

QUILLEN'S CONJECTURE FOR GROUPS OF p -RANK 3

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ABSTRACT. Let G be a finite group and $\mathcal{A}_p(G)$ be the poset of nontrivial elementary abelian p -subgroups of G . Quillen conjectured that $O_p(G)$ is nontrivial if $\mathcal{A}_p(G)$ is contractible. We prove Quillen's conjecture for groups of p -rank 3.

1. INTRODUCTION

The poset $\mathcal{S}_p(G)$ of nontrivial p -subgroups of G was introduced by K.S. Brown in [4], where he proved that the Euler characteristic $\chi(\mathcal{K}(\mathcal{S}_p(G)))$ of its order complex is 1 modulo the greatest power of p dividing the order of G . Some years later, Quillen [7] studied some homotopy properties of $\mathcal{K}(\mathcal{S}_p(G))$. In that article, Quillen considered the subposet $\mathcal{A}_p(G)$ of nontrivial elementary abelian p -subgroups and proved that it is homotopy equivalent to $\mathcal{S}_p(G)$ [7, Proposition 2.1].

Quillen also proved that if $O_p(G)$, the greatest normal p -subgroup of G , is nontrivial then $\mathcal{A}_p(G)$ is contractible [7, Proposition 2.4] and conjectured that the converse should hold. In this paper we consider the following stronger version of Quillen's conjecture, stated by Aschbacher and Smith [2].

Conjecture 1.1 (Quillen's conjecture). *If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$.*

Quillen proved some cases of this conjecture. For example, he proved it for solvable groups [7, Theorem 12.1]. In [2], M. Aschbacher and S.D. Smith made a huge progress on the study of this conjecture. By using the classification of finite simple groups, they proved that Quillen's conjecture holds if $p > 5$ and G does not contain certain unitary components. Previously, Aschbacher and Kleidman [1] had proved Quillen's conjecture for almost simple groups (i.e. finite groups G such that $L \leq G \leq \text{Aut}(L)$ for some simple group L).

In this paper we prove Quillen's conjecture 1.1 for groups of p -rank 3. Recall that the p -rank of G is the maximum possible rank of an elementary abelian p -subgroup of G . The p -rank 2 case was considered by Quillen [7, Proposition 2.10] and follows from the fact that an action of a finite group on a tree has a fixed point.

C. Casacuberta and W. Dicks conjectured that every finite group acting on a contractible 2-complex has a fixed point [5]. This conjecture was studied by Aschbacher and Segev in [3]. Posteriorly, Oliver and Segev classified groups that can act without fixed points on an acyclic 2-complex. The results of their work [6] are the basis of our proof of the p -rank 3 case of

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Quillen's conjecture. Our main result, Corollary 3.2 can also be seen as a special case of the Casacuberta-Dicks conjecture.

Note that Corollary 3.2 was not known, for the results of [2] do not apply for $p < 5$.

2. THE RESULTS OF OLIVER AND SEGEV

In this section we review the results of [6] needed in the proof of Corollary 3.2. If X is a poset, $\mathcal{K}(X)$ denotes the order complex of X (i.e. the simplicial complex whose simplices are the finite nonempty totally ordered subsets of X). By a G -complex we mean a G -CW complex. Note that the order complex of a G -poset is always a G -complex.

Definition 2.1 ([6]). A G -complex X is *essential* if there is no normal subgroup $1 \neq N \triangleleft G$ such that for each $H \subseteq G$, the inclusion $X^{HN} \rightarrow X^H$ induces an isomorphism on integral homology.

The main results of [6] are the following two theorems.

Theorem 2.2 ([6, Theorem A]). *For any finite group G , there is an essential fixed point free 2-dimensional (finite) \mathbb{Z} -acyclic G -complex if and only if G is isomorphic to one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod{8}$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Furthermore, the isotropy subgroups of any such G -complex are all solvable.*

Theorem 2.3 ([6, Theorem B]). *Let G be any finite group, and let X be any 2-dimensional \mathbb{Z} -acyclic G -complex. Let N be the subgroup generated by all normal subgroups $N' \triangleleft G$ such that $X^{N'} \neq \emptyset$. Then X^N is \mathbb{Z} -acyclic; X is essential if and only if $N = 1$; and the action of G/N on X^N is essential.*

The set of subgroups of G will be denoted by $\mathcal{S}(G)$.

Definition 2.4 ([6]). By a *family* of subgroups of G we mean any subset $\mathcal{F} \subseteq \mathcal{S}(G)$ which is closed under conjugation. A nonempty family is said to be *separating* if it has the following three properties: (a) $G \notin \mathcal{F}$; (b) if $H' \subseteq H$ and $H \in \mathcal{F}$ then $H' \in \mathcal{F}$; (c) for any $H \triangleleft K \subseteq G$ with K/H solvable, $K \in \mathcal{F}$ if $H \in \mathcal{F}$.

For any family \mathcal{F} of subgroups of G , a (G, \mathcal{F}) -complex will mean a G -complex all of whose isotropy subgroups lie in \mathcal{F} . A (G, \mathcal{F}) -complex is *H -universal* if the fixed point set of each $H \in \mathcal{F}$ is acyclic.

