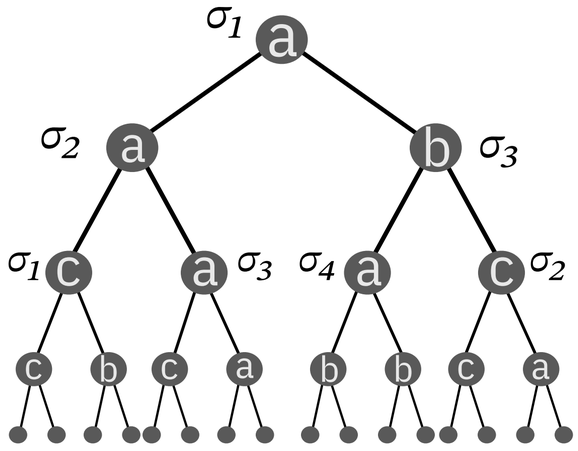

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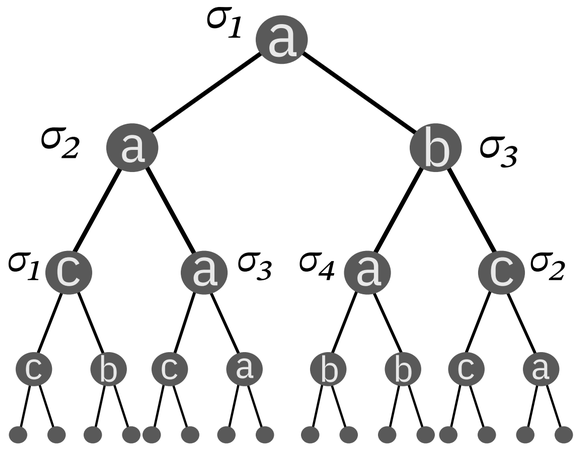

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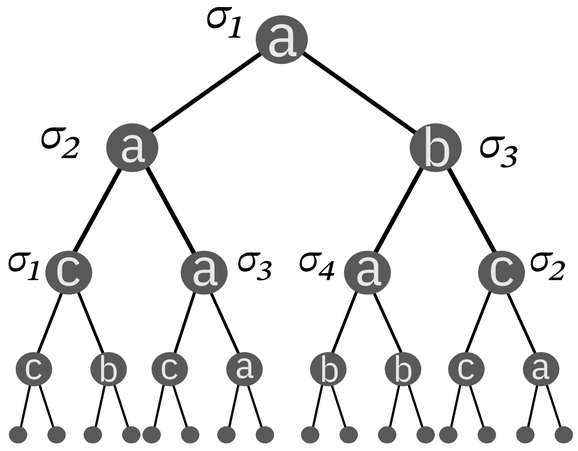

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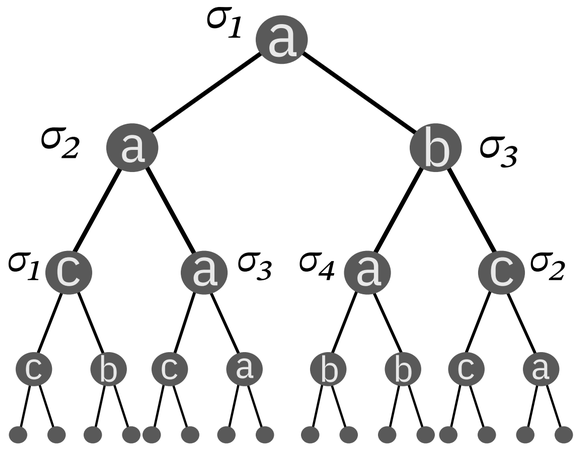

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↓ *projection in the Σ - component*

(*P_{reject} guesses a positional strategy*)

ND automaton for $L(\mathcal{A})^c$

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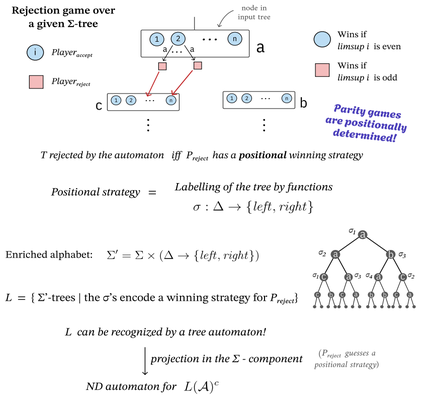

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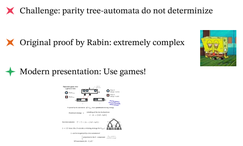

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Some classic results

Büchi, 1962

Rabin, 1969

MSO theory of $(\mathbb{N}, <)$

MSO theory of $(\mathbb{Q}, <)$ and the full binary tree

THEOREM *(Büchi, 1962)*

The MSO theory of $(\mathbb{N}, <)$ is decidable.

Büchi proved it using . . . ***automata!***

$(\mathbb{N}, <) \;=\;$ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ● ● ●

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I don't like automata

Shelah, 1975

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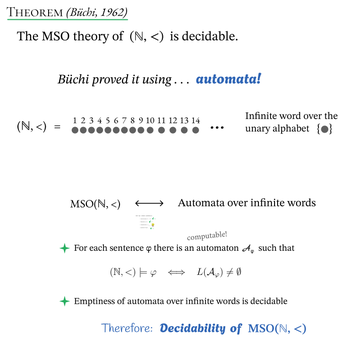

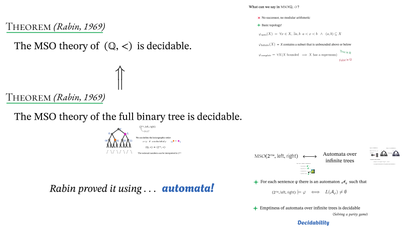

Our methods are model-theoretic, and we do not use automaton theory.

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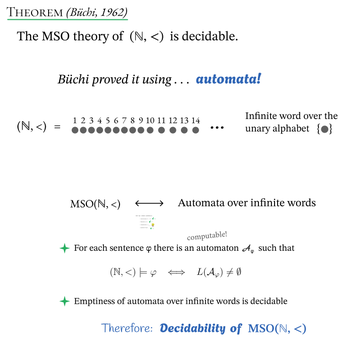

