Spiking Networks
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Exercise 3: Multi-interval Distribution

Subset of Poisson Events I (Erlang Process)

For a Poisson distribution with $\nu = \lambda$, selecting every n^{th} event of the process will have the following cumulative probability function (CPF) $F_a^*(t)$, and hence the multi-interval probability density function (PDF) $f_a^*(t)$ and count probability $P_a^*(t)$:

$$F_a^*(t) = P[t \geq t_a^*] = P[t \geq t_{na}] = F_{na}(t)$$

$$f_a^*(t) = \frac{d}{dt}F_a^*(t) = f_{na}(t)$$

$$P_a^*(t) = F_a^*(t) - F_{a+1}^*(t) = \sum_{k=na}^{na+n-1} P_k(t)$$

The expected count, the expected interval, and the integration of the interval densities are as followed:

$$E[n(t)] = \sum_{k=0}^{\infty} kP_k^*(t)$$

$$E[n(t)] = \sum_{a=0}^{\infty}\sum_{k=na}^{na+n-1} a\,P_k(t)$$

$$E[n(t)] = (\sum_{k=0}^{\infty} kP_k(t) - \sum_{i=0}^{\infty}\sum_{j=1}^{n-1} jP_{in+j}(t))/n$$

$$E[n(t)] \approx \sum_{k=0}^{\infty} kP_k(t)\,/n$$

$$E[n(t)] \approx \sum_{k=0}^{\infty} kP_k(t)\,/n$$

$$E[n(t)] \approx \lambda t/n$$

Therefore, $E[t_{isi}] = \frac{t}{E[n(t)]}$

$$E[t_{isi}] = \frac{n}{\lambda}$$

$$\int_0^{\infty} f_a^*(t)\,dt = \int_0^{\infty} f_{na}(t)\,dt$$

$$\int_0^{\infty} f_a^*(t)\,dt = \int_0^{\infty} \lambda P_{na-1}(t)\,dt$$

$$\int_0^{\infty} f_a^*(t)\,dt = 1$$

Subset of Poisson Events II (Random Process)

For the same Poisson distribution with $\nu = \lambda$, selecting event randomly with a probability of $\frac{1}{n}$ will have a binomial combination of choosing and un-choosing, and hence the following count probability:

$$P[N_1(t) = p, N_2(t) = q | N(t) = p+q] = \frac{(p+q)!}{p!\,q!}\left(\frac{1}{n}\right)^p (1-\frac{1}{n})^q$$

$$since\ P[N(t) = p+q] = \frac{(\lambda t)^{p+q}}{(p+q)!}e^{-\lambda t}$$

$$P[N_1(t) = p, N_2(t) = q] = \frac{(p+q)!}{p!\,q!}\left(\frac{1}{n}\right)^p (1-\frac{1}{n})^q \frac{(\lambda t)^{p+q}}{(p+q)!}e^{-\lambda t}$$

$$P[N_1(t) = p, N_2(t) = q] = \frac{1}{p!}\left(\frac{\lambda t}{n}\right)^p e^{-\frac{\lambda t}{n}} \cdot \frac{1}{q!}[\left(1 - \frac{1}{n}\right)\lambda t]^q e^{-\left(1-\frac{1}{n}\right)\lambda t}$$

Hence, the newly selected and unselected sequences formed two independent Poisson distribution with an average rate of $\frac{\lambda}{n}$and$\left(1 - \frac{1}{n}\right)\lambda$. As a result, the multi-interval PDF $f_a^*(t)$:

$$f_a^*(t) = \frac{(\frac{\lambda}{n})^a t^{a-1}}{(a-1)!} e^{-\frac{\lambda}{n}t}$$

The expected count, the expected interval, and the integration of the interval densities are as followed:

$$E[n(t)] = \frac{\lambda t}{n}$$

Therefore, $E[t_{isi}] = \frac{t}{E[n(t)]}$

$$E[t_{isi}] = \frac{n}{\lambda}$$

$$\int_0^\infty f_a^*(t)\, dt = \int_0^\infty \frac{\lambda}{n} P_a^*(t)\, dt$$

$$\int_0^\infty f_a^*(t)\, dt = 1$$