Spiking Network
Teacher: Prof. Jochen Braun
Name: Lee Po Shing
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Exercise 8: State Space of Stochastic System

Introduction

In a model where an excitatory population $E_1$and an inhibitory population E_2(I(t) in *Fig. 1*) interacts in a stochastic manner as shown in *Fig. 1*, we would like to study the dynamics of the activities of the two populations with different initial states.

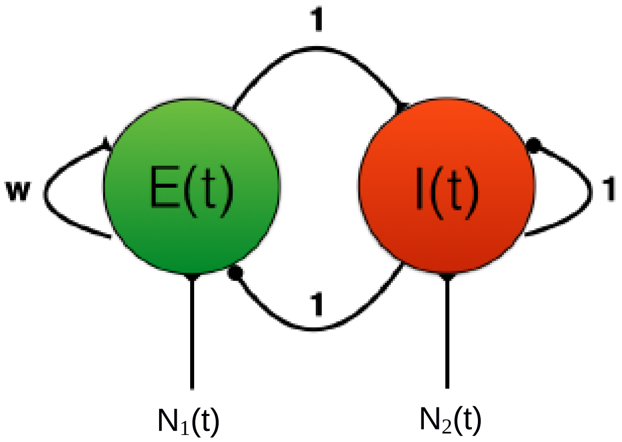


Fig. 1 Schematic diagram of a network of E_1 and E_2 (I(t) in the diagram). For both populations, they receive stochastic input from independent external sources N_1 and N_2, together with auto-activity and interaction between the two populations. Notably, the strength of auto-excitation of $E_1$is determined by $w = 3.6$.

The dynamic equations used were as followed:

$$\tau\frac{dE_1}{dt} = -E_1 + \Phi[wE_1 - E_2 - 0.5] + N_1$$

$$\tau\frac{dE_2}{dt} = -E_2 + \Phi[E_1 - E_2 - 0.5] + N_2$$

$$\Phi(x) = \frac{x^2}{\kappa^2 + x^2}$$

In this assignment, the model described above was studied using MatLab. In particular, the behaviour of the stochastic processes was studied under different initial states of the system.

Method

In this exercise, the main script *SN_Exercise_6_LeePoShing.m* was divided into four parts. In the first part, with $\mathrm{w} = 3.6$, the time-developments of the stochastic process starting at (0.1, 0.1) were plotted on the state-space with the corresponding flow-field, isoclines and steady-points generated by the function *plotstatespace.m*. In the second and third parts, with an initial state at high-state (0.9, 0.1) or at low-state (0.2, 0.6), the first-passage time reaching a low-state was studied. In the last part, the power spectrums of the processes around each of the two above states were studied.

For the first part, the time-development was calculated with the help of the function *noisttimetrajectory.m*. In this function, the connectivity was set inside (A and b in the code). With the time parameters (t, dt, tau, tau_n in the code), the standard deviation $\sigma_n = 1$(rho_n in the code) and the initial state (E_0 in the code), the state at each time point will be calculated iteratively and output (E in the code) together with the Brownian noise (N in the code). The value of the parameters were as followed: $\tau = 10$, $\kappa = 0.75$, $\tau_n = 2$, $\mathrm{dt} = 0.05\tau_n$. In each realization, the Brownian noise was calculated iteratively based on the following equation:

$$N_i(t + dt) = N_i(t)e^{\frac{-dt}{\tau}} + \sqrt{\frac{\sigma^2}{2}\left(1 - e^{\frac{-2dt}{\tau}}\right)}n(t), \qquad n(t) \in N(0,1)$$

For the properties of the state-space, including the gradient or flow-field F_1 and F_2 (F_1, and F_2 in the code), the isoclines for $\frac{dE_1}{dt} = 0$ and $\frac{dE_2}{dt} = 0$, and the steady-point(s) (steadypoint in the code) were calculated and plotted by the function *plotstatespace.m* previously described.

In the second part, the first-passage time of realizations from one steady-point to another was studied. Here, firstly a high noise strength $\sigma_n = 2$, a high-state $E_0 = (0.9, 0.1)$, and a short length of the iteration $t_{end} = 20\tau$ (t_end in the code) was used. The first-passage time was calculated with the function *noisytime.m*. Its structure is similar to *noisytimetrajectory.m*, except there is a conditional stage in the for-loop to check whether the iteration has reached ± 0.1 of the low-state steady-point (0.1974, 0.5815). If true, then the for-loop would be halted and the time-point would be recorded and exported as t_temp in the code. Then the cumulative exported value from 150 trials from high to low. The same was repeated with a low-state $E_0 = (0.2, 0.6)$ to a high-state steady-point (0.9279, 0.1335). The two sets of recorded first-passage time were compared in terms of cumulative distribution.

The third part was the same comparison of first-passage time, but with a lower level of noise $\sigma_n = 1$, and a longer length of iteration $t_{end} = 50\tau$.

In the last part, the power spectral densities of processes around the initial state were compared. The empirical values were calculated as $S(\omega) = \frac{1}{T}|\tilde{x}(\omega)|$ (S_omega in the code). We compared firstly the Brownian noise with the theoretical value of $S(\omega) = \frac{\tau_n \sigma_n^2}{1+\tau_n^2\omega^2}$. Next, the differential spectrum between processes around the high steady-state and that around the low one was computed. Noise level was $\sigma_n = 1$, and a length of iteration $t_{end} = 20dt$.

Results & Discussion

Trajectories of a stochastic system fluctuated between two different states around the steady-points

In *Fig. 1*, the 25 realizations according to the parameter defined were shown. In general, two clouds of trajectories could be observed around the two steady-points of the system. The Brownian noise inputted to the system provided the trajectories energy to move from one state to another than simply staying at the steady point..

