QUILLEN'S CONJECTURE FOR GROUPS OF p-RANK 3

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Abstract. Let G be a finite group and $\mathcal{A}_p(G)$ be the poset of nontrivial elementary abelian p-subgroups of G. Quillen conjectured that $O_p(G)$ is nontrivial if $\mathcal{A}_p(G)$ is contractible. We prove Quillen's conjecture for groups of p-rank 3.

1. Introduction

The poset $\mathcal{S}_p(G)$ of nontrivial p-subgroups of G was introduced by K.S. Brown in [Bro75], where he proved that the Euler characteristic $\chi(\mathcal{K}(\mathcal{S}_p(G)))$ of its order complex is 1 modulo the greatest power of p dividing the order of G. Some years later, Quillen [Qui78] studied some homotopy properties of $\mathcal{K}(\mathcal{S}_p(G))$. In that article, Quillen considered the subposet $\mathcal{A}_p(G)$ of nontrivial elementary abelian p-subgroups and proved that it is homotopy equivalent to $\mathcal{S}_p(G)$ [Qui78, Proposition 2.1].

Quillen also proved that if $O_p(G)$, the largest normal p-subgroup of G, is nontrivial then $\mathcal{A}_p(G)$ is contractible [Qui78, Proposition 2.4] and conjectured that the converse should hold. In this paper we consider the following stronger version of Quillen's conjecture, stated by Aschbacher and Smith [AS93b].

Conjecture 1.1 (Quillen's conjecture). *If $O_p(G) = 1$ then $\widetilde{H}_*(\mathcal{A}_p(G)) \neq 0$.*

Quillen proved some cases of this conjecture. For example, he proved it for solvable groups [Qui78, Theorem 12.1]. In [AS93b], M. Aschbacher and S.D. Smith made a huge progress on the study of this conjecture. By using the classification of finite simple groups, they proved that Quillen's conjecture holds if $p > 5$ and G does not contain certain unitary components. Previously, Aschbacher and Kleidman [AK90] had proved Quillen's conjecture for almost simple groups (i.e. finite groups G such that $L \leq G \leq \mathrm{Aut}(L)$ for some simple group L).

In this paper we prove Quillen's conjecture 1.1 for groups of p-rank 3. Recall that the *p-rank* of G is the maximum possible rank of an elementary abelian p-subgroup of G. The p-rank 2 case was considered by Quillen [Qui78, Proposition 2.10] and follows from the fact that an action of a finite group on a tree has a fixed point.

C. Casacuberta and W. Dicks conjectured that every finite group acting on a contractible 2-complex has a fixed point [CD92]. This conjecture was studied by Aschbacher and Segev in [AS93a]. Posteriorly, Oliver and Segev classified groups that can act without fixed points on an

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acyclic 2-complex. The results of their work [OS02] are the basis of our proof of the p-rank 3 case of Quillen's conjecture. Our main result, Corollary 3.2 can also be seen as a special case of the Casacuberta-Dicks conjecture.

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Note that Corollary 3.2 was not known, for the results of [AS93b] do not apply for $p < 5$.

2. The results of Oliver and Segev

In this section we review the results of [OS02] needed in the proof of Corollary 3.2. If X is a poset, $\mathcal{K}(X)$ denotes the order complex of X (i.e. the simplicial complex whose simplices are the finite nonempty totally ordered subsets of X). By a G-complex we mean a G-CW complex. Note that the order complex of a G-poset is always a G-complex.

Definition 2.1 ([OS02])**.** A G-complex X is *essential* if there is no normal subgroup $1 \neq N \triangleleft G$ such that for each $H \subseteq G$, the inclusion $X^{HN} \to X^H$ induces an isomorphism on integral homology.

The main results of [OS02] are the following two theorems.

Theorem 2.2 ([OS02, Theorem A])**.** *For any finite group G, there is an essential fixed point free 2-dimensional (finite) $\mathbb{Z}$-acyclic G-complex if and only if G is isomorphic to one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Furthermore, the isotropy subgroups of any such G-complex are all solvable.*

Theorem 2.3 ([OS02, Theorem B])**.** *Let G be any finite group, and let X be any 2-dimensional $\mathbb{Z}$-acyclic G-complex. Let N be the subgroup generated by all normal subgroups $N' \triangleleft G$ such that $X^{N'} \neq \emptyset$. Then X^N is $\mathbb{Z}$-acyclic; X is essential if and only if $N = 1$; and the action of G/N on X^N is essential.*

The set of subgroups of G will be denoted by $\mathcal{S}(G)$.

