7.21 Construct the ϕ_k function to be used for a mixed equality–inequality constrained problem in the interior penalty function approach.

7.22 What is a parametric constraint?

7.23 Match the following methods:

(a)	Zoutendijk method	Heuristic method
(b)	Cutting plane method	Barrier method
(c)	Complex method	Feasible directions method
(d)	Projected Lagrangian method	Sequential linear programming method
(e)	Penalty function method	Gradient projection method
(f)	Rosen's method	Sequential unconstrained minimization method
(g)	Interior penalty function method	Sequential quadratic programming method

7.24 Answer true or false:

(a) The Rosen's gradient projection method is a method of feasible directions.
(b) The starting vector can be infeasible in Rosen's gradient projection method.
(c) The transformation methods seek to convert a constrained problem into an unconstrained one.
(d) The ϕ_k function is defined over the entire design space in the interior penalty function method.
(e) The sequence of unconstrained minima generated by the interior penalty function method lies in the feasible space.
(f) The sequence of unconstrained minima generated by the exterior penalty function method lies in the feasible space.
(g) The random search methods are applicable to convex and nonconvex optimization problems.
(h) The GRG method is related to the method of elimination of variables.
(i) The sequential quadratic programming method can handle only equality constraints.
(j) The augmented Lagrangian method is based on the concepts of penalty function and Lagrange multiplier methods.
(k) The starting vector can be infeasible in the augmented Lagrangiam method.

PROBLEMS

7.1 Find the solution of the problem:

$$\text{Minimize } f(\mathbf{X}) = x_1^2 + 2x_2^2 - 2x_1x_2 - 14x_1 - 14x_2 + 10$$

subject to

$$4x_1^2 + x_2^2 - 25 \le 0$$

using a graphical procedure.

7.2 Generate four feasible design vectors to the welded beam design problem (Section 7.22.3) using random numbers.

7.3 Generate four feasible design vectors to the three-bar truss design problem (Section 7.22.1) using random numbers.

7.4 Consider the tubular column described in Example 1.1. Starting from the design vector ($d = 8.0$ cm, $t = 0.4$ cm), complete two steps of reflection, expansion, and/or contraction of the complex method.

7.5 Consider the problem:

$$\text{Minimize } f(\mathbf{X}) = x_1 - x_2$$

subject to

$$3x_1^2 - 2x_1x_2 + x_2^2 - 1 \le 0$$

(a) Generate the approximating LP problem at the vector, $\mathbf{X}_1 = \begin{Bmatrix} -2 \\ 2 \end{Bmatrix}$.

(b) Solve the approximating LP problem using graphical method and find whether the resulting solution is feasible to the original problem.

7.6 Approximate the following optimization problem as **(a)** a quadratic programming problem, and **(b)** a linear programming problem at $\mathbf{X} = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$.

$$\text{Minimize } f(\mathbf{X}) = 2x_1^3 + 15x_2^2 - 8x_1x_2 + 15$$

subject to

$$x_1^2 + x_1x_2 + 1 = 0$$
$$4x_1 - x_2^2 \le 4$$

7.7 The problem of minimum volume design subject to stress constraints of the three-bar truss shown in Fig. 7.21 can be stated as follows:

$$\text{Minimize } f(\mathbf{X}) = 282.8x_1 + 100.0x_2$$

subject to

$$\sigma_1 - \sigma_0 = \frac{20(x_2 + \sqrt{2}x_1)}{2x_1x_2 + \sqrt{2}x_1^2} - 20 \le 0$$
$$-\sigma_3 - \sigma_0 = \frac{20x_2}{2x_1x_2 + \sqrt{2}x_1^2} - 20 \le 0$$
$$0 \le x_i \le 0.3, \quad i = 1, 2$$

where σ_i is the stress induced in member i, $\sigma_0 = 20$ the permissible stress, x_1 the area of cross section of members 1 and 3, and x_2 the area of cross section of member 2. Approximate the problem as a LP problem at ($x_1 = 1, x_2 = 1$).

7.8
$$\text{Minimize } f(\mathbf{X}) = x_1^2 + x_2^2 - 6x_1 - 8x_2 + 10$$

subject to

$$4x_1^2 + x_2^2 \le 16$$
$$3x_1 + 5x_2 \le 15$$
$$x_i \ge 0, \quad i = 1, 2$$

with the starting point $\mathbf{X}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$. Using the cutting plane method, complete one step of the process.

