Reflection equation

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We will derive an equation for reflections in a scene following an iterative approach, allowing us to later implement it using a for loop. We will note N the maximum number of reflections, $\alpha_i \in [0, 1]$ gives how much reflective the $i + 1$th material we intersect first with is, c_b the resulting color for a given pixel, c_i the color obtained at the intersection of the ith reflected ray without considering reflections and c^i the color due to further reflections using the following relation:

$$c^i = (1 - \alpha_i)c_i + \alpha_i c^{i+1} \tag{1}$$

We can compute the pixel color c_b as :

$$\begin{aligned} c_b &= (1 - \alpha_0)c_0 + \alpha_0 c^1 \\ &= (1 - \alpha_0)c_0 + \alpha_0((1 - \alpha_1)c_1 + \alpha_1 c^2) \\ &= (1 - \alpha_0)c_0 + \alpha_0(1 - \alpha_1)c_1 + \alpha_0\alpha_1 c^2 \\ &= (1 - \alpha_0)c_0 + \alpha_0(1 - \alpha_1)c_1 + \alpha_0\alpha_1((1 - \alpha_2)c_2 + \alpha_2 c^3) \\ &= (1 - \alpha_0)c_0 + \alpha_0(1 - \alpha_1)c_1 + \alpha_0\alpha_1(1 - \alpha_2)c_2 + \alpha_0\alpha_1\alpha_2 c^3 \\ &= \dots \\ &= \sum_{i=0}^{N}(1 - \alpha_i)c_i \prod_{k=0}^{i-1}\alpha_k + \prod_{i=0}^{N}\alpha_i c^{N+1} \end{aligned} \tag{2}$$

If we concider a maximum of N reflections, then :

$$c^{N+1} = (1 - \alpha_{N+1})c_{N+1} + \alpha_{N+1}c^{N+2} \tag{3}$$

We define $c_i = 0$ if the intersection number i doesn't occur. Thus we have :

$$\forall\, i > N,\ c_i = 0 \implies \prod_{i=0}^{N}\alpha_i c^{N+1} = 0 \tag{4}$$

We can then simplify the equation for the resulting color of a given pixel c_b after N relfections as :

$$c_b = \sum_{i=0}^{N}(1 - \alpha_i)c_i \prod_{k=0}^{i-1}\alpha_k \tag{5}$$

We can expand this reasoning when $N \to +\infty$ to obtain :

$$c_b = \sum_{i=0}^{+\infty} (1 - \alpha_i) c_i \prod_{k=0}^{i-1} \alpha_k \tag{6}$$

For our implementation, for each reflection point, we first compute the pixel color *pix_color* without any reflections. We then introduced a *reflection_weight* representing $\prod_{k=0}^{i-1} \alpha_k$ in our equation that we multiply by $(1 - m.mirror)$, where $m.mirror$ is the reflectiveness of the current materiel α_i. We add this new pixel value to the one computed without an reflections. Finally, we find the next reflection point and start the process over again.