Lemma 2.5 ([6, Lemma 1.2]). *Let X be any 2-dimensional acyclic G -complex without fixed points. Let \mathcal{F} be the set of subgroups $H \subseteq G$ such that $X^H \neq \emptyset$. Then \mathcal{F} is a separating family of subgroups of G , and X is an H -universal (G, \mathcal{F}) -complex.*

If G is not solvable, the separating family of solvable subgroups of G is denoted by \mathcal{SLV} .

Proposition 2.6 ([6, Proposition 6.4]). *Assume that L is one of the simple groups $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $G \subseteq \mathrm{Aut}(L)$ be any subgroup containing L , and let \mathcal{F} be a separating family for G . Then there is a 2-dimensional \mathbb{Z} -acyclic (G, \mathcal{F}) -complex if and only if $G = L$, $\mathcal{F} = \mathcal{SLV}$, and q is a power of 2 or $q \equiv \pm 3 \pmod{8}$.*

Definition 2.7 ([6, Definition 2.1]). For any family \mathcal{F} of subgroups of G define

$$i_{\mathcal{F}}(H) = \frac{1}{[N_G(H) : H]} (1 - \chi(\mathcal{K}(\mathcal{F}_{>H}))).$$

Lemma 2.8 ([6, Lemma 2.3]). *Fix a separating family \mathcal{F} , a finite H -universal (G, \mathcal{F}) -complex X , and a subgroup $H \subseteq G$. For each n , let $c_n(H)$ denote the number of orbits of n -cells of type G/H in X . Then $i_{\mathcal{F}}(H) = \sum_{n \geq 0} (-1)^n c_n(H)$.*

I: ahora tenemos ejemplos para afirmar esto con más confianza no?

Proposition 2.9 ([6, Tables 2,3,4]). *Let G be one of the simple groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod{8}$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Then $i_{\mathcal{SLV}}(1) = 1$.*

3. THE CASE OF p -RANK 3

Theorem 3.1. *If X is an acyclic and 2-dimensional G -invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, then $O_p(G) \neq 1$.*

Recordar que la acción de G en $\mathcal{S}_p(G)$ es por conjugación.

Proof. Suppose $O_p(G) = 1$. Then G acts fixed point freely on X . Consider the subgroup N generated by the subgroups $N' \triangleleft G$ such that $X^{N'} \neq \emptyset$. Clearly N is normal in G . By Theorem 2.3 $Y = X^N$ is acyclic (in particular it is nonempty) and the action of G/N on Y is essential and fixed point free. By Lemma 2.5 $\mathcal{F} = \{H \leq G/N : Y^H \neq \emptyset\}$ is a separating family and Y is an H -universal $(G/N, \mathcal{F})$ -complex. Thus, Theorem 2.2 asserts that G/N must be one of the groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod{8}$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. In any case, by Proposition 2.6 we must have $\mathcal{F} = \mathcal{SLV}$. By Proposition 2.9, $i_{\mathcal{SLV}}(1) = 1$. Finally by Lemma 2.8, Y must have at least one free G/N -orbit. Therefore X has a G -orbit of type G/N . Let $\sigma = (A_0 < \dots < A_j)$ be a simplex of X with stabilizer N . Since $A_0 \triangleleft N$, we have that $O_p(N)$ is nontrivial. Since $N \triangleleft G$ and $O_p(N) \text{ char } N$ we have $O_p(N) \triangleleft G$ and therefore $O_p(N) \leq O_p(G)$. So $O_p(G)$ is nontrivial, a contradiction. \square

Since the p -rank of G is equal to $\dim \mathcal{K}(\mathcal{A}_p(G)) + 1$ we obtain:

Corollary 3.2. *Let G be a finite group of p -rank 3. If $\tilde{H}_*(\mathcal{A}_p(G)) = 0$ then $O_p(G) \neq 1$.*

By Theorem 3.1, Quillen's conjecture also holds when $\mathcal{K}(\mathcal{B}_p(G))$ is 2-dimensional. Recall that the subposet $\mathcal{B}_p(G) = \{Q \in \mathcal{S}_p(G) : Q = O_p(N_G(Q))\}$ is homotopy equivalent to $\mathcal{S}_p(G)$. See [8] for an account of the relations between the different p -group complexes.

Finally we mention that a possible approach to prove Conjecture 1.1 is to find an acyclic and G -invariant 2-dimensional subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$. If Quillen's conjecture were true, then this would be possible. Therefore, by Theorem 3.1 we have the following equivalent version of the conjecture.

Conjecture 3.3 (Restatement of Quillen's conjecture). *Assume $\mathcal{K}(\mathcal{S}_p(G))$ is acyclic. Then there exists a G -invariant acyclic subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$ of dimension at most 2.*

I: no me convence poner este restatement.

4. SOME EXAMPLES

I: acá irían los ejemplos.

$$p = 3,$$

$$G = (\mathrm{Sz}(8)^3 \times U_3(8)) \rtimes (C_3 \times C_3)$$

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