MACHINE LEARNING AND PATTERN RECOGNITION REPORT

Fingerprint Spoofing Detection

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Who?
Where?
When?

Contents

Introduction The project task consists of a binary classification problem. The goal is to perform fingerprint spoofing detection, i.e. to identify genuine vs counterfeit fingerprint images. The dataset consists of labeled samples corresponding to the genuine (True, label 1) class and the fake (False, label 0) class. The samples are computed by a feature extractor that summarizes high-level characteristics of a fingerprint image. The data is 6-dimensional.

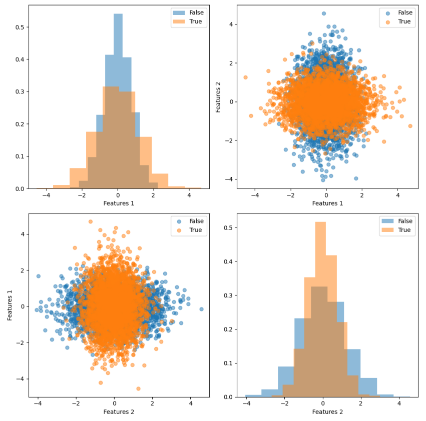
1 Dataset Analysis

In our analysis process, one first begins to represent what are the data related to the various features and how among the various features the data are distributed by making a visual representation in pairs of features.

1. Starting with the analysis of the first two features and creating a histogram and a scatter **??**, one can see:

    - Both features overlap
    - Follow a Gaussian distribution
    - Feature 1 has a peak at [-0.213, 0.276] and it is worth 0.541 for the false class, instead for Feature 2 the peak at [-0.402, 0.165] and it is worth 0.516 for the true class.

    Seeing **??** again, it would appear that **??** and **??** appear to be visually the same but in b they are represented centred which as can be seen is quite similar because Feature 1 has $\mu = 0.00170711$ and $\sigma^2 = 1.00134304$, instead Feature 2 has $\mu = 0.00503903$ and $\sigma^2 = 0.9983527$


(a) Without mean

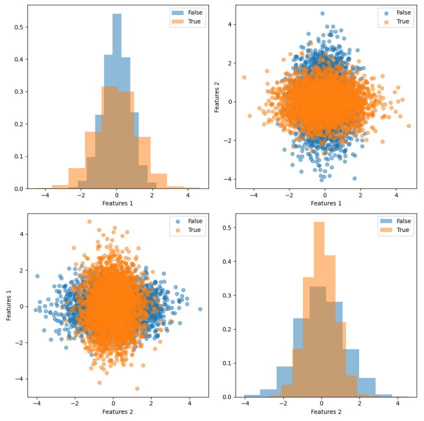

(b) With mean

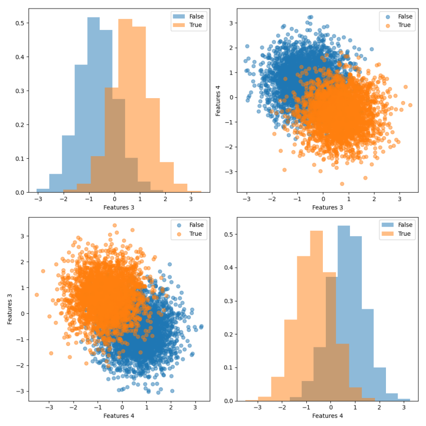
Figure 1: Feature 1 vs Feature 2 - Without centering the data relative to the average (a) and with centering the data relative to the average (b)

2. For features 3 and 4, observed in **??**, on the other hand, they have:

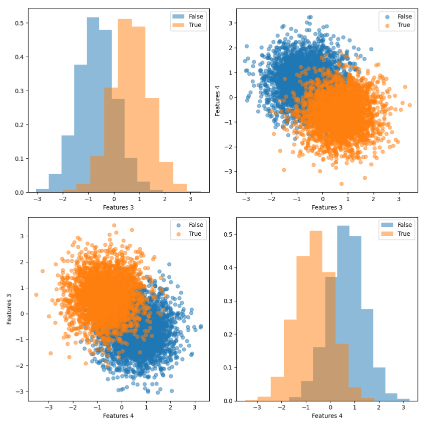
    - do not overlap like the previous two
    - Follow a Gaussian distribution but the true and false labels are centred at different points

- Feature 3 has a peak at [-1.063, -0.568] and it is worth 0.517 for the false class, instead for Feature 4 the peak at [0.290, 0.783] and it is worth 0.525 for the false class.

