Mathematical Analysis Homework

Seminar XI

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Seminar XI – Homework

Problem Statement:

6. ★[Python] Let A be a 2×2 matrix and let the quadratic function $f : \mathbb{R}^2 \to R$, $f(x) = \frac{1}{2}x^T Ax$.

    (a) Give a matrix A such that f has a unique minimum.

    (b) Give a matrix A such that f has a unique maximum.

    (c) Give a matrix A such that f has a unique saddle point.

    In each case plot the 3d surface, three contour lines and the gradient at three different points.

Solution:

To find the unique minimum, maximum and saddle point we will take in consideration the local extremum conditions using eigenvalues.

The unicity of these points will be given by a zero gradient in a single point of R^2.

Following exercise 4) from the seminar, we take A a symmetric 2x2 matrix:

4. Let A be a symmetric $n \times n$ matrix and the quadratic function $f : \mathbb{R}^n \to R$, $f(x) = \frac{1}{2}x^T Ax$. Prove that $\nabla f(x) = Ax$ and $H(x) = A$. *Hint: use the Taylor expansion.*

We see that the gradient of our function is represented by the term Ax, whilst the Hessian Matrix is represented by the matrix A itself.

Hence:

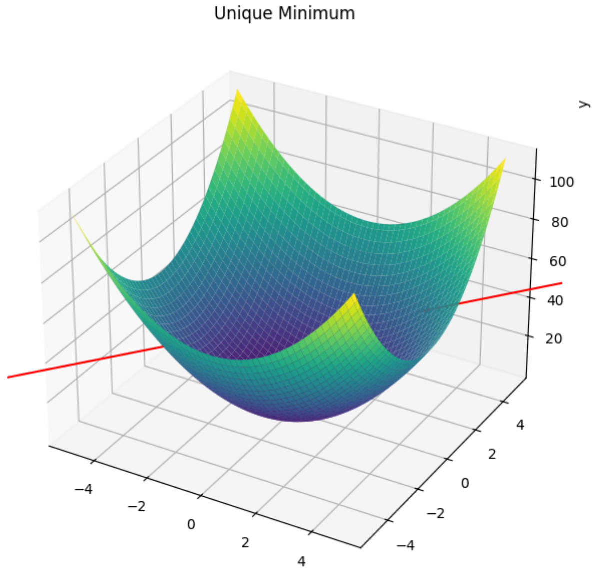
- for the minimum point we need both eigenvalues of A to be positive (e.g. ((4,0)(0,5)))
- for the maxium point we need both eigenvalues of A to be negative (e.g. ((-4,0)(0,-5)))
- for the saddle point we need the eigenvalues of A to be of different sign (e.g. ((4,0)(0,-5)))

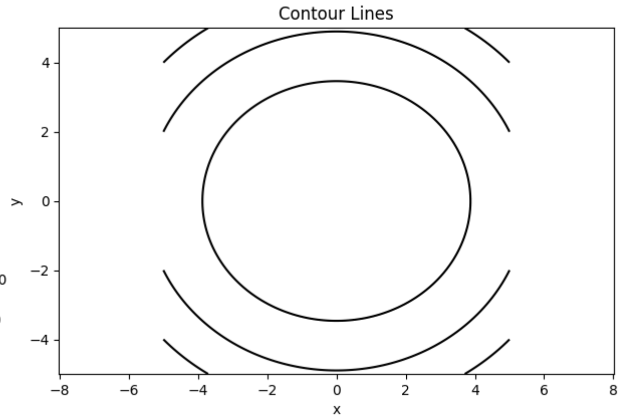
All given matrices are symmetric.

We can verify for each of these points the unicity of the extremum points by calculating the gradient of the function at zero (Ax = 0) which will result in x = (0,0) – unique point.

