CS512 — Computer Vision — First assignment

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1 Part A: Vector Operations

Given:

$$\mathbf{p} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} \quad \mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Resolve:

1. $\mathbf{3p + 2q}$

Answer:

$$\mathbf{3p + 2q} = 3 * \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 12 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 22 \end{bmatrix}$$

2. $\hat{\mathbf{p}}$ **: a unit vector in the direction of p**

Answer:

$$\|\mathbf{p}\| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21} \rightarrow \hat{\mathbf{p}} = \frac{\mathbf{p}}{\|\mathbf{p}\|} = \begin{bmatrix} \frac{2}{\sqrt{21}} \\ \frac{-1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \end{bmatrix}$$

3. $\|\mathbf{p}\|$ **and the angle of p relative to the positive** $\mathbf{y-axis}$

Answer:

3.1 $\|p\| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$

3.2 $cos(\theta) = \frac{p \cdot d}{\|p\|\|d\|}$ being $d = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow cos(\theta) = \frac{p}{\|p\|} = \frac{2*0+(-1)*1+4*0}{\sqrt{21}*1} = \frac{-1}{\sqrt{21}}$

$\boldsymbol{\theta} = cos^{-1} \frac{-1}{\sqrt{21}}$

4. **The direction cosines of p**

Answer:

4.1 $cos\alpha = \frac{2}{\sqrt{21}}$

4.2 $cos\beta = \frac{-1}{\sqrt{21}}$

4.3 $cos\gamma = \frac{4}{\sqrt{21}}$

5. **The angle between p and q**

Answer:
$\|q\| = \sqrt{0^2 + (3)^2 + 5^2} = \sqrt{34}$

$cos(\theta) = \frac{p \cdot q}{\|p\|\|q\|} = \frac{2*0+(-1)*3+4*5}{\sqrt{21}\sqrt{34}} = \frac{17}{\sqrt{21}\sqrt{34}}$

$\boldsymbol{\theta} = cos^{-1}\frac{17}{\sqrt{21}\sqrt{34}}$

6. $\mathbf{p} \cdot \mathbf{q}$ **and** $\mathbf{q} \cdot \mathbf{p}$

Answer:

6.1 $\mathbf{p} \cdot \mathbf{q} = 2 * 0 + (-1) * 3 + 4 * 5 = 17$

6.2 $\mathbf{q} \cdot \mathbf{p} = 0 * 2 + 3 * (-1) + 5 * 4 = 17$

7. $\mathbf{p} \cdot \mathbf{q}$ **using the angle between p and q**

Answer:
$\mathbf{p} \cdot \mathbf{q} = \|p\|\|q\|cos\theta = \sqrt{21}\sqrt{34}cos\theta$

8. **The scalar projection of q onto** $\hat{\mathbf{p}}$

Answer:
$\hat{p} = \frac{p}{\|p\|} = \begin{bmatrix} \frac{2}{\sqrt{21}} \\ \frac{-1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \end{bmatrix}$

$\mathbf{q} \cdot \hat{\mathbf{p}} = 0 * \frac{2}{\sqrt{21}} + 3 * \frac{-1}{\sqrt{21}} + 5 * \frac{4}{\sqrt{21}} = \frac{17}{\sqrt{21}}$

9. **A vector that is perpendicular to p**

Answer:
$p \cdot x = 0 \rightarrow \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow 2 * x_1 + (-1) * x_2 + 4 * x_3 = 0 \rightarrow$ being x_1 and $x_2 = 1$

A perpendicular vector to p $= \begin{bmatrix} 1 \\ 1 \\ \frac{-1}{4} \end{bmatrix}$

10. **p x q and q x p**

Answer:

(a) **p x q =** $det \begin{bmatrix} i & j & k \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{bmatrix} = -5i + 6k - 10j - 12i = -17i - 10j + 6k = \begin{bmatrix} -17 \\ -10 \\ 6 \end{bmatrix}$

(b) **q x p =** $det \begin{bmatrix} i & j & k \\ 0 & 3 & 5 \\ 2 & -1 & 4 \end{bmatrix} = 12i + 10j - 6k + 5i = 17i + 10j - 6k = \begin{bmatrix} 17 \\ 10 \\ -6 \end{bmatrix}$

11. **A vector that is perpendicular to both p and q**

Answer:
p x q is a perpendicular vector to p and q

12. **The linear dependency between p, q and r**

Answer:
There is linear dependence between p, q and r if the determinant of the matrix that merges them is zero.
$det \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \\ 4 & 5 & 2 \end{bmatrix} = 0 \rightarrow 12-5-12-20 = -25 \rightarrow 25! = 0 \rightarrow$ **There is not linear dependence**