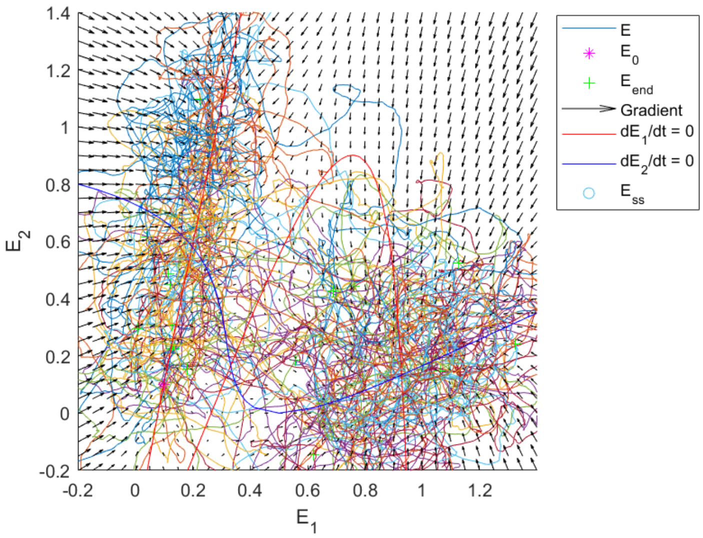


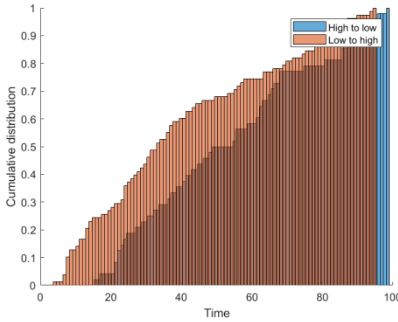
Fig. 1 Time-evolution of trajectories initiated at (0.1, 0.1) formed two clusters around the high and low steady-points. The two clouds around the steady-points can be seen as the high state and the low state of the system, in which the system fluctuated between the two states.

Cumulative probability density from the first-passage time revealed the lower stability of the lower state

In *Fig. 2*, the first-passage time of the trajectories successfully transited from a high state to the low was compared with the reversed direction. Under a high-noise input, the transition from low state to high state used on average less time than the other way around (*Fig. 2(a)*). The shortest time for the transition of low-to-high transition was less than that of high-to-low. The difference is less when a low-noise input was used (*Fig. 2(b)*). Still, as one can see, the curve for low-to-high transition was more concave. This indicates that the required time for such a transition was less than the other direction. Thus, it can be concluded that the low state is less stable than the high state. When the system entered the high state, it would take a longer time for it to have escape to the low state.

Under a less noisy situation, the average time that a transition occurred increased (*Fig. 2(b)*). This behaviour resembled the deterministic system, in which the trajectory will go to only one steady-point without any transition between them, i.e. the time for a transition to occur is infinitely long.

(a)



(b)

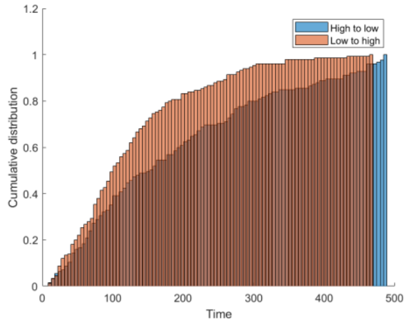


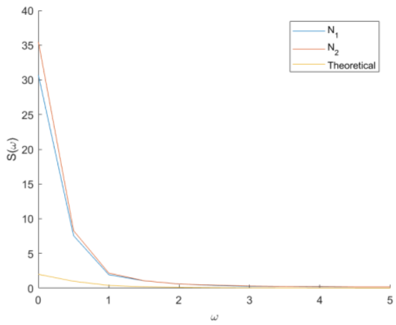
Fig. 2 First-passage time from the low-to-high transition was on average faster than that of high-to-low. In general, with both high-noise and low-noise input, the low-to-high transition required on average less time compared with that of high-to-low. This can be understood as the graph for low-to-high transition was more shifted to the left. For a lower noise situation, the time required for a transition generally decreased. Nevertheless, the transition from low state to high was on average than the other one.

Power spectral analysis of Brownian noise did not align with the expected value

In *Fig. 3*, the power spectral densities calculated from the Brownian noise and the differential spectrums of the processes around the initial states were shown. That of Brownian noise didn't align with the theoretical value calculated (*Fig. 3(a)*). Interestingly, the empirical value increases with the number of steps in the iteration, but it couldn't be simply normalized by the value (data not shown). Further adjustment needed to be made.

For the differential spectrum showed a larger difference in power in the low frequency region (*Fig. 3(b)*). However, no significant difference was noticed for the high-frequency power. This is not as expected, since the stability at a high-state is higher, the high frequency power of that should be lower, as the state should be dominated by the drift.

(a)



(b)

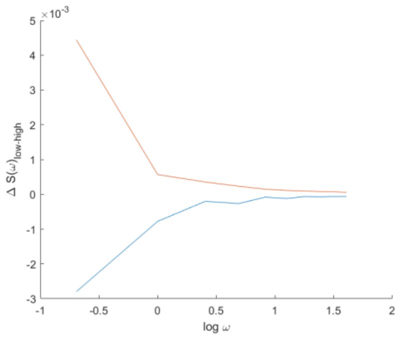


Fig. 3 Power spectral density of the Brownian noise and the processes around the initial state.