Seeing **??** and **??**, data are already similar because, the mean calculated with reference to the two classes is close to 0, in fact: Feature 3 has $\mu = -0.00560753$ and $\sigma^2 = 1.0024818$, instead Feature 4 has $\mu = 0.00109537$ and $\sigma^2 = 0.99029389$



(a) Without mean



(b) With mean

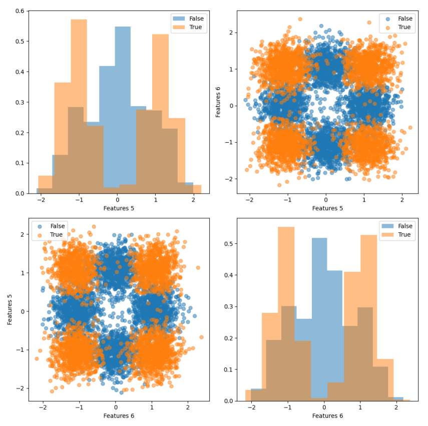
Figure 2: Feature 3 vs Feature 4 - Without centering the data relative to the average (a) and with centering the data relative to the average (b)

3. For features 5 and 6, observed in **??**, one can see:
   - Do not totally overlap
   - For both features, the true labels don't follow a Gaussian distribution as opposed to the false ones, which could be more approximate
   - Feature 5 has a peak at [-1.211, -0.783] and it is worth 0.572 for the true class, instead for Feature 6 has a peak at [-1.273, -0.817] and it is worth 0.553 for the true class.

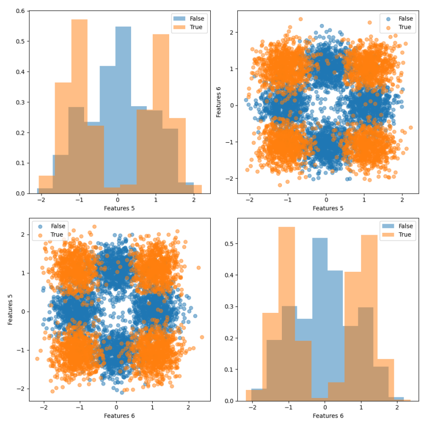
   Also in this other case, **??** and **??** are similar because again the average is close to 0. Feature 5 has $\mu = -0.00700025$ and Feature 6 has $\mu = 0.00910515$

2 Dimensionality Reduction

Before proceeding with classification, two techniques of dimensionality reduction PCA and LDA can be analysed. The goal is to find a subspace of the feature space that preserves most of the useful information, that is, mapping from the *n–dimensional* feature space to *m–dimensional* space, with $m \ll n$



(a) Without mean



(b) With mean

Figure 3: Feature 5 vs Feature 6 - Without centering the data relative to the average (a) and with centering the data relative to the average (b)

2.1 PCA

is an unsupervised technique. Where starting from a dataset $X = \{x_1, \dots, x_k\}$ and calculated average. It starts with the empirical covariance matrix:

$$C = \frac{1}{K} \sum (x_i - \bar{x})(x_i - \bar{x})^T \tag{1}$$

We compute the eigen-decomposition of $C = U \Sigma U^T$ and project the data in the subspace spanned by the m columns of U corresponding to the m largest eigenvalues.

$$y_i = P^T (x_i - \bar{x}) \tag{2}$$

where P is the matrix of the m columns of U associated to the m highest eigenvalues of C. A cross-validation approach can be used to figure out the optimal value of m to be selected. To evaluate each eigenvalue, one would have to calculate the variance corresponding to the axis. The percentage can be calculated as the rate between the sum of the m eigenvalues and the sum of all of them. In **??** we can see how it changes in the project. A good m, corresponds to that value which allows a percentage greater than 95%, so in our case we would need all 6 features.

2.2 LDA

is a supervised technique. To find a direction that has the best separation between classes, we measure spread between classes in terms of class covariance. The objective is to maximize the $between - class$ variability over $within - class$ variability ratio for the transformed samples:

$$\max_w \frac{w^T S_B w}{w^T S_W w} \tag{3}$$

where:

$$S_B \triangleq \frac{1}{N} \sum_{c=1}^{K} n_c (\mu_c - \mu)(\mu_c - \mu)^T \tag{4}$$

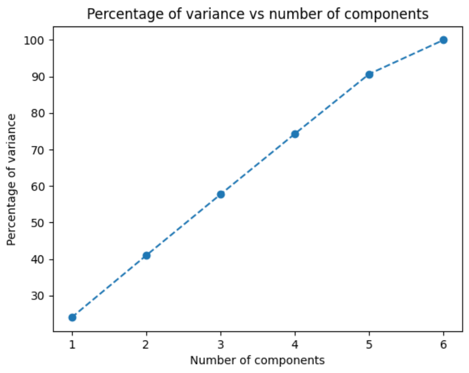


Figure 4: Cross validation for PCA impact evaluation

$$S_W \triangleq \frac{1}{N}\sum_{c=1}^{K}\sum_{i=1}^{n_c}(x_{c,i}-\mu_c)(x_{c,i}-\mu_c)^T \tag{5}$$

μ is dataset mean μ_c is class mean

2.3 Our project

PCA and LDA are applied to the dataset, in particular, m = 6 is used, and in **??** we can observe what are the outcomes for the indicated directions.

