QUILLEN'S CONJECTURE FOR GROUPS OF p-RANK 3

KEVIN IVÁN PITERMAN*, IVÁN SADOFSCHI COSTA**, AND ANTONIO VIRUEL***

Abstract. Let G be a finite group and $\mathcal{A}_p(G)$ be the poset of nontrivial elementary abelian p-subgroups of G. Quillen conjectured that $O_p(G)$ is nontrivial if $\mathcal{A}_p(G)$ is contractible. We prove Quillen's conjecture for groups of p-rank 3.

1. Introduction

The poset $\mathcal{S}_p(G)$ of nontrivial p-subgroups of G was introduced by K.S. Brown in [4], where he proved that the Euler characteristic $\chi(\mathcal{K}(\mathcal{S}_p(G)))$ of its order complex is 1 modulo the greatest power of p dividing the order of G. Some years later, Quillen [7] studied some homotopy properties of $\mathcal{K}(\mathcal{S}_p(G))$. In that article, Quillen considered the subposet $\mathcal{A}_p(G)$ of nontrivial elementary abelian p-subgroups and proved that it is homotopy equivalent to $\mathcal{S}_p(G)$ [7, Proposition 2.1].

Quillen also proved that if $O_p(G)$, the greatest normal p-subgroup of G, is nontrivial then $\mathcal{A}_p(G)$ is contractible [7, Proposition 2.4] and conjectured that the converse should hold. In this paper we consider the following stronger version of Quillen's conjecture, stated by Aschbacher and Smith [2].

Conjecture 1.1 (Quillen's conjecture)**.** *If $O_p(G) = 1$ then $\widetilde{H}_*(\mathcal{A}_p(G)) \neq 0$.*

Quillen proved some cases of this conjecture. For example, he proved it for solvable groups [7, Theorem 12.1]. In [2], M. Aschbacher and S.D. Smith made a huge progress on the study of this conjecture. By using the classification of finite simple groups, they proved that Quillen's conjecture holds if $p > 5$ and G does not contain certain unitary components. Previously, Aschbacher and Kleidman [1] had proved Quillen's conjecture for almost simple groups (i.e. finite groups G such that $L \leq G \leq \mathrm{Aut}(L)$ for some simple group L).

In this paper we prove Quillen's conjecture 1.1 for groups of p-rank 3. Recall that the *p-rank* of G is the maximum possible rank of an elementary abelian p-subgroup of G and equals $\dim \mathcal{K}(\mathcal{A}_p(G))+1$. The p-rank 2 case was considered by Quillen [7, Proposition 2.10] and follows from the fact that an action of a finite group on a tree has a fixed point.

C. Casacuberta and W. Dicks conjectured that every finite group acting on a contractible 2-complex has a fixed point [5]. This conjecture was studied by Aschbacher and Segev in [3]. Posteriorly, Oliver and Segev classified groups that can act without fixed points on an acyclic 2-complex. The results of their work [6] are the basis of our proof of the p-rank 3 case of

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Quillen's conjecture. Our main result, Theorem 3.1 can also be seen as a special case of the Casacuberta-Dicks conjecture.

Quizás es una afirmación demasiado fuerte. Ejemplos concretos.

Note that Theorem 3.1 was not known, for the results of [2] do not apply for $p < 5$.

2. The results of Oliver and Segev

In this section we review the results of [6] needed in the proof of Theorem 3.1. If X is a poset, $\mathcal{K}(X)$ denotes the order complex of X (i.e. the simplicial complex whose simplices are the finite nonempty totally ordered subsets of X). By a G-complex we mean a G-CW complex. Note that the order complex of a G-poset is always a G-complex.

Definition 2.1 ([6])**.** A G-complex X is *essential* if there is no normal subgroup $1 \neq N \triangleleft G$ such that for each $H \subseteq G$, the inclusion $X^{HN} \to X^H$ induces an isomorphism on integral homology.

The main results of [6] are the following two theorems.

Theorem 2.2 ([6, Theorem A])**.** *For any finite group G, there is an essential fixed point free 2-dimensional (finite) $\mathbb{Z}$-acyclic G-complex if and only if G is isomorphic to one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Furthermore, the isotropy subgroups of any such G-complex are all solvable.*

Theorem 2.3 ([6, Theorem B])**.** *Let G be any finite group, and let X be any 2-dimensional $\mathbb{Z}$-acyclic G-complex. Let N be the subgroup generated by all normal subgroups $N' \triangleleft G$ such that $X^{N'} \neq \emptyset$. Then X^N is $\mathbb{Z}$-acyclic; X is essential if and only if $N = 1$; and the action of G/N on X^N is essential.*

The set of subgroups of G will be denoted by $\mathcal{S}(G)$.

Definition 2.4 ([6])**.** By a *family* of subgroups of G we mean any subset $\mathcal{F} \subseteq \mathcal{S}(G)$ which is closed under conjugation. A nonempty family is said to be *separating* if it has the following three properties: (a) $G \notin \mathcal{F}$; (b) if $H' \subseteq H$ and $H \in \mathcal{F}$ then $H' \in \mathcal{F}$; (c) for any $H \triangleleft K \subseteq G$ with K/H solvable, $K \in \mathcal{F}$ if $H \in \mathcal{F}$.

For any family $\mathcal{F}$ of subgroups of G, a $(G, \mathcal{F})$*-complex* will mean a G-complex all of whose isotropy subgroups lie in $\mathcal{F}$. A $(G, \mathcal{F})$-complex is H*-universal* if the fixed point set of each $H \in \mathcal{F}$ is acyclic.

