Faculty of Physics
Physics Degree
2020-21
Bachelor's Dissertation

Correlation and critical ballasting of coupled superconducting quantum states

Author:
Antonio Álvarez López

Supervisor:
Manuel Vázquez Ramallo
Department of Particle Physics, Area of Condensed Matter Physics, QMatterPhotonics Research Group

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• **Abstract:**

This work presents original theoretical contributions on the possibility of using in a superconducting system the effects of proximity coupling (Josephson coupling) and the increase of relaxation time near the transition (critical slowdown) to amplify its quantum coherence time, a crucial parameter in any implementation of physical systems for quantum information and communication technologies. Our results suggest that such a kind of critical ballasting can increase the quantum coherence time significantly (about a factor 3.5 considering a simple Josephson coupling and numerical values for the involved parameters typical of high-temperature superconductors).

• **Resumo:**

Este traballo presenta contribucións teóricas orixinais sobre a posibilidade de utilizar nun sistema supercondutor os efectos de acoplamento por proximidade (acoplamento Josephson) e da desaceleración crítica preto da transición (critical slowdown) para aumentar o seu tempo de coherencia cuántica, un parámetro crucial en calquera implementación de sistemas físicos para tecnoloxías de información e comunicación cuánticas. Os nosos resultados suxiren que esta forma de lastrado crítico pode aumentar significativamente o tempo de coherencia cuántica (aproximadamente un factor 3.5 tendo en conta un simple acoplamento de Josephson e valores numéricos para os parámetros implicados típicos dos supercondutores de alta temperatura).

• **Resumen:**

Este trabajo presenta contribuciones teóricas originales sobre la posibilidad de utilizar en un sistema superconductor los efectos de acoplamiento por proximidad (acoplo Josephson) y de ralentización crítica cerca de la transición (critical slowdown) para aumentar su tiempo de coherencia cuántica, un parámetro crucial en toda implementación de sistemas físicos para las tecnologías de información y comunicación cuánticas. Nuestros resultados sugieren que dicha forma de lastrado crítico puede aumentar el tiempo de coherencia cuántica significativamente (aproximadamente un factor 3.5 considerando un acoplo Josephson simple y valores numéricos para los parámetros implicados típicos de los superconductores de alta temperatura).

Contents

1 Introduction

1.1 General Introduction

In the recent years, one of the most exciting lines of research in Physics is the one known as "quantum information and communication technologies" (or Q-ICT, a probably more appropriate name than the popular term "quantum computing", that is really only a subset of these technologies). The processing of information by manipulating and measuring quantum wave functions of various types has been shown to be not only possible, but promising in some areas, to far exceed the possibilities of current technologies. The interest in this line of work is not only academic: multiple multinational companies are now heavily investing in Q-ICT. The area is also the subject of important European Flagship Actions and will probably be a main subject in the Next Generation funds of the European Union, also in Galicia.

In order to advance in the design and implementation of useful Q-ICT devices, one of the requirements to be optimized is the *coherence time* of the quantum state, defined in general as the time in which this state is destroyed or altered by external excitations, in principle random. Hence, for example, Q-TIC devices based on superconductors (that are the vast majority of current commercial implementations, including, for example, those of Google and IBM and also the future quantum computer to be installed by the latter company at CESGA's premises in Santiago de Compostela) require an extreme cooling, even higher than the level that can normally be obtained with liquid ^{4}He. These devices need to operate at temperatures of the order of mK, which are achievable only through dilution refrigerations or similar, in order to decrease the k_BT excitations to values that allow coherence times of the superconducting quantum states compatible with the treatment of these wave functions for the purposes of Q-ICT. Therefore, the study of any mechanism that allows the increase of coherence times is of interest. In this sense, many researchers around the world have been studying various possibilities, such as redundant hardware with error correction that allows discarding time deviations, new topological superconducting materials, superconducting devices containing a quantum protectorate (vortices, Josephson tricrystals, etc).

In this work, we intend to explore a way (that to our knowledge has not yet been considered) to increase the coherence time of a superconducting wave function. The mechanism consists in slowing down (ballasting) a superconductor by means of its Josephson coupling (tunnel or proximity coupling) with a much larger superconductor that is very close to its critical temperature. The idea is to take advantage of the phenomenon known as *critical slowing down* that such a superconductor near its transition would experience. The critical slowing down is a well-known phenomenon in phase transitions although it rarely has measurable consequences, except in transitions as clean as the superconducting one. The study of this ballast is part of the efforts that the QMatterPhotonics group at USC has been making to integrate their superconductivity and integrated optics laboratories into a common group that has Q-ICT as one of its main lines of work. In this sense, funding has been received from both the Xunta de Galicia and the Plan Nacional. The aim of the present work is not the complete design of a Q-ICT device, but simply a first study of the

mentioned critical effects of quantum ballasting (which we have called *critical ballasting*) in a specially simplified configuration that allows the first theoretical estimation studies. Therefore the specific system-problem to be studied in this short work will be the one described in Figure 1.

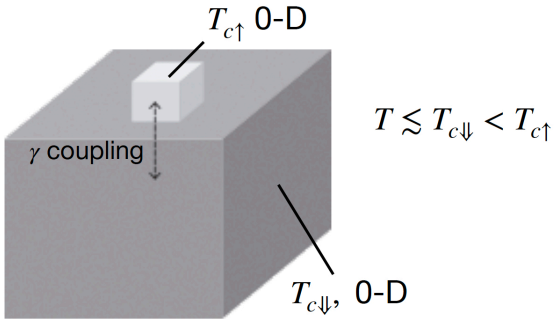

Figure 1: Representation of the system formed by the two superconductors

In Figure 1, a small superconductor (the one to be slowed down) is placed in contact with a much larger superconductor (the superconducting ballast) with a weak coupling, also called tunnel or Josephson coupling. The large superconductor will be considered to have a lower superconducting critical temperature than the small one. In the remainder of the paper we will denote both superconductors by the indices $\uparrow$ for the small superconductor of higher T_c and $\Downarrow$ for the large superconductor of lower T_c. We will study the system for temperatures close to $T_{c\Downarrow}$ at which the energetic coupling by proximity between both superconductors can be expected to increase the time inertia of the superconductor "$\uparrow$" due to the critical slowing down that the superconductor "$\Downarrow$" will be experiencing, which therefore acts as a ballast. Thus we are interested in calculating under these circumstances what would be the coherence time of the superconducting wave functions in the "$\uparrow$" superconductor, which is where the Q-TIC manipulation and measurement processes would be performed.

1.2 Overview and original contributions of the present work

To the best of our knowledge, the problem described above has not been considered by any researcher, so our calculations, although corresponding to a very simplified situation, provide original contributions to the subject. In the following we summarize the organization and contents of the present work to facilitate its reading, emphasizing the original contributions given in it:

In Chapter 2, the system-problem with the characteristics described above, as well as its energy functional, which depends on the macroscopic wave functions of both superconductors, their temperatures and several parameters among which the coupling constant γ stands out, will be presented in a formal and detailed way. In this chapter we also compare the wave functions of the two superconductors in the zero coupling regime and solve the equilibrium calculation under these conditions, as well as considering the effect

that coupling has on their behavior. Finally, the way in which the coherence time can be computed is discussed.

In Chapter 3, we present our first original results on the system-problem introduced in the previous chapter. In particular, three developments of analytical calculations are carried out: The first one aims to characterize the equilibrium solution, i.e., the two non-zero wave functions that minimize the energy functional of the zero coupling system. The second development computes the energy of the Gaussian fluctuations of the system around the equilibrium from the spectrum of the Hessian matrix of the functional. Thirdly, the coherence times of the two superconductors are obtained by studying the behavior in the limit of weak coupling (small γ) regime. The author is not aware that these calculations have been performed before, so they can be considered novel.