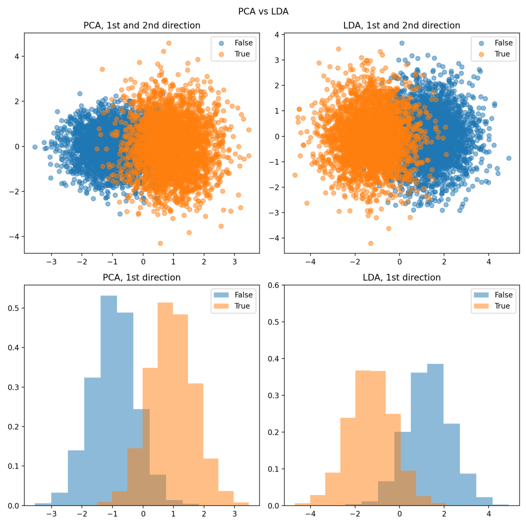


Figure 5: Comparing resulta between PCA and LDA

At a later stage, they were used to carry out a classification. The available dataset was divided into two sub-portions one for training and the other for validation. In **??** we can see the error of the classification, errors made in the classification when varying m (only for some values) and the threshold were reported

Method	**Num Samples**	**Error**	**Error Rate** (%)
LDA - First threshold	2000	186	9.30
LDA - Second threshold	2000	186	9.30
PCA (m=5) + LDA - First threshold	2000	186	9.30
PCA (m=5) + LDA - Second threshold	2000	185	9.25
PCA (m=6) + LDA - First threshold	2000	186	9.30
PCA (m=6) + LDA - Second threshold	2000	184	9.20

Table 1: Table showing the results of the LDA and PCA + LDA method for classification.

3 Multivariate Gaussian Density

Multivariate Gaussian Density is an extension of the Gaussian Density to multiple dimensions. It is used to describe the distribution of a vector of random variables in a $multi-dimensional$ space, and it could be defined as:

$$N(x \mid \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{M}{2}} |\Sigma|^{\frac{1}{2}}} \exp^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)} \tag{6}$$

where M is the size of the feature vector x, and $|\Sigma|$ is the determinant of Σ. Since the computation of the exponential could cause problems, the logarithm is applied, so from **??** we get **??**:

$$\log N(x \mid \mu, \Sigma) = -\frac{M}{2}\log(2\pi) - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) \tag{7}$$

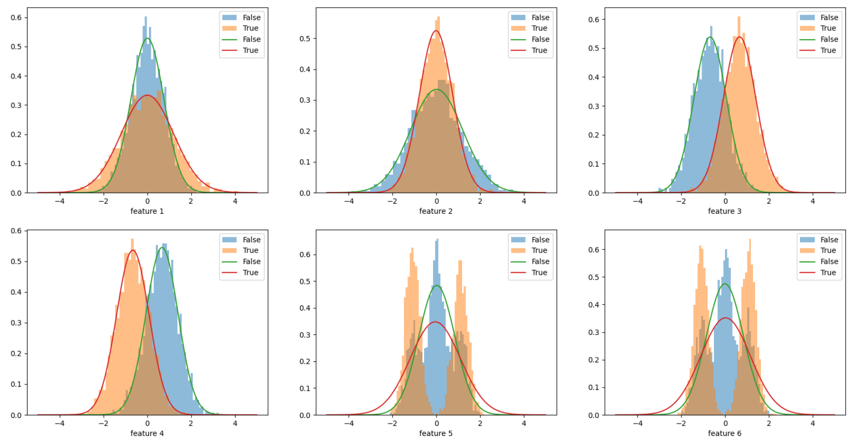


Figure 6: Gaussian Density

Thus, applying Gaussian probability density to the 6 features in our dataset, the following results in **??**, are obteinded.
It can observe that **features 1 - 2 - 3 - 4** fit the Gaussian distribution, the histogram of the data is well approximated by the Gaussian curve. For **features 5 - 6** this does not happen and therefore could not be a good model.

4 Model evaluation for classification

To assess whether one model classifies correctly or it is better than another, it is important to introduce what may be a unit of measurement. In particular, the evaluation of a model is done using the **DCF (Detection Cost Function)** or also called **empirical Bayes Risk**. But before delving into this method of valuation, we must consider what is called the working point of a binary classification application. This point is characterised by a triplet of values:

$$(\pi_T, C_{fn}, C_{fp}) \tag{8}$$

where π_T is the prior probability of the target class, C_{fn} is the cost of a false negative and C_{fp} is the cost of a false positive.

But from π_T, one can have attention on a particular triplet that is defined by:

$$(\tilde{\pi}, C_{fn}, C_{fp}) = (\tilde{\pi}, 1, 1) \tag{9}$$

where $\tilde{\pi}$ in **??** is called **effective prior** probability of the target class, defined as:

$$\tilde{\pi} = \frac{\pi_T C_{fn}}{\pi_T C_{fn} + (1 - \pi_T) C_{fp}} \tag{10}$$

So we can define DCF_u as:

$$B_{emp} = DCF_u = \sum_{c=1}^{K} \frac{\pi_c}{N_c} \sum_{i|c_i=c} C(a(x_i, R) \mid c) \tag{11}$$

From **??**, we can obtain:

$$DCF_u(\pi_T, C_{fn}, C_{fp}) = \pi_T C_{fn} P_{fn} + (1 - \pi_T) C_{fp} P_{fp} \tag{12}$$

where:

$$P_{fn} = \frac{FN}{FN + TP} \ , \quad P_{fp} = \frac{FP}{FP + TN} \tag{13}$$

From the **??**, it is possible to find **minDCF** and **actDCF**. When calculating minDCF, the threshold used is the one that minimize DCF, and there are various methods for finding it. Whereas for actDCF, the threshold used is:

$$t' = -\log \frac{\tilde{\pi}}{1 - \tilde{\pi}} \tag{14}$$

5 Classification Models Analysis

To perform the classification, the dataset must first be divided into two sub-portions, the training and validation sub-portions.

