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Lista 2. Otimização não-Linear

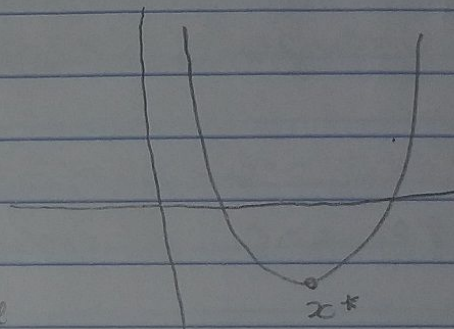
Review questions.

2.1) if $f'(x^*) = f''(x^*) = \dots = f^{(n-1)}(x^*) = 0$ and
$f^{(n)}(x^*) > 0$

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b = 0$

$x^* = -\frac{b}{2a}$



$f''(x) = 2a$, if $a > 0$ we have a convex function, so $f(x^*)$ will be minimum global

2.2) $\frac{df(x)}{dx} = 0$, represent the inflection point (stationary point) from $f(x)$, so if have multiple stationary points, the operation df/dx can't be use to find minimum of the $f(x)$.

2.3)

2.4) If $f(x^*)$ is continuous and differentiable, so taylor series of $f(x)$ around x^*:

$$f(x^*) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x^*).(x-x^*)^n$$

2.5) if $\nabla f(x^*) = 0$ and $H(x^*) < 0$ (negative), so x^* is a maximum point.

2.6)

2.7) The Hessian matrix A, will be positive if all its eigenvalues are positive, and negative if all eigenvalues are negative, with,

$$|A - \lambda I| = 0$$

Otherwise A will be indefiniteness

2.8) Saddle point:

$$\nabla f(x^*) = 0 \text{ or } \frac{d}{dx} f(x^*) = \frac{d}{dy} f(x^*) = 0$$

$$x^* = [x \; y]$$

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2.4) Solution using Method of Lagrange
Solution by Direct Substitution

2.11) The method of Lagrange is a strategy for finding the minimum-maximum of a function subject to constraints:

$$g(x) = 0 \quad , \text{ constraints}$$

$$\mathcal{L}(x, \lambda) = f(x) - \lambda g(x) \quad , \text{ Lagrange function}$$

2.14) The Kuhn tucker Conditions can't be understood like a general case of Lagrange Method.

$$\frac{\partial f}{\partial x_i} + \sum_{j \in J_1} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \; ; \; i \in [1, m]$$

$$\lambda_j > 0, \quad j \in J_1$$

Four conditions for optimal primal (X) and dual (λ) variables.

1. $g_i(x^*) - b_i$, is feasible

2. $\nabla f(x^*) - \sum_{j=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$, no feasible descent

3. $\lambda_i^* (g_i(x^*) - b_i) = 0$, complementary slackness

4. $\lambda_i^* \geq 0$, Positive Lagrange Multipliers

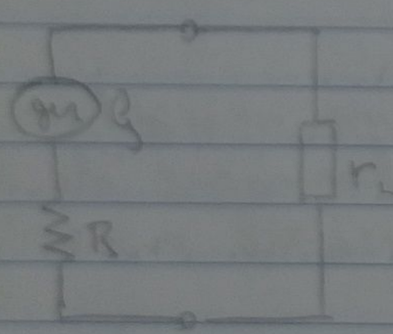
2.15) Active constraint are the optimum point that satisfied with an equality sign $g_j = 0$

2.18) Analysis of Hessian Matrix:
a - neither, b - positive, c - positive
d - neither, e - negative

2.19) a - concave, b - convex, c - neither,
d - concave, e - convex, f - concave
g - neither, h - neither

Problem.

2.1)



$P = i^2 r_L$

$i = \frac{V}{R + r_L}$

$P(r) = r_L \frac{V^2}{(R + r_L)^2}$

$P(r) = V^2 \frac{r_L}{R^2 + 2Rr_L + r_L^2} \rightarrow \frac{dP}{dr_L} = 0$ and $\frac{d^2P}{dr_L^2} < 0$

