Fix a finite group G and a prime number p dividing its order $|G|$. Denote by $\mathcal{A}_p(G)$ the poset of nontrivial elementary abelian p-subgroups of G and by $\mathcal{S}_p(G)$ the poset of all nontrivial p-subgroups of G. Let $\Omega_1(G)$ denotes the subgroup generated by all the elements of order p in G. We prove the following theorem by using the results of Aschbacher and Smith [AS93].

Theorem 0.1. *If G has p-rank at most 3 and $O_p(G) = 1$, then $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$.*

We perform a serie of reductions first by using the results of Aschbacher and Smith. Recall that a component of G is a subnormal quasisimple subgroup of G. A group H is termed quasisimple if it is perfect and $H/Z(H)$ is simple. Equivalently, H is a central extension of a simple group. The subgroup $E(G)$ of G is the subgroup generated by all the components of G. It is known that different components of G commute, and so $E(G)$ is the central product of the components of G. Denote by $\mathcal{C}(G)$ the set of components of G. Let $F(G)$ denotes the Fitting subgroup of G and $F^*(G) = F(G)E(G)$ the generalized Fitting subgroup of G. Recall that $C_G(F^*(G)) \leq F^*(G)$. Denote by $r_p(G)$ the p-rank of the group G.

(1) By [Qui78] we may assume that $r_p(G) = 3$.
(2) $\Omega_1(G) = G$: since $\mathcal{A}_p(G) = \mathcal{A}_p(\Omega_1(G))$.
(3) $O_{p'}(G) = 1$: by [AS93, Proposition 1.6].
(4) By 3 and the hypothesis $O_p(G) = 1$, $F(G) = 1$, so $F^*(G) = E(G) = L_1 \times \ldots \times L_n$ is the direct product of the components of G, which are simple. See also [AS93, Proposition 1.5].
(5) G does not have a strongly p-embedded subgroup. That is, $\mathcal{A}_p(G)$ is a connected poset (see [Qui78]).
(6) By [AK90] we may assume that G is not almost simple, so $n \geq 2$.
(7) Note that $p \mid |L_i|$ for all i since $O_{p'}(G) = 1$. In particular, $F^*(G) = L_1 \times \ldots \times L_n$ contains an elementary abelian p-subgroup of rank n. Therefore, $n \leq 3$.
(8) Since $n = 2$ or 3, some L_i has p-rank equal 1. The only simple group of 2-rank 1 is C_2 (see [Gor83]). Since $\langle L_i : L_i \simeq C_2 \rangle \leq O_2(G) = 1$, we deduce $p \neq 2$.
(9) By the above reasoning, p is an odd prime, so the Sylow p-subgroups of those components of p-rank 1, are cyclic.
(10) If G can be decomposed as a direct product $G_1 \times G_2$ with $p \mid |G_i|$ for $i = 1, 2$, then $\mathcal{A}_p(G) \simeq \mathcal{A}_p(G_1) * \mathcal{A}_p(G_2)$ (see [Qui78]), and we can apply inductive hypothesis. Therefore, we may assume G is an indecomposable group.
(11) $\Omega_1(F^*(G)) = F^*(G) < G = \Omega_1(G)$, so we can always get an element x of order p such that $x \in G - F^*(G)$.

At this point we divide the proof in two cases: $n = 2$ and $n = 3$.

Assume $F^*(G) = L_1 \times L_2 \times L_3$.

Therefore, $r_p(L_i) = 1$ for all i. If $x \in G - F^*(G)$ is an element of order p, then x acts in the set $\mathcal{C}(G)$ of components of G. Since x has primer order p, the action is either regular and $p = 3$ or x normalizes each component L_i.

If x normalizes each component L_i, then x acts on the set of Sylow p-subgroups $\mathrm{Syl}_p(L_i)$ for each i. Since the number of Sylow p-subgroups is coprime with p, there exists a Sylow $S_i \in \mathrm{Syl}_p(L_i)$ normalized by x, for each $i = 1, 2, 3$. Therefore, there is an element $y_i \in S_i$ of order p, such that $[x, y_i] = 1$. Hence, $\langle x, y_1, y_2, y_3 \rangle$ is an elementary abelian p-subgroup of G of rank 4, a contradiction.

