PROBLEMS

6.1 A bar is subjected to an axial load, P_0, as shown in Fig. 6.17. By using a one-finite-element model, the axial displacement, $u(x)$, can be expressed as [6.1]

$$u(x) = \{N_1(x) \quad \mathrm{N}_2(x)\} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

where $N_i(x)$ are called the shape functions:

$$N_1(x) = 1 - \frac{x}{l}, \quad N_2(x) = \frac{x}{l}$$

and u_1 and u_2 are the end displacements of the bar. The deflection of the bar at point Q can be found by minimizing the potential energy of the bar (f), which can be expressed as

$$f = \frac{1}{2} \int_0^l EA \left(\frac{\partial u}{\partial x} \right)^2 dx - P_0 u_2$$

where E is Young's modulus and A is the cross-sectional area of the bar. Formulate the optimization problem in terms of the variables u_1 and u_2 for the case $P_0 l/EA = 1$.

6.2 The natural frequencies of the tapered cantilever beam (ω) shown in Fig. 6.18, based on the Rayleigh-Ritz method, can be found by minimizing the function [6.34]:

$$f(c_1, c_2) = \frac{\dfrac{Eh^3}{3l^2}\left(\dfrac{c_1^2}{4} + \dfrac{c_2^2}{10} + \dfrac{c_1 c_2}{5}\right)}{\rho h l \left(\dfrac{c_1^2}{30} + \dfrac{c_2^2}{280} + \dfrac{2c_1 c_2}{105}\right)}$$

with respect to c_1 and c_2, where $f = \omega^2$, E is Young's modulus, and ρ is the density. Plot the graph of $3f\rho l^3/Eh^2$ in (c_1, c_2) space and identify the values of ω_1 and ω_2.

6.3 The Rayleigh's quotient corresponding to the three-degree-of-freedom spring–mass system shown in Fig. 6.19 is given by [6.34]

$$R(\mathbf{X}) = \frac{\mathbf{X}^T[K]\mathbf{X}}{\mathbf{X}^T[M]\mathbf{X}}$$

where

$$[K] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad [M] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

It is known that the fundamental natural frequency of vibration of the system can be found by minimizing $R(\mathbf{X})$. Derive the expression of $R(\mathbf{X})$ in terms of x_1, x_2, and x_3 and suggest a suitable method for minimizing the function $R(\mathbf{X})$.

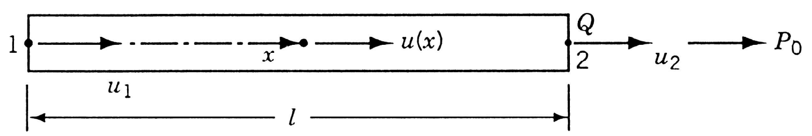

Figure 6.17 Bar subjected to an axial load.

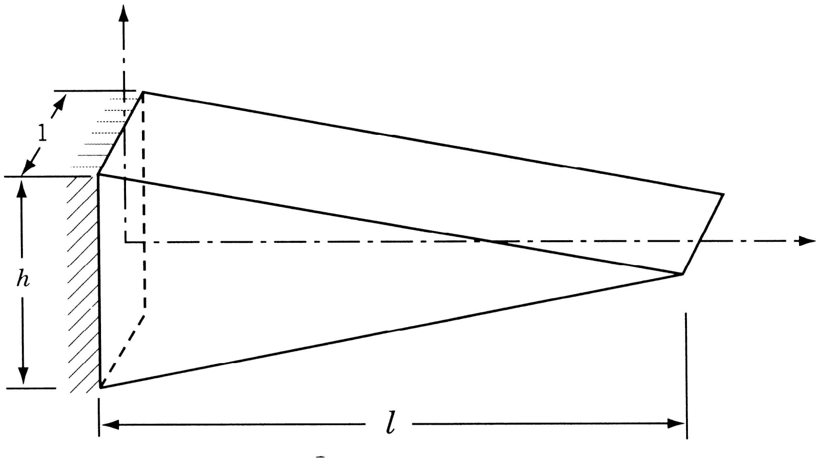

Figure 6.18 Tapered cantilever beam.