Definition 2.4 ([OS02])**.** By a *family* of subgroups of G we mean any subset $\mathcal{F} \subseteq \mathcal{S}(G)$ which is closed under conjugation. A nonempty family is said to be *separating* if it has the following three properties: (a) $G \notin \mathcal{F}$; (b) if $H' \subseteq H$ and $H \in \mathcal{F}$ then $H' \in \mathcal{F}$; (c) for any $H \triangleleft K \subseteq G$ with K/H solvable, $K \in \mathcal{F}$ if $H \in \mathcal{F}$.

For any family $\mathcal{F}$ of subgroups of G, a $(G, \mathcal{F})$-*complex* will mean a G-complex all of whose isotropy subgroups lie in $\mathcal{F}$. A $(G, \mathcal{F})$-complex is H-*universal* if the fixed point set of each $H \in \mathcal{F}$ is acyclic.

Lemma 2.5 ([OS02, Lemma 1.2])**.** *Let X be any 2-dimensional acyclic G-complex without fixed points. Let $\mathcal{F}$ be the set of subgroups $H \subseteq G$ such that $X^H \neq \emptyset$. Then $\mathcal{F}$ is a separating family of subgroups of G, and X is an H-universal $(G, \mathcal{F})$-complex.*

If G is not solvable, the separating family of solvable subgroups of G is denoted by $\mathcal{SLV}$.

Proposition 2.6 ([OS02, Proposition 6.4])**.** *Assume that L is one of the simple groups $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $G \subseteq \mathrm{Aut}(L)$ be any subgroup containing L, and let $\mathcal{F}$ be a separating family for G. Then there is a 2-dimensional $\mathbb{Z}$-acyclic $(G, \mathcal{F})$-complex if and only if $G = L$, $\mathcal{F} = \mathcal{SLV}$, and q is a power of 2 or $q \equiv \pm 3 \pmod 8$.*

Definition 2.7 ([OS02, Definition 2.1])**.** For any family $\mathcal{F}$ of subgroups of G define

$$i_{\mathcal{F}}(H) = \frac{1}{[N_G(H) : H]}(1 - \chi(\mathcal{K}(\mathcal{F}_{>H}))).$$

Lemma 2.8 ([OS02, Lemma 2.3])**.** *Fix a separating family $\mathcal{F}$, a finite H-universal $(G,\mathcal{F})$-complex X, and a subgroup $H \subseteq G$. For each n, let $c_n(H)$ denote the number of orbits of n-cells of type G/H in X. Then $i_{\mathcal{F}}(H) = \sum_{n\geq 0}(-1)^n c_n(H)$.*

Proposition 2.9 ([OS02, Tables 2,3,4])**.** *Let G be one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Then $i_{\mathcal{SLV}}(1) = 1$.*

3. The case of p-rank 3

Theorem 3.1. *If X is an acyclic and 2-dimensional G-invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, then $O_p(G) \neq 1$.*

Proof. Suppose $O_p(G) = 1$. Then G acts fixed point freely on X. Consider the subgroup N generated by the subgroups $N' \lhd G$ such that $X^{N'} \neq \emptyset$. Clearly N is normal in G. By Theorem 2.3 $Y = X^N$ is acyclic (in particular it is nonempty) and the action of G/N on Y is essential and fixed point free. By Lemma 2.5 $\mathcal{F} = \{H \leq G/N : Y^H \neq \emptyset\}$ is a separating family and Y is an H-universal $(G/N, \mathcal{F})$-complex. Thus, Theorem 2.2 asserts that G/N must be one of the groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. In any case, by Proposition 2.6 we must have $\mathcal{F} = \mathcal{SLV}$. By Proposition 2.9, $i_{\mathcal{SLV}}(1) = 1$. Finally by Lemma 2.8, Y must have at least one free G/N-orbit. Therefore X has a G-orbit of type G/N. Let $\sigma = (A_0 < \ldots < A_j)$ be a simplex of X with stabilizer N. Since $A_0 \lhd N$, we have that $O_p(N)$ is nontrivial. Since $N \lhd G$ and $O_p(N)\,\mathrm{char}\,N$ we have $O_p(N) \lhd G$ and therefore $O_p(N) \leq O_p(G)$. So $O_p(G)$ is nontrivial, a contradiction. □

Since the p-rank of G is equal to $\dim \mathcal{K}(\mathcal{A}_p(G)) + 1$ we obtain:

Corollary 3.2. *Let G be a finite group of p-rank 3. If $\widetilde{H}_*(\mathcal{A}_p(G)) = 0$ then $O_p(G) \neq 1$.*

By Theorem 3.1, Quillen's conjecture also holds when $\mathcal{K}(\mathcal{B}_p(G))$ is 2-dimensional. Recall that the subposet $\mathcal{B}_p(G) = \{Q \in \mathcal{S}_p(G) : Q = O_p(N_G(Q))\}$ is homotopy equivalent to $\mathcal{S}_p(G)$. See [Smi11] for an account of the relations between the different p-group complexes.