Lemma 2.5 ([6, Lemma 1.2])**.** *Let X be any 2-dimensional acyclic G-complex without fixed points. Let $\mathcal{F}$ be the set of subgroups $H \subseteq G$ such that $X^H \neq \emptyset$. Then $\mathcal{F}$ is a separating family of subgroups of G, and X is an H-universal $(G, \mathcal{F})$-complex.*

If G is not solvable, the separating family of solvable subgroups of G is denoted by $\mathcal{SLV}$.

Proposition 2.6 ([6, Proposition 6.4])**.** *Assume that L is one of the simple groups $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $G \subseteq \mathrm{Aut}(L)$ be any subgroup containing L, and let $\mathcal{F}$ be a separating family for G. Then there is a 2-dimensional $\mathbb{Z}$-acyclic $(G, \mathcal{F})$-complex if and only if $G = L$, $\mathcal{F} = \mathcal{SLV}$, and q is a power of 2 or $q \equiv \pm 3 \pmod 8$.*

Definition 2.7 ([6, Definition 2.1])**.** For any family $\mathcal{F}$ of subgroups of G define

$$i_{\mathcal{F}}(H) = \frac{1}{[N_G(H) : H]}(1 - \chi(\mathcal{K}(\mathcal{F}_{>H}))).$$

Lemma 2.8 ([6, Lemma 2.3])**.** *Fix a separating family $\mathcal{F}$, a finite H-universal $(G, \mathcal{F})$-complex X, and a subgroup $H \subseteq G$. For each n, let $c_n(H)$ denote the number of orbits of n-cells of type G/H in X. Then $i_{\mathcal{F}}(H) = \sum_{n \geq 0} (-1)^n c_n(H)$.*

Proposition 2.9 ([6, Tables 2,3,4])**.** *Let G be one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Then $i_{\mathcal{SLV}}(1) = 1$.*

3. The case of p-rank 3

Now using the results of Oliver and Segev [6] we prove Quillen's conjecture for groups of p-rank 3.

Theorem 3.1. *Let G be a finite group of p-rank 3. If $\widetilde{H}_*(\mathcal{A}_p(G)) = 0$ then $O_p(G) \neq 1$.*

Proof. Suppose the statement is false and consider a counterexample G. Then $X = \mathcal{K}(\mathcal{A}_p(G))$ is a 2-dimensional acyclic complex. Equipped with the conjugation action of G, X is a G-complex. Since we are assuming $O_p(G) = 1$, the action is fixed point free. Consider the subgroup N generated by the subgroups $N' \triangleleft G$ such that $X^{N'} \neq \emptyset$. Clearly N is normal in G. By Theorem 2.3 $Y = X^N$ is acyclic (in particular it is nonempty) and the action of G/N on Y is essential and fixed point free. By Lemma 2.5 $\mathcal{F} = \{H \leq G/N \ : \ Y^H \neq \emptyset\}$ is a separating family and Y is an H-universal $(G/N, \mathcal{F})$-complex. Thus, Theorem 2.2 asserts that G/N must be one of the groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. In any case, by Proposition 2.6 we must have $\mathcal{F} = \mathcal{SLV}$. By Proposition 2.9, $i_{\mathcal{SLV}}(1) = 1$. Finally by Lemma 2.8, Y must have at least one free G/N-orbit. Therefore X has a G-orbit of type G/N. Let $\sigma = (A_0 < \ldots < A_j)$ be a simplex of X with stabilizer N. Since $A_0 \triangleleft N$, we have that $O_p(N)$ is nontrivial. Since $N \triangleleft G$ and $O_p(N) \operatorname{char} N$ we have $O_p(N) \triangleleft G$ and therefore $O_p(N) \leq O_p(G)$. So $O_p(G)$ is nontrivial, a contradiction. $\square$

With the same argument we can prove the following generalization of Theorem 3.1.

Theorem 3.2. *If K is an acyclic and G-invariant 2-dimensional subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, then $O_p(G) \neq 1$. In particular, $\mathcal{K}(\mathcal{S}_p(G))$ is contractible.*

By Theorem 3.2, Quillen's conjecture also holds when $\mathcal{K}(\mathcal{B}_p(G))$ is 2-dimensional. Recall that the subposet $\mathcal{B}_p(G) = \{Q \in \mathcal{S}_p(G) : Q = O_p(N_G(Q))\}$ is homotopy equivalent to $\mathcal{S}_p(G)$. See [8] for an account of the relations between the different p-group complexes.

Finally we mention that a possible approach to prove Conjecture 1.1 is to find an acyclic and G-invariant 2-dimensional subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$. If Quillen's conjecture were true, then this would be possible. Therefore, by Theorem 3.2 we have the following equivalent version of the conjecture.

Conjecture 3.3 (Restatement of Quillen's conjecture)**.** *Assume $\mathcal{K}(\mathcal{S}_p(G))$ is acyclic. Then there exists a G-invariant acyclic subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$ of dimension at most 2.*

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Universidad de Buenos Aires. Facultad de Ciencias Exactas y Naturales. Departamento de Matemática. Buenos Aires, Argentina.

CONICET-Universidad de Buenos Aires. Instituto de Investigaciones Matemáticas Luis A. Santaló (IMAS). Buenos Aires, Argentina.

E-mail address: kpiterman@dm.uba.ar
E-mail address: isadofschi@dm.uba.ar
E-mail address: viruel@uma.es