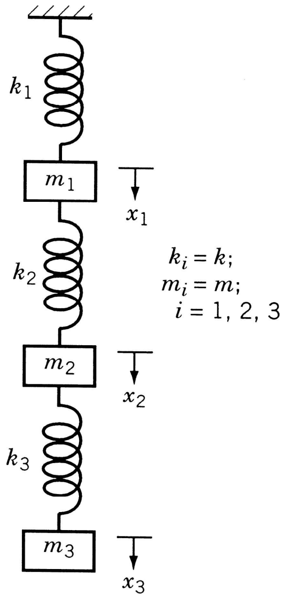

Figure 6.19 Three-degree-of-freedom spring–mass system.

6.4 The steady-state temperatures at points 1 and 2 of the one-dimensional fin (x_1 and x_2) shown in Fig. 6.20 correspond to the minimum of the function [6.1]:

$$f(x_1, x_2) = 0.6382x_1^2 + 0.3191x_2^2 - 0.2809x_1x_2 \\ - 67.906x_1 - 14.290x_2$$

Plot the function f in the (x_1, x_2) space and identify the steady-state temperatures of points 1 and 2 of the fin.

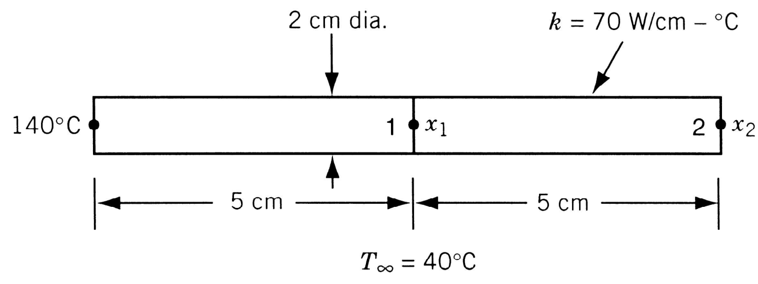

Figure 6.20 Straight fin.

6.5 Figure 6.21 shows two bodies, A and B, connected by four linear springs. The springs are at their natural positions when there is no force applied to the bodies. The displacements x_1 and x_2 of the bodies under any applied force can be found by minimizing the potential energy of the system. Find the displacements of the bodies when forces of 1000 lb and 2000 lb are applied to bodies A and B, respectively, using Newton's method. Use the starting vector, $\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$. *Hint:*

Potential energy of the system = strain energy of springs − potential of applied loads

where the strain energy of a spring of stiffness k and end displacements x_1 and x_2 is given by $\frac{1}{2}k(x_2 - x_1)^2$ and the potential of the applied force, F_i, is given by $x_i F_i$.

6.6 The potential energy of the two-bar truss shown in Fig. 6.22 under the applied load P is given by

$$f(x_1, x_2) = \frac{EA}{s}\left(\frac{l}{2s}\right)^2 x_1^2 + \frac{EA}{s}\left(\frac{h}{s}\right)^2 x_2^2 - Px_1 \cos\theta - Px_2 \sin\theta$$

where E is Young's modulus, A the cross-sectional area of each member, l the span of the truss, s the length of each member, h the depth of the truss, θ the angle at which load is applied, x_1 the horizontal displacement of free node, and x_2 the vertical displacement of the free node.

(a) Simplify the expression of f for the data $E = 207 \times 10^9$ Pa, $A = 10^{-5}$ m^2, $l = 1.5$ m, $h = 4$ m, $P = 10{,}000$ N, and $\theta = 30^\circ$.

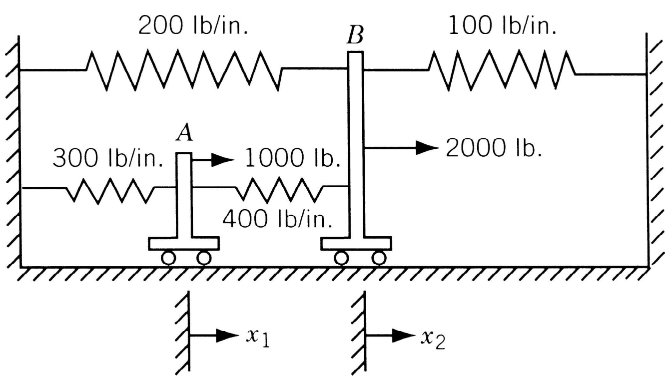

Figure 6.21 Two bodies connected by springs.