5.1 Gaussian models

Since, it is dealing with a binary classification task, it will assign a probabilistic score to each sample in terms of the class-posterior log-ratio:

$$\log r(x_t) = \log \frac{P(C = h_1 \mid x_t)}{P(C = h_0 \mid x_t)} \tag{15}$$

Analysing **??** in more detail, it becomes:

$$\log r(x_t) = \log \frac{f_{X|C}(x_t \mid h_1)}{f_{X|C}(x_t \mid h_0)} + \log \frac{P(C = h_1)}{P(C = h_0)} \tag{16}$$

The first addend of the equation is called the *llr* or *log-likelihood ratio* and an optimal decision is given by **??**.

$$\log r(x_t) \gtrless 0 \tag{17}$$

Considering $P(C = h_1) = \pi$ and $P(C = h_0) = 1 - \pi$, from **??** and **??**, it is possible to write that the class assignment is based on **??** and **??**, to obtain **??**.

$$llr(x_t) = \log \frac{f_{X|C}(x_t \mid h_1)}{f_{X|C}(x_t \mid h_0)} \gtrless -\log \frac{\pi}{1 - \pi} \tag{18}$$

The optimal class decision is based on a comparison between the and a threshold , if the llr is greater than th the sample is assigned to class h_1, otherwise to class h_0. It is necessary to find the parameters θ, μ_c, Σ_c; this can be done by maximising the log-likelihood. Parameter estimation is part of the training phase and this therefore performed on the training part of the dataset, then an estimation of the error rate can be performed on the validation part.

5.1.1 Multivariate Gaussian Classifier

The first classifier is MVG and it is given by the empirical mean and covariance matrix for each class,

$$\mu_c^* = \frac{1}{N_c} \sum_{i|c_i=c} x_i \ , \quad \Sigma_c^* = \frac{1}{N_c} \sum_{i|c_i=c} (x_i - \mu_c^*)(x_i - \mu_c^*)^T \tag{19}$$

5.1.2 Naive Bayes Gaussian Classifier

This model makes an important assumption that simplifies the number of parameters to be estimated, it assumes that the features are independent given their class. This causes the covariance matrix to be a diagonal matrix, consequently, matching MVG with a diagonal covariance matrix. However, the assumption of independence may be too restrictive and lead to inferior performance if the features are indeed correlated.

$$\mu_{c,[j]}^* = \frac{1}{N_c} \sum_{i|c_i=c} x_{i,[j]} \ , \quad \sigma_{c,[j]}^2 = \frac{1}{N_c} \sum_{i|c_i=c} (x_{i,[j]} - \mu_{c,[j]}^*)^2 \tag{20}$$

5.1.3 Tied Covariance Gaussian Classifier

The assumption of the latter model consists of its own average for each class, but an equal covariance matrix for all classes.

$$\mu_c^* = \frac{1}{N_c} \sum_{i|c_i=c} x_i \ , \quad \Sigma^* = \frac{1}{N} \sum_{c} \sum_{i|c_i=c} (x_i - \mu_c)(x_i - \mu_c)^T \tag{21}$$

The characteristic of this model is that it is strongly correlated to LDA.

5.1.4 Gaussian Models Comparison

A threshold of 0 was used to perform our results, which means that $P(C = 1) = P(C = 0) = 1/2$. This model was applied and the outcomes can be seen in the **??**.

Features	**Model**	**Error Rate** (%)
	no PCA	
1 to 6	MVG	7.00
1 to 6	Naive Bayes	7.20
1 to 6	Tied Covariance	9.30
1 to 4	MVG	7.95
1 to 4	Naive Bayes	7.65
1 to 4	Tied Covariance	9.50
1 - 2	MVG	36.50
1 - 2	Naive Bayes	36.30
1 - 2	Tied Covariance	49.45
3 - 4	MVG	9.45
3 - 4	Naive Bayes	9.45
3 - 4	Tied Covariance	9.40
	PCA $m = 5$	
1 to 6	MVG	7.10
1 to 6	Naive Bayes	8.75
1 to 6	Tied Covariance	9.30
	PCA $m = 6$	
1 to 6	MVG	7.00
1 to 6	Naive Bayes	8.90
1 to 6	Tied Covariance	9.30

Table 2: Table showing the results of the Error Rate for different Models and Features.

Comparing the results with the **??**, we can see that for some configurations there were improvements in terms of error rate. This means that Gaussian models are better able to classify the data. If we go into the details of how the error rate changes as a function of the observed features, we can see that:

- **1 to 6**: in the case we consider all 6 features, the error rate is quite low and its range goes from 7.00% to 9.30%. This means that they all provide useful information.
- **1 to 4**: if we consider features from 1 to 4, we can see that the error rate increases

slightly. This allow us to say that features 5 and 6 have useful but not fundamental information to change the outcome.

- **1 - 2**: features 1 and 2 have a rather high error rate, meaning that they don't contain relevant information.
- **3 - 4**: on the other hand, the latter two features considered have a rather low error rate, a value close to the case where all features are considered. This means that the information contained in these two features is relevant for making the classification.