In Chapter 4, in order to better examine the results obtained above, the parameters involved will be given physically reasonable numerical values, and several computations will be carried out. These have as objectives, on the one hand, to determine whether the equilibrium states are well defined in the given range of temperatures and coupling parameters; and on the other hand, to represent the variation of the coherence time of the small superconductor (which is the one of major interest) versus γ, and to determine approximately the time increment resulting from the ballasting of the large superconductor. The γ limit beyond which the weak coupling approximation is no longer valid is also explored.

Finally, the present work has a chapter of Conclusions.

2 Preliminary considerations

2.1 Initial approach

As a starting point, we use the uniform or zero-dimensional (0-D) Ginzburg-Landau (GL) free energy functional. For the theoretical basis we recommend [1] or the rest of the bibliography of the *Superconductors and Superfluids* course of the USC Physics degree. Here, we will only mention that the F functional can be seen as a power series development of the macroscopic wave function Ψ. The Lawrence-Doniach functional is also a power series development of the difference between the wave functions of two systems in contact.

Let a superconductor "$\uparrow$" coupled by tunnel effect (Josephson coupling, with γ as the coupling parameter) with another superconductor "$\Downarrow$" such that their critical temperatures satisfy that $T_{c\uparrow} > T_{c\Downarrow}$. The superconductor "$\uparrow$" will have a much smaller volume than the superconductor "$\Downarrow$", although both can be considered 0-D, so that the Ginzburg-Landau functional of the total system will be F (Lawrence-Doniach type functional) given by:

$$F = F_{\uparrow} + F_{\Downarrow} + F_{\uparrow\Downarrow},$$

where $F_{\uparrow}$, $F_{\Downarrow}$ and $F_{\uparrow\Downarrow}$ are, respectively, the individual free energies of both superconductors in the absence of magnetic fields and gradients and the coupling term:

$$F_{\uparrow} = a_{\uparrow}\epsilon_{\uparrow}|\Psi_{\uparrow}|^2 + \frac{b_{\uparrow}}{2}|\Psi_{\uparrow}|^4, \quad F_{\Downarrow} = a_{\Downarrow}\epsilon_{\Downarrow}|\Psi_{\Downarrow}|^2 + \frac{b_{\Downarrow}}{2}|\Psi_{\Downarrow}|^4, \quad F_{\uparrow\Downarrow} = \gamma|\Psi_{\uparrow} - \Psi_{\Downarrow}|^2.$$

In these expressions, $\Psi_{\uparrow}$ and $\Psi_{\Downarrow}$ denote the wave functions of both superconductors,

$$\epsilon_{\uparrow} = \frac{T - T_{c\uparrow}}{T_{c\uparrow}} \quad \text{y} \quad \epsilon_{\Downarrow} = \frac{T - T_{c\Downarrow}}{T_{c\Downarrow}}$$

are the reduced temperatures, and the other parameters are constants that satisfy

$$a_{\uparrow}, a_{\Downarrow}, b_{\uparrow}, b_{\Downarrow}, \gamma > 0.$$

The Figure 1 graphically represents the system. Although one superconductor is larger in volume than the other, the zero-dimensionality condition (0-D) implies that the size of both is much smaller than their respective coherence lengths, so that the contributions of the $\xi^2_{\uparrow,\Downarrow}(0)|\nabla\Psi_{\uparrow,\Downarrow}|^2$ term to the free energy functional will be negligible. This property, achievable in low critical temperature superconductors such as aluminum or lead, greatly simplifies the equilibrium study.

Therefore we are interested in the case when the volumes satisfy $V_{\Downarrow} \gg V_{\uparrow}$ and the critical temperatures are $T_{c\uparrow} > T_{c\Downarrow}$. In this situation, when $T \lesssim T_{c\Downarrow} < T_{c\downarrow}$, the individual free energies $F_{\uparrow}$ and $F_{\Downarrow}$ will be as in the following graph:

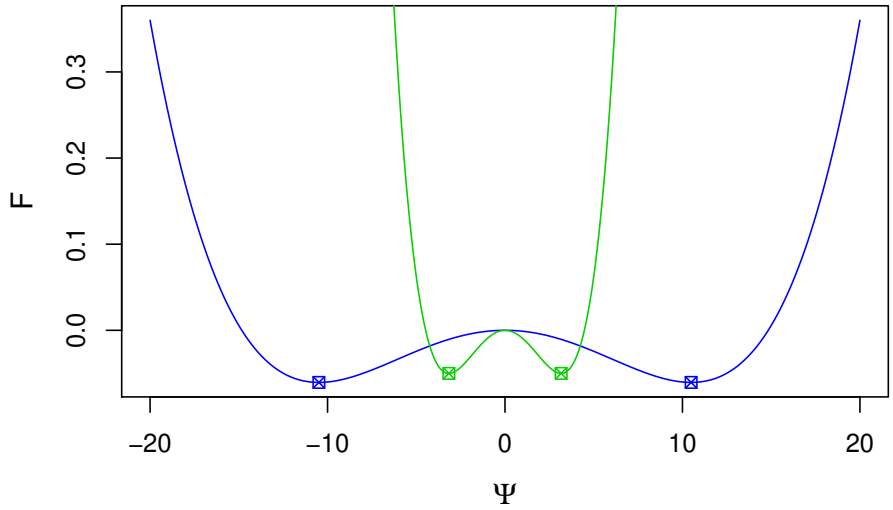

Figure 2: In blue, $F_\uparrow$ versus $\Psi_\uparrow$ with the two real equilibrium points marked. In green, ditto for $F_\Downarrow$ and $\Psi_\Downarrow$. In both cases we consider $\Psi \in \mathbb{R}$.

The two plots correspond to the usual Ginzburg-Landau potential: when $T \to T_c$, the minimum of F, and thus the most energetically favorable wave function Ψ, tends to $\Psi = 0$, and so the curve flattens and F decreases.

The coupling term $F_{\uparrow\Downarrow} = \gamma|\Psi_\uparrow - \Psi_\Downarrow|^2$ energetically favors that $\Psi_\uparrow \approx \Psi_\Downarrow$ and will link the two Ψ states in which the superconductors " $\uparrow$ " and " $\Downarrow$ " are found, which would otherwise have graphs as above and independent of each other. For the energetics of $\Psi_\Downarrow$ to be relevant we therefore need $V_\uparrow \ll V_\Downarrow$, as this increases the values of $F_\uparrow$ since $a_{\uparrow,\Downarrow}, b_{\uparrow,\Downarrow}$ are proportional to $V_{\uparrow,\Downarrow}$ in the Ginzburg-Landau functionals.

Now, in the case $V_\uparrow \ll V_\Downarrow$, both energetics are relevant, and the wave functions $\Psi_\uparrow, \Psi_\Downarrow$ must seek a fit between their equilibrium states without either dominating the other. To find the equilibrium values, we will follow a common method of calculation in Ginzburg-Landau theory, which basically consists of requiring $\left(\frac{\partial F}{\partial|\Psi|^2}\right)_{\Psi=\Psi_{eq}} = 0$.

2.2 Solution without coupling

In the case of a single superconductor, we proceed as follows:

$$F = a\epsilon|\Psi|^2 + \frac{b}{2}|\Psi|^4, \tag{1}$$

so

$$\frac{\partial F}{\partial|\Psi|^2} = a\epsilon + b|\Psi|^2,$$

which implies $|\Psi^{eq}|^2 = \frac{-a\epsilon}{b} = \frac{a|\epsilon|}{b}$, as we are considering the case $T < T_c$, that is equivalent to $\epsilon < 0$.