13. $\mathbf{p^T q}$ **and** $\mathbf{pq^T}$

Answer:
$p^T = \begin{bmatrix} 2 \text{ -1 } 4 \end{bmatrix}$
$q^T = \begin{bmatrix} 0 \ 3 \ 5 \end{bmatrix}$
$\mathbf{p^T q} = \begin{bmatrix} 2 \text{ -1 } 4 \end{bmatrix} * \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} = 2 * 0 + (-1) * 3 + 4 * 5 = 17$
$\mathbf{pq^T} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} * \begin{bmatrix} 0 \ 3 \ 5 \end{bmatrix} = 2 * 0 + (-1) * 3 + 4 * 5 = 17$

2 Part B: Matrix Operations

Given:

$$\mathbf{X} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 4 & 2 & -2 \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} \quad \mathbf{Z} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix} \quad \mathbf{s} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

Resolve:

1. **X + 2Y**

Answer:
$\mathbf{X} + \mathbf{2Y} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 4 & 2 & -2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 8 & -2 & 4 \\ 6 & 0 & -6 \\ 2 & 4 & 2 \end{bmatrix} =$
$\begin{bmatrix} 10 & -1 & 4 \\ 5 & 3 & -2 \\ 6 & 6 & 0 \end{bmatrix}$

2. **XY and YX**

Answer:

2.1 $\mathbf{XY} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 4 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & -2 & 1 \\ 9 & 9 & -7 \\ 20 & -8 & 0 \end{bmatrix}$

2.2 $\mathbf{YX} = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 4 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 5 & -8 \\ -6 & -3 & -6 \\ 4 & 9 & 6 \end{bmatrix}$

3. **$(\mathbf{XY})^\mathbf{T}$ and $\mathbf{Y^T X^T}$**

Answer:

3.1 $(\mathbf{XY})^\mathbf{T} = \begin{bmatrix} 11 & -2 & 1 \\ 9 & 9 & -7 \\ 20 & -8 & 0 \end{bmatrix}^T = \begin{bmatrix} 11 & 9 & 20 \\ -2 & 9 & -8 \\ 1 & -7 & 0 \end{bmatrix}$

3.2 $\mathbf{Y^T X^T} = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}^T \cdot \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 4 & 2 & -2 \end{bmatrix}^T = \begin{bmatrix} 4 & 3 & 1 \\ -1 & 0 & 2 \\ 2 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 2 \\ 0 & 4 & -2 \end{bmatrix} =$
$\begin{bmatrix} 11 & 9 & 20 \\ -2 & 9 & -8 \\ 1 & -7 & 0 \end{bmatrix}$

4. **$|\mathbf{X}|$ and $|\mathbf{Z}|$ (see question A-12)**

Answer:

4.1 $|\mathbf{X}| = det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 4 & 2 & -2 \end{bmatrix} = 2{*}3(-2){+}1{*}4{*}4{+}(-1){*}2{*}0{-}(4{*}3{*}0{+}2{*}2{*}4{+}(-1)1{*}(-2)) =$
$-12 + 16 + 0 + 0 - 16 - 2 = -14$

4.2 $|\mathbf{Z}| = det \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix} = 2*4*2+0*5*3+1*1*(-1)-(3*4*(-1)0*1*2+2*5*1) =$
$16 + 0 + (-1) + 12 + 0 - 10 = 17$

5. **The matrix (either X,Y or Z) in which the row vectors form an orthogonal set**

Answer:
We need to check whether the dot product between each pair of row is zero.

5.1 **Matrix X: Does not form an orthogonal set**

Row 1 and row 2: $2*(-1)+1*3+0*4=1$

Row 1 and row 3: $2*4+1*2+0*(-2)=10$

Row 2 and row 3: $(-1)*4+3*2+4*(-2)=-6$

5.2 **Matrix Y: does not form an orthogonal set**

Row 1 and row 2: $4*3+(-1)*0+2*(-3)=6$

Row 1 and row 3: $4*1+(-1)*2+2*1=4$

Row 2 and row 3: $3*1+0*2+(-3)*1=0$

5.3 **Matrix Z: does not form an orthogonal set**

Row 1 and row 2: $2*1+0*4+(-1)*5=-3$

Row 1 and row 3: $2*3+0*1+(-1)*2=4$

Row 2 and row 3: $1*3+4*1+5*2=17$

6. **$\mathbf{X^{-1}}$ and $\mathbf{Y^{-1}}$ (see question A-12)**

Answer:
Both matrix have a determinant unequal to zero, so both matrix have an inverse.