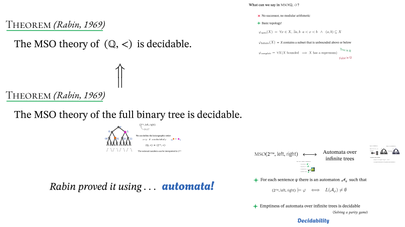

- New decidability proof for MSO $(\mathbb{Q}, <)$. ***Compositional method***

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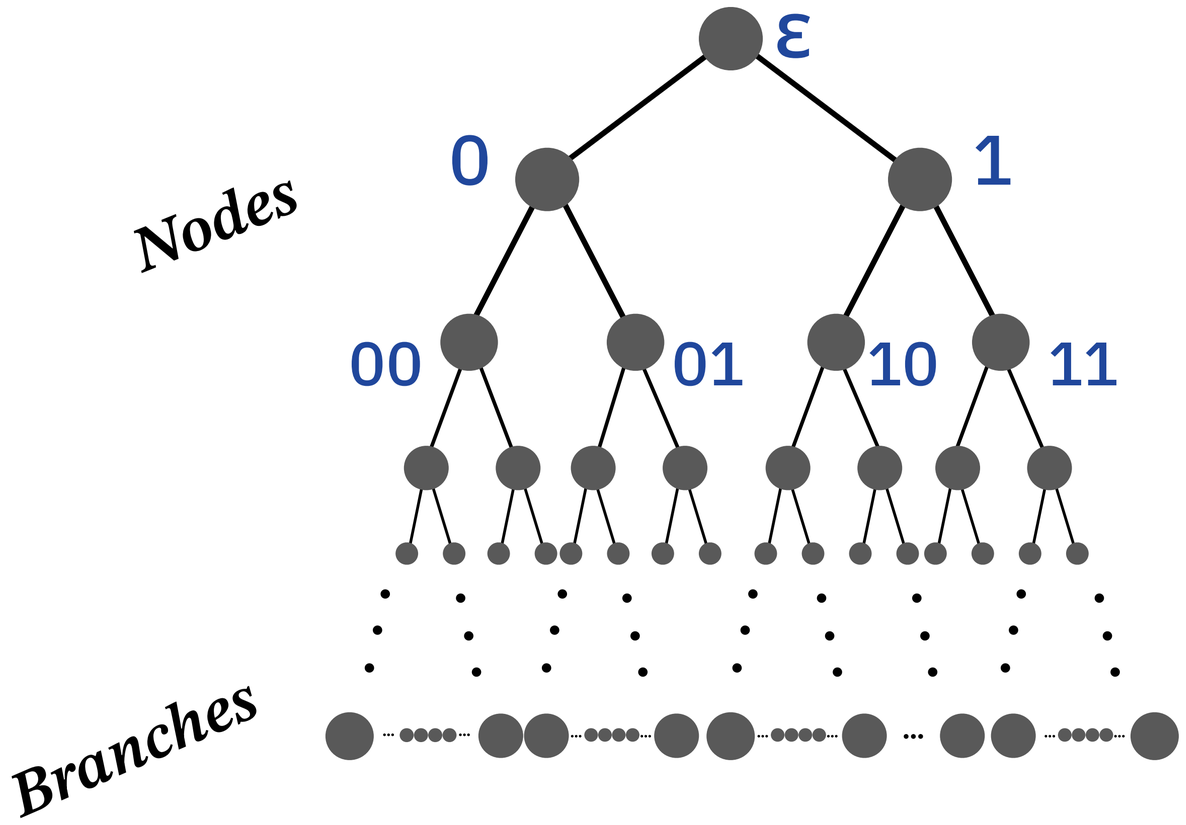
COROLLARY

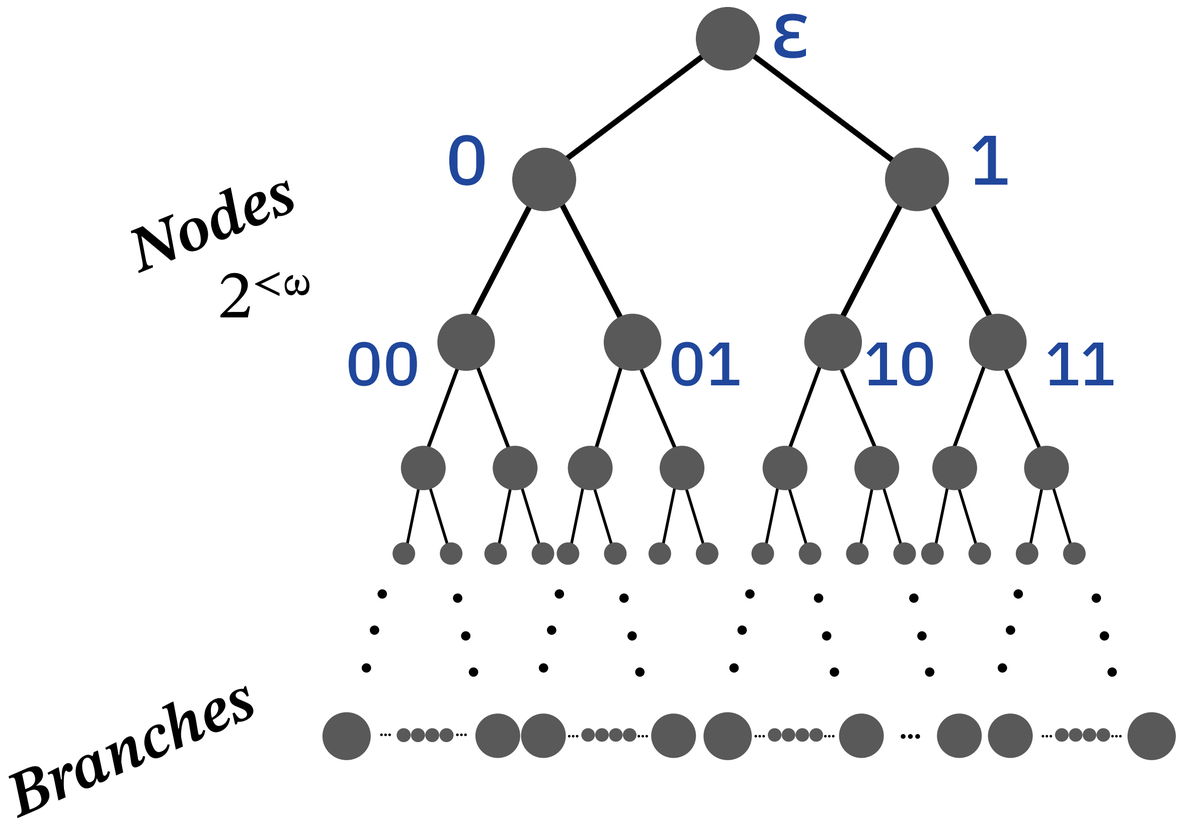
The MSO theory of the ***full binary tree with branches*** is ***undecidable***.

full binary tree with branches



full binary tree with branches

$(2^{\leq\omega}, <, \sqsubseteq)$



$2^{\omega} \simeq$ *Cantor space*

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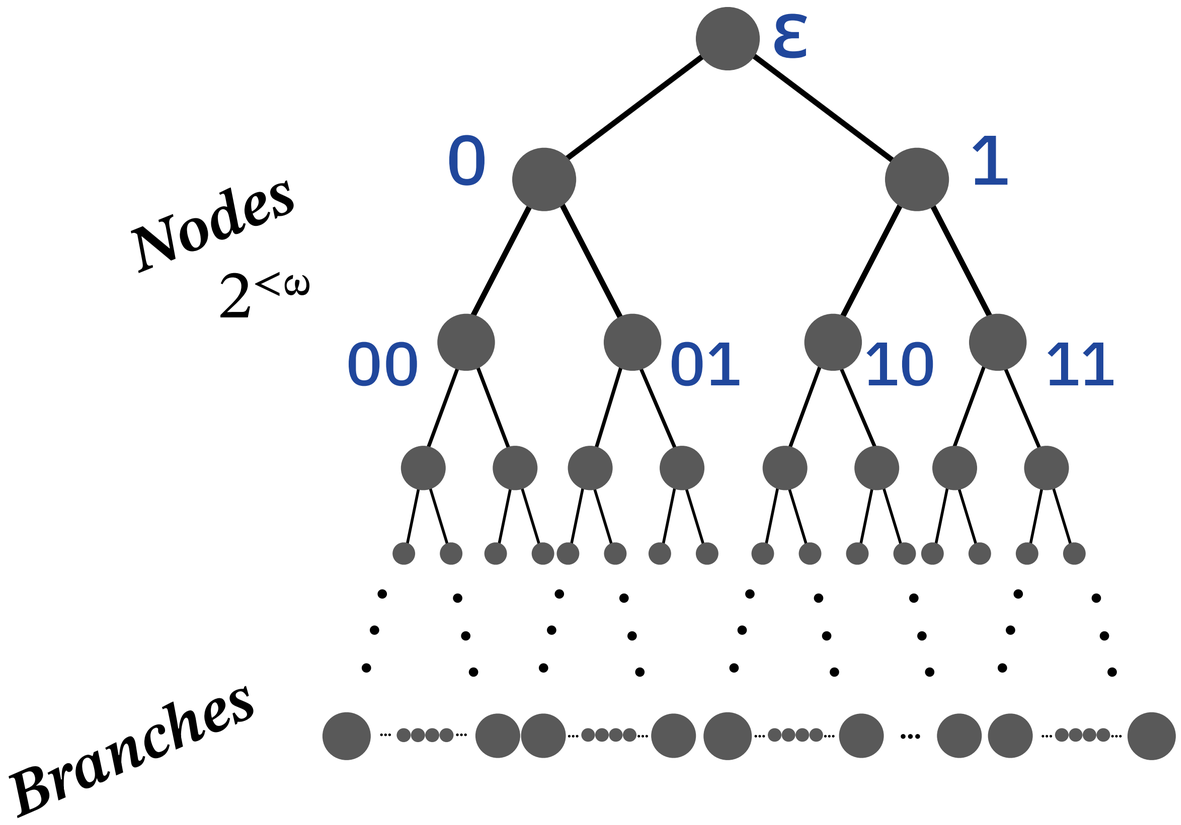