Therefore, in the uncoupled system, the wave functions of both superconductors at equilibrium are given by

$$\Psi_{\uparrow}^{eq} = \sqrt{\frac{a_{\uparrow}|\epsilon_{\uparrow}|}{b_{\uparrow}}}, \qquad \Psi_{\Downarrow}^{eq} = \sqrt{\frac{a_{\Downarrow}|\epsilon_{\Downarrow}|}{b_{\Downarrow}}}. \tag{2}$$

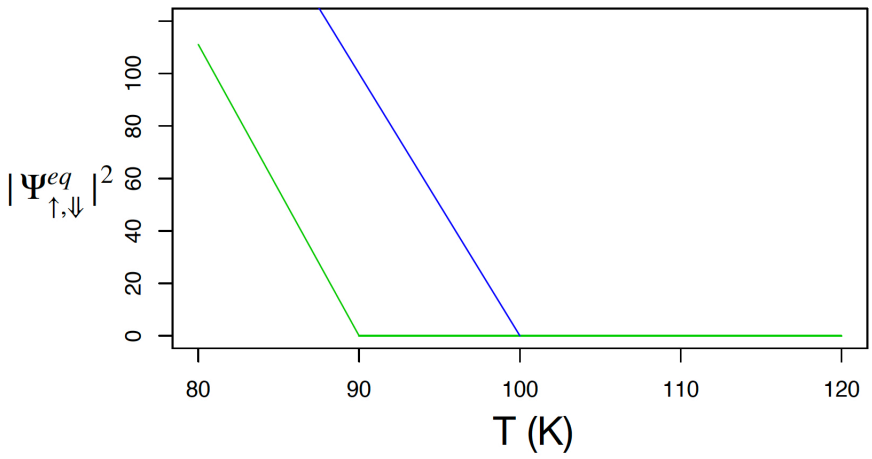

Figure 3: Variation of $|\Psi_{eq}|^2$ with the temperature (in Kelvins). In blue, for the "↑" superconductor, for which we take $T_{c\uparrow} = 100$ K; in green, for the "⇓" superconductor, for which we take $T_{c\Downarrow} = 90$ K.

It can be seen that, if $T \to T_{c\Downarrow}$, the superconductor "⇓" approaches its transition while "↑" may still be far from it. Therefore, if they were independent, since $|\Psi|^2$ is proportional to the concentration of superconducting carriers ("Cooper pairs"), "↑" would have many more such carriers than "⇓" because it is immersed in the superconducting state. Since they are not independent, $\Psi_{\uparrow}$ and $\Psi_{\Downarrow}$ must synchronize their values to also minimize $F_{\uparrow\Downarrow} \propto |\Psi_{\uparrow} - \Psi_{\Downarrow}|^2$.

2.3 Coherence time

We can easily compute the coherence time τ from the time uncertainty principle

$$\tau \Delta E \sim \hbar.$$

In a Gaussian fluctuation $\delta\Psi$ without coupling terms $|\Psi_{\uparrow} - \Psi_{\Downarrow}|^2$, the energy input would be $2a|\epsilon|$, as we will see in this paper. Thus, in the uncoupled system,

$$\tau \propto \frac{k_B T}{2a|\epsilon|}, \tag{3}$$

where $k_B T$ is the energy available for excitations. Therefore we see that if $\epsilon_{\Downarrow} \to 0$ (i.e. $T \to T_{c\Downarrow}$), then we have $\tau_{\Downarrow} \to \infty$. This means that, due to the existence of a critical transition, the coherence time slows down (diverges), which is precisely what is known as the critical slowdown of isolated systems.

The calculation of τ for the coupled system, which includes the term proportional to $|\Psi_{\uparrow} - \Psi_{\Downarrow}|^2$, involves calculating the energy spectrum of the fluctuations that replaces the $2a|\epsilon|$ of the uncoupled case. For this purpose, we are going to use a calculation technology inspired by the evaluation of critical phenomena in Lawrence-Doniach (LD) type models for laminar superconductors. These consider a number, in principle infinite, of parallel, equispaced, two-dimensional layers with a coupling to first neighbors that is proportional to $\gamma|\Psi_{\text{layer}} - \Psi_{\text{next layer}}|^2$. The original LD studies are a common tool among those used to consider interlaminar coupling mechanisms in high critical temperature superconductors. It is therefore a well supported theoretical tool.

3 Analytical calculation

3.1 Writing the Ginzburg-Landau Functional

Although our main objective is the calculation of the energy spectrum associated with the non-Gaussian fluctuations $\delta\Psi$ in the coupled system, a necessary previous step is the calculation of the equilibrium state $\left(\Psi_{\uparrow}^{eq}, \Psi_{\Downarrow}^{eq}\right)$ of the coupled system from the equations given by the condition

$$\left(\frac{\partial F}{\partial|\Psi|^2}\right)_{\Psi=\Psi^{eq}} = 0,$$

or, equivalently, we can use $\frac{\partial F}{\partial|\Psi|}$ instead of $\frac{\partial F}{\partial|\Psi|^2}$ since, by the chain rule, we have

$$\frac{\partial F}{\partial|\Psi|} = \frac{\partial|\Psi|^2}{\partial|\Psi|}\frac{\partial F}{\partial|\Psi|^2} = 2|\Psi|\frac{\partial F}{\partial|\Psi|^2},$$

and the equivalence is satisfied if we eliminate the trivial null solution. Subsequently, we consider the energetics of the fluctuations as.

$$\delta F = F(\Psi_{\uparrow}^{eq} + \delta\Psi_{\uparrow}, \Psi_{\Downarrow}^{eq} + \delta\Psi_{\Downarrow}) - F(\Psi_{\uparrow}^{eq}, \Psi_{\Downarrow}^{eq}).$$

If we use the Taylor development around the equilibrium state and apply the Gaussian approximation, which consists of disregarding all terms of order greater than 2, we are left with the following

$$\delta F = \frac{1}{2}(\delta\Psi_{\uparrow}, \delta\Psi_{\Downarrow})\left(\frac{\partial^2 F}{\partial|\Psi_i|\partial|\Psi_j|}\right)_{\Psi=\Psi^{eq}}(\delta\Psi_{\uparrow}, \delta\Psi_{\Downarrow})^T.$$

Thus the energy spectrum sought is simply the set of eigenvalues of the Hessian matrix HF of the Ginzburg-Landau functional F multiplied by a factor of $\frac{1}{2}$.

Before starting the equilibrium state calculation, let us recall the expression of the energy functional for the coupled system:

$$F = a_{\uparrow}\epsilon_{\uparrow}|\Psi_{\uparrow}|^2 + \frac{b_{\uparrow}}{2}|\Psi_{\uparrow}|^4 + a_{\Downarrow}\epsilon_{\Downarrow}|\Psi_{\Downarrow}|^2 + \frac{b_{\Downarrow}}{2}|\Psi_{\Downarrow}|^4 + \gamma|\Psi_{\uparrow} - \Psi_{\Downarrow}|^2. \tag{4}$$

The value of this functional is independent of the phases of the two wave functions, i.e., it depends only on their moduli. This phase symmetry is reflected mathematically in that the function F is invariant by the action

$$\begin{array}{rcl} \mathbb{S}^1 \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (z, (\Psi_{\uparrow}, \Psi_{\Downarrow})) & \longmapsto & z \cdot (\Psi_{\uparrow}, \Psi_{\Downarrow}) = (z\Psi_{\uparrow}, z\Psi_{\Downarrow}). \end{array}$$

Therefore the functional does not have critical points, but critical orbits that are contained in the Cartesian product of two circumferences in $\mathbb{C}^2$. This allows us to deduce that the energy will be identical in all pairs of wave functions with equal moduli. Consequently, we can restrict our study of equilibrium to the pairs $(\Psi_{\uparrow}, \Psi_{\Downarrow})$ with imaginary null parts.

Despite this, we will begin our analysis with a general approach and subsequently apply this simplification.