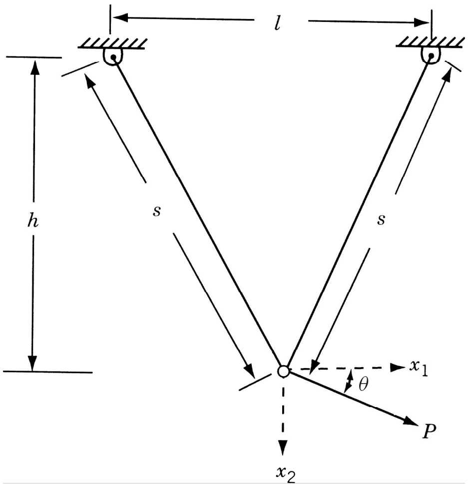

Figure 6.22 Two-bar truss.

(b) Find the steepest descent direction, $\mathbf{S}_1$, of f at the trial vector $\mathbf{X}_1 = \left\{\begin{smallmatrix}0\\0\end{smallmatrix}\right\}$.

(c) Derive the one-dimensional minimization problem, $f(\lambda)$, at $\mathbf{X}_1$ along the direction $\mathbf{S}_1$.

(d) Find the optimal step length λ^* using the calculus method and find the new design vector $\mathbf{X}_2$.

6.7 Three carts, interconnected by springs, are subjected to the loads P_1, P_2, and P_3 as shown in Fig. 6.23. The displacements of the carts can be found by minimizing the potential energy of the system (f):

$$f(\mathbf{X}) = \tfrac{1}{2}\mathbf{X}^{\mathrm{T}}[K]\mathbf{X} - \mathbf{X}^{\mathrm{T}}\mathbf{P}$$

where

$$[K] = \begin{bmatrix} k_1 + k_4 + k_5 & -k_4 & -k_5 \\ -k_4 & k_2 + k_4 + k_6 & -k_6 \\ -k_5 & -k_6 & k_3 + k_5 + k_6 + k_7 + k_8 \end{bmatrix}$$

$$\mathbf{P} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Derive the function $f(x_1, x_2, x_3)$ for the following data: $k_1 = 5000$ N/m , $k_2 = 1500$ N/m, $k_3 = 2000$ N/m, $k_4 = 1000$ N/m, $k_5 = 2500$ N/m, $k_6 = 500$ N/m, $k_7 = 3000$ N/m, $k_8 = 3500$ N/m, $P_1 = 1000$ N, $P_2 = 2000$ N, and $P_3 = 3000$ N. Complete one iteration of Newton's method and find the equilibrium configuration of the carts. Use $\mathbf{X}_1 = \{0\ 0\ 0\}^{\mathrm{T}}$.

6.8 Plot the contours of the following function over the region $(-5 \le x_1 \le 5, -3 \le x_2 \le 6)$ and identify the optimum point:

$$f(x_1, x_2) = (x_1 + 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2$$

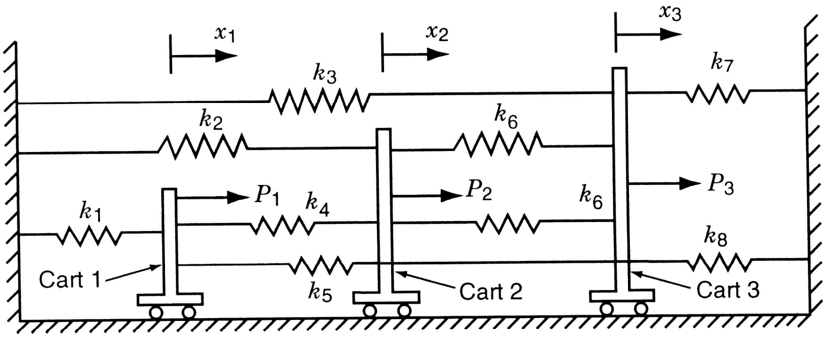

Figure 6.23 Three carts interconnected by springs.

6.9 Plot the contours of the following function in the two dimensional (x_1, x_2) space over the region $(-4 \le x_1 \le 4, -3 \le x_2 \le 6)$ and identify the optimum point:

$$f(x_1, x_2) = 2(x_2 - x_1^2)^2 + (1 - x_1)^2$$

6.10 Consider the problem

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

Plot the contours of f over the region $(-4 \le x_1 \le 4, -3 \le x_2 \le 6)$ and identify the optimum point.