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prefix relation

(*allows to say node* $\in$ *branch*)

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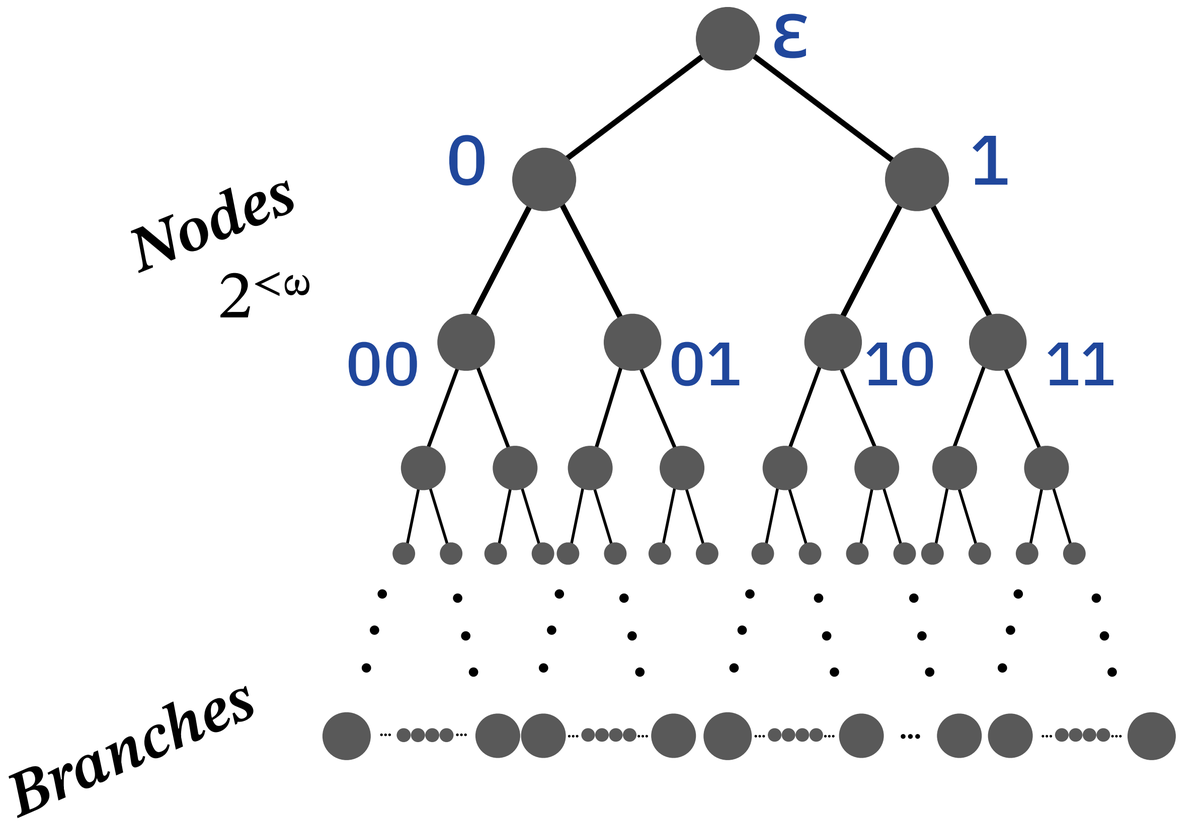

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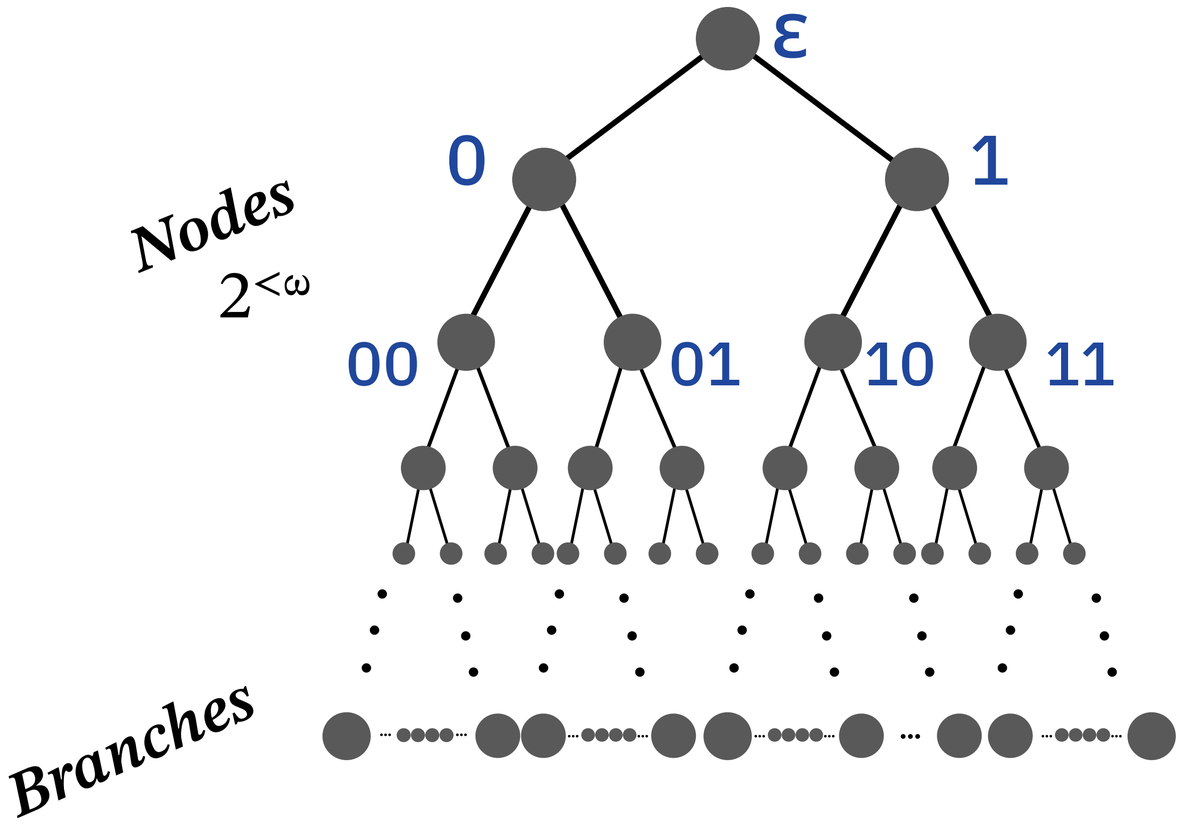

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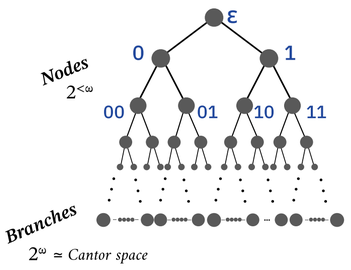