For ease of computation, we rename the superconductor wave functions to

$$\Psi_\uparrow = R + iI, \quad \Psi_\Downarrow = V + iW,$$

so that the energy functional is written as

$$F = a_\uparrow\epsilon_\uparrow(R^2+I^2) + a_\Downarrow\epsilon_\Downarrow(V^2+W^2) + \frac{b_\uparrow}{2}(R^4+I^4+2R^2I^2) + \\ + \frac{b_\Downarrow}{2}(V^4+W^4+2V^2W^2) + +\gamma(R^2+V^2-2RV) + \gamma(I^2+W^2-2IW).$$

3.2 The equilibrium solution

With these variables, the equilibrium condition is now:

$$\begin{cases} \partial_R F = 2a_\uparrow\epsilon_\uparrow R + 2b_\uparrow R^3 + 2b_\uparrow I^2 R + 2\gamma R - 2\gamma V = 0 \\ \partial_I F = 2a_\uparrow\epsilon_\uparrow I + 2b_\uparrow I^3 + 2b_\uparrow R^2 I + 2\gamma I - 2\gamma W = 0 \\ \partial_V F = 2a_\Downarrow\epsilon_\Downarrow V + 2b_\Downarrow V^3 + 2b_\Downarrow W^2 V + 2\gamma V - 2\gamma R = 0 \\ \partial_W F = 2a_\Downarrow\epsilon_\Downarrow W + 2b_\Downarrow W^3 + 2b_\Downarrow V^2 W + 2\gamma W - 2\gamma I = 0. \end{cases} \tag{5}$$

Simplifying, this system of equations can be equivalently written as

$$\begin{cases} \gamma V = R(a_\uparrow\epsilon_\uparrow + b_\uparrow R^2 + b_\uparrow I^2 + \gamma) \\ \gamma W = I(a_\uparrow\epsilon_\uparrow + b_\uparrow I^2 + b_\uparrow R^2 + \gamma) \\ \gamma R = V(a_\Downarrow\epsilon_\Downarrow + b_\Downarrow V^2 + b_\Downarrow W^2 + \gamma) \\ \gamma I = W(a_\Downarrow\epsilon_\Downarrow + b_\Downarrow W^2 + b_\Downarrow V^2 + \gamma), \end{cases}$$

or, if $R \neq 0 \neq V$,

$$\begin{cases} W = I\dfrac{V}{R} \\ \gamma\dfrac{V}{R} = b_\uparrow(R^2+I^2) + a_\uparrow\epsilon_\uparrow + \gamma \\ \gamma\dfrac{R}{V} = b_\Downarrow(V^2+W^2) + a_\Downarrow\epsilon_\Downarrow + \gamma. \end{cases}$$

Now, we will make the following change of variables:

$$\begin{cases} \alpha = R^2 + I^2 \\ \beta = V/R. \end{cases}$$

Therefore

$$W^2 + V^2 = I^2\frac{V^2}{R^2} + V^2 = \frac{I^2+R^2}{R^2}V^2 = \alpha\beta^2,$$

and we can rewrite the equilibrium condition as

$$\begin{cases} W = \beta I \\ \gamma\beta = b_\uparrow \alpha + a_\uparrow \epsilon_\uparrow + \gamma \\ \dfrac{\gamma}{\beta} = b_\Downarrow \alpha\beta^2 + a_\Downarrow \epsilon_\Downarrow + \gamma. \end{cases} \tag{6}$$

As we can see, β is a constant of proportionality between the real and imaginary parts of $\Psi_\uparrow$ and $\Psi_\Downarrow$, i.e., $\Psi_\Downarrow = \beta\Psi_\uparrow$. On the other hand, α is nothing but the squared modulus of the $\Psi_\uparrow$ wave function, i.e., the probability distribution for the Cooper pairs of such a superconductor.

If we assume that $b_\uparrow \neq 0$, we can write α as a function of β from the second equation of (6) as

$$\alpha = \frac{1}{b_\uparrow}(\gamma\beta - a_\uparrow \epsilon_\uparrow - \gamma), \tag{7}$$

and, since α is a squared modulus, this value must be greater than or equal to zero for the solution to be valid, i.e., it must be verified that

$$\beta \geq \frac{a_\uparrow \epsilon_\uparrow}{\gamma} + 1. \tag{8}$$

Substituting the expression (7) of α into the third equation of (6) and developing, we obtain the following polynomial equation of fourth degree:

$$p(\beta) := \beta^4 - \left(\frac{a_\uparrow \epsilon_\uparrow}{\gamma} + 1\right)\beta^3 + \frac{b_\uparrow}{b_\Downarrow}\left(\frac{a_\Downarrow \epsilon_\Downarrow}{\gamma} + 1\right)\beta - \frac{b_\uparrow}{b_\Downarrow} = 0. \tag{9}$$

The roots of this polynomial can be computed explicitly, for example, by the Ferrari method:

Let p_0, p_1 and p_3 be the coefficients of the monomials of degree 0, 1 and 3 respectively of the polynomial $p(\beta)$. Then, we define

$$\begin{cases} m = \sqrt{\dfrac{p_3^2}{4} + y}, \\ n = \sqrt{\dfrac{y^2}{4} - p_0}, \end{cases}$$

where y is a real root of the equation

$$y^3 + (p_3 p_1 - 4p_0)y - (p_3^2 p_0 + p_1^2) = 0,$$

The solutions of (9) will then be the following four values:

$$\begin{cases} \beta_1 = \frac{1}{2}\left[(-\frac{p_3}{2} - m) + \sqrt{\frac{p_3^2}{4} + p_3 m + m^2 - 2y + 4n}\right] \\ \beta_2 = \frac{1}{2}\left[(-\frac{p_3}{2} - m) - \sqrt{\frac{p_3^2}{4} + p_3 m + m^2 - 2y + 4n}\right] \\ \beta_3 = \frac{1}{2}\left[(-\frac{p_3}{2} + m) + \sqrt{\frac{p_3^2}{4} - p_3 m + m^2 - 2y - 4n}\right] \\ \beta_4 = \frac{1}{2}\left[(-\frac{p_3}{2} + m) - \sqrt{\frac{p_3^2}{4} - p_3 m + m^2 - 2y - 4n}\right]. \end{cases}$$

Due to the characteristics of the equation (9) (polynomial of fourth degree with positive leading coefficient and negative independent term), we know that there must be at least two real roots, one positive and one negative. Moreover, each of the real solutions is valid for the study of equilibrium if it satisfies the condition (8). We can obtain a sufficient condition for the existence of at least one valid solution by simply requiring that

$$p(\frac{a_\uparrow \epsilon_\uparrow}{\gamma} + 1) < 0.$$

Developing this expression, it is concluded that a sufficient condition for the existence of a root β of the equation (9) satisfying (8) is

$$\epsilon_\uparrow \epsilon_\Downarrow < -\gamma \frac{a_\Downarrow \epsilon_\Downarrow + a_\uparrow \epsilon_\uparrow}{a_\uparrow a_\Downarrow},$$

or, since $\epsilon_\uparrow < \epsilon_\Downarrow < 0$ in the temperature range we are interested, the above condition can be equivalently written as

$$1 < \gamma \cdot \left(\frac{1}{a_\Downarrow |\epsilon_\Downarrow|} + \frac{1}{a_\uparrow |\epsilon_\uparrow|}\right). \tag{10}$$

Suppose we have a root β satisfying the condition (8). In such a case, we can obtain α from (7) and obtain an equilibrium state from any wave function $\Psi_\uparrow = R + iI$ satisfying $R^2 + I^2 = \alpha$ (the other wave function at equilibrium would be $\Psi_\Downarrow = \beta \Psi_\uparrow$). This is where phase freedom arises. As we explained previously, we will continue our study by fixing a zero-phase equilibrium point, i.e., $I_{\text{eq}} = 0$ and $R_{\text{eq}} = \sqrt{\alpha}$. Then, from the equilibrium condition (6) we obtain that $W_{\text{eq}} = 0$ and $V_{\text{eq}} = \beta\sqrt{\alpha}$, so both wave functions $(\Psi_\uparrow^{\text{eq}}, \Psi_\Downarrow^{\text{eq}})$ are real and are either in phase or in phase opposition depending on the sign of the root β considered.