6.11 It is required to find the solution of a system of linear algebraic equations given by $[A]\mathbf{X} = \mathbf{b}$, where $[A]$ is a known $n \times n$ symmetric positive-definite matrix and $\mathbf{b}$ is an n-component vector of known constants. Develop a scheme for solving the problem as an unconstrained minimization problem.

6.12 Solve the following equations using the steepest descent method (two iterations only) with the starting point, $\mathbf{X}_1 = \{0\ 0\ 0\}$:

$$2x_1 + x_2 = 4, \quad x_1 + 2x_2 + x_3 = 8, \quad x_2 + 3x_3 = 11$$

6.13 An electric power of 100 MW generated at a hydroelectric power plant is to be transmitted 400 km to a stepdown transformer for distribution at 11 kV. The power dissipated due to the resistance of conductors is i^2c^{-1}, where i is the line current in amperes and c is the conductance in mhos. The resistance loss, based on the cost of power delivered, can be expressed as $0.15i^2c^{-1}$ dollars. The power transmitted (k) is related to the transmission line voltage at the power plant (e) by the relation $k = \sqrt{3}ei$, where e is in kilovolts. The cost of conductors is given by $2c$ millions of dollars, and the investment in equipment needed to accommodate the voltage e is given by $500e$ dollars. Find the values of e and c to minimize the total cost of transmission using Newton's method (one iteration only).

6.14 Find a suitable transformation of variables to reduce the condition number of the Hessian matrix of the following function to one:

$$f = 2x_1^2 + 16x_2^2 - 2x_1x_2 - x_1 - 6x_2 - 5$$

6.15 Find a suitable transformation or scaling of variables to reduce the condition number of the Hessian matrix of the following function to one:

$$f = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + 10$$

6.16 Determine whether the following vectors serve as conjugate directions for minimizing the function $f = 2x_1^2 + 16x_2^2 - 2x_1x_2 - x_1 - 6x_2 - 5$.

(a) $\mathbf{S}_1 = \begin{Bmatrix} 15 \\ -1 \end{Bmatrix}, \quad \mathbf{S}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

(b) $\mathbf{S}_1 = \begin{Bmatrix} -1 \\ 15 \end{Bmatrix}, \quad \mathbf{S}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

6.17 Consider the problem:

$$\text{Minimize } f = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

Find the solution of this problem in the range $-10 \le x_i \le 10$, $i = 1, 2$, using the random jumping method. Use a maximum of 10,000 function evaluations.

6.18 Consider the problem:

$$\text{Minimize } f = 6x_1^2 - 6x_1x_2 + 2x_2^2 - x_1 - 2x_2$$

Find the minimum of this function in the range $-5 \le x_i \le 5$, $i = 1, 2$, using the random walk method with direction exploitation.

6.19 Find the condition number of each matrix.

(a) $[A] = \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}$

(b) $[B] = \begin{bmatrix} 3.9 & 1.6 \\ 6.8 & 2.9 \end{bmatrix}$

6.20 Perform two iterations of the Newton's method to minimize the function

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

from the starting point $\begin{Bmatrix} -1.2 \\ 1.0 \end{Bmatrix}$.

6.21 Perform two iterations of univariate method to minimize the function given in Problem 6.20 from the stated starting vector.

6.22 Perform four iterations of Powell's method to minimize the function given in Problem 6.20 from the stated starting point.

6.23 Perform two iterations of the steepest descent method to minimize the function given in Problem 6.20 from the stated starting point.

6.24 Perform two iterations of the Fletcher–Reeves method to minimize the function given in Problem 6.20 from the stated starting point.

6.25 Perform two iterations of the DFP method to minimize the function given in Problem 6.20 from the stated starting vector.

6.26 Perform two iterations of the BFGS method to minimize the function given in Problem 6.20 from the indicated starting point.

6.27 Perform two iterations of the Marquardt's method to minimize the function given in Problem 6.20 from the stated starting point.

6.28 Prove that the search directions used in the Fletcher–Reeves method are $[A]$-conjugate while minimizing the function

$$f(x_1, x_2) = x_1^2 + 4x_2^2$$

6.29 Generate a regular simplex of size 4 in a two-dimensional space using each base point:

(a) $\begin{Bmatrix} 4 \\ -3 \end{Bmatrix}$ **(b)** $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ **(c)** $\begin{Bmatrix} -1 \\ -2 \end{Bmatrix}$

6.30 Find the coordinates of the vertices of a simplex in a three-dimensional space such that the distance between vertices is 0.3 and one vertex is given by (2, −1, −8).