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$$\Uparrow$$

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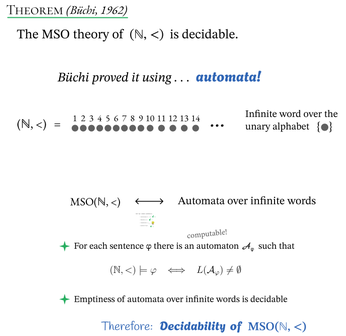
Some classic results

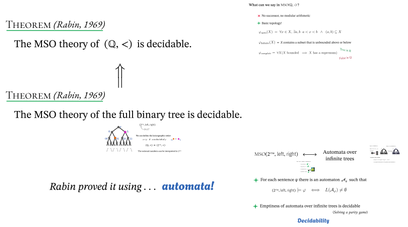
Büchi, 1962

MSO theory of $(\mathbb{N}, <)$



Rabin, 1969

MSO theory of $(\mathbb{Q}, <)$ and the full binary tree



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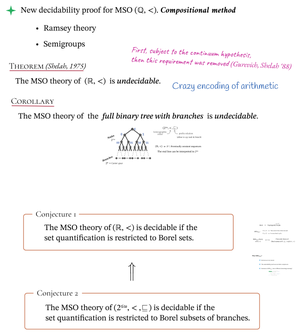

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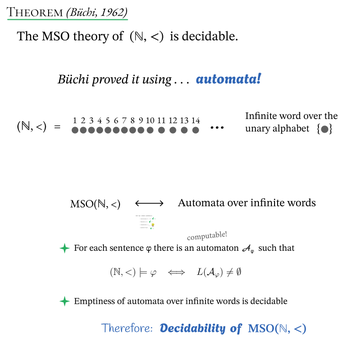

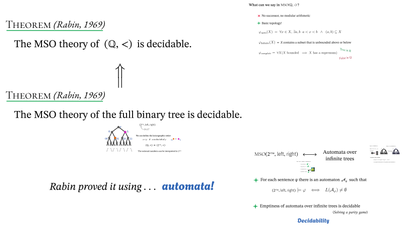

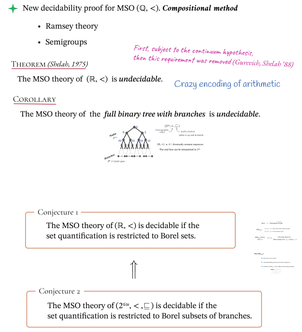

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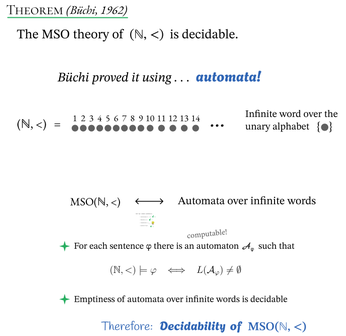

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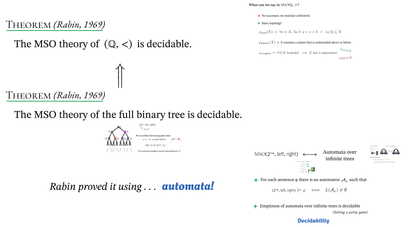

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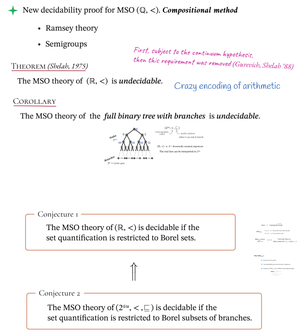

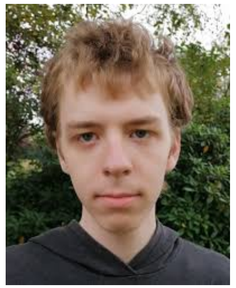

Manthe, 2024

MSO$_{\text{Borel}}$ theory of $(\mathbb{R}, <)$

THEOREM *(Manthe, 2024 , under review)*

The $\mathrm{MSO}_{\mathrm{Borel}}$ theory of $(\mathbb{R}, <)$ is decidable.

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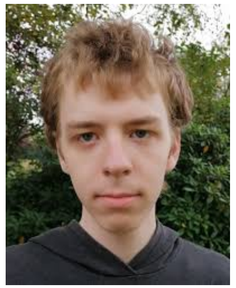

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THEOREM *(Büchi, 1962)*

The MSO theory of $(\mathbb{N}, <)$ is decidable.

Büchi proved it using . . . ***automata!***

$(\mathbb{N}, <) = \underset{\bullet}{1}\ \underset{\bullet}{2}\ \underset{\bullet}{3}\ \underset{\bullet}{4}\ \underset{\bullet}{5}\ \underset{\bullet}{6}\ \underset{\bullet}{7}\ \underset{\bullet}{8}\ \underset{\bullet}{9}\ \underset{\bullet}{10}\ \underset{\bullet}{11}\ \underset{\bullet}{12}\ \underset{\bullet}{13}\ \underset{\bullet}{14}\ \cdots$ Infinite word over the unary alphabet $\{\bullet\}$

$\mathrm{MSO}(\mathbb{N}, <) \longleftrightarrow$ Automata over infinite words

computable!

+ For each sentence φ there is an automaton $\mathcal{A}_\varphi$ such that

$$(\mathbb{N}, <) \models \varphi \iff L(\mathcal{A}_\varphi) \neq \emptyset$$

+ Emptiness of automata over infinite words is decidable

Therefore: ***Decidability of*** $\mathrm{MSO}(\mathbb{N}, <)$

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$\Uparrow$

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(Solving a parity game)

Decidability

+ New decidability proof for MSO $(\mathbb{Q}, <)$. ***Compositional method***
  - Ramsey theory
  - Semigroups

First, subject to the continuum hypothesis; then this requirement was removed (Gurevich, Shelah '88)

THEOREM *(Shelah, 1975)*

The MSO theory of $(\mathbb{R}, <)$ is ***undecidable.***

Crazy encoding of arithmetic

COROLLARY

The MSO theory of the ***full binary tree with branches*** is ***undecidable.***

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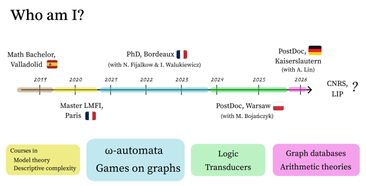
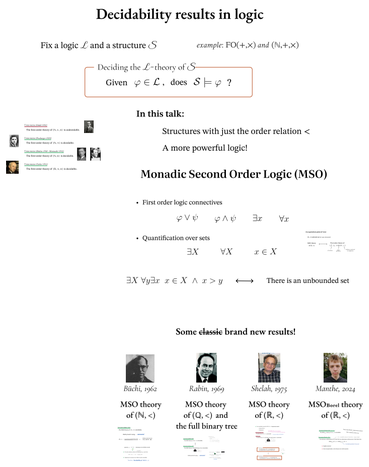
An automata model for Borel-MSO

Antonio Casares · RPTU Kaiserslautern



Joint work with: Mikołaj Bojańczyk, Sven Manthe and Paweł Parys



Séminaire Plume, 24 November 2025

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and this theory is decidable.