$P'(r_L) = \frac{V^2 (R^2 - r_L^2)}{(R^2 + 2Rr_L + r_L^2)^2} = 0$

$\rightarrow r_L = R \quad , R, r_L \geq 0$

$P''(r_L) = -\frac{V^2}{8R^3} < 0$, so $r_L = R \ \forall r_L \in R^+$

2.3) $f(x) = 4x^3 - 18x^2 + 27x - 7$

$f'(x) = 12x^2 - 36x + 27 = (4x^2 - 12x + 9).3$

$f''(x) = 24x - 36 = (2x - 3)12$

$4x^2 - 12x + 9 = 0$

$\Delta = 144 - 144 = 0$

$x^* = 1.5, 1.5$

$f'(1.5) = 0$

$f''(1.5) = 0$

$f'''(x^*) = 24 > 0 \ \forall x$

x^* is a inflection point

2.12) Positive, $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$

2.13) neither, $\lambda_1 = 0, \lambda_{2,3} = 6 \pm \sqrt{12}$

2.14) negative, $\lambda_1 \approx -0.3, \lambda_2 \approx -0.64, \lambda_3 \approx -5.0$

2.18) $f(X) = -x_1^2 - x_2^2 + 2x_1x_2 - x_3^2 + 6x_1x_3 + 4x_1 - 5x_3 + 2$, in form matrix as:

$$f(X) = \frac{1}{2} X^t [A] X + B^t [X] + C$$

comparing term by term from both equations

$$f(X) = C + [b_1 \ b_2 \ b_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{2} [x_1 \ x_2 \ x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[A] = \begin{bmatrix} -2 & 2 & 6 \\ 2 & -2 & 0 \\ 6 & 0 & -2 \end{bmatrix} ; [B] = [4 \ 0 \ -5]$$

$$[C] = [2]$$

2.22) $p(x_1, x_2) = 20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2$

$\max p(x_1, x_2)$

$\nabla p(x_1, x_2) = (20 + 4x_2 - 8x_1, 26 + 4x_1 - 6x_2)$
$\nabla p(x^*) = 0 \Rightarrow (7, 9)$

$H(p(x_1, x_2)) = \begin{bmatrix} -8 & 4 \\ 4 & -6 \end{bmatrix}$; $48 - 16 = 32$

$H(p(x^*)) > 0$, negative

$x^* = (7, 9)$ is the maximum arg to $f(x_1, x_2)$

2.24) $f(x_1, x_2) = (x_1 - 1)^2 e^{x_2} + x_1$
$(x_1, x_2) = (0,0)$ and $(x_1, x_2) = (1,1)$

$f(x) = f'(x_0) + \frac{1}{2} f''(x_0)$

$f'(x_0) = [h_1 \ h_2] \nabla f(x_0)$

$2f''(x_0) = [h_1 \ h_2] \, H(f(x_0)) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

$\nabla f(x) = \begin{bmatrix} 2(x_1 - 1)e^{x_2} + 1 \\ (x_1 - 1)^2 e^{x_2} \end{bmatrix}$

$H(f(x)) = \begin{bmatrix} 2e^{x_2} & 2(x_1 - 1)e^{x_2} \\ 2(x_1 - 1)e^{x_2} & (x_1 - 1)^2 e^{x_2} \end{bmatrix}$

2) $\nabla f(0,0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$H(f) = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$

b) $\nabla f(1,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$H(f(x^*)) = \begin{bmatrix} 2e & 0 \\ 0 & 0 \end{bmatrix}$

7.26) $f = 12xy - x^2 - 3y^2$

constraint $\{ 1000(x+y) \leq 48000$

$1000(x+y) \leq 48000$

$x + y \leq 48 \; ; \; y = 48 - x$

$f(x) = 12x(48-x) - x^2 - 3(48-x)^2$

$= -16x^2 + 864x - 6912$

$f'(x) = -32x + 864 = 0$

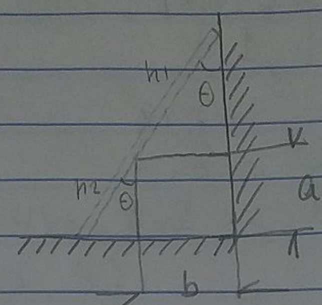
$x^* = 27$

$y^* = 48 - 27 = 21$

$f''(x) = -32 < 0 \;\forall x$, so (x^*, y^*) is a max

$f_{max}(27, 21) = 4752$

2.38)