Starting with the previous performance, it may be interesting to analyse how performance changes as three main parameters vary:

- $\tilde{\pi}$: represents prior probability of the positive class
- C_{fn}: misclassification cost of a sample predicted as negative but it is positive
- C_{fp}: misclassification cost of a sample predicted as positive but it is negative

	MVG	**Naive Bayes**	**Tied Covariance**
	Application $(\tilde{\pi}, C_{fn}, C_{fp}) = (0.5, 1, 1)$		
actDCF	0.1399	0.1439	0.1860
minDCF	0.1302	0.1311	0.1812
	Application $(\tilde{\pi}, C_{fn}, C_{fp}) = (0.9, 1, 1)$		
actDCF	0.4001	0.3893	0.4626
minDCF	0.3423	0.3509	0.4421
	Application $(\tilde{\pi}, C_{fn}, C_{fp}) = (0.1, 1, 1)$		
actDCF	0.3051	0.3022	0.4061
minDCF	0.2629	0.2569	0.3628
	Application $(\tilde{\pi}, C_{fn}, C_{fp}) = (0.5, 1, 9)$		
actDCF	0.3051	0.3022	0.4061
minDCF	0.2629	0.2569	0.3628
	Application $(\tilde{\pi}, C_{fn}, C_{fp}) = (0.5, 9, 1)$		
actDCF	0.4001	0.3893	0.4626
minDCF	0.3423	0.3509	0.4421

Table 3: Table showing minDCF and actDCF for different models and applications.

Analysing the results of the **??**, it is possible to observe:

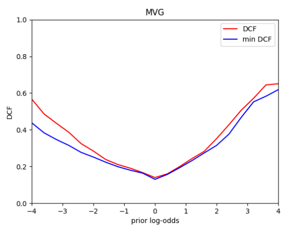
- Observing how the $\tilde{\pi}$ varies, it can be seen that the best outcome is obtained when it takes the value 0.5. On the other hand, when it takes value 0.1 and 0.9, the outcome gets worse because it penalises false negatives and false positives respectively
- Observing how the values of C_{fn} and C_{fp} change when they assume value 9. The outcomes worsen, in particular there is a greater impact on the model when the cost of false negatives increases.

Starting from the result obtained in **??**, it is possible to consider the application of PCA as pre-processing technique focusing on the cases of $\tilde{\pi}$ equal to 0.1, 0.5 and 0.9 and $C_{fn} = C_{fp} = 1$, obtaining the outcomes shown in **??**

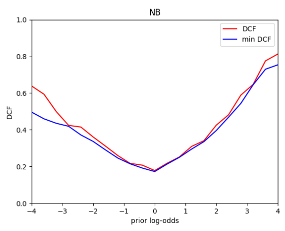
	MVG	**Naive Bayes**	**Tied Covariance**
	Application $(\tilde{\pi}, C_{fn}, C_{fp}) = (0.5, 1, 1)$		
	no PCA		
actDCF	0.1399	0.1439	0.1860
minDCF	0.1302	0.1311	0.1812
	PCA $m = 5$		
actDCF	0.1419	0.1749	0.1860
minDCF	0.1331	0.1737	0.1812
	$m = 6$		
actDCF	0.1399	0.1780	0.1860
minDCF	0.1302	0.1727	0.1812
	Application $(\tilde{\pi}, C_{fn}, C_{fp}) = (0.9, 1, 1)$		
	no PCA		
actDCF	0.4001	0.3893	0.4626
minDCF	0.3423	0.3509	0.4421
	PCA $m = 5$		
actDCF	0.3980	0.4660	0.4626
minDCF	0.3512	0.4340	0.4451
	$m = 6$		
actDCF	0.4001	0.4512	0.4626
minDCF	0.3423	0.4359	0.4421
	Application $(\tilde{\pi}, C_{fn}, C_{fp}) = (0.1, 1, 1)$		
	no PCA		
actDCF	0.3051	0.3022	0.4061
minDCF	0.2629	0.2569	0.3628
	PCA $m = 5$		
actDCF	0.3042	0.3930	0.4051
minDCF	0.2738	0.3545	0.3648
	$m = 6$		
actDCF	0.3051	0.3920	0.4061
minDCF	0.2629	0.3535	0.3628

Table 4: Show minDCF and actDCF for different models and applications before and after applying PCA.

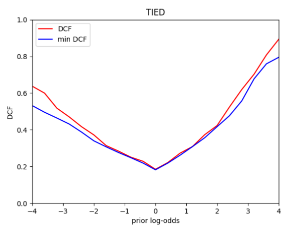
From the results obtained in the **??**, it can be seen that the application of PCA was not very helpful because in no case did the outcomes improve, instead they remained the same or



(a) MVG



(b) Naive Bayes



(c) Tied Covariance

Figure 7: Error Bayes plots

even worsened. Analysing overall, it can be said that the model that tends to perform worse is the Tied Covariance, whereas MVG and Naive Bayes are rather similar. In particular for MVG and Naive Bayes, the best result is obtained for the $0.5, 1, 1$ configuration whether applying PCA or not. In addition, it can be said that there is a good calibration for this configuration because applying or not applying PCA, minDCF and actDCF doesn't change what is not the case for the other configurations.