6.31 Generate a regular simplex of size 3 in a three-dimensional space using each base point.

(a) $\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ **(b)** $\begin{Bmatrix} 4 \\ 3 \\ 2 \end{Bmatrix}$ **(c)** $\begin{Bmatrix} 1 \\ -2 \\ 3 \end{Bmatrix}$

6.32 Find a vector $\mathbf{S}_2$ that is conjugate to the vector

$$\mathbf{S}_1 = \begin{Bmatrix} 2 \\ -3 \\ 6 \end{Bmatrix}$$

with respect to the matrix:

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

6.33 Compare the gradients of the function $f(\mathbf{X}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$ at $\mathbf{X} = \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix}$ given by the following methods:

(a) Analytical differentiation
(b) Central difference method
(c) Forward difference method
(d) Backward difference method

Use a perturbation of 0.005 for x_1 and x_2 in the finite-difference methods.

6.34 It is required to evaluate the gradient of the function

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

at point $\mathbf{X} = \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix}$ using a finite-difference scheme. Determine the step size Δx to be used to limit the error in any of the components, $\partial f/\partial x_i$, to 1 % of the exact value, in the following methods:

(a) Central difference method
(b) Forward difference method
(c) Backward difference method

6.35 Consider the minimization of the function

$$f = \frac{1}{x_1^2 + x_2^2 + 2}$$

Perform one iteration of Newton's method from the starting point $\mathbf{X}_1 = \begin{Bmatrix} 4 \\ 0 \end{Bmatrix}$ using Eq. (6.86). How much improvement is achieved with $\mathbf{X}_2$?

6.36 Consider the problem:

$$\text{Minimize } f = 2(x_1 - x_1^2)^2 + (1 - x_1)^2$$

If a base simplex is defined by the vertices

$$\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \mathbf{X}_2 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \mathbf{X}_3 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

find a sequence of four improved vectors using reflection, expansion, and/or contraction.

6.37 Consider the problem:

$$\text{Minimize } f = (x_1 + 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2$$

If a base simplex is defined by the vertices

$$\mathbf{X}_1 = \begin{Bmatrix} -2 \\ -2 \end{Bmatrix}, \quad \mathbf{X}_2 = \begin{Bmatrix} -3 \\ 0 \end{Bmatrix}, \quad \mathbf{X}_3 = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix}$$

find a sequence of four improved vectors using reflection, expansion, and/or contraction.

6.38 Consider the problem:

$$f = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

Find the solution of the problem using grid search with a step size $\Delta x_i = 0.1$ in the range $-3 \le x_i \le 3$, $i = 1, 2$.

6.39 Show that the property of quadratic convergence of conjugate directions is independent of the order in which the one-dimensional minimizations are performed by considering the minimization of

$$f = 6x_1^2 + 2x_2^2 - 6x_1x_2 - x_1 - 2x_2$$

using the conjugate directions $\mathbf{S}_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$ and $\mathbf{S}_2 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ and the starting point $\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$.

6.40 Show that the optimal step length λ_i^* that minimizes $f(\mathbf{X})$ along the search direction $\mathbf{S}_i = -\nabla f_i$ is given by Eq. (6.75).

6.41 Show that β_2 in Eq. (6.76) is given by Eq. (6.77).

6.42 Minimize $f = 2x_1^2 + x_2^2$ from the starting point $(1, 2)$ using the univariate method (two iterations only).

6.43 Minimize $f = 2x_1^2 + x_2^2$ by using the steepest descent method with the starting point $(1, 2)$ (two iterations only).

6.44 Minimize $f = x_1^2 + 3x_2^2 + 6x_3^2$ by the Newton's method using the starting point as $(2, -1, 1)$.

6.45 Minimize $f = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$ starting from point (0, 0) using Powell's method. Perform four iterations.

6.46 Minimize $f(x_1, x_2) = x_1^4 - 2x_1^2x_2 + x_1^2 + x_2^2 + 2x_1 + 1$ by the simplex method. Perform two steps of reflection, expansion, and/or contraction.