$h_1 = b/\sin\theta$

$h_2 = a/\cos\theta \qquad L = h_1 + h_2$

$$L(\theta) = \frac{a}{\cos(\theta)} + \frac{b}{\sin(\theta)}$$

$$L'(\theta) = \frac{a\sin(\theta)}{\cos^2(\theta)} - \frac{b\cos(\theta)}{\sin^2(\theta)} = 0$$

$$\frac{a\sin(\theta)}{\cos^2(\theta)} = \frac{b\cos(\theta)}{\sin^2(\theta)}$$

$$a\sin^3(\theta) = b\cos^3(\theta)$$

$$tg^3(\theta) = \frac{b}{a}$$

$$\rightarrow \theta^* = tg^{-1}\left[\left(\frac{b}{a}\right)^{1/3}\right]$$

$$L''(\theta) = (1+2\tan^2(\theta))\frac{a}{\cos(\theta)} + (1+2\cot^2(\theta))\frac{b}{\sin(\theta)}$$

$$L''(\theta) > 0 \quad \forall\ \theta \in [0, 2\pi]$$

$$\rightarrow \min L(\theta^*) = a/\cos(\theta^*) + b/\sin(\theta^*)$$

2.50) arg max $f(X)$

$$f(X) = \frac{6xyz}{x+2y+2z} , \quad xyz = 16$$

Substitution

$$x = 16/yz$$

$$f(y,z) = \frac{96yz}{16 + 2y^2z + 2yz^2}$$

$$f_y = \frac{96z(16+2y^2z+2yz^2) - 96yz(4z + 2z^2)}{(16+2y^2z+2yz^2)^2}$$

$$f_y = 0 \; ; \; (16+2y^2z+2yz^2)^2 \neq 0$$

$$192y^2z^2 - 1536z = 0$$

$$\rightarrow y^2z = 8$$

$$f_z = \frac{96y(16+2y^2z+2yz^2) - 96yz(2y^2 + 4yz)}{(16+2y^2z+2yz^2)^2}$$

$$f_z = 0 \; , \; (16+2y^2z+2yz^2)^2 \neq 0$$

$$f_z = 192y^2z^2 - 1536y = 0$$

$$yz^2 = 8$$

$$yz = \frac{16}{x} = \frac{8}{2} = \frac{8}{4} \; ; \; z = y = 2x$$

$$2z \cdot z\, z = 16$$

$$z^3 = 8$$

$$z = 2$$

$$X^* = (4,2,2)$$

$$f(X^*) = 8$$

2.61) $\min f = (x_1-2)^2 + (x_2-1)^2$

$$\begin{cases} x_1 + x_2 - 2 \leq 0 \to g_1 \\ x_1^2 - x_2 \leq 0 \to g_2 \end{cases}$$

using Kuhn-Tucker to:

$$X_1 = \begin{Bmatrix} 1.5 \\ 0.5 \end{Bmatrix} ; X_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} ; X_3 = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

KT. $-\frac{\partial f}{\partial x_i} = \sum_{j \in J} \lambda_j \frac{\partial g_j}{\partial x_i} \quad i = 1, 2$

$\frac{\partial f}{\partial x_1} = 2(x_1 - 2)$ | $\frac{\partial g_1}{\partial x_1} = 1$ | $\frac{\partial g_2}{\partial x_1} = 2x_1$

$\frac{\partial f}{\partial x_2} = 2(x_2 - 1)$ | $\frac{\partial g_1}{\partial x_2} = 1$ | $\frac{\partial g_2}{\partial x_2} = -1$

X_1: $g_1 = 0$
$g_2 = 2.25 - 0.5 > 0$ (violated g_2)

X_2: $g_1 = 0$
$g_2 = 0$ (active to both)

$-2(x_1-2) = \lambda_1 + \lambda_2(2x_1) \Rightarrow 2 = \lambda_1 + 2\lambda_2$
$-2(x_2-1) = \lambda_1 - \lambda_2 \Rightarrow 2 = \lambda_1 - \lambda_2$

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 2/3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} > 0$$

X_2: minimum Local

x_3: $g_1 = 0$

$g_2 = 4 - 0 = 4$ (violated)

2.63) $\max f = -x_1 - x_2 \leftrightarrow \min -f = x_1 + x_2$

$$\begin{cases} -x_2^2 - x_2 + 2 \leq 0 \\ -x_1 - 3x_2 + 4 \leq 0 \\ x_1 + x_2^4 - 30 \leq 0 \end{cases}$$

a) K-T to $X = [1\ 1]^t$

$g_1 = 0,\ g_2 = 0,\ g_3 = -28$ (g_1 and g_2 active)