In **??** the Bayes error was calculated for a prior log odds in the range $(-4, +4)$, for the three models with a configuration having a $\tilde{\pi} = 0.1$ and applying the PCA as pre-processing.

5.2 Logistic Regression Classifier

Logistic Regression is a discriminative classification model, directly evaluating the posterior probability $C \mid X$. In particular by determining that hyperplane which maximises the posterior probability. Starting from the results obtained from the Tied Gaussian that provides log-likelihood ratios that are linear functions of our data, where log-posterior probability ratio is:

$$\log \frac{P(C = h_1 \mid X)}{P(C = h_0 \mid X)} = \log \frac{f_{X|C}(x \mid h_1)}{f_{X|C}(x \mid h_0)} + \log \frac{\pi}{1 - \pi} = \omega^T x + b \tag{22}$$

where prior information has been absorbed in the bias term b of the **??**. So from this point we can define the score function as:

$$s(x) = \omega^T x + b = 0 \tag{23}$$

where it is positive for samples of class h_1 and negative for samples of class h_0. Given ω and b we can compute the posterior class probability as:

$$P(C = h_1 \mid x, \omega, b) = \sigma(\omega^T x + b) = \sigma(s(x)) \tag{24}$$

where σ is sigmoid function defined as:

$$\sigma(x) = \frac{1}{1 + e^{-x}} \tag{25}$$

This approach assumes that the decision rules will be hyperplanes orthogonal to vector w.

5.2.1 Binary Logistic Regression

Binary Logistic Regression Not Prior-Weighted

The objective is to minimise the loss function $J(\omega, b)$, but to this is introduced what is a penalty term, so the new function becomes:

$$J(\omega, b) = \frac{\lambda}{2} \|w\|^2 + \frac{1}{n} \sum_{i=1}^{n} \log(1 + e^{-z_i(\omega^T x_i + b)}), \quad z_i = \begin{cases} 1 & \text{if } c_i = 1 \\ -1 & \text{if } c_i = 0 \end{cases} \tag{26}$$

where λ of **??** is the regularization term, this term has been introduced to make problem solvable in case of linearly separable classes.

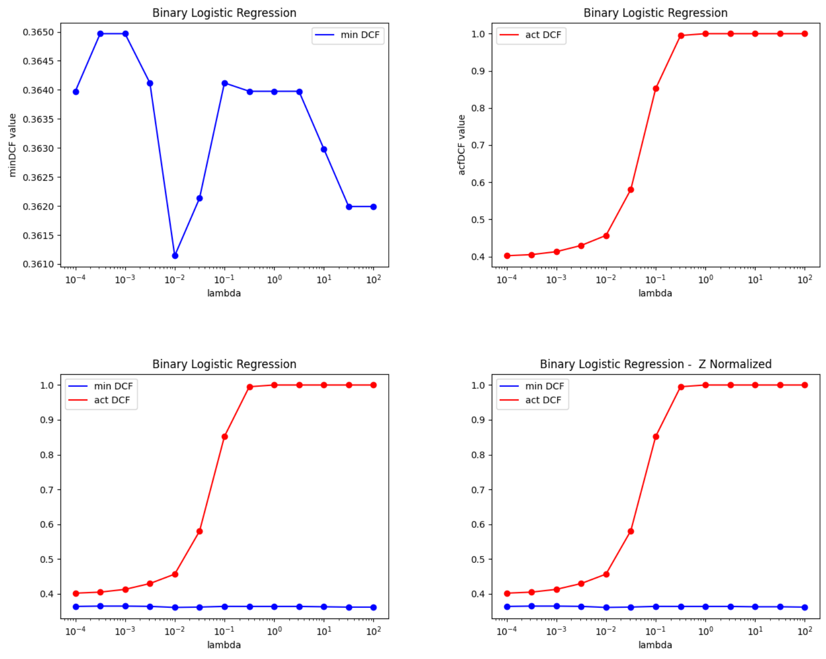


Figure 8: Binary Logistic Regression not Prior-Weighted

In this model $\pi_T = 0.1$ is used. In **??**, it can be seen how the values of minDCF and actDCF vary when λ changes, Z-normalization is applied or not and if the whole training set or a portion was used. It can be deduced from the values obtained that the application of z-normalisation brings no advantage. On the other hand, by using only 50 samples, it can see that using a limited number of samples can significantly influence the model and could lead to misleading results that are not representative of the entire sample. Consequently, as many samples as possible should be used for training to obtain a more accurate model. **??** and **??** give a graphic representation of how minDCF and actDCF vary with λ