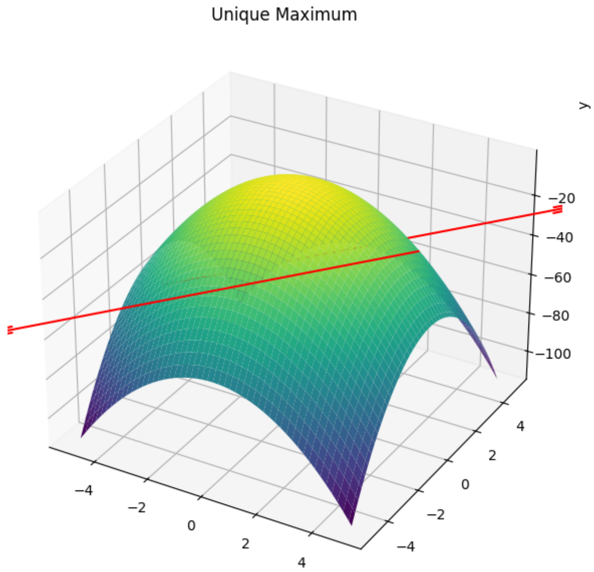
Let's now plot all these:

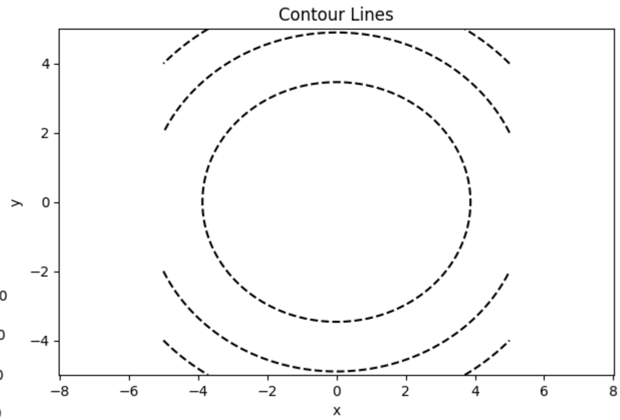
I. Minimum Point – given by A = ((4,0)(0,5))



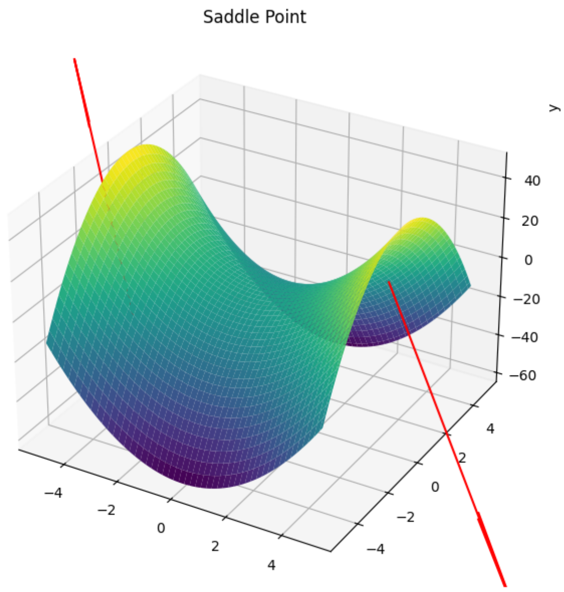


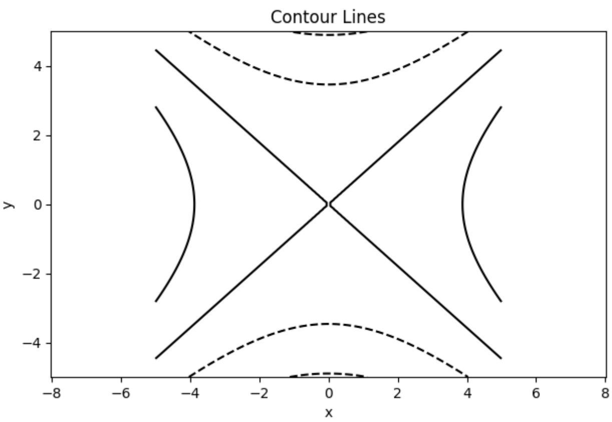
II. Maximum Point – given by A = ((-4,0)(0,-5))





III. Saddle Point – given by A = ((4,0)(0,-5))





In these representations the plane is presented in tones of blue, green and yellow, the red lines plot the gradient, and the contour lines are symbolized on the right