6.1 $\mathbf{X^{-1}} = \frac{1}{|X|}Adj(X^T) = \frac{1}{-14}\begin{bmatrix} det\begin{bmatrix}3 & 2\\ 4 & -2\end{bmatrix} & -det\begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix} & det\begin{bmatrix}1 & 3\\ 0 & 4\end{bmatrix}\\ -det\begin{bmatrix}-1 & 4\\ 4 & -2\end{bmatrix} & det\begin{bmatrix}2 & 4\\ 0 & -2\end{bmatrix} & -det\begin{bmatrix}2 & -1\\ 0 & 4\end{bmatrix}\\ det\begin{bmatrix}-1 & 4\\ 3 & 2\end{bmatrix} & -det\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix} & det\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}\end{bmatrix} =$

$\frac{1}{-14}\begin{bmatrix}-14 & 2 & 4\\ 14 & -4 & -8\\ -14 & 0 & 7\end{bmatrix} = \begin{bmatrix}1 & \frac{2}{-14} & \frac{4}{-14}\\ -1 & \frac{4}{14} & \frac{8}{14}\\ 1 & 0 & \frac{7}{-14}\end{bmatrix} = \begin{bmatrix}1 & -\frac{1}{7} & -\frac{2}{7}\\ -1 & \frac{2}{7} & \frac{4}{7}\\ 1 & 0 & -\frac{1}{2}\end{bmatrix}$

6.2 $\mathbf{Y^{-1}} = \frac{1}{|Y|}Adj(Y^T) = \begin{bmatrix}\frac{1}{6} & \frac{5}{36} & \frac{1}{12}\\ -\frac{1}{6} & \frac{1}{36} & \frac{5}{12}\\ \frac{1}{6} & -\frac{7}{36} & \frac{1}{12}\end{bmatrix}$

7. **$\mathbf{Z^{-1}}$ (see question B-4)**

Answer:
$\mathbf{Z^{-1}} = \frac{1}{|Z|}Adj(Z^T) = \begin{bmatrix}\frac{3}{17} & -\frac{1}{17} & \frac{4}{17}\\ \frac{13}{17} & \frac{7}{17} & -\frac{11}{17}\\ -\frac{11}{17} & -\frac{2}{17} & \frac{8}{17}\end{bmatrix}$

8. **The product Xs**

Answer:

$$\mathbf{Xs} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 4 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 4 \end{bmatrix}$$

9. **The scalar projection of the rows of X onto the vector s (with s normalized)**

Answer:

$\|s\| = \sqrt{(-1)^2 + 4^2 + 0^2} = \sqrt{17}$

$$\hat{s} = \frac{1}{\|s\|} = \begin{bmatrix} -\frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \\ 0 \end{bmatrix}$$

9.1 **First row of X:** $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \\ 0 \end{bmatrix} = \frac{-2+4}{\sqrt{17}} = 0.485$

9.2 **Second row of X:** $\begin{bmatrix} -1 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \\ 0 \end{bmatrix} = \frac{-1+12}{\sqrt{17}} = 3.153$

9.3 **Third row of X:** $\begin{bmatrix} 4 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \\ 0 \end{bmatrix} = \frac{-4+8}{\sqrt{17}} = 0.97$

10. **The vector projection of the rows of X onto the vector s (with s normalized)**

Answer:

$$Proj_s v = \frac{v \cdot s}{s \cdot s} s$$

$s \cdot s = \|s\|^2 = \sqrt{17}^2 = 17$

9.1 **First row of X:** $\frac{\begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}}{17} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \frac{2}{17} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{17} \\ \frac{8}{17} \\ 0 \end{bmatrix}$

9.2 **Second row of X:** $\frac{\begin{bmatrix} -1 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}}{17} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \frac{13}{17} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{13}{17} \\ \frac{42}{17} \\ 0 \end{bmatrix}$

9.3 **Third row of X:** $\frac{\begin{bmatrix} 4 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}}{17} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \frac{4}{17} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{17} \\ \frac{16}{17} \\ 0 \end{bmatrix}$

11. **The linear combination of the columns of X using the elements of s**

Answer:
$v = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 4 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 4 \end{bmatrix}$