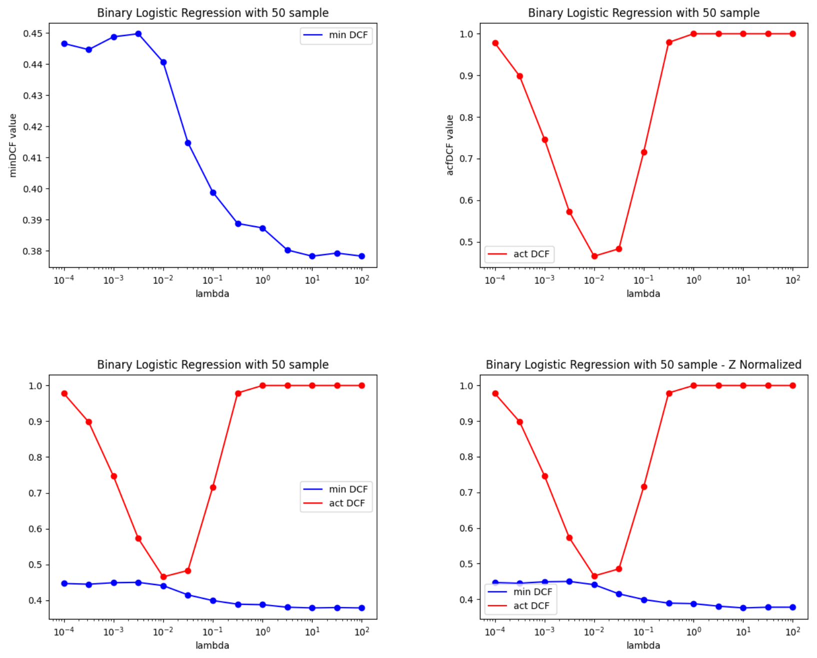


Figure 9: Binary Logistic Regression not Prior-Weighted with 50 Samples

Binary Logistic Regression Not Prior-Weighted				
λ	**minDCF**		**actDCF**	
	no z-norm	**z-norm**	**no z-norm**	**z-norm**
10^{-4}	0.3640	0.3640	0.4021	0.4021
10^{-3}	0.3650	0.3650	0.4130	0.4130
10^{-2}	0.3611	0.3611	0.4568	0.4568
10^{-1}	0.3641	0.3641	0.8522	0.8522

Binary Logistic Regression Not Prior-Weighted (50 Samples)				
λ	**minDCF**		**actDCF**	
	no z-norm	**z-norm**	**no z-norm**	**z-norm**
10^{-4}	0.4466	0.4466	0.9780	0.9780
10^{-3}	0.4487	0.4487	0.7466	0.7466
10^{-2}	0.4407	0.4407	0.4652	0.4652
10^{-1}	0.3988	0.3988	0.7164	0.7164

Table 5: Show minDCF and actDCF for Binary Logistic Regression Not Prior-Weighted model

Binary Logistic Regression Prior-Weighted

Another possible Logistic Regression approach is that Prior-Weighted; it allows to simulate different priors for class 1. Therefore, the objective function becomes:

$$J(\omega, b) = \frac{\lambda}{2}\left\|w\right\|^2 + \sum_{i=1}^{n} \xi_i \log(1 + e^{-z_i(\omega^T x_i + b)}), \quad \xi_i = \begin{cases} \frac{\pi_t}{n_T} & \text{if } z_i = +1 (c_i = 1) \\ \frac{1-\pi_T}{n_F} & \text{if } z_i = -1 (c_i = 0) \end{cases} \tag{27}$$

Now it is possible analyze the results obtained from the Prior-Weighted model with $\pi_T = 0.1$.

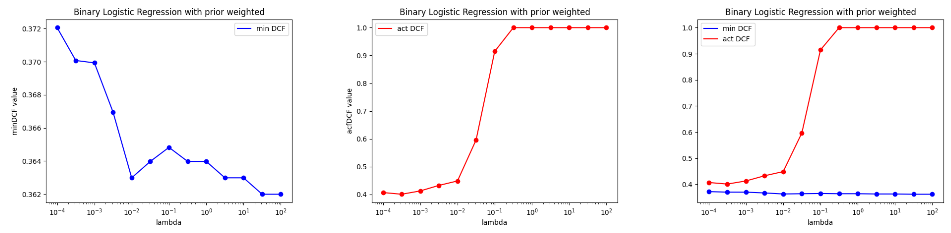


Figure 10: Binary Logistic Regression Prior-Weighted

Binary Logistic Regression Prior-Weighted		
	$\pi_T = 0.1$	
λ	**minDCF**	**actDCF**
10^{-4}	0.3721	0.4071
10^{-3}	0.3699	0.4129
10^{-2}	0.3630	0.4487
10^{-1}	0.3648	0.9147

Table 6: Show minDCF and actDCF for Binary Logistic Regression Prior-Weighted

Binary Logistic Regression with Pre-processing(PCA)

5.2.2 Quadratic Logistic Regression

In this step we can analyze training on a Quadratic Logistic Regression model by performing features expansion, so it's possible write log-likelihood ratio as:

$$\log \frac{P(C = h_1|x)}{P(C = h_0|x)} = x^T Ax + b^T x + c = s(\mathbf{x}, \mathbf{A}, \mathbf{b}, \mathbf{c}) \tag{28}$$