3.3 Energy of an excitation (calculation of the spectrum of fluctuations)

To continue our study, we compute the Hessian matrix HF of the functional of the coupled system at the equilibrium point $(\Psi_\uparrow^{\text{eq}}, \Psi_\Downarrow^{\text{eq}})$ considered. To do this, it is sufficient

to calculate the second derivatives of the functional:

$$\begin{array}{cccc} \partial_{RR}F = 2a_\uparrow\epsilon_\uparrow + 6b_\uparrow R^2 + 2b_\uparrow I^2 + 2\gamma & \partial_{RI}F = 4b_\uparrow IR & \partial_{RV}F = -2\gamma & \partial_{RW}F = 0 \\ \partial_{II}F = 2a_\uparrow\epsilon_\uparrow + 6b_\uparrow I^2 + 2b_\uparrow R^2 + 2\gamma & \partial_{IV}F = 0 & \partial_{IW}F = -2\gamma & \\ \partial_{VV}F = 2a_\Downarrow\epsilon_\Downarrow + 6b_\Downarrow V^2 + 2b_\Downarrow W^2 + 2\gamma & \partial_{VW}F = 0 & & \\ \partial_{WW}F = 2a_\Downarrow\epsilon_\Downarrow + 6b_\Downarrow W^2 + 2b_\Downarrow V^2 + 2\gamma & & & \end{array}$$

The remaining elements are obtained from the previous ones by the symmetry of HF. Consequently, the Hessian matrix evaluated at our equilibrium point is:

$$HF(R_{\text{eq}}, 0, V_{\text{eq}}, 0) = \begin{bmatrix} 2a_\uparrow\epsilon_\uparrow + 6b_\uparrow\alpha + 2\gamma & 0 & -2\gamma & 0 \\ 0 & 2a_\uparrow\epsilon_\uparrow + 2b_\uparrow\alpha + 2\gamma & 0 & -2\gamma \\ -2\gamma & 0 & 2a_\Downarrow\epsilon_\Downarrow + 6b_\Downarrow\beta^2\alpha + 2\gamma & 0 \\ 0 & -2\gamma & 0 & 2a_\Downarrow\epsilon_\Downarrow + 2b_\Downarrow\beta^2\alpha + 2\gamma \end{bmatrix} \tag{11}$$

Recall that we are considering Gaussian fluctuations around the equilibrium, so the associated energy variation is

$$\begin{aligned} \delta F(\Psi_\uparrow, \Psi_\Downarrow) = (\Psi_\uparrow, \Psi_\Downarrow) \cdot \frac{1}{2} HF(\Psi_\uparrow^{\text{eq}}, \Psi_\Downarrow^{\text{eq}}) \cdot (\Psi_\uparrow, \Psi_\Downarrow)^T \\ = (R, I, V, W) \cdot \frac{1}{2} HF(R_{\text{eq}}, 0, V_{\text{eq}}, 0) \cdot (R, I, V, W)^T. \end{aligned}$$

The spectrum of energies is the set of eigenvalues of $\frac{1}{2}HF(R_{\text{eq}}, 0, V_{\text{eq}}, 0)$, which we proceed to calculate.

Since the critical orbits are actually a pair of circumferences in $\mathbb{C}^2$ coupled by the proportionality constant β obtained, there exists for any equilibrium state a direction in which the functional will not change its value. This direction is the tangent to the equilibrium orbit, which for our point $(R_{\text{eq}}, 0, V_{\text{eq}}, 0)$ is the one given by the vector $(0, 1, 0, \beta)$, which will therefore be an eigenvector with zero eigenvalue. We check this below:

$$\begin{aligned} \frac{1}{2} HF(R_{\text{eq}}, 0, V_{\text{eq}}, 0) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \beta \end{pmatrix} &= \begin{pmatrix} 0 \\ a_\uparrow\epsilon_\uparrow + b_\uparrow\alpha + \gamma(1-\beta) \\ 0 \\ \beta(a_\Downarrow\epsilon_\Downarrow + b_\Downarrow\beta^2\alpha) + \gamma(\beta - 1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ a_\uparrow\epsilon_\uparrow + \gamma\beta - a_\uparrow\epsilon_\uparrow - \gamma + \gamma(1-\beta) \\ 0 \\ \beta(\frac{\gamma}{\beta} - \gamma) + \gamma(\beta - 1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where we have used $\alpha = \frac{1}{b_\uparrow}(\gamma\beta - a_\uparrow\epsilon_\uparrow - \gamma)$ and $\frac{\gamma}{\beta} = b_\Downarrow\alpha\beta^2 + a_\Downarrow\epsilon_\Downarrow + \gamma$, equalities from the equilibrium condition (6).

The fact that there is a null eigenvalue implies that there is a direction that does not entail any energy cost for the system. The physical interpretation of the displacement of the system in this direction is the appearance of a Goldstone boson.

In our study, we are not interested in this eigenvalue, so we should analyze the eigenvalues of HF in the orthogonal complement to that direction, $\langle(0,1,0,\beta)\rangle^{\perp}$, in the $\mathbb{C}^2$ phase space of the coupled system. The orthogonal complement $\langle(0,1,0,\beta)\rangle^{\perp}$ is generated by $\{(0,\beta,0,-1),(1,0,0,0),(0,0,1,0)\}$. Let us compute the images of these vectors by HF:

$$\frac{1}{2}HF(R_{\text{eq}},0,V_{\text{eq}},0)\cdot\begin{pmatrix}0\\ \beta\\ 0\\ -1\end{pmatrix}=\begin{pmatrix}0\\ \beta(a_\uparrow\epsilon_\uparrow+b_\uparrow\alpha+\gamma)+\gamma\\ 0\\ -\beta\gamma-a_\Downarrow\epsilon_\Downarrow-b_\Downarrow\beta^2\alpha-\gamma\end{pmatrix}=$$
$$=\begin{pmatrix}0\\ \beta(a_\uparrow\epsilon_\uparrow+b_\uparrow\alpha+\gamma(1+1/\beta))\\ 0\\ -a_\Downarrow\epsilon_\Downarrow-b_\Downarrow\beta^2\alpha-\gamma(\beta+1)\end{pmatrix}=\gamma(\beta+1/\beta)\begin{pmatrix}0\\ \beta\\ 0\\ -1\end{pmatrix},$$

where we have again used the equalities of the equilibrium condition (6). We see that $(0,\beta,0,-1)$ is also an eigenvector of HF and the associated eigenvalue is $\gamma(\beta+1/\beta)=\gamma(V_{\text{eq}}/R_{\text{eq}}+R_{\text{eq}}/V_{\text{eq}})$.

To simplify the notation, we define

$$\begin{cases}K_\uparrow=a_\uparrow\epsilon_\uparrow+3b_\uparrow\alpha+\gamma,\\ K_\Downarrow=a_\Downarrow\epsilon_\Downarrow+3b_\Downarrow\beta^2\alpha+\gamma.\end{cases}\tag{12}$$

Then:

$$\frac{1}{2}HF(R_{\text{eq}},0,V_{\text{eq}},0)\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}K_\uparrow\\0\\-\gamma\\0\end{pmatrix}=K_\uparrow\begin{pmatrix}1\\0\\0\\0\end{pmatrix}-\gamma\begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$
$$\frac{1}{2}HF(R_{\text{eq}},0,V_{\text{eq}},0)\cdot\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}-\gamma\\0\\K_\Downarrow\\0\end{pmatrix}=-\gamma\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+K_\Downarrow\begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

Thus $H_1:=\frac{1}{2}HF(R_{\text{eq}},0,V_{\text{eq}},0)\big|_{\langle(1,0,0,0),(0,0,1,0)\rangle}\equiv\begin{pmatrix}K_\uparrow & -\gamma\\ -\gamma & K_\Downarrow\end{pmatrix}$. The characteristic polynomial of H_1 is

$$(K_\uparrow-\lambda)(K_\Downarrow-\lambda)-\gamma^2=K_\uparrow K_\Downarrow+\lambda^2-K_\uparrow\lambda-K_\Downarrow\lambda-\gamma^2,$$

which, when equal to 0, gives the equation

$$\lambda^2-(K_\uparrow+K_\Downarrow)\lambda+K_\uparrow K_\Downarrow-\gamma^2=0,$$

whose roots are

$$\lambda=\frac{K_\uparrow+K_\Downarrow\pm\sqrt{(K_\uparrow+K_\Downarrow)^2-4(K_\uparrow K_\Downarrow-\gamma^2)}}{2}=\frac{K_\uparrow+K_\Downarrow\pm\sqrt{(K_\uparrow-K_\Downarrow)^2+4\gamma^2}}{2}.$$

Therefore, compiling the results, the spectrum of energy fluctuations to be considered is:

$$\begin{cases} \lambda_0 = \gamma(\beta + 1/\beta), \\ \lambda_1 = \bar{K} + \sqrt{(\Delta K)^2 + \gamma^2}, \\ \lambda_2 = \bar{K} - \sqrt{(\Delta K)^2 + \gamma^2}, \end{cases} \tag{13}$$

where $\bar{K} = \frac{K_\uparrow + K_\Downarrow}{2}$, $\Delta K = \frac{|K_\uparrow - K_\Downarrow|}{2}$ and $K_{\uparrow,\Downarrow}$ are given by (12).