7.9 Minimize $f(\mathbf{X}) = 9x_1^2 + 6x_2^2 + x_3^2 - 18x_1 - 12x_2 - 6x_3 - 8$
subject to

$$x_1 + 2x_2 + x_3 \leq 4$$

$$x_i \geq 0, \quad i = 1, 2, 3$$

Using the starting point $\mathbf{X}_1 = \{0, 0, 0\}^{\mathrm{T}}$, complete one step of sequential linear programming method.

7.10 Complete one cycle of the sequential linear programming method for the truss of Section 7.22.1 using the starting point, $\mathbf{X}_1 = \left\{\begin{smallmatrix}1\\1\end{smallmatrix}\right\}$.

7.11 A flywheel is a large mass that can store energy during coasting of an engine and feed it back to the drive when required. A solid disk-type flywheel is to be designed for an engine to store maximum possible energy with the following specifications: maximum permissible weight = 150 lb, maximum permissible diameter (d) = 25 in., maximum rotational speed = 3000 rpm, maximum allowable stress $(\sigma_{\max})$ = 20,000 psi, unit weight $(\gamma) = 0.283$ lb/in^3, and Poisson's ratio $(\nu) = 0.3$. The energy stored in the flywheel is given by $\frac{1}{2}I\omega^2$, where I is the mass moment of inertia and ω is the angular velocity, and the maximum tangential and radial stresses developed in the flywheel are given by

$$\sigma_t = \sigma_r = \frac{\gamma(3+\nu)\omega^2 d^2}{8g}$$

where g is the acceleration due to gravity and d the diameter of the flywheel. The distortion energy theory of failure is to be used, which leads to the stress constraint

$$\sigma_t^2 + \tau_r^2 - \sigma_t\sigma_r \leq \sigma_{\max}^2$$

Considering the diameter (d) and the width (w) as design variables, formulate the optimization problem. Starting from ($d = 15$ in., $w = 2$ in.), complete one iteration of the SLP method.

7.12 Derive the necessary conditions of optimality and find the solution for the following problem:

$$\text{Minimize } f(\mathbf{X}) = 5x_1x_2$$

subject to

$$25 - x_1^2 - x_2^2 \geq 0$$

7.13 Consider the following problem:

$$\text{Minimize } f = (x_1 - 5)^2 + (x_2 - 5)^2$$

subject to

$$x_1 + 2x_2 \leq 15$$

$$1 \leq x_i \leq 10, \quad i = 1, 2$$

Derive the conditions to be satisfied at the point $\mathbf{X} = \left\{\begin{smallmatrix}1\\7\end{smallmatrix}\right\}$ by the search direction $\mathbf{S} = \left\{\begin{smallmatrix}s_1\\s_2\end{smallmatrix}\right\}$ if it is to be a usable feasible direction.

7.14 Consider the problem:

$$\text{Minimize } f = (x_1-1)^2 + (x_2-5)^2$$

subject to

$$g_1 = -x_1^2 + x_2 - 4 \le 0$$

$$g_2 = -(x_1 - 2)^2 + x_2 - 3 \le 0$$

Formulate the direction-finding problem at $\mathbf{X}_i = \begin{Bmatrix} -1 \\ 5 \end{Bmatrix}$ as a linear programming problem (in Zoutendijk method).

7.15 Minimize $f(\mathbf{X}) = (x_1 - 1)^2 + (x_2 - 5)^2$
subject to

$$-x_1^2 + x_2 \le 4$$

$$-(x_1 - 2)^2 + x_2 \le 3$$

starting from the point $\mathbf{X}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ and using Zoutendijk's method. Complete two one-dimensional minimization steps.

7.16 Minimize $f(\mathbf{X}) = (x_1 - 1)^2 + (x_2 - 2)^2 - 4$
subject to

$$x_1 + 2x_2 \le 5$$

$$4x_1 + 3x_2 \le 10$$

$$6x_1 + x_2 \le 7$$

$$x_i \ge 0, \quad i = 1, 2$$

by using Zoutendijk's method from the starting point $\mathbf{X}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$. Perform two one-dimensional minimization steps of the process.

7.17 Complete one iteration of Rosen's gradient projection method for the following problem:

$$\text{Minimize } f = (x_1 - 1)^2 + (x_2 - 2)^2 - 4$$

subject to

$$x_1 + 2x_2 \le 5$$

$$4x_1 + 3x_2 \le 10$$

$$6x_1 + x_2 \le 7$$

$$x_i \ge 0, \quad i = 1, 2$$

Use the starting point, $\mathbf{X}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$.