This expression is quadratic in x but it's linear in A and b. We could rewrite it to obtain a decision function that is linear for the expanded features space but quadratic in original features space.
We can write features expansion as:

$$\Phi(x) = \begin{bmatrix} vec(xx^T) \\ x \end{bmatrix}, \quad w = \begin{bmatrix} vec(A) \\ b \end{bmatrix} \tag{29}$$

where vec(X) is the operator that stacks the columns of X into a single column vector. In this way we can write the posterio log-likelihood as:

$$s(x, w, c) = s^T \phi(x) + c \tag{30}$$

Tabella

In conclusion, the results obtained from the Quadratic Logistic Regression model show that

5.3 Support Vector Machine Classifier

5.3.1 Linear Support Vector Machines

Support Vector Machines are linear classifiers that look for maximum margin separation hyperplanes. The primal formulation of the soft-margin SVM problem consists in minimizing the function:

$$\mathbf{J}(w, b) = \frac{1}{2}||w||^2 + C \sum_{i=1}^{N} max(0, 1 - z_i(w^T x_i + b)) \tag{31}$$

where N is the number of training samples, C is the regularization parameter, and z_i is the margin of the i-th sample.
The dual formulation of the problem is:

$$\mathbf{J}(\alpha) = -\frac{1}{2}\alpha^T \, \mathbf{H} \, \alpha + \alpha^T \mathbf{1} \qquad 1 \leq \alpha_i \leq C, \;\; \forall i \in \{1, ..., N\}, \;\; \sum_{i=1}^{n} \alpha_i z_i = 0 \tag{32}$$

where H is $H_{ij} = z_i z_j x_i^T x_j$ and the dual solution is the maximizer of $J^D(\alpha)$.
Primal and dual solutions are releted through:

$$w^* = \sum_{i=1}^{N} \alpha_i^* z_i x_i \tag{33}$$

In addition it's possible to rewrite dual problem as minimization of:

$$\hat{\mathbf{L}}(\alpha) = -\mathbf{J}(\alpha) = \frac{1}{2}\alpha^T \, \mathbf{H} \, \alpha - \alpha^T \mathbf{1} \tag{34}$$

and it can be minimize by L-BFGS-B algorithm. After that we have calculated the optimal α we can compute w^*.

5.3.2 Kernel Support Vector Machines

It's possible in Support Vector Machines to use kernels to allow nonlinear classification. In this case there is no explicit expansion of the feature space; we only can calculate the scalar product between the expanded features:$k(x_1, x_2) = \phi(x_1)_T \phi(x_2)$ where k is the kernel function. To do this we need to go and replace H as we saw in the previous section with $\hat{H} = z_i z_j k(x_1, x_2)$. During our project we see two different types of kernels:

- **Polynomial kernel of degree d**: $k(x_1, x_2) = (x_1^T x_2 + c)^d$
- **Radial Basis Function kernel(RBF)**: $k(x_1, x_2) = e^{-\gamma ||x1 - x_2||^2}$

We can now apply the polynomial kernel to the SVM with $d = 2, \quad c = 1, \quad \xi = 0$ and see how minDCF and actDCF vary as C changes.

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6 Gaussian Mixture Models Classifier

The last model we are going to consider is a generative model, the Gaussian Mixture Model (GMM). This model is based on the assumption that the data is generated by a mixture of K Gaussian distributions. The GMM density consists of a weighted sum of K Gaussians:

$$\mathbf{X} \sim GMM(\mathbf{M}, \mathcal{S}, w) \implies f_x(x) = \sum_{c=1}^{K} \mathcal{N}(x|\mu_g, \Sigma_g) w_g \tag{35}$$

where $M = [\mu_1 ... \mu_k]$, $\mathcal{S} = [\Sigma_1 ... \Sigma_k]$ and $w = [w_1 ... w_k]$ are the parameters of the model. Gaussian components can be viewed as clusters to which the samples belong (hard or soft), and the cluster label is an unobserved latent random variable. We can also define in this case the responsability term that represents the posterior probability that a sample belongs to a certain cluster:

$$\gamma(z_{n,i}) = P(G_i = g | X_i = x) = \frac{f_{x_i, G_i}(x_i, g)}{f_{x_i}(x_i)} = \frac{\mathcal{N}(x_i|\mu_g, \Sigma_g) w_g}{\sum_{g'} \mathcal{N}(x_i|\mu_{g'}, \Sigma_{g'}) w_{g'}} \tag{36}$$

Then assign the sample to the cluster label for which the liability is highest and re-estimate the model parameters based on the cluster assignment. We can apply the Expectation-Maximization algorithm to estimate the model parameters. The EM algorithm is an iterative algorithm that consists of two steps:

- Expectation stage: estimation of the responsability (given the model parameters (M_t, S_t, w_t))
- Maximization step: estimation of new model parameters using the above statistics, estimation continues from an initial value of the model parameters until a certain criterion is met.

The EM algorithm then requires an initial estimate for the GMM parameters, so we use the LBG algorithm to incrementally construct a GMM with 2G components from a GMM with G components. The starting point will be $(1, \mu, \sigma)$, so we use the empirical mean and covariance matrix of the data set. Then it builds a 2-component model starting from one and from each of the new components 2 more components are generated and so on. GMM can have differents versions as:

- **The diagonal covariance model:** in this setup, the covariance matrix of each component is assumed to be diagonal, which means the variables are considered independent.
- **The full covariance model:** in this case each component has a full covariance matrix, which means that all possible covariances between the variables are considered.

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