$\frac{\partial f}{\partial x_1} = 1,\ \frac{\partial f}{\partial x_2} = 1$ | $\frac{\partial g_2}{\partial x_1} = 1$

$\frac{\partial g_1}{\partial x_1} = 2x_1,\ \frac{\partial g_1}{\partial x_2} = 1$ | $\frac{\partial g_2}{\partial x_2} = 3$

$$1 - 2\lambda_1 - \lambda_2 = 0 \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$1 - \lambda_1 - 3\lambda_2 = 0$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix} > 0$$

X^t is optimum.

b) $\lambda_1 = 0.4,\ \lambda_2 = 0.2$

2.67) $\min f = x_1^3 - 6x_1^2 + 11x_1 + x_3$

$$\begin{cases} x_1^2 + x_2^2 - x_3^2 \leq 0 \\ 4 - x_1^2 - x_2^2 - x_3^2 \leq 0 \\ x_3 - 5 \leq 0 \end{cases} \quad x_i > 0,\ 1, 2, 3$$

$\nabla f = (3x_1^2 - 12x_1 + 11, 0, 1)$

$\nabla g_1 = (2x_1, 2x_2, -2x_3)$ $\quad \nabla g_4 = (-1, 0, 0)$

$\nabla g_2 = (-2x_1, -2x_2, -2x_3)$ $\quad \nabla g_5 = (0, -1, 0)$

$\nabla g_3 = (0, 0, 1)$ $\quad \nabla g_6 = (0, 0, -1)$

$x^* = [0 \ \sqrt{2} \ \sqrt{2}]^t$

$\nabla g_1(x^*) = [0 \ \ 2\sqrt{2} \ \ -2\sqrt{2}]$

$\nabla g_2(x^*) = [0 \ \ -2\sqrt{2} \ \ -2\sqrt{2}]$

$\nabla g_3(x^*) = [0 \ \ 0 \ \ 1]$

$\nabla g_4(x^*) = [-1 \ \ 0 \ \ 0]$

$\nabla f(x^*) = [11, 0, 1]$

$$\begin{bmatrix} 0 & 0 & 1 \\ -2\sqrt{2} & 2\sqrt{2} & -1 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 1 \end{bmatrix}$$

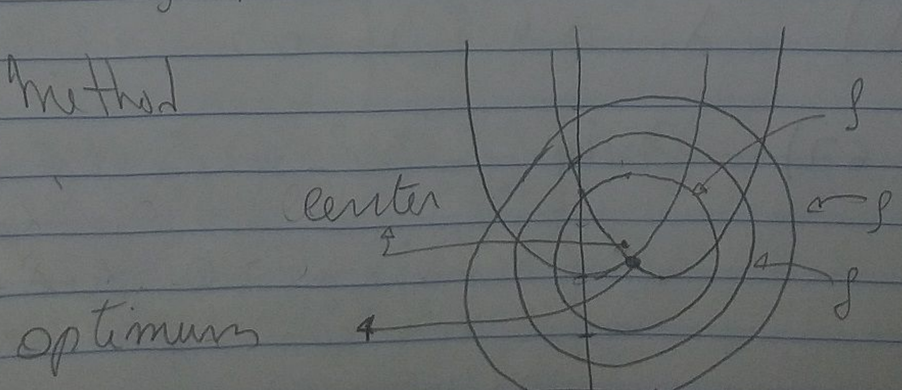
$[\lambda_1 \ \lambda_2 \ \lambda_3]^t = [\sqrt{2}/8 \ \ \sqrt{2}/8 \ \ 11]^t$

X^* is optimum value.

2.69) $\min f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 5)^2$

$\begin{cases} g_1 \to -x_1^2 + x_2 - 4 \leq 0 \\ g_2 \to -(x_1 - 2)^2 + x_2 - 3 \leq 0 \end{cases}$

a) graphical method



$x_2 = x_1^2 + 4$

$x_2 = (x_1 - 2)^2 + 3$

$x_1^2 + 4 = (x_1 - 2)^2 + 3$

$\cancel{x_1^2} = \cancel{x_1^2} - 4x_1 + 3$

$x_1 = 3/4$ $\quad x^* = (0.75, 4.56)$

$x_2 = 73/16$ $\quad f^* = 0.256$

$\nabla f = (2x_1 - 2, 2x_2 - 10)$

$\nabla g_1 = (-2x_1, 1)$

$\nabla g_2 = (-2x_1 + 4x_1, 1)$

KTC: $\dfrac{\partial f}{\partial x_i} + \lambda_1 \dfrac{\partial g_1}{\partial x_i} + \lambda_2 \dfrac{\partial g_2}{\partial x_i} = 0$