7.18 Complete one iteration of the GRG method for the problem:

$$\text{Minimize } f = x_1^2 + x_2^2$$

subject to

$$x_1 x_2 - 9 = 0$$

starting from $\mathbf{X}_1 = \begin{Bmatrix} 2.0 \\ 4.5 \end{Bmatrix}$.

7.19 Approximate the following problem as a quadratic programming problem at ($x_1 = 1$, $x_2 = 1$):

$$\text{Minimize } f = x_1^2 + x_2^2 - 6x_1 - 8x_2 + 15$$

subject to

$$4x_1^2 + x_2^2 \leq 16$$

$$3x_1^2 + 5x_2^2 \leq 15$$

$$x_i \geq 0, \quad i = 1, 2$$

7.20 Consider the truss structure shown in Fig. 7.25. The minimum weight design of the truss subject to a constraint on the deflection of node S along with lower bounds on the cross sectional areas of members can be started as follows:

$$\text{Minimize } f = 0.1847x_1 + 0.1306x_2$$

subject to

$$\frac{26.1546}{x_1} + \frac{30.1546}{x_2} \leq 1.0$$

$$x_i \geq 25 \text{ mm}^2, \quad i = 1, 2$$

Complete one iteration of sequential quadratic programming method for this problem.

7.21 Find the dimensions of a rectangular prism type parcel that has the largest volume when each of its sides is limited to 42 in. and its depth plus girth is restricted to a maximum value of 72 in. Solve the problem as an unconstrained minimization problem using suitable transformations.

7.22 Transform the following constrained problem into an equivalent unconstrained problem:

$$\text{Maximize } f(x_1, x_2) = [9 - (x_1 - 3)^2]\frac{x_2^3}{27\sqrt{3}}$$

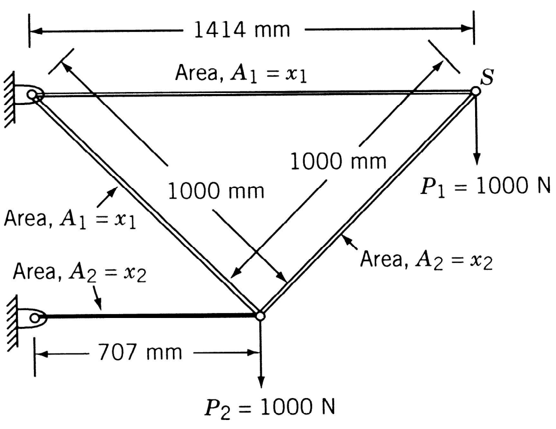

Figure 7.25 Four-bar truss.

subject to

$$0 \le x_1$$
$$0 \le x_2 \le \frac{x_1}{\sqrt{3}}$$
$$0 \le x_1 + \sqrt{3}x_2 \le 6$$

7.23 Construct the ϕ_k function, according to **(a)** interior and **(b)** exterior penalty function methods and plot its contours for the following problem:

$$\text{Maximize } f = 2x$$

subject to

$$2 \le x \le 10$$

7.24 Construct the ϕ_k function according to the exterior penalty function approach and complete the minimization of ϕ_k for the following problem.

$$\text{Minimize } f(x) = (x-1)^2$$

subject to

$$g_1(x) = 2 - x \le 0, \quad g_2(x) = x - 4 \le 0$$

7.25 Plot the contours of the ϕ_k function using the quadratic extended interior penalty function method for the following problem:

$$\text{Minimize } f(x) = (x-1)^2$$

subject to

$$g_1(x) = 2 - x \le 0, \quad g_2(x) = x - 4 \le 0$$

7.26 Consider the problem:

$$\text{Minimize } f(x) = x^2 - 10x - 1$$

subject to

$$1 \le x \le 10$$

Plot the contours of the ϕ_k function using the linear extended interior penalty function method.

7.27 Consider the problem:

$$\text{Minimize } f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 2)^2$$

subject to

$$2x_1 - x_2 = 0 \quad \text{and} \quad x_1 \le 5$$

Construct the ϕ_k function according to the interior penalty function approach and complete the minimization of ϕ_1.