Finally we mention that a possible approach to prove Conjecture 1.1 is to find an acyclic and G-invariant 2-dimensional subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$. If Quillen's conjecture were true, then this would be possible. Therefore, by Theorem 3.1 we have the following equivalent version of the conjecture.

Conjecture 3.3 (Restatement of Quillen's conjecture)**.** *Assume $\mathcal{K}(\mathcal{S}_p(G))$ is acyclic. Then there exists a G-invariant acyclic subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$ of dimension at most 2.*

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4. Some examples

One of the first reductions in the proof of the main theorem of Aschbacher-Smith [AS93b] is to assume that $O_{p'}(G) = 1$ (see [AS93b, Proposition 1.6]). In order to perform such reduction, they use [AS93b, Theorems 2.3 & 2.4]. The hypothesis $p > 5$ is the key of why they can carry on the proof and use such theorems. More precisely, they are not allowed to apply [AS93b, Theorem 2.3] if a component of $C_G(O_{p'}(G))$ is isomorphic to $L_2(8)$, $U_3(8)$ or $Sz(32)$ with $p = 3$, 3 or 5 respectively.

In view of this, we present some examples such that the reductions of [AS93b] cannot be carried out for $p = 3$ or 5 due to the presence of those components in $C_G(O_{p'}(G))$. Nevertheless, our theorem does hold in these examples.

Example 4.1. Let $p = 3$ and let $G = (\mathrm{Sz}(2^{3^n}) \times U_3(8)) \rtimes C_{3^n}$, where C_{3^n} acts by field automorphism on $\mathrm{Sz}(2^{3^n})$ and trivially on $U_3(8)$. Note that $O_{p'}(G) = \mathrm{Sz}(2^{3^n})$ and $C_G(O_{p'}(G)) = U_3(8)$. Therefore, the reduction to $O_{p'}(G) = 1$ cannot be performed in this group by using the techniques of [AS93b].

Moreover, $O_p(G/O_{p'}(G)) = C_{3^n} \neq 1$ and $O_p(G) = 1$. By [AS93b, Lemma 0.12], there is an inclusion $\tilde{H}_*(\mathcal{A}_p(G/O_{p'}(G))) \subseteq \tilde{H}_*(\mathcal{A}_p(G))$. However, we have $\tilde{H}_*(\mathcal{A}_p(G/O_{p'}(G))) = 0$, so we cannot produce non-trivial homology on $\mathcal{A}_p(G)$ in this way.

The p-rank of G is $m_p(G) = m_p(U_3(8) \times C_{3^n}) = m_p(U_3(8)) + 1 = 3$. Hence, the main theorem of [AS93b] does not apply but, by Corollary 3.2, Quillen's conjecture holds for G.

Example 4.2. Let $p = 3$ and consider the following group. Take $A = C_{3^n}$ and $B = C_3$. Let $G = (\mathrm{Sz}(2^{3^n})^3 \times U_3(8)) \rtimes (A \times B)$ be the split extension with the following actions. The group A acts on $\mathrm{Sz}(2^{3^n})^3$ by field automorphism on each copy of the Suzuki group, while B acts by permuting the three copies of the Suzuki groups. The action of B on $U_3(8)$ is also by field automorphism but we let A act trivially on this group.

Now note that $O_p(G) = 1$, $O_{p'}(G) = \mathrm{Sz}(2^{3^n})^3$ and $C_G(O_{p'}(G)) = U_3(8)$. Therefore, the reduction of [AS93b] to $O_{p'}(G) = 1$ cannot be done in this group. Also, the conclusion of [AS93b, Lemma 0.12] does not produce non-trivial homology in $\mathcal{A}_p(G)$ since $O_p(G/O_{p'}(G)) = A \neq 1$.