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- Atomic formulas
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✦ Proof in some subcases

Borel tree automata

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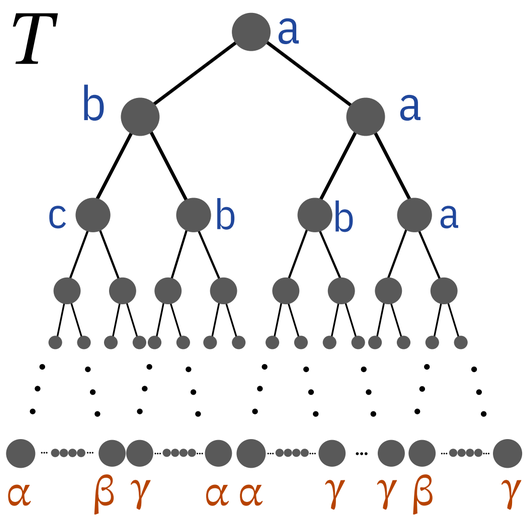

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Same branch labeling

States: $Q = \{1, 2, 3, \ldots, n\}$

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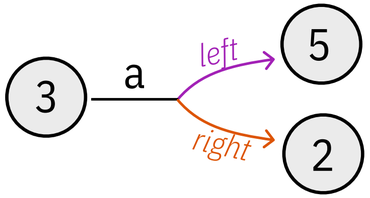

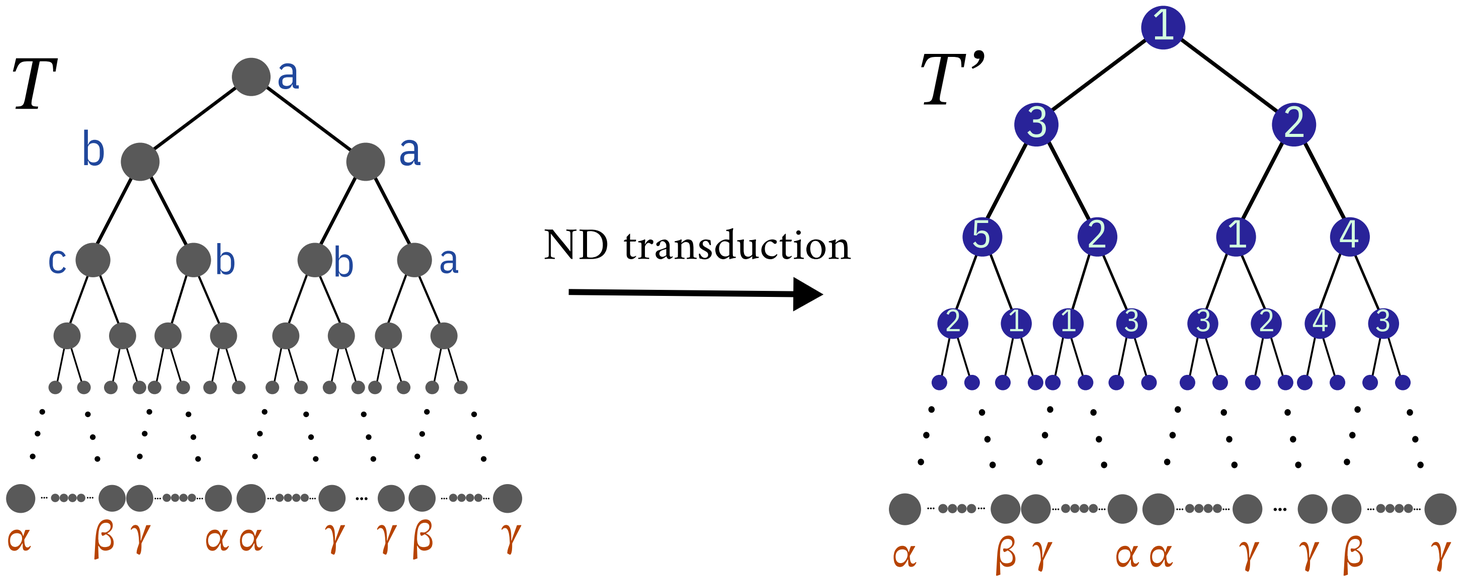

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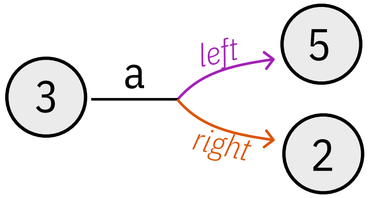

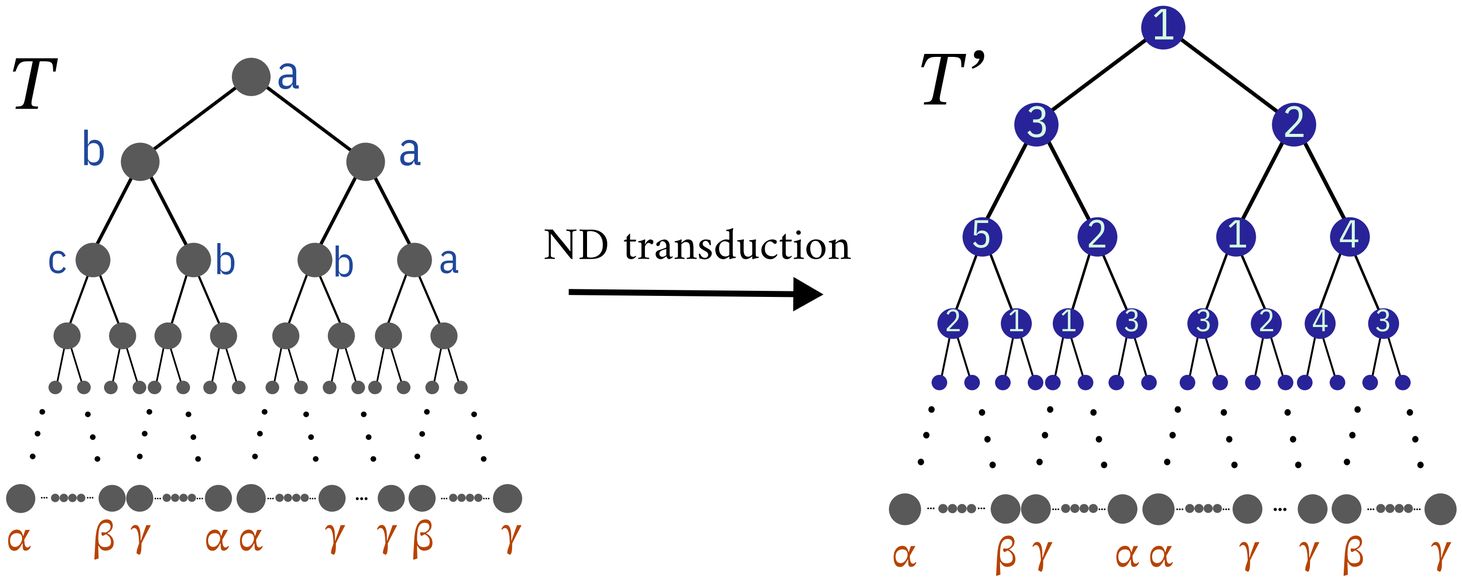

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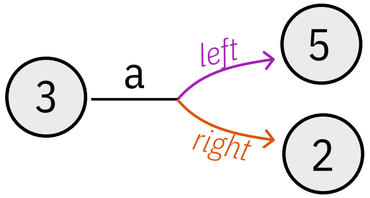