7.28 Solve the following problem using an interior penalty function approach coupled with the calculus method of unconstrained minimization:

$$\text{Minimize } f = x^2 - 2x - 1$$

subject to

$$1 - x \geq 0$$

Note: Sequential minimization is not necessary.

7.29 Consider the problem:

$$\text{Minimize } f = x_1^2 + x_2^2 - 6x_1 - 8x_2 + 15$$

subject to

$$4x_1^2 + x_2^2 \geq 16, \quad 3x_1 + 5x_2 \leq 15$$

Normalize the constraints and find a suitable value of r_1 for use in the interior penalty function method at the starting point $(x_1, x_2) = (0, 0)$.

7.30 Determine whether the following optimization problem is convex, concave, or neither type:

$$\text{Minimize } f = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$$

subject to

$$2x_1 + x_2 \leq 6, \quad x_1 - 4x_2 \leq 0, \quad x_i \geq 0, \quad i = 1, 2$$

7.31 Find the solution of the following problem using an exterior penalty function method with classical method of unconstrained minimization:

$$\text{Minimize } f(x_1, x_2) = (2x_1 - x_2)^2 + (x_2 + 1)^2$$

subject to

$$x_1 + x_2 = 10$$

Consider the limiting case as $r_k \to \infty$ analytically.

7.32 Minimize $f = 3x_1^2 + 4x_2^2$ subject to $x_1 + 2x_2 = 8$ using an exterior penalty function method with the calculus method of unconstrained minimization.

7.33 A beam of uniform rectangular cross section is to be cut from a log having a circular cross section of diameter $2a$. The beam is to be used as a cantilever beam to carry a concentrated load at the free end. Find the cross-sectional dimensions of the beam which will have the maximum bending stress carrying capacity using an exterior penalty function approach with analytical unconstrained minimization.

7.34 Consider the problem:

$$\text{Minimize } f = \tfrac{1}{3}(x_1 + 1)^3 + x_2$$

subject to

$$1 - x_1 \leq 0, \quad x_2 \geq 0$$

The results obtained during the sequential minimization of this problem according to the exterior penalty function approach are given below:

Value of k	r_k	Starting point for minimization of $\phi(\mathbf{X}, r_k)$	Unconstrained minimum of $\phi(\mathbf{X}, r_k) = \mathbf{X}_k^*$	$f(\mathbf{X}_k^*) = f_k^*$
1	1	(−0.4597, −5.0)	(0.2361, −0.5)	0.1295
2	10	(0.2361, −0.5)	(0.8322, −0.05)	2.0001

Estimate the optimum solution, $\mathbf{X}^*$ and f^*, using a suitable extrapolation technique.

7.35 The results obtained in an exterior penalty function method of solution for the optimization problem stated in Problem 7.15 are given below:

$$r_1 = 0.01, \quad \mathbf{X}_1^* = \begin{Bmatrix} -\ 0.80975 \\ -50.0 \end{Bmatrix}, \quad \phi_1^* = -24.9650, \quad f_1^* = -49.9977$$

$$r_2 = 1.0, \quad \mathbf{X}_2^* = \begin{Bmatrix} 0.23607 \\ -0.5 \end{Bmatrix}, \quad \phi_2^* = 0.9631, \quad f_2^* = 0.1295$$

Estimate the optimum design vector and optimum objective function using an extrapolation method.

7.36 The following results have been obtained during an exterior penalty function approach:

$$r_1 = 10^{-10}, \quad \mathbf{X}_1^* = \begin{Bmatrix} 0.66 \\ 28.6 \end{Bmatrix}$$

$$r_2 = 10^{-9}, \quad \mathbf{X}_2^* = \begin{Bmatrix} 1.57 \\ 18.7 \end{Bmatrix}$$

Find the optimum solution, $\mathbf{X}^*$, using an extrapolation technique.

7.37 The results obtained in a sequential unconstrained minimization technique (using an exterior penalty function approach) from the starting point $\mathbf{X}_1 = \begin{Bmatrix} 6.0 \\ 30.0 \end{Bmatrix}$ are

$$r_1 = 10^{-10}, \quad \mathbf{X}_1^* = \begin{Bmatrix} 0.66 \\ 28.6 \end{Bmatrix}; \quad r_2 = 10^{-9}, \quad \mathbf{X}_2^* = \begin{Bmatrix} 1.57 \\ 18.7 \end{Bmatrix}$$

$$r_3 = 10^{-8}, \quad \mathbf{X}_3^* = \begin{Bmatrix} 1.86 \\ 18.8 \end{Bmatrix}$$

Estimate the optimum solution using a suitable extrapolation technique.