3.4 Study of the behaviour at the weak coupling limit

We are going to analyze the behavior of the fluctuations spectrum when $\gamma \to 0$, and we will compare the results with those obtained from the study of the energy fluctuations in the two superconductors "$\uparrow$" and "$\Downarrow$" separately, that is, in the uncoupled case.

We have seen that, in the uncoupled system, the generic functional of a zero-dimensional superconductor is given by (1), and the wave function at equilibrium is of the form (2). The energy associated with the fluctuations is

$$\frac{1}{2}\left(\frac{\partial^2 F}{(\partial|\Psi|)^2}\right)_{\Psi=\Psi^{eq}} = \frac{1}{2}(2a\epsilon + 6b|\Psi^{eq}|^2) = a\epsilon + 3b\frac{-a\epsilon}{b} = -2a\epsilon = 2a|\epsilon|.$$

Consequently, for our uncoupled system, the energies of the fluctuations in the superconductors "$\uparrow$" y "$\Downarrow$" will be, respectively, $2a_\uparrow|\epsilon_\uparrow|$ and $2a_\Downarrow|\epsilon_\Downarrow|$.

Now, we compute the limit when $\gamma \to 0$ in the energies spectrum λ_0, λ_1 ans λ_2 of the coupled system:

$$\lim_{\gamma\to 0} \lambda_0 = \lim_{\gamma\to 0} \gamma(\beta + 1/\beta) = 0.$$

Consequently, we must omit this eigenvalue in the calculation of the coherence time as we did with the null eigenvalue, since it has another physical meaning. This "energy" is proportional to the coupling between the two superconductors and, for a fixed γ, has a minimum when $\beta = 1$, that is, when the two wave functions at equilibrium are equal.

$$\lim_{\gamma\to 0} \lambda_1 = \lim_{\gamma\to 0} \left(\bar{K} + \sqrt{(\Delta K)^2 + \gamma^2}\right) = \lim_{\gamma\to 0}(\bar{K} + \Delta K) = \lim_{\gamma\to 0}(\max\{K_\uparrow, K_\Downarrow\}),$$

and, similarly,

$$\lim_{\gamma\to 0} \lambda_2 = \lim_{\gamma\to 0} \left(\bar{K} - \sqrt{(\Delta K)^2 + \gamma^2}\right) = \lim_{\gamma\to 0}(\bar{K} - \Delta K) = \lim_{\gamma\to 0}(\min\{K_\uparrow, K_\Downarrow\}).$$

Now, applying $\alpha = \frac{1}{b_\uparrow}(\gamma\beta - a_\uparrow\epsilon_\uparrow - \gamma)$ and $\frac{\gamma}{\beta} = b_\Downarrow\alpha\beta^2 + a_\Downarrow\epsilon_\Downarrow + \gamma$ in the expressions (12) of $K_\uparrow$ and $K_\Downarrow$, we can express those parameters as

$$K_\uparrow = -2a_\uparrow\epsilon_\uparrow - 2\gamma + 3\gamma\beta = 2a_\uparrow|\epsilon_\uparrow| - 2\gamma + 3\gamma\beta$$

and

$$K_\Downarrow = -2a_\Downarrow\epsilon_\Downarrow - 2\gamma + 3\gamma/\beta = 2a_\Downarrow|\epsilon_\Downarrow| - 2\gamma + 3\gamma/\beta.$$

For the range of values we consider (superconductor "$\uparrow$" with much smaller volume and lower critical temperature than the superconductor "$\Downarrow$"), we have that $K_\uparrow < K_\Downarrow$. Consequently,

$$\lim_{\gamma\to 0}\lambda_1 = \lim_{\gamma\to 0} K_\Downarrow = 2a_\Downarrow|\epsilon_\Downarrow|,$$

and

$$\lim_{\gamma\to 0}\lambda_2 = \lim_{\gamma\to 0} K_\uparrow = 2a_\uparrow|\epsilon_\uparrow|.$$

These two values coincide with the energies of the fluctuations in the two superconductors separately. It is thus reasonable to assume that the eigenvalues "λ_1" and "λ_2" are, respectively, the energies of the fluctuations in the "$\Downarrow$" and "$\uparrow$" superconductors when they are coupled.

3.5 Coherence time $\tau_\uparrow$

Having computed the energy spectrum of the Gaussian fluctuations, we can now obtain the coherence time of both superconductors using the conclusions of the previous section. Following a reasoning analogous to the one used in (3), we can deduce that the time of each fluctuation mode will be given by

$$\tau_{\text{mode}} \propto \frac{k_B T}{\epsilon_{\text{mode}}}.$$

In our system, we have two modes whose energies are given by λ_1 and λ_2. As studied, the mode that corresponds to the superconductor "$\uparrow$" in the limit $\gamma \to 0$ is λ_2, so it is reasonable to assume, for sufficiently small γ, that

$$\tau_\uparrow = \tau_2 \propto \frac{1}{\lambda_2}. \tag{14}$$

Note that, if γ grows, there will come a time when the above expression will no longer be valid, and indeed in our results we will see that there comes a time when $\tau_\uparrow$ decreases with γ.

For the sake of detail, we give the proportionality constant for the expression (14) as the accepted value for the BCS model, which is $\frac{\pi}{8}\frac{k_B T_c}{\hbar}$. However, this does not influence what follows, where we compare the coherence time for different cases (in particular, we will work with an indeterminate reference value τ_0).

4 Numerical evaluation of the result

4.1 Numerical evaluation with typical parameters of high critical temperature superconductors.

We are going to apply the obtained formulas to compute the coherence time (in relation to a reference value τ_0) for some physically reasonable values of the different parameters. With these parameters, the coefficients of the polynomial (9) are obtained and its four roots are computed. We check that the largest real root β satisfies the condition (8) and compute α and the matrix (11). Subsequently, we determine its eigenvalues spectrum, and we will finally obtain from (14) the coherence time $\tau_\uparrow$ associated with the highest critical temperature superconductor "$\uparrow$". If the assumptions are true, $\tau_\uparrow$ should increase due to the ballast on this medium produced by the superconductor "$\Downarrow$". On the other hand, we also compute the coherence time of the isolated superconductor "$\Downarrow$" simply from

$$\tau_\uparrow(\gamma = 0) = \frac{1}{2a_\uparrow|\epsilon_\uparrow|},$$

which is an expression that grows when $T \to T_{c\Downarrow}^-$ up to the value

$$\tau_\uparrow(\gamma = 0) = \frac{T_{c\uparrow}}{2a_\uparrow|T_{c\Downarrow} - T_{c\uparrow}|}.$$

Therefore we can compare this value with the maximum coherence time obtained considering the coupling in order to determine, approximately, the increase factor thanks to the ballasting. The values of the fixed parameters to be taken for the computations are shown below.

