Homework 3

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ECO532

Time Series Econometrics

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HW3

Due: Friday, March 13

Directions: Students are encouraged to work together. However, students should think for themselves; do not simply copy what your classmate does.

1.) Consider the following $AR(1)$ process:

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t,$$
$$t = -\infty, ..., -1, 0, 1, ..., \infty$$

where ε_t is white noise and $var(\varepsilon_t) = \sigma^2$.

- (a) Using the "solution" to the first-order difference equation that we've covered in class, write this $AR(1)$ process as an $MA(\infty)$ process.

$$y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$$

- (b) Let $a_1 = 1$ so that y_t has a unit root. Calculate $var(y_t)$. (Hint: let your answer to part (a) assist you.)

$$var(y_t) = var(\frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i})$$
$$var(y_t) = var(\frac{a_0}{1-a_1}) + var(\sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}) + 2cov(\frac{a_0}{1-a_1}, \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i})$$
$$var(y_t) = 0 + \infty * \sigma^2 + 0$$
$$var(y_t) = \infty$$

2.) Consider the following $ARMA(1,2)$ process:

$$y_t = a_o + a_1 y_{t-1} + \varepsilon_t + b_1\varepsilon_{t-1} + b_2\varepsilon_{t-2}$$
$$t = -\infty, ..., -1, 0, 1, ..., \infty$$

- (a) Apply the "solution" to the first-order difference equation and write this $ARMA(2,1)$ process as an $MA(\infty)$ process (Hint: let $x_t = \varepsilon_t + b_1\varepsilon_{t-1} + b_2\varepsilon_{t-2}$).

$$y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i x_{t-i}$$

$$y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i x_{t-i}$$

$$y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i(\varepsilon_{t-i} + b_1\varepsilon_{t-i-1} + b_2\varepsilon_{t-i-2})$$

- (b) Calculate the following: $\frac{\partial y_t}{\partial \varepsilon_t}$, $\frac{\partial y_{t+1}}{\partial \varepsilon_t}$, $\frac{\partial y_{t+2}}{\partial \varepsilon_t}$, $\frac{\partial y_{t+3}}{\partial \varepsilon_t}$, and $\frac{\partial y_{t+k}}{\partial \varepsilon_t}$ for a general k.

$$\frac{\partial y_t}{\partial \varepsilon_t} = 1$$
$$\frac{\partial y_{t+1}}{\partial \varepsilon_t} = \frac{\partial y_t}{\partial \varepsilon_{t-1}} = b_1 + a_1$$
$$\frac{\partial y_{t+2}}{\partial \varepsilon_t} = \frac{\partial y_t}{\partial \varepsilon_{t-2}} = b_2 + a_1 b_1 + a_1^2$$
$$\frac{\partial y_{t+3}}{\partial \varepsilon_t} = \frac{\partial y_t}{\partial \varepsilon_{t-3}} = a_1 b_2 + a_1^2 b_1 + a_1^3$$
$$\frac{\partial y_{t+k}}{\partial \varepsilon_t} = \frac{\partial y_t}{\partial \varepsilon_{t-k}} = a_1^{k-2} b_2 + a_1^{k-1} b_1 + a_1^k$$

- (c) If $a_1 = 0.5$, $b_1 = 0.5$, and $b_2 = 2$ what does $\frac{\partial y_{t+k}}{\partial \varepsilon_t}$ approach as $k \to \infty$?

$$\lim_{k\to\infty}(2 * 0.5^{k-2} + 0.5 * 0.5^{k-1} + 0.5^k) = 0$$

- (d) If $a_1 = 0.5$, $b_1 = 2$, and $b_2 = 0.5$ what does $\frac{\partial y_{t+k}}{\partial \varepsilon_t}$ approach as $k \to \infty$?

$$\lim_{k\to\infty}(0.5 * 0.5^{k-2} + 2 * 0.5^{k-1} + 0.5^k) = 0$$

- (e) If $a_1 = 2$, $b_1 = 0.5$, and $b_2 = 0.5$ what does $\frac{\partial y_{t+k}}{\partial \varepsilon_t}$ approach as $k \to \infty$?

$$\lim_{k\to\infty}(0.5 * 2^{k-2} + 0.5 * 2^{k-1} + 2^k) = \infty$$

- (f) Are there finite values of b_1 or b_2 that would lead to explosive impulse response functions? Given this, do you think the moving average coefficients matters for stationarity?