7.38 The two-bar truss shown in Fig. 7.26 is acted on by a varying load whose magnitude is given by $P(\theta) = P_0 \cos 2\theta;\ 0^\circ \le \theta \le 360^\circ$. The bars have a tubular section with mean diameter d and wall thickness t. Using $P_0 = 50{,}000$ lb, $\sigma_{\text{yield}} = 30{,}000$ psi, and $E = 30 \times 10^6$ psi, formulate the problem as a parametric optimization problem for minimum volume design subject to buckling and yielding constraints. Assume the bars to be pin connected for the purpose of buckling analysis. Indicate the procedure that can be used for a graphical solution of the problem.

7.39
$$\text{Minimize } f(\mathbf{X}) = (x_1 - 1)^2 + (x_2 - 2)^2$$
subject to
$$x_1 + 2x_2 - 2 = 0$$

using the augmented Lagrange multiplier method with a fixed value of $r_p = 1$. Use a maximum of three iterations.

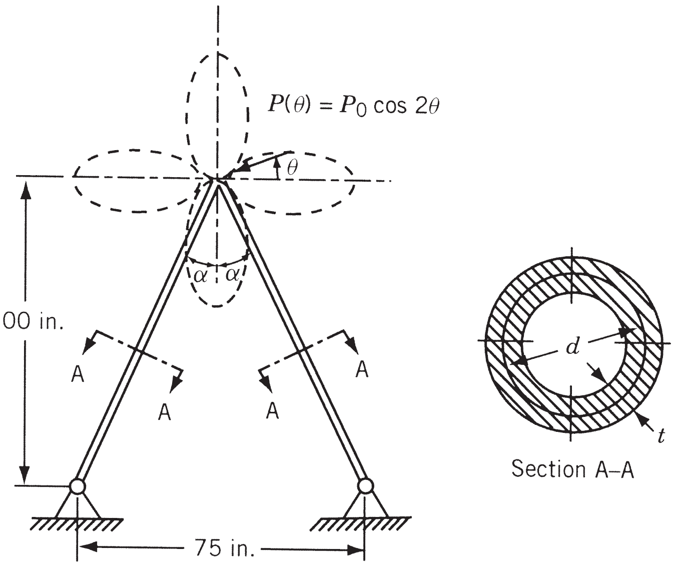

Figure 7.26 Two-bar truss subjected to a parametric load.

7.40 Solve the following optimization problem using the augmented Lagrange multiplier method keeping $r_p = 1$ throughout the iterative process and $\lambda^{(1)} = 0$:

$$\text{Minimize } f = (x_1 - 1)^2 + (x_2 - 2)^2$$

subject to

$$-x_1 + 2x_2 = 2$$

7.41 Consider the problem:

$$\text{Minimize } f = (x_1 - 1)^2 + (x_2 - 5)^2$$

subject to

$$x_1 + x_2 - 5 = 0$$

(a) Write the expression for the augmented Lagrange function with $r_p = 1$.

(b) Start with $\lambda_1^{(1)} = 0$ and perform two iterations.

(c) Find $\lambda_1^{(3)}$.

7.42 Consider the optimization problem:

$$\text{Minimize } f = x_1^3 - 6x_1^2 + 11x_1 + x_3$$

subject to

$$x_1^2 + x_2^2 - x_3^2 \le 0, \quad 4 - x_1^2 - x_2^2 - x_3^2 \le 0, \quad x_3 \le 5,$$

$$x_i \ge 0, \quad i = 1, 2, 3$$

Determine whether the solution

$$\mathbf{X} = \begin{Bmatrix} 0 \\ \sqrt{2} \\ \sqrt{2} \end{Bmatrix}$$

is optimum by finding the values of the Lagrange multipliers.

7.43 Determine whether the solution

$$\mathbf{X} = \begin{Bmatrix} 0 \\ \sqrt{2} \\ \sqrt{2} \end{Bmatrix}$$

is optimum for the problem considered in Example 7.8 using a perturbation method with $\Delta x_i = 0.001$, $i = 1, 2, 3$.