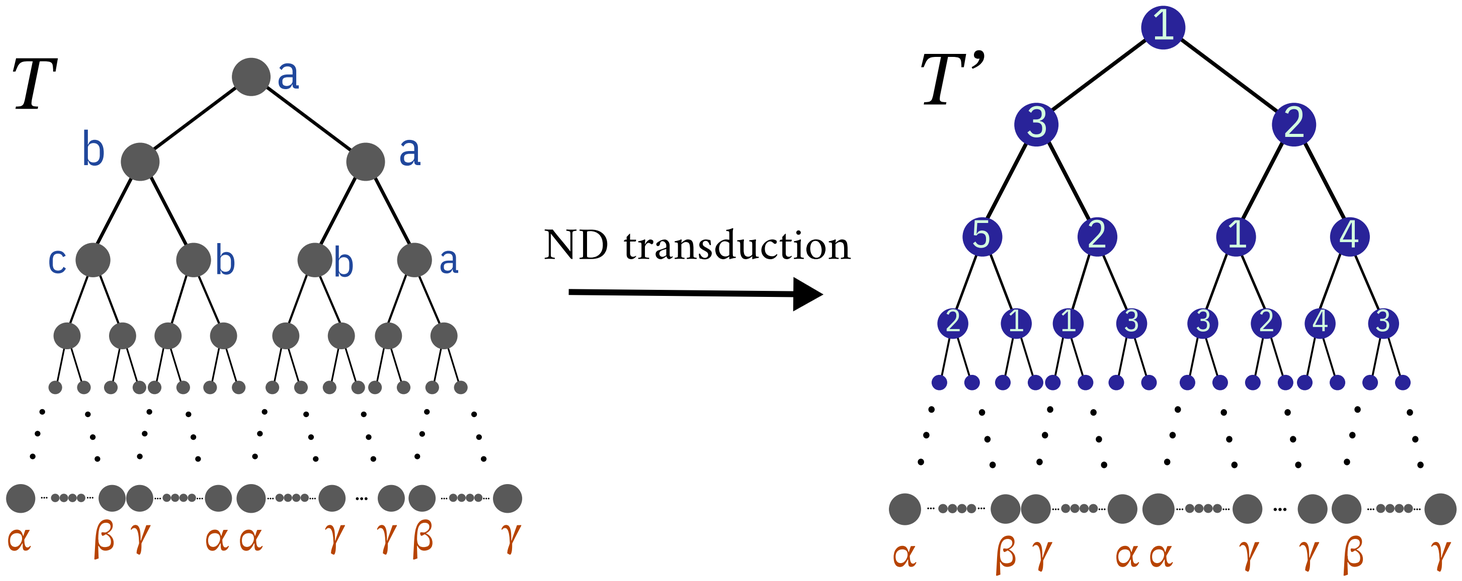

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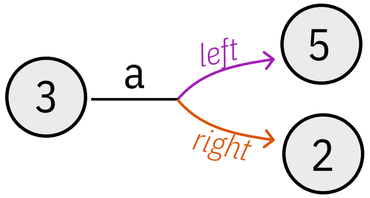

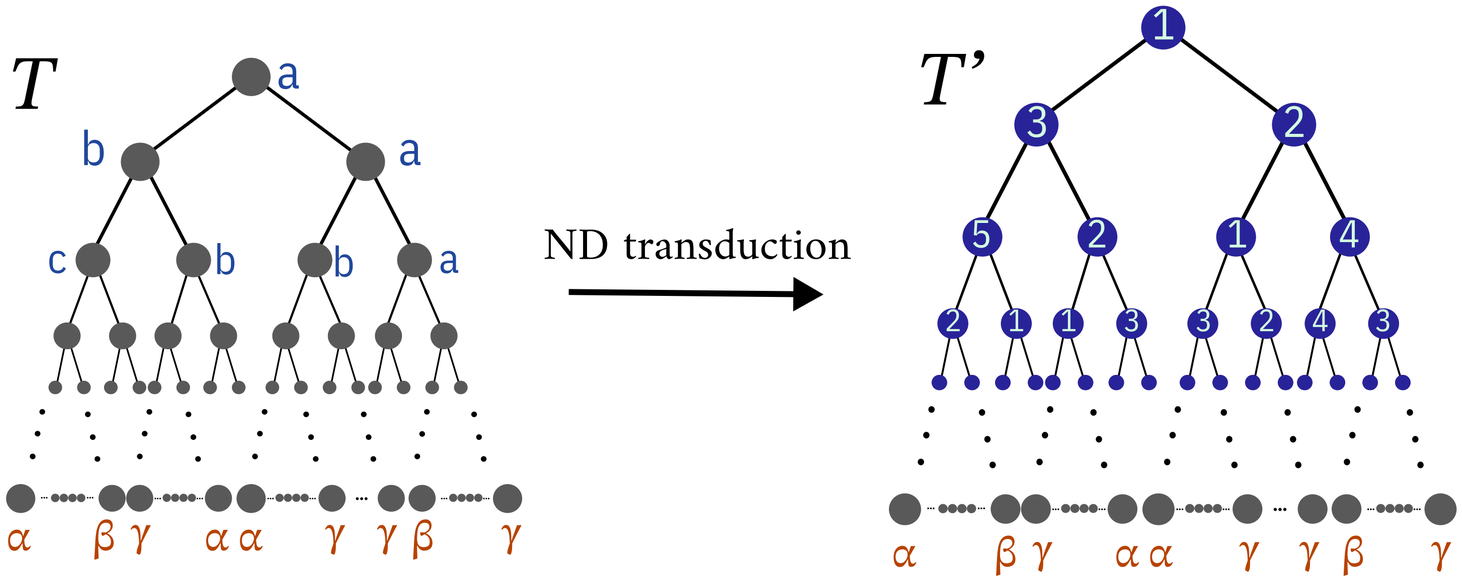

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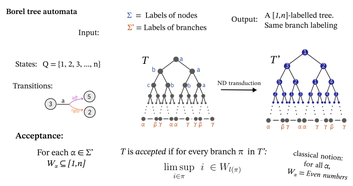
classical notion:
for all α,
W_α = *Even numbers*

Contributions



- Introduction of Borel-tree automata
- Introduction of prioritized Borel games

THEOREM *(Bojańczyk, C., Manthe, Parys)*

If prioritized Borel-games are finite-memory determined, then

$$\mathrm{MSO}_{\mathrm{Borel}}(2^{\leq\omega}) \equiv \mathrm{MSO}_{\mathrm{Bool}(\Sigma 2)}(2^{\leq\omega}) \equiv \text{Borel-tree-automata}$$

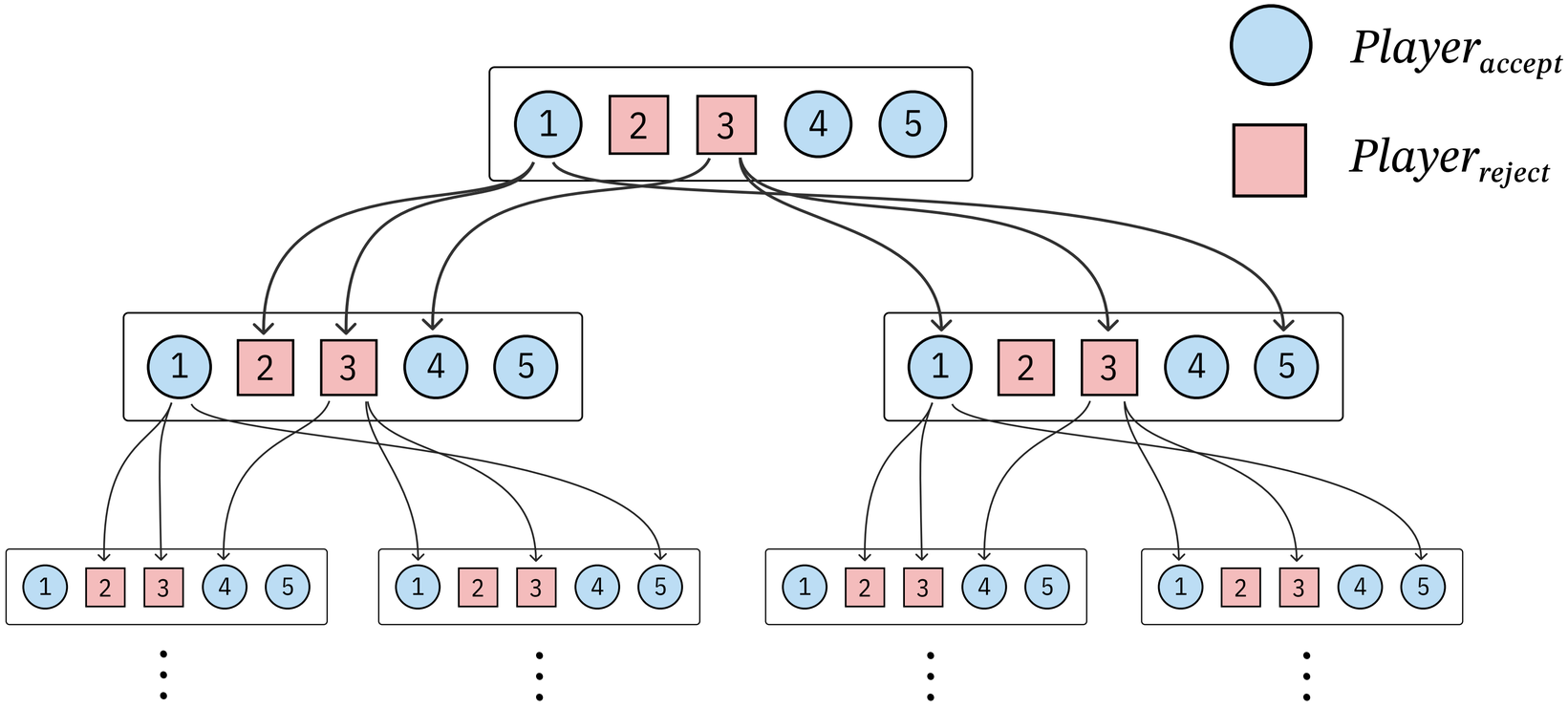
and this theory is decidable.