The group G has p-rank 4 given that

$$m_p(G) = m_p(U_3(8) \rtimes (A \times B)) = m_p(A \times (U_3(8) \rtimes B)) = m_p(A) + m_p(U_3(8) \rtimes B) = 1 + 3 = 4.$$

In order to see that Quillen's conjecture holds for G and $p = 3$, the aim is to find a G-invariant 2-dimensional subcomplex K of $\mathcal{K}(\mathcal{S}_p(G))$ which is also homotopy equivalent to $\mathcal{K}(\mathcal{S}_p(G))$. Note that $\mathcal{K}(\mathcal{A}_p(G))$ is 3-dimensional in this case. Suppose we find such subcomplex K. If Quillen's conjecture fails for G, then $\mathcal{K}(\mathcal{S}_p(G))$ is acyclic. Therefore, K is acyclic. By Theorem 3.1, $O_p(G) \neq 1$, a contradiction.

One candidate for K is $\mathcal{K}(\mathcal{B}_p(G))$. However, we do not know yet if this subcomplex has dimension 2.

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Further examples can be built in a similar way for $p = 5$.

Example 4.3. Let $p = 5$. Let r be a prime number such that $r \equiv 2$ or $3 \mod 5$ and let $q = r^{5^n}$. Then, the simple groups $L_2(q)$, $G_2(q)$, ${}^3D_4(q^3)$ and ${}^2G_2(3^{5^n})$ have order prime to 5.

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We construct G similar to the previous examples. Let L be one of the simple groups above and let $A = C_{5^n}$ and $B = C_5$. Let $G = (L^5 \times \mathrm{Sz}(32)^2) \rtimes (A \times B)$, where A acts on each copy of L by field automorphism and trivially on $\mathrm{Sz}(32)^2$, and B acts by permuting the copies L and by field automorphism on each copy of $\mathrm{Sz}(32)$.

Note that $O_{p'}(G) = L^5$ and $C_G(O_{p'}(G)) = \mathrm{Sz}(32)^2$. Therefore, the reduction to $O_{p'}(G) = 1$ cannot be carried out in this example by using [AS93b, Proposition 1.6].

On the other hand, $O_p(G) = 1$ and $O_p(G/O_{p'}(G)) = A \neq 1$. Also, the p-rank of G is

$$m_p(G) = m_p(\mathrm{Sz}(32)^2 \rtimes (A \times B)) = m_p(A) + m_p(\mathrm{Sz}(32)^3 \rtimes B) = 1 + 3 = 4.$$

Now the aim is to find a G-invariant 2-dimensional homotopy equivalent subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$ to proceed like in Example 4.2.

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Example 4.4. Similar to Example 4.1, with the notations of Example 4.3, if $G = (L \times \mathrm{Sz}(32)^2) \rtimes A$ then $m_p(G) = 3$, $O_p(G) = 1$, $O_{p'}(G) = L$ and $O_p(G/O_{p'}(G)) = A \neq 1$. Therefore, the reductions of Aschbacher-Smith do not apply but by Corollary 3.2, Quillen's conjecture holds for G.

Finally, we remark that the prime 2 is not treated in the principal theorems used in [AS93b]. Therefore, our result does present new cases of Quillen's conjecture that does not follow from the techniques and reductions of [AS93b].

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References

[AK90] Michael Aschbacher and Peter B. Kleidman, *On a conjecture of Quillen and a lemma of Robinson*, Arch. Math. (Basel) **55** (1990), no. 3, 209–217. MR 1075043

[AS93a] Michael Aschbacher and Yoav Segev, *A fixed point theorem for groups acting on finite* 2*-dimensional acyclic simplicial complexes*, Proc. London Math. Soc. (3) **67** (1993), no. 2, 329–354. MR 1226605

[AS93b] Michael Aschbacher and Stephen D. Smith, *On Quillen's conjecture for the p-groups complex*, Ann. of Math. (2) **137** (1993), no. 3, 473–529. MR 1217346

[Bro75] Kenneth S. Brown, *Euler characteristics of groups: the p-fractional part*, Invent. Math. **29** (1975), no. 1, 1–5. MR 0385008

[CD92] Carles Casacuberta and Warren Dicks, *On finite groups acting on acyclic complexes of dimension two*, Publ. Mat. **36** (1992), no. 2A, 463–466 (1993). MR 1209816

[OS02] Bob Oliver and Yoav Segev, *Fixed point free actions on* $\mathbf{Z}$*-acyclic 2-complexes*, Acta Math. **189** (2002), no. 2, 203–285. MR 1961198

[Qui78] Daniel Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Adv. in Math. **28** (1978), no. 2, 101–128. MR 493916

[Smi11] Stephen D. Smith, *Subgroup complexes*, Mathematical Surveys and Monographs, vol. 179, American Mathematical Society, Providence, RI, 2011. MR 2850680

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