12. **The solution t for the equation Y t=s**

Answer:
$Yt = s \rightarrow \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \rightarrow t_1 = 1/3, t_2 = 1/3, t_3 = -1$

13. **The solution t for the equation Z t=s and the reason for it (see question B-7)**

Answer:
$Zt = s \rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \rightarrow t_1 = \frac{-7}{17}, t_2 = \frac{15}{17}, t_3 = \frac{3}{17}$

3 Part C: Eigenvalues and eigenvectors

Let:

$$\mathbf{M} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

Find:

1. **The eigenvalues and corresponding eigenvectors of M**

> **Answer:**
> First: We have to calculate the eigenvalue associated using the next formula:
>
> $$det|M - \lambda I| = 0$$
>
> Then: $det|\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}| = 0 \rightarrow det|\begin{bmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = 0 \rightarrow (3-\lambda) * (4-\lambda) + 2 = 0 \rightarrow$ $12 - 3\lambda - 4\lambda + \lambda^2 + 2 = 0 \rightarrow \lambda^2 - 7\lambda + 14 = 0 \rightarrow \lambda = \frac{7}{2} + -i\frac{\sqrt{7}}{2}$
>
> Second: Get the eigenvector associated to each eigenvalue using the next formula:
>
> $$(M - \lambda I)v = 0$$
>
> 1.1 For $\lambda = \frac{7}{2} + i\frac{\sqrt{7}}{2}$: $(\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \begin{bmatrix} v_1 \\ v2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} * \begin{bmatrix} v_1 \\ v2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{i\sqrt{7}+1}{2} \\ 1 \end{bmatrix}$
>
> 1.2 For $\lambda = \frac{7}{2} - i\frac{\sqrt{7}}{2}$: $(\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \begin{bmatrix} v_1 \\ v2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} * \begin{bmatrix} v_1 \\ v2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{-i\sqrt{7}+1}{2} \\ 1 \end{bmatrix}$

2. **The dot product between the eigenvectors of M**

> **Answer:**
> $\begin{bmatrix} \frac{i\sqrt{7}+1}{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{-i\sqrt{7}+1}{2} \\ 1 \end{bmatrix} = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$

3. **The dot product between the eigenvectors of N**

> **Answer:**
>
> 3.1 $\lambda_1 = \frac{11+\sqrt{37}}{2} \rightarrow v_1 = \begin{bmatrix} \frac{\sqrt{37}+1}{6} \\ 1 \end{bmatrix}$
>
> 3.2 $\lambda_2 = \frac{11-\sqrt{37}}{2} \rightarrow v_2 = \begin{bmatrix} \frac{\sqrt{37}-1}{6} \\ 1 \end{bmatrix}$
>
> $v_1 \cdot v_2 = \begin{bmatrix} \frac{\sqrt{37}+1}{6} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{37}-1}{6} \\ 1 \end{bmatrix} = \frac{\sqrt{37}+1}{6} \cdot \frac{\sqrt{37}-1}{6} - 1 = 1 - 1 = 0$

4. **The property of the eigenvectors of N and the reason for it (see question C-4)**

> **Answer:**
> The property of the eigenvectors is that all of them are orthogonal (perpendicular) to each other. For this reason, the dot product of any pair of eigenvectors is zero.

5. **The value of a trivial solution t to the equation P t=0**

Answer:
$Pt = 0 \rightarrow \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = 0 \rightarrow$

$$\begin{cases} 2t_1 + 4t_2 = 0 \\ 4t_1 + 8t_2 = 0 \end{cases} \tag{1}$$

Resolving the system, we got:

$$t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

6. **The value of two non-trivial solutions t to the equation P t=0**

Answer:
If we want to get two non-trivial solutions for t, one of them should not be zero.

6.1 For $t_1 = 1 \rightarrow 2 * 1 + 4t_2 = 0 \rightarrow t_2 = \frac{-2}{4} \rightarrow t_2 = -0.5$

6.2 For $t_2 = 1 \rightarrow 2t_1 + 4 * 1 = 0 \rightarrow t_1 = \frac{-4}{2} \rightarrow t_1 = -2$

7. **The only solution t to the equation M t=0 and the reason for having a single solution**

Answer:
$Mt = 0 \rightarrow \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = 0 \rightarrow$

$$\begin{cases} 3t_1 + 2t_2 = 0 \\ -t_1 + 4t_2 = 0 \end{cases} \tag{2}$$

Resolving this system, we got:

$$t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The reason for having a single solution is that the matrix M is non-singular. M is invertible, and therefore, there are no non-trivial solutiions to this system.

