221 Spiking Networks
Exercise 8: State space of stochastic system

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Due Jan 2019

Stochastic system

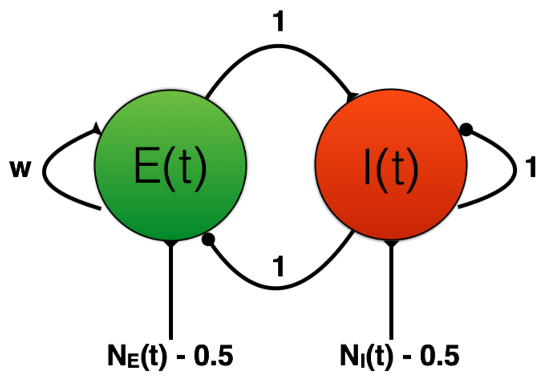
We model the interaction between an excitatory population E_1 and an inhibitory population E_2 with a stochastic model. To this end, we add Brownian noise to the deterministic model of Exercise 4.

The dynamic equations are

$$\begin{aligned}
\tau \frac{dE_1}{dt} &= -E_1 + \Phi\left[w\, E_1 - E_2 - 0.5\right] + N_1 \\
\tau \frac{dE_2}{dt} &= -E_2 + \Phi\left[E_1 - E_2 - 0.5\right] + N_2 \\
\tau_n \frac{dN_1}{dt} &= -N_1 + \sigma_n \sqrt{\tau_n}\, \xi(t) \\
\tau_n \frac{dN_2}{dt} &= -N_2 + \sigma_n \sqrt{\tau_n}\, \xi(t)
\end{aligned}$$

where $\xi(t)$ is white noise and N_1 and N_2 are independent realisations of Brownian noise (Ornstein-Uhlenbeck processes at steady-state).

As previously, the non-linearity is $\Phi(x) = \frac{x^2}{\kappa^2+x^2}$. We choose $\tau = 10$, $\kappa = 0,75$, $w = 3.6$, and $\tau_n = 2$, leaving σ_n as a free parameter.



Assignment I: numerical simulation

Simulate the time-evolution of the system by iterative integration. Choose $T_{end} = 20\,\tau$, $dt = 0.05\,\tau_n$, and $\sigma_n = 1.0$.

Visualize the time-evolution in the state space (E_1, E_2), starting either at $(0.1, 0.1)$ or at $(1.2, 0.8)$. Add the isoclines (from Exercise 4) to your plot, if you wish. Repeat to accumulate approximately 20 trajectories.

$$E_1 = \frac{(wE_1 - E_2 - 0.5)^2}{\kappa^2 + (wE_1 - E_2 - 0.5)^2} \qquad \Rightarrow \qquad E_2 = wE_1 - 0.5 \pm \kappa \sqrt{\frac{E_1}{1 - E_1}}$$

$$E_2 = \frac{(E_1 - E_2 - 0.5)^2}{\kappa^2 + (E_1 - E_2 - 0.5)^2} \qquad \Rightarrow \qquad E_1 = E_2 + 0.5 \pm \kappa \sqrt{\frac{E_2}{1 - E_2}}$$

Assignment II: analysis of stability

Compute approximately 100 trajectories starting in the high state $(0.9, 0.1)$, with $T_{end} = 10\,\tau$ and $\sigma_n = 2$. For each trajectory, determine the **first-passage time** to the low state (*i.e.*, the time at which it first crosses a suitable threshold). Plot the cumulative distribution of first-passage times from *high* to *low*!

Repeat, but this time starting at the low state $(0.2, 0.6)$. Plot the cumulative distribution of first-passage times from *low* to *high*!

Repeat with $T_{end} = 50\,\tau$ and $\sigma_n = 1$!

Discuss your results! How would you characterise the respective stability of *high* and *low* stable states?

Assignment III: spectral analysis

Choose a simulation duration T_{end} and a noise level σ_n such that at least 95% of all trajectories remain in the initial state for the duration of the simulation.

Compute and save the average spectrum of $E_1(t)$ and $E_2(t)$ over approximately 100 realisations, starting either in the high or the low state. Simultaneously, compute the average spectrum of $N_1(t)$ and $N_2(t)$. Compare the latter results to the theoretical spectrum of Brownian noise $N(t)$

$$S(\omega) = \frac{|\tilde{N}(\omega)|^2}{T} = \frac{\tau_n\,\sigma_n^2}{1 + \tau_n^2\,\omega^2}$$

where $\tilde{N}(\omega)$ is the Fourier transform of $N(t)$.

Finally, compute the *differential spectrum* between *high* and *low* steady-state! In other words, subtract the spectrum in the low state from that of the high state!

Discuss your results! Which state exhibits more high-frequency power and which more low-frequency power? How does this relate to the stability of the two states?

Assignment IV (optional)

Linearize system around fixed points to obtain theoretical spectra for each fixed point. Compare empirical with empirical spectra. What do the differences reveal about stability?