Thus, none $x \in G - F^*(G)$ of order p can normalize the components of G. In particular, $p = 3$ and the action of x in $F^*(G)$ is regular on the components. That is, $F^*(G) \simeq L_1 \wr \langle x \rangle$. We may assume $L_i^x = L_{i+1}$ for all i, with $i+1 = 1$ if $i = 3$.

Recall the following result of [Asc86].

Proposition 0.2. [Asc86, 31.18.1]. *Let $O_{p'}(G) = 1$, x of order p in G, $L \in \mathcal{C}(G)$, and $Y = O_{p'}(C_G(x))$. If $L \neq [L, x]$ then $[L, Y] = 1$ and one of the following holds:*

(1) *$L \in \mathcal{C}(C_G(x))$, or*
(2) *$L \neq L^x$ and $C_{[L,x]}(x)' = K \in \mathcal{C}(C_G(x))$ with K a homomorphic image of L.*

We apply this result to our situation with $L = L_1$. If $l \in L_1$, then $[l, x] = lxl^{-1}x^{-1}$. Note that $u = xl^{-1}x^{-1} \in L_2$ since ${}^xL_1 = L_1^{x^2} = L_2$. Thus, $[l, x] = lu$. If $[L, x] \leq L_1$ then $u \in L_1 \cap L_2 = 1$. That is, $u = {}^xl = 1$, which means $l = 1$, a contradiction since $l \in L_1$ is any element. Therefore $L_1 \neq [L_1, x]$ and we are in the hypotheses of the above proposition. Since $L_1 \not\leq C_G(x)$ because of the action of x, the second case of the proposition holds and $C_{[L_1,x]}(x)' = K \in \mathcal{C}(C_G(x))$ and K is a homomorphic image of L_1. Given that L_1 is a simple group and that the components are nontrivial groups, we deduce $K = L_1$ and $L_1 \leq C_{[L_1,x]}(x)' \leq C_G(x)$, a contradiction.

Assume $F^*(G) = L_1 \times L_2$

In this case, $p \geq 3$, so any order p element $x \in G - F^*(G)$ normalizes L_1 and L_2 the components of G. Thus, $L_i \trianglelefteq G$ for $i = 1, 2$. Note that if $r_p(L_i) \geq 2$, then we may construct an elementary abelian p-subgroup of p-rank 4 just as the previous case. Therefore, $r_p(L_i) = 1$ and each L_i has a strongly p-embedded subgroup.

Assume that $H_1(\mathcal{A}_p(G)) = 0$. By the proof of [Asc93, 10.3], $G = (L_1 \times L_2)X$ for some subgroup $X \leq G$ of order p inducing outer automorphisms on L_1 and L_2. Moreover, L_i is of Lie type and Lie rank 1 in characteristic p and X induces field automorphisms. That is, $L_i \simeq L_2(q)$, $U_3(q^p)$ or ${}^2G_2(q)$ with q a power of p. Since each L_i has p-rank 1, their Sylow p-subgroups must by cyclic. Therefore, $L_i \simeq L_2(p)$ and $p > 3$ or $L_i \simeq {}^2G_2(3)'$ and $p = 3$. Since $\mathrm{Out}(L_2(p)) = C_2$, $L_2(p)$ does not have outer automorphisms of primer order $p > 3$. Therefore $L_i \simeq {}^2G_2(3)'$ and $p = 3$. Note that ${}^2G_2(3)' \simeq \mathrm{PSL}_2(8)$. However, in this case $\mathcal{A}_p(F^*(G)) \simeq \mathcal{A}_p(G)$ are homotopy equivalent, and $H_1(\mathcal{A}_p(F^*(G))) \neq 0$, contradicting our assumption (see the proof of [Asc93, 10.3]).

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