4 Part D: Neural Networks Basics

1. Given a neuron with input values $x_1 = 0.5, x_2 = -0.8$ and $x_3 = 0.3$, weights $w_1 = 0.4, w_2 = -0.6$ and $w_3 = 0.9$ and a bias $b = 0.2$, compute the output of the neuron before applying any activation function.

Answer:

$$y = w_1x_1 + w_2x_2 + w_3x_3 + b$$

so, then:

$$y = 0.5 \cdot 0.4 + (-0.8) \cdot (-0.6) + 0.3 \cdot 0.9 + 0.2$$

$$y = 1.15$$

2. Using the output computed in the previous question, apply the sigmoid activation function to find the activation value.

Answer:
Sigmoid activation function $= \sigma(x) = \frac{1}{1+e^{-x)}} \rightarrow \frac{1}{1+e^{-1.15}} \rightarrow \sigma(1.15) = 0.7595$

3. Given the same input values and weights as in question 1, compute the output of the neuron after applying a ReLU activation function.

Answer:
ReLU formula is: $f(x) = max(0, x) \rightarrow max(0, 1.15) \rightarrow f(1.15) = 1.15$

4. Given a neural network layer with weights W_h, bias b_h, and input x, as follows:

$$W_h = \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, b_h = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

compute the output of the layer using sigmoid activation.

Answer:

$$y = W_h \cdot x + b$$

so, then: $y = \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} \rightarrow$

$$\begin{cases} 0.4 + 0.6 + 0.1 = y_1 \\ 0.2 + 1.4 - 0.1 = y_2 \end{cases} \tag{3}$$

Therefore $y_1 = 1.1$ and $y_2 = 1.5$. Applying sigmoid activation function for each output element, we get:

4.1 $\sigma(1.1) = \frac{1}{1+e^{-1.1}} = 0.75$

4.2 $\sigma(1.5) = \frac{1}{1+e^{-1.5}} = 0.8175$

5. Feed the output of the layer from the previous question into an output layer with weights W_h, bias b_h, as follows:

$$W_h = \begin{bmatrix} 0.5 \\ -0.3 \end{bmatrix}, b_h = \begin{bmatrix} 0.1 \end{bmatrix}$$

Answer:

$$y = w_1 x_1 + w_2 x_2 + b$$

so, then:

$$y = 0.5 * 0.75 + (-0.3) * 0.8175 + 0.1$$

$$y = 0.22975$$

6. Assume a neural network with a single hidden layer and an output layer. The hidden layer consists of 2 units, with weights W_h, biases b_h and outputs from the hidden layer z (after activation). The output to the network is a 2-dimensional vector x = $[x_1, x_2]$. Let the loss function be denoted by L. Write the chain rule to compute the gradient of the loss L with respect to the input components x_1,i.e, $\frac{\partial L}{\partial x_1}$

Answer:

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial y} \cdot (w_{o1} \cdot \sigma^2(w_{11}x_1 + w_{12}x2 + b_{h1}) \cdot w_{11} + w_{o2} \cdot \sigma^2(w_{21}x_1 + w_{22}x2 + b_{h1}) \cdot w_{21}$$

5 Part E: Gradient Calculations (do not submit in Python)

Let:

$$f(x) = 2x^2 - 1$$
$$g(x) = 3x^2 + 4$$
$$h(x, y) = x^2 + y^2 + xy$$

Find:

1. The first and second derivatives of f(x) with respect to x: f'(x) and f"(x)

Answer:

1.1 $f'(x) = 4x$

1.2 $f''(x) = 4$

2. The partial derivatives $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$

Answer:

2.1 $\frac{\partial h}{\partial x} = 2x + y$

2.2 $\frac{\partial h}{\partial y} = 2y + x$

3. The gradient vector $\nabla h(x, y)$

Answer:

$$\nabla h = \frac{\partial h}{\partial x} i + \frac{\partial x}{\partial y} j = (2x + y)i + (2y + x)j = \begin{bmatrix} 2x + y \\ 2y + x \end{bmatrix}$$

4. The derivative $\frac{d}{dx} f(g(x))$ with and without using the chain rule for derivatives

Answer:
$f(g(x)) = 2(3x^2 + 4)^2 - 1 = 2(9x^4 + 24x^2 + 16) - 1 = 18x^4 + 48x^2 + 31$

$\frac{d}{dx} f(g(x)) = 72x^3 + 96x$