7.44 The following results are obtained during the minimization of

$$f(\mathbf{X}) = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$$

subject to

$$x_1 + x_2 + 2x_3 \leq 3$$

$$x_i \geq 0, \quad i = 1, 2, 3$$

using the interior penalty function method:

Value of r_i	Starting point for minimization of $\phi(\mathbf{X}, r_i)$	Unconstrained minimum of $\phi(\mathbf{X}, r_i) = \mathbf{X}_i^*$	$f(\mathbf{X}_i^*) = f_i^*$
1	$\begin{Bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{Bmatrix}$	$\begin{Bmatrix} 0.8884 \\ 0.7188 \\ 0.7260 \end{Bmatrix}$	0.7072
0.01	$\begin{Bmatrix} 0.8884 \\ 0.7188 \\ 0.7260 \end{Bmatrix}$	$\begin{Bmatrix} 1.3313 \\ 0.7539 \\ 0.3710 \end{Bmatrix}$	0.1564
0.0001	$\begin{Bmatrix} 1.3313 \\ 0.7539 \\ 0.3710 \end{Bmatrix}$	$\begin{Bmatrix} 1.3478 \\ 0.7720 \\ 0.4293 \end{Bmatrix}$	0.1158

Use an extrapolation technique to predict the optimum solution of the-problem using the following relations:

(a) $\mathbf{X}(r) = \mathbf{A}_0 + r\mathbf{A}_1$; $f(r) = a_0 + ra_1$

(b) $\mathbf{X}(r) = \mathbf{A}_0 + r^{1/2}\mathbf{A}_1$; $f(r) = a_0 + r^{1/2}a_1$

Compare your results with the exact solution

$$\mathbf{X}^* = \begin{Bmatrix} \frac{12}{9} \\ \frac{7}{9} \\ \frac{4}{9} \end{Bmatrix}, \qquad f_{\min} = \frac{1}{9}$$

7.45 Find the extrapolated solution of Problem 7.44 by using quadratic relations for $\mathbf{X}(r)$ and $f(r)$.

7.46 Give a proof for the convergence of exterior penalty function method.

7.47 Write a computer program to implement the interior penalty function method with the DFP method of unconstrained minimization and the cubic interpolation method of one-dimensional search.

7.48 Write a computer program to implement the exterior penalty function method with the BFGS method of unconstrained minimization and the direct root method of one-dimensional search.

7.49 Write a computer program to implement the augmented Lagrange multiplier method with a suitable method of unconstrained minimization.

7.50 Write a computer program to implement the sequential linear programming method.

7.51 Find the solution of the welded beam design problem formulated in Section 7.22.3 using the MATLAB function `fmincon` with the starting point $\mathbf{X}_1 = \{0.4, 6.0, 9.0, 0.5\}^{\mathrm{T}}$

7.52 Find the solution of the following problem (known as Rosen–Suzuki problem) using the MATLAB function `fmincon` with the starting point $\mathbf{X}_1 = \{0, 0, 0, 0\}^{\mathrm{T}}$:

Minimize

$$f(\mathbf{X}) = x_1^2 + x_2^2 + 2x_3^2 - x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4 + 100$$

subject to

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 100 \le 0$$

$$x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10 \le 0$$

$$2x_1^2 + x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5 \le 0$$

$$-100 \le x_i \le 100, \quad i = 1, 2, 3, 4$$

7.53 Find the solution of the following problem using the MATLAB function `fmincon` with the starting point $\mathbf{X}_1 = \{0.5, 1.0\}^{\mathrm{T}}$:

Minimize

$$f(\mathbf{X}) = x_1^2 + x_2^2 - 4x_1 - 6x_2$$

subject to

$$x_1 + x_2 \le 2$$

$$2x_1 + 3x_2 \le 12$$

$$x_i \ge 0, \quad i = 1, 2$$

7.54 Find the solution of the following problem using the MATLAB function `fmincon` with the starting point: $\mathbf{X}_1 = \{0.5, 1.0, 1.0\}$:

Minimize $f(\mathbf{X}) = x_1^2 + 3x_2^2 + x_3$
subject to

$$x_1^2 + x_2^2 + x_3^2 = 16$$

7.55 Find the solution of the following problem using the MATLAB function `fmincon` with the starting point: $\mathbf{X}_1 = \{1.0,\ 1.0\}^{\mathrm{T}}$:

Minimize $f(\mathbf{X}) = x_1^2 + x_2^2$
subject to

$$4 - x_1 - x_2^2 \le 0$$

$$3x_2 - x_1 \le 0$$

$$-3x_2 - x_1 \le 0$$