Since a_1 is the only of the three values being raised to the $k'th$ power, it is the only one that can lead to explosive impulse response functions. Therefore, b_1 and b_2 do not affect function explosiveness and I do not think moving average coefficients matter for stationarity, overall.

3.) Consider the following process:

$$y_t = 2.5y_{t-1} - 2y_{t-2} + 0.5y_{t-3} + \varepsilon_t$$
$$t = -\infty, ..., -1, 0, 1, ..., \infty$$

where ε_t is white noise. Note that y_t is non-stationary.

- (a) Take the first difference of y_t and write a time series model expression for $\Delta y_t = y_t - y_{t-1}$. (Hint: Δy_t will follow an $AR(2)$ process.)

$$\Delta y_t = y_t - y_{t-1}$$
$$\Delta y_t = 2.5y_{t-1} - 2y_{t-2} + 0.5y_{t-3} + \varepsilon_t - y_{t-1}$$
$$\Delta y_t = 1.5y_{t-1} - 1.5y_{t-2} - 0.5y_{t-2} + 0.5y_{t-3} + \varepsilon_t$$
$$\Delta y_t = 1.5(y_{t-1} - y_{t-2}) - 0.5(y_{t-2} - y_{t-3}) + \varepsilon_t$$
$$\Delta y_t = 1.5\Delta y_{t-1} - 0.5\Delta y_{t-2} + \varepsilon_t$$

- (b) We know that for an $AR(2)$ process to be stationary, $|a_1 \pm a_2| < 1$ and $|a_2| < 1$. Using your answer to part (a), is Δy_t stationary?

Since $|1.5 - (-0.5)| = 1$, we can conclude that Δy_t is not stationary.

- (c) Take the second difference of y_t and write a time series model expression for $\Delta\Delta y_t = \Delta y_t - \Delta y_{t-1}$. (Hint: $\Delta\Delta y_t$ will follow an $AR(1)$ process.)

$$\Delta\Delta y_t = \Delta y_t - \Delta y_{t-1}$$
$$\Delta\Delta y_t = 1.5\Delta y_{t-1} - 0.5\Delta y_{t-2} + \varepsilon_t - \Delta y_{t-1}$$
$$\Delta\Delta y_t = 0.5\Delta y_{t-1} - 0.5\Delta y_{t-2} + \varepsilon_t$$
$$\Delta\Delta y_t = 0.5\Delta\Delta y_{t-1} + \varepsilon_t$$

- (d) Based on your answer to part (c), do you think $\Delta\Delta y_t$ is stationary?

In this case, $a_1 = 0.5$ so $|a_1| < 1$ which confirms that $\Delta\Delta y_t$ is stationary.

- (e) Is y_t an $ARIMA(p, d, q)$ process? If so, what are p, d, and q equal to?

After taking two differences of y_t, it becomes stationary, and thus an $ARIMA(1, 2, 0)$ process. Notation simply indicates that there is one autoregressive lag, no moving average lags, and that it took two differences for stationarity to be reached.

4.) During the US presidential primaries, journalists frequently say that “politician [X] has significant ‘momentum’ moving forward because he/she won the previous primaries.” One way of interpreting this “momentum” is that winning a state’s primary at time t makes it more likely for that politician to win future state primaries. Let y_t be the margin of victory for a particular politician. For example, if the politician wins the time t state primary by a 15% margin, then $y_t = 0.15$; if the politician loses the primary by a 10% margin, then $y_t = -0.1$. Assume, for simplicity purposes, that there are only two politicians in the race, that only one state holds its primary at time t, and that each primary is equally spaced apart. Suppose the true/population expression for y_t is

$$y_t = 0.4y_{t-1} + 0.1y_{t-2} + \varepsilon_t$$

where ε_t is white noise and $var(\varepsilon_t) = 0.2$.