CONJECTURE

Prioritized Borel-games are finite-memory determined.

- Proof in some subcases

Prioritized Borel games

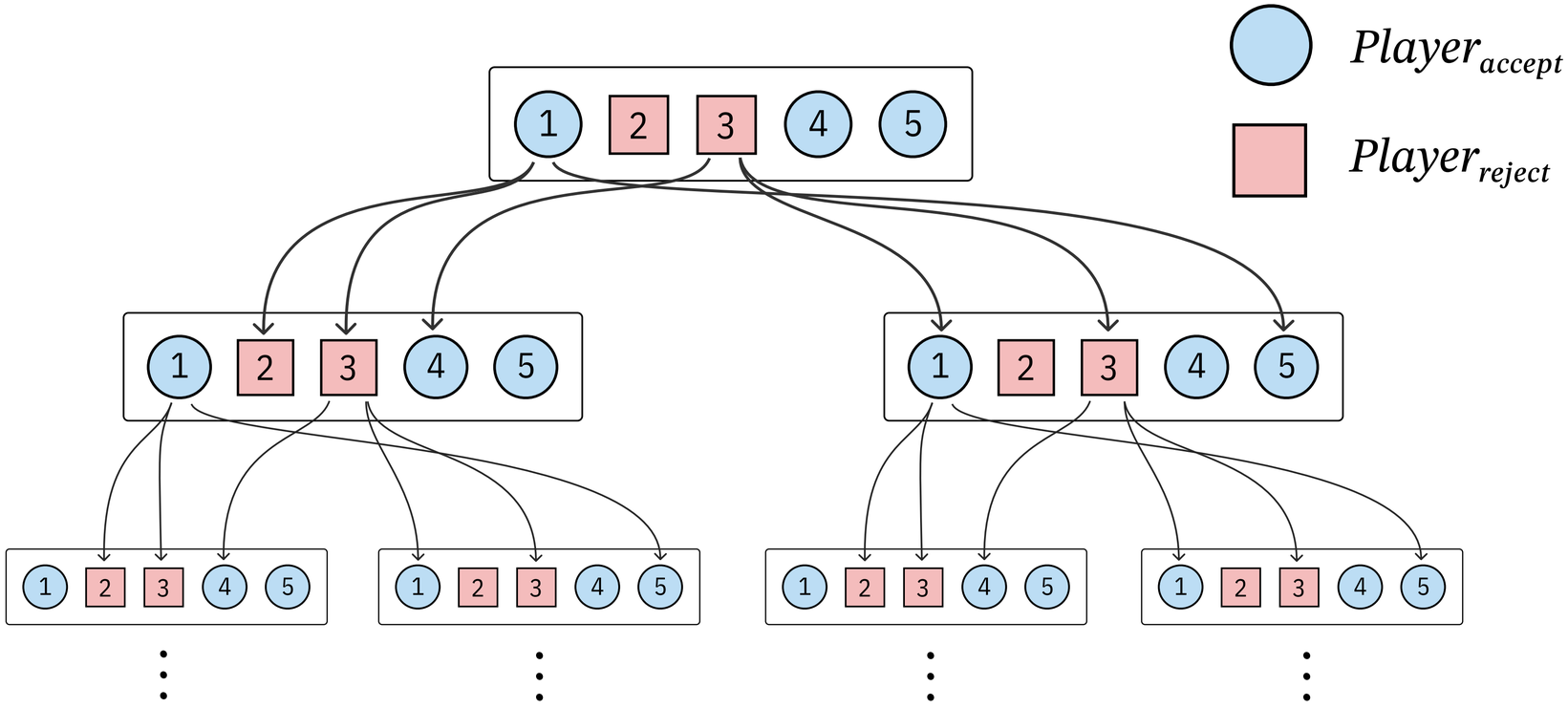


Prioritized Borel games

Winning condition

For each $i \in [1,n]$

W_i a subset of branches



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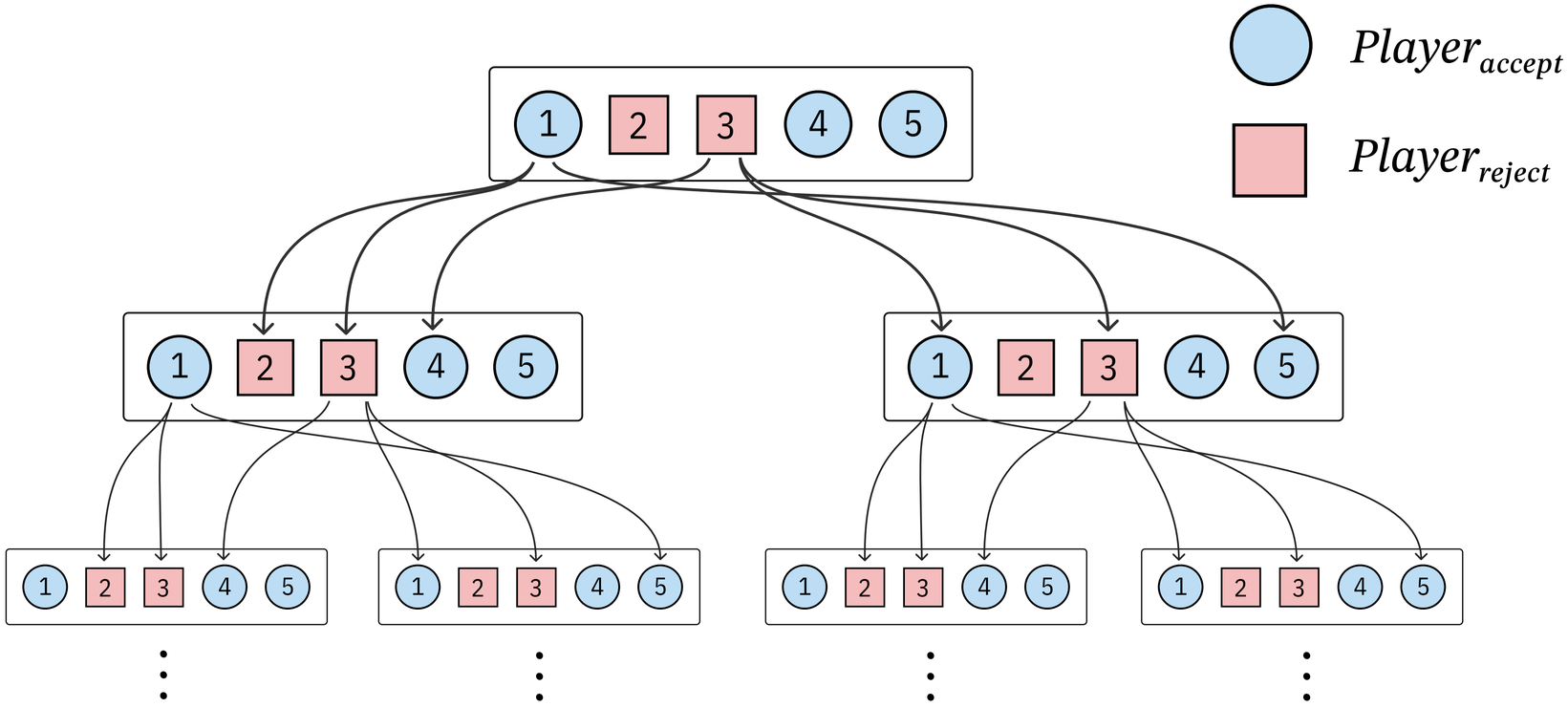