6.47 Solve the following system of equations using Newton's method of unconstrained minimization with the starting point

$$\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$2x_1 - x_2 + x_3 = -1, \quad x_1 + 2x_2 = 0, \quad 3x_1 + x_2 + 2x_3 = 3$$

6.48 It is desired to solve the following set of equations using an unconstrained optimization method:

$$x^2 + y^2 = 2, \quad 10x^2 - 10y - 5x + 1 = 0$$

Formulate the corresponding problem and complete two iterations of optimization using the DFP method starting from $\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$.

6.49 Solve Problem 6.48 using the BFGS method (two iterations only).

6.50 The following nonlinear equations are to be solved using an unconstrained optimization method:

$$2xy = 3, \quad x^2 - y = 2$$

Complete two one-dimensional minimization steps using the univariate method starting from the origin.

6.51 Consider the two equations

$$7x^3 - 10x - y = 1, \quad 8y^3 - 11y + x = 1$$

Formulate the problem as an unconstrained optimization problem and complete two steps of the Fletcher–Reeves method starting from the origin.

6.52 Solve the equations $5x_1 + 3x_2 = 1$ and $4x_1 - 7x_2 = 76$ using the BFGS method with the starting point (0, 0).

6.53 Indicate the number of one-dimensional steps required for the minimization of the function $f = x_1^2 + x_2^2 - 2x_1 - 4x_2 + 5$ according to each scheme:

(a) Steepest descent method
(b) Fletcher–Reeves method
(c) DFP method
(d) Newton's method
(e) Powell's method
(f) Random search method
(g) BFGS method
(h) Univariate method

6.54 Same as Problem 6.53 for the following function:

$$f = (x_2 - x_1^2)^2 + (1 - x_1)^2$$

6.55 Verify whether the following search directions are $[A]$-conjugate while minimizing the function

$$f = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

(a) $\mathbf{S}_1 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}, \quad \mathbf{S}_2 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

(b) $\mathbf{S}_1 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}, \quad \mathbf{S}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

6.56 Solve the equations $x_1 + 2x_2 + 3x_3 = 14$, $x_1 - x_2 + x_3 = 1$, and $3x_1 - 2x_2 + x_3 = 2$ using Marquardt's method of unconstrained minimization. Use the starting point $\mathbf{X}_1 = \{0, 0, 0\}^{\mathrm{T}}$.

6.57 Apply the simplex method to minimize the function f given in Problem 6.20. Use the point $(-1.2, 1.0)$ as the base point to generate an initial regular simplex of size 2 and go through three steps of reflection, expansion, and/or contraction.

6.58 Write a computer program to implement Powell's method using the golden section method of one-dimensional search.

6.59 Write a computer program to implement the Davidon–Fletcher–Powell method using the cubic interpolation method of one-dimensional search. Use a finite-difference scheme to evaluate the gradient of the objective function.

6.60 Write a computer program to implement the BFGS method using the cubic interpolation method of one-dimensional minimization. Use a finite-difference scheme to evaluate the gradient of the objective function.

6.61 Write a computer program to implement the steepest descent method of unconstrained minimization with the direct root method of one-dimensional search.

6.62 Write a computer program to implement the Marquardt method coupled with the direct root method of one-dimensional search.

6.63 Find the minimum of the quadratic function given by Eq. (6.141) starting from the solution $\mathbf{X}_1 = \{0, 0\}^{\mathrm{T}}$ using MATLAB.

6.64 Find the minimum of the Powell's quatic function given by Eq. (6.142) starting from the solution $\mathbf{X}_1 = \{3, -1, 0, 1\}^{\mathrm{T}}$ using MATLAB.

6.65 Find the minimum of the Fletcher and Powell's helical valley function given by Eq. (6.143) starting from the solution $\mathbf{X}_1 = \{-1, 0, 0\}^{\mathrm{T}}$ using MATLAB.

6.66 Find the minimum of the nonlinear function given by Eq. (6.144) starting from the solution $\mathbf{X}_1 = \{0, 1, 2\}^{\mathrm{T}}$ using MATLAB.

6.67 Find the minimum of the Wood's function given by Eq. (6.149) starting from the solution $\mathbf{X}_1 = \{-3, -1, -3, -1\}^{\mathrm{T}}$ using MATLAB.