$a_\uparrow$	$a_\Downarrow$	$b_{\uparrow\Downarrow}$	$T_{c\uparrow}(K)$	$T_{c\Downarrow}(K)$
0.01	1	$a_{\uparrow\Downarrow}/10^3$	100	90

Table 1: Values assigned to the model parameters for numerical evaluation

The temperature T will take values between 80 and 89.99 K, a range that, taking into account that $T_{c\Downarrow} = 90$ K, will correctly describe what happens in the phase transition of the superconductor "$\Downarrow$".

The range of values over which we will vary the coupling parameter γ will be the interval $(10^{-6}, 10^{-2})$, based on the experimental results [3], [4][1].

[1]In determining the interval it has been taken into account that in most literature this parameter actually takes the expression $a_\uparrow\gamma$, i.e., there is a dependence with the size of the small superconductor, which follows from the laminar version of the energy functional. This is experimentally determined in high critical temperature cuprates through studies of interlaminar couplings, where usually each lamella is of the same area. Now, if one lamella has much smaller area than the adjacent one, the coupling energy will logically be the same surface energy density as before, multiplied by the contact area, which is the area of the small superconductor. If we identify the $|\Psi_{\uparrow,\Downarrow}|^2$ as a carrier density, we see that the energy dependence with sample size is in the coefficients $a_\uparrow$ and $a_\Downarrow$, i.e., these would be the product of a

4.1 Numerical evaluation with typical parameters of high critical temperature superconductors.

We will start by computing the dependence of α on the coupling parameter for different temperatures, in order to check whether the condition (8) is satisfied in the working range and therefore the equilibrium solutions are valid:

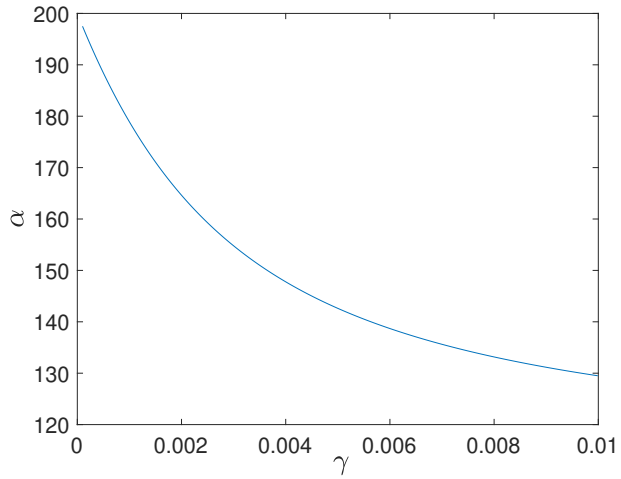

Figure 4: α versus γ for $T = 80$ K.

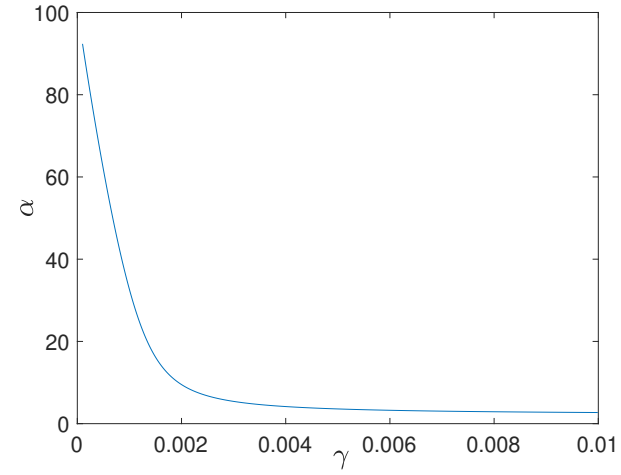

Figure 5: α versus γ for $T = 89.9$ K.

In figures 4 and 5 we see that, for the temperatures taken, we have $\alpha > 0$ over the entire range of coupling parameters, so the condition (8) is satisfied. The dependence appears to be in the form of a constant term plus a decreasing exponential, and when $T \to T_{c\Downarrow}$, the

pair of size-independent constants $\alpha_\uparrow$ and $\alpha_\Downarrow$ multiplied by the volumes (or areas per thickness) of the superconductors "$\uparrow$" and "$\Downarrow$", respectively. Thus, in cuprates with two lamellae 1 and 2, if we increase the size of 2, there is only one interaction between sheets in the common area of 1 and 2, which is that of 1, and it will be

$$\Delta F = a_1|\Psi_1|^2 + a_2|\Psi_2|^2 + a_1\gamma|\Psi_1 - \Psi_2|^2,$$

with $a_2 \gg a_1$. In our case, $a_\Downarrow/a_\uparrow = 100 \gg 1$, and therefore when choosing the range for the parameter γ we must incorporate a factor $a_\uparrow$ with respect to the usual ranges for γ.

constant tends to be zero, so that α gets closer and closer to 0 as γ increases. However, for high values of γ, the two superconductors are strongly coupled and quantum effects start to appear that make our model unsuitable to describe the system.

We plot below the behavior of the coherence time $\tau_\uparrow$ with the coupling parameter γ for different temperatures approaching $T_{c_\Downarrow}$.

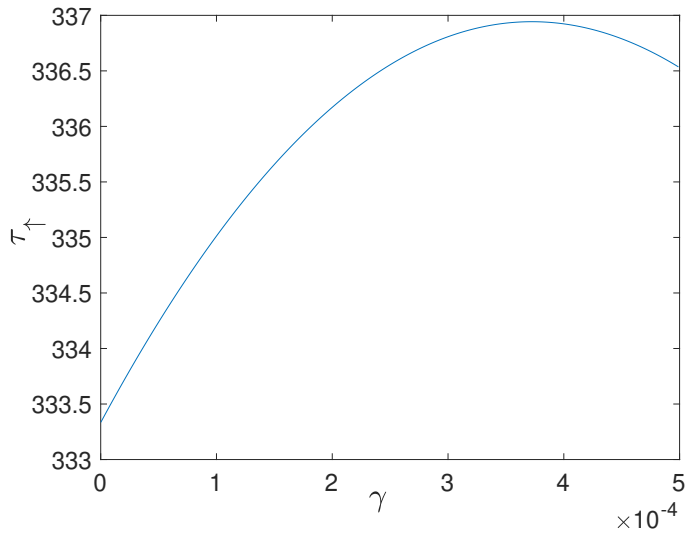

Figure 6: $\tau_\uparrow$ versus γ for $T = 85$ K.

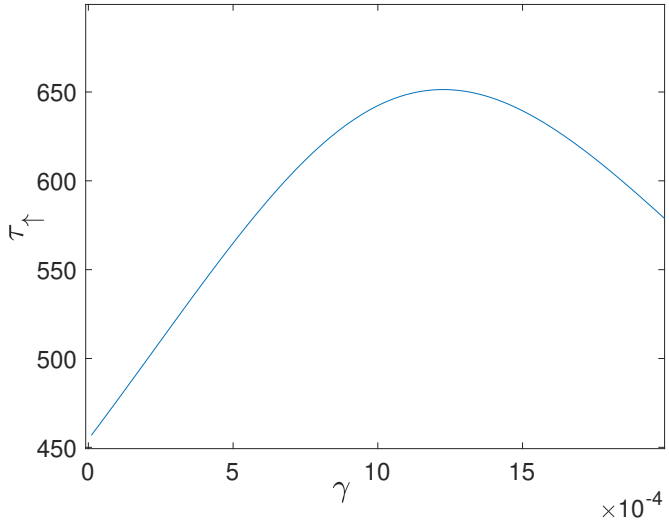

Figure 7: $\tau_\uparrow$ versus γ for $T = 89$ K.