$\lambda_j g_j = 0$

$g_j \leq 0$

$\lambda_j \geq 0$

Final Solution

$-0.5 - 1.5\lambda_1 + 2.5\lambda_2 = 0$

$-0.875 + \lambda_1 + \lambda_2 = 0$

$[\lambda_1 \ \lambda_2] = [0.4219 \ 0.4531]$

$\rightarrow -4x_1 + 3 = 0$

$x_1 = 0.75$ → same solution from graphic method

$x_2 = 4.56$

7.10) $\max f(x_1,x_2) = 8x_1 + 4x_2 + x_1x_2 - x_1^2 - x_2^2$

$$2x_1 + 3x_2 \leq 24$$
$$-5x_1 + 12x_2 \leq 24$$
$$x_2 \leq 5$$

$$\max \quad 8x_1 + 4x_2 + x_1x_2 - x_1^2 - x_2^2$$
$$\begin{cases} 2x_1 + 3x_2 - 24 \leq 0 \\ -5x_1 + 12x_2 - 24 \leq 0 \\ x_2 - 5 \leq 0 \end{cases}$$

$$\nabla f(x^*) + [\nabla g_i(x^*)][\lambda_i] = 0 \qquad (I)$$

$$\nabla f = [8 + x_2 - 2x_1,\ 4 + x_1 - 2x_2]$$

$$\nabla g_1 = [2,\ 3], \quad \nabla g_2 = [-5,\ 12], \quad \nabla g_3 = [0,\ 1]$$

$$\begin{bmatrix} 8 - 2x_1 + x_2 \\ 4 + x_1 - 2x_2 \end{bmatrix} + \begin{bmatrix} 2 & -5 & 0 \\ 3 & 12 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = 0 \qquad (I)$$

$$\lambda_j g_j = 0$$

$$\begin{bmatrix} 2x_1 + 3x_2 - 24 & -5x_1 + 12x_2 - 24 & -5 + x_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = 0 \qquad (II)$$

$$g_j \leq 0 \qquad (III)$$

$$\begin{bmatrix} 2 & 3 & -24 \\ -5 & 12 & -24 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (III)$$

$$\lambda_j \leq 0, \quad j = 1, 2, 3$$

Solution that satisfies all equations

$X^* = [4.21 \quad 5.68]$
$\lambda^* = [0 \quad 0.42]$
$f^* = 36.21$

2.73) $\min f(X) = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$

$$\begin{cases} 2x_1 + x_2 - 6 \leq 0 \\ x_1 - 4x_2 \leq 0 \\ -x_1, -x_2 \leq 0 \end{cases}$$

$\nabla f = [-4 + 2x_1 - 2x_2 \quad -2x_1 + 4x_2]$

$H_f = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$; $J_1 = 2$, $J_2 = 4$ > 0

positive

$\rightarrow f =$ convex

2.75) $\min f(X) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$

$$\begin{cases} x_2^2 - x_1 \geq 0 \\ x_1^2 - x_2 \geq 0 \\ -0.5 \leq x_1 \leq 0.5, \ x_2 \leq 1 \end{cases}$$

$\nabla f = [-400(x_1x_2 - x_1^3) + 2(x_1 - 1), 200(x_2 - x_1^2)]$
$\nabla g_1 = [1, 2x_2]$
$\nabla g_2 = [-2x_1, 1]$
$\nabla g_3 = [-1, 0]$ $\quad$ f_1 e f_2 satisfied
$\nabla g_4 = [1, 0]$
$\nabla g_5 = [0, 1]$

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a) $X_1 = [0 \quad 0]^t$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$[\lambda_1 \quad \lambda_2] = [2 \quad 0]$, not positive

b) $X_2 = [0 \quad -1]^t$

$\nabla f(X_2) = [-2, -200] < 0$, not satisfied

c) $X_3 = [-0.5, 0.25]$

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$[\lambda_2 \quad \lambda_3] = [0 \quad -3] < 0$

not satisfied

X_1, X_2 and X_3 not are optimum solution to $f(x_1, x_2, x_3)$.

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