A play produces:

$1342342342\ldots \in [1,n]^{\omega}$

and

a branch π

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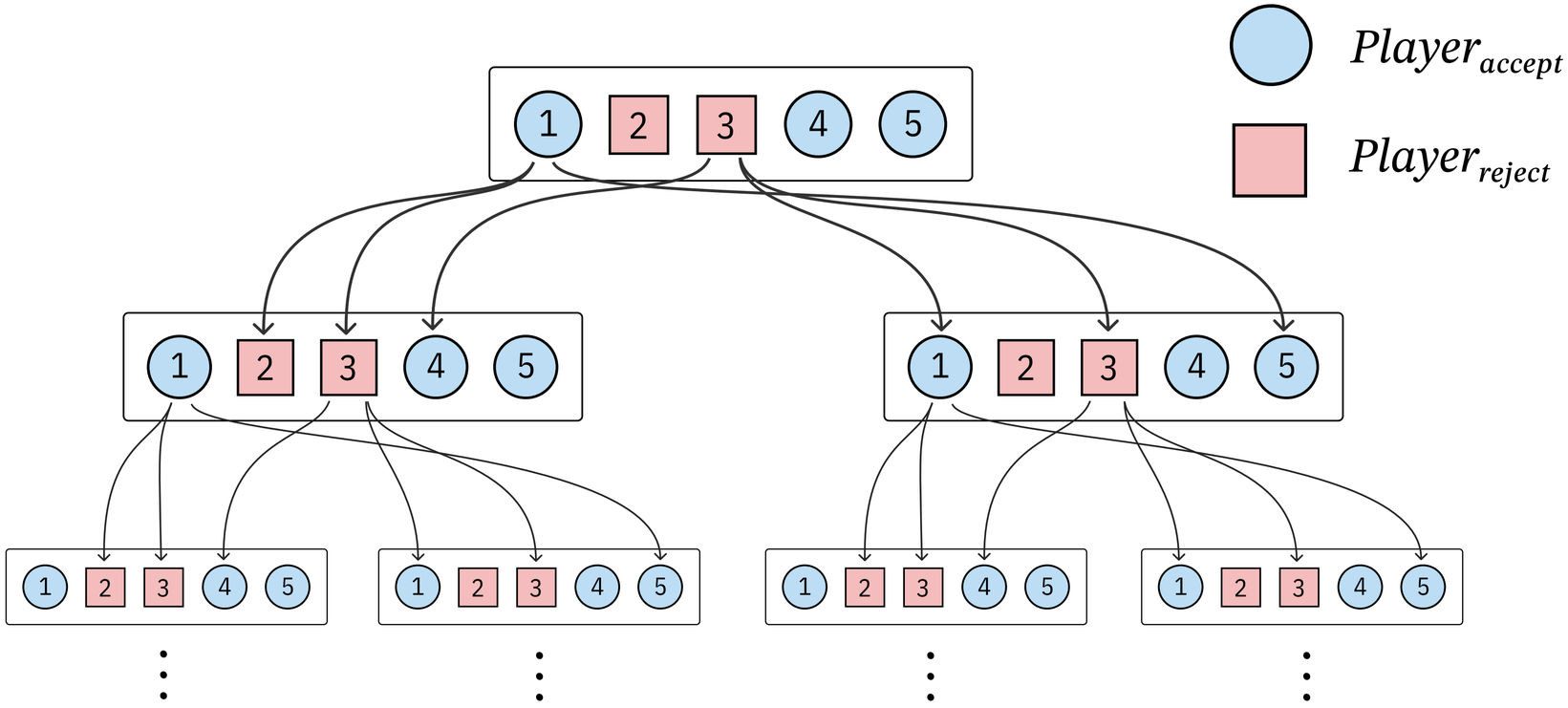


A play produces:

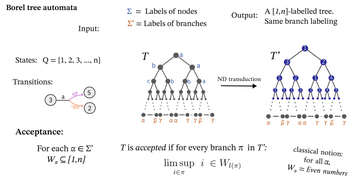
$1342342342\ldots \in [1,n]^\omega \xrightarrow{limsup} i$

and

a branch π

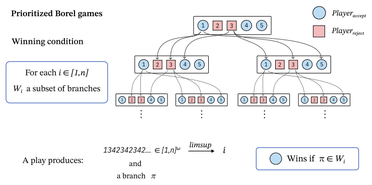
Wins if $\pi \in W_i$

Contributions



✦ Introduction of Borel-tree automata

✦ Introduction of prioritized Borel games



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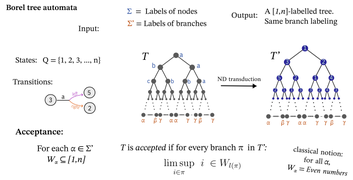

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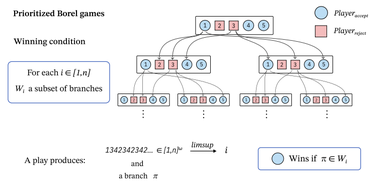

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CONJECTURE

Prioritized Borel-games are finite-memory determined.

✦ Proof in some subcases

Thanks for your attention!

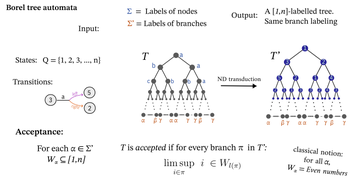
- Prioritized Borel games are not positional

- Finite-memory = Winning strategies can be implemented by a finite automaton

Conjecture

There exists a function $f(n)$ such that the winner in prioritized Borel games with n internal nodes can win with a strategy given by an automaton of size $\leq f(n)$.

Contributions



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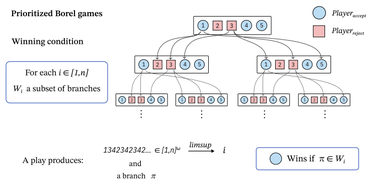

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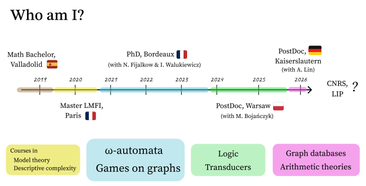
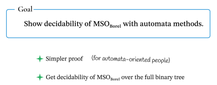
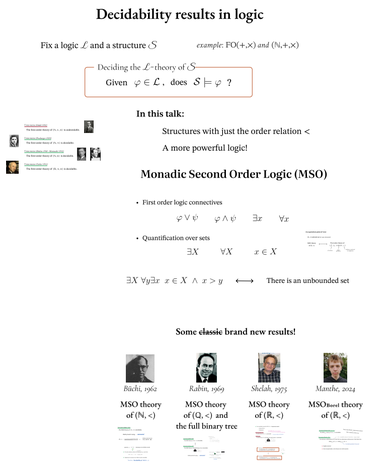
An automata model for Borel-MSO

Antonio Casares · RPTU Kaiserslautern

Joint work with: Mikołaj Bojańczyk, Sven Manthe and Paweł Parys

Séminaire Plume, 24 November 2025