- (a) As functions of the information available at time t, predict the margins of victory of the next three state primaries. In other words, compute the one-step, two-step, and three-step forecasts: $\hat{y}_{t+1}$, $\hat{y}_{t+2}$, and $\hat{y}_{t+3}$.

$$\hat{y}_{t+1} = E[y_{t+1}|y_t, y_{t-1}, y_{t-2}, ...]$$
$$\hat{y}_{t+1} = 0.4y_t + 0.1y_{t-1}$$
$$\hat{y}_{t+2} = E[y_{t+2}|y_{t+1}, y_t, y_{t-1}, ...]$$
$$\hat{y}_{t+2} = 0.4\hat{y}_{t+1} + 0.1y_t$$
$$\hat{y}_{t+2} = 0.4(0.4y_t + 0.1y_{t-1}) + 0.1y_t$$
$$\hat{y}_{t+2} = 0.26y_t + 0.04y_{t-1}$$
$$\hat{y}_{t+3} = E[y_{t+3}|y_{t+2}, y_{t+1}, y_t, ...]$$
$$\hat{y}_{t+3} = 0.4\hat{y}_{t+2} + 0.1\hat{y}_{t+1}$$
$$\hat{y}_{t+3} = 0.4(0.26y_t + 0.04y_{t-1}) + 0.1(0.4y_t + 0.1y_{t-1})$$
$$\hat{y}_{t+3} = 0.144y_t + 0.026y_{t-1}$$

- (b) Compute the one-step, two-step, and three-step forecast errors: $u_{t+1} = y_{t+1} - \hat{y}_{t+1}$, $u_{t+2} = y_{t+2} - \hat{y}_{t+2}$, and $u_{t+3} = y_{t+3} - \hat{y}_{t+3}$

$$u_{t+1} = y_{t+1} - \hat{y}_{t+1}$$
$$u_{t+1} = 0.4y_t + 0.1y_{t-1} + \varepsilon_{t+1} - 0.4y_t - 0.1y_{t-1}$$
$$u_{t+1} = \varepsilon_{t+1}$$
$$u_{t+2} = y_{t+2} - \hat{y}_{t+2}$$
$$u_{t+2} = 0.4y_{t+1} + 0.1y_t + \varepsilon_{t+2} - 0.4\hat{y}_{t+1} - 0.1y_t$$
$$u_{t+2} = 0.4(y_{t+1} - \hat{y}_{t+1}) + \varepsilon_{t+2}$$
$$u_{t+2} = 0.4\varepsilon_{t+1} + \varepsilon_{t+2}$$
$$u_{t+3} = y_{t+3} - \hat{y}_{t+3}$$
$$u_{t+3} = 0.4y_{t+2} + 0.1y_{t+1} + \varepsilon_{t+3} - 0.4\hat{y}_{t+2} - 0.1\hat{y}_{t+1}$$
$$u_{t+3} = 0.4(y_{t+2} - \hat{y}_{t+2}) + 0.1(y_{t+1} - \hat{y}_{t+1}) + \varepsilon_{t+3}$$
$$u_{t+3} = 0.4(\varepsilon_{t+1} + \varepsilon_{t+2}) + 0.1\varepsilon_{t+1} + \varepsilon_{t+3}$$
$$u_{t+3} = 0.26\varepsilon_{t+1} + 0.4\varepsilon_{t+2} + \varepsilon_{t+3}$$

- (c) Give specific numerical values for the one-step, two-step, and three-step forecast error variances: $var(u_{t+1})$, $var(u_{t+2})$, and $var(u_{t+3})$.

$$var(u_{t+1}) = var(\varepsilon_{t+1}) = 0.2$$
$$var(u_{t+2}) = var(0.4\varepsilon_{t+1} + \varepsilon_{t+2}) = 0.232$$
$$var(u_{t+2}) = var(0.26\varepsilon_{t+1} + 0.4\varepsilon_{t+2} + \varepsilon_{t+3}) = 0.24552$$

- (d) Say the politician won the time t state's primary by a margin of 20% and lost the time $t-1$ state's primary by a margin of 5%. Give specific numerical forecasts for the margins of victory of the next three state primaries: $\hat{y}_{t+1}$, $\hat{y}_{t+2}$, and $\hat{y}_{t+3}$.

$$\hat{y}_{t+1} = 0.4y_t + 0.1y_{t-1} = 0.4(0.2) + 0.1(-0.05) = 0.075$$
$$\hat{y}_{t+2} = 0.26y_t + 0.04y_{t-1} = 0.26(0.2) + 0.04(-0.05) = 0.05$$
$$\hat{y}_{t+3} = 0.144y_t + 0.026y_{t-1} = 0.144(0.2) + 0.026(-0.05) = 0.0275$$

According to the y_t model, the politican is expected to win the next three primaries by margins of 7.5, 5, and 2.75 percent, respectively.

More at github.com/antoniojurlina/ECO532