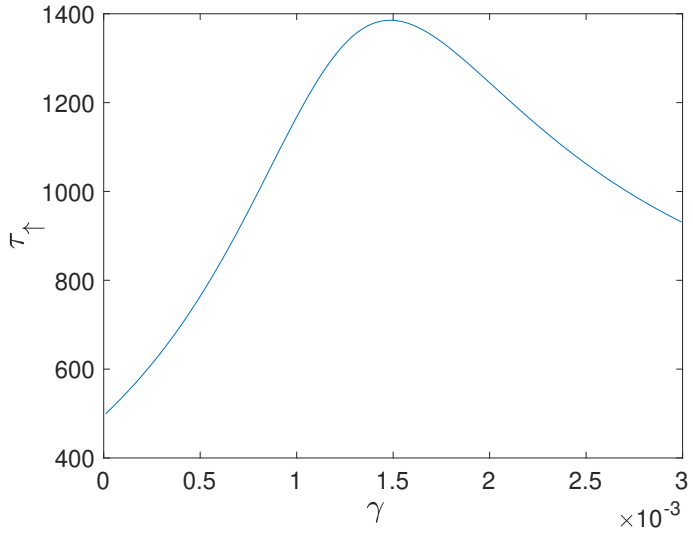

Figure 8: $\tau_{\uparrow}$ versus γ for $T = 89.9$ K.

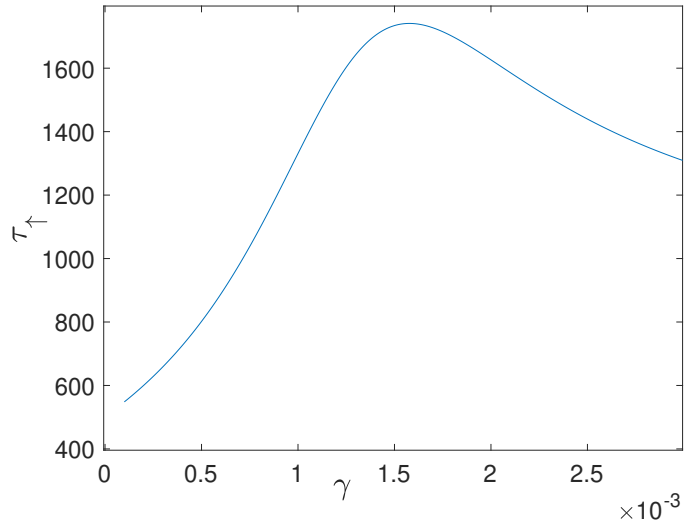

Figure 9: $\tau_{\uparrow}$ versus γ for $T = 89.99$ K.

We can observe:

- $\tau_{\uparrow}$ increasingly grows with γ as $T \to T_{c\Downarrow}$.
- Note that, above a certain γ, the depicted relation fails.

4.2 Discussion of the equation (14) and limit of validity as γ increases

The computed coherence time is, in general, associated to a linear combination of fluctuations $\delta\Psi_\uparrow$ and $\delta\Psi_\Downarrow$ obtained by diagonalizing the matrix HF at the equilibrium state. Since the large superconductor "$\Downarrow$" is going to have a larger weight on the energy of the system than the small one, the overall coherence time is expected to be dominated by the time of the large superconductor $\tau_\uparrow$. The same is true for the contributions to that coherence time when these are the result of linear combinations that mix the physics of the large and small superconductor.

The exception is when we identify some contribution that can be associated with the small superconductor "$\uparrow$". In that case, we can take that contribution as representative of what is happening to the small superconductor. This is the case of the time $\tau_\uparrow$ that we have obtained by studying the behaviour when $\gamma \to 0$. This value coincides with the coherence time of the superconductor "$\uparrow$" in the decoupled case $\gamma = 0$, so, by continuity, it seems reasonable that for small values of γ the parameter $\tau_\uparrow$ is still dominated by the contribution of the small superconductor. Moreover, its dependence on γ is as qualitatively as one would expect, since it is increasing when γ is sufficiently small.

When the coupling parameter γ increases in value, the contributions of the two superconductors are expected to become progressively mixed, since their interaction, and thus their entanglement, is increasing. This is why it is not so clear that the time $\tau_\uparrow$ still represents the behaviour of the small superconductor. The numerical results confirm this, since from a certain value of γ, the coherence time $\tau_\uparrow$ starts to decrease, which is well shown in Figures 6, 7, 8 and 9. We thus see that, starting from such a γ, the small coupling condition may not be satisfied for the purposes of our expressions for $\tau_\uparrow$.

4.3 Relative increase of $\tau_\uparrow$ due to the critical ballasting

We compile the results of Figures 6, 7, 8 and 9, which include, for each temperature, the maximum value of the coherence time $\tau_\uparrow$ with the corresponding coupling parameter, the theoretical coherence time when $\gamma = 0$, and the ratio between both times,

$$\Delta\tau = \frac{\max_\gamma \tau_\uparrow}{\tau_\uparrow(\gamma = 0)},$$

which indicates the increase in this parameter caused by ballasting.

T (K)	85 K	89 K	89.9 K	89.99 K
$(\gamma)_{\tau_{\max}}$	$3.9970 \cdot 10^{-4}$	0.0012	0.0015	0.0016
$\max \tau_\uparrow/\tau_0$	336.9238	651.2357	1385.2	1740.6
$\tau_\uparrow(\gamma = 0)/\tau_0$	333.3333	454.5455	495.0495	499.5005
$\Delta\tau = \max_\gamma \tau_\uparrow/\tau_\uparrow(\gamma = 0)$	1.0108	1.4327	2.7981	3.4847

Table 2: Results of computations for $\tau_\uparrow$

Of great interest is the ratio $\Delta\tau$ of the maximum and uncoupled coherence times, as this number describes the effects of the ballast relative to an unballasted system. A plot of the dependence of this ratio on temperature is shown below: The result of this figure

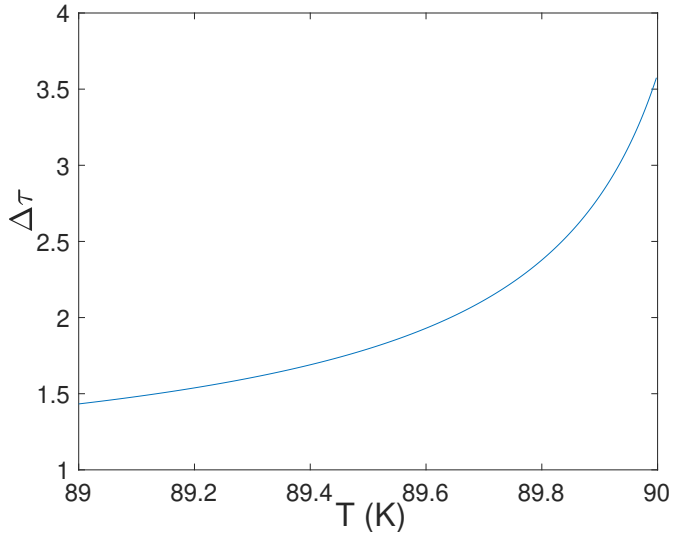

Figure 10: $\Delta\tau = \frac{\max_\gamma \tau_\uparrow}{\tau_\uparrow(\gamma=0)}$ versus T (K)

can be seen as the main practical conclusion of this work: the critical ballasting described by Figure 1 can produce, taking numerical parameters of the order of the experimental ones, an increase of the coherence times of the superconductor "↑" when $T \to T_{c\Downarrow}$ of approximately a factor $\Delta\tau = 3.5$.

5 Conclusions

In this paper we have developed Ginzburg-Landau calculations for a simple system formed by two weakly coupled superconductors and we have provided original theoretical contributions on the possibility of using, in this system, the proximity effects (Josephson coupling) and critical slowing down near the transition to increase its quantum coherence time. This is a crucial parameter in any implementation of physical systems for quantum information and communication technologies. Through a study of the equilibrium for the free energy functional of the system, we have arrived at results suggesting that this kind of critical slowing down can increase the quantum coherence time significantly. Specifically, it increases by about a factor 3.5 when considering numerical values for the involved parameters typical of high-temperature superconductors.

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