Spiking Networks
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Exercise 2: Renewal Processes

Renewal Process with Linearly Increasing Hazard

With $\rho(t) = \frac{t}{a^2}$, the survivor fraction $S(t)$, the interval density $P(t)$, the cumulative fraction $C(t)$ and the incerse survivor fraction $t = S^{-1}(u)$ can be calculated as followed:

$$\rho(t) = \frac{d}{dt}\ln S(t)$$

$$\frac{t}{a^2} = -\frac{d}{dt}\ln S(t)$$

$$-\int_0^\infty \frac{t}{a^2}dt = \ln S(t)$$

$$-\frac{t^2}{2a^2} + C = \ln S(t)$$

$$since\ S(0) = 1, C = 0$$

$$S(t) = e^{\frac{-t^2}{2a^2}}$$

$$C(t) = 1 - S(t)$$

$$C(t) = 1 - \frac{t}{a^2}e^{\frac{-t^2}{2a^2}}$$

$$P(t) = -\frac{d}{dt}S(t)$$

$$P(t) = -\frac{d}{dt}e^{\frac{-t^2}{2a^2}}$$

$$P(t) = -\frac{-2t}{2a^2}e^{\frac{-t^2}{2a^2}}$$

$$P(t) = \frac{t}{a^2}e^{\frac{-t^2}{2a^2}}$$

$$S(t) = e^{\frac{-t^2}{2a^2}}$$

$$-\frac{t^2}{2a^2} = \ln S(t)$$

$$t = \sqrt{-2a^2\ln S(t)}$$

$$t = S^{-1}(u) = \sqrt{-2a^2\ln u}$$

Moments of the Interval Density

With $P(t) = \frac{t}{a^2}e^{\frac{-t^2}{2a^2}}$, the first moment and the second moment will be

$$E(t) = \int_0^\infty tP(t)dt$$

$$By\ integration\ by\ part, E(t) = [-tS(t)]_0^\infty + \int_0^\infty S(t)dt$$

$$E(t) = [-te^{\frac{-t^2}{2a^2}}]_0^\infty + \int_0^\infty e^{\frac{-t^2}{2a^2}}dt$$

$$E(t) = 0 + \int_0^\infty e^{\frac{-t^2}{2a^2}}dt$$

$$E(t) = \frac{1}{2}\sqrt{2\pi a^2}$$

$$E(t^2) = \int_0^\infty t^2P(t)dt$$

$$By\ integration\ by\ part, E(t^2) = [-t^2S(t)]_0^\infty + \int_0^\infty 2tS(t)dt$$

$$E(t^2) = \int_0^\infty 2te^{\frac{-t^2}{2a^2}}dt$$

$$E(t^2) = -2a^2\int_0^\infty \frac{-2t}{2a^2}e^{\frac{-t^2}{2a^2}}dt$$

$$E(t^2) = -2a^2[e^{\frac{-t^2}{2a^2}}]_0^\infty$$

$$E(t) = \sqrt{\frac{\pi a^2}{2}} \qquad\qquad E(t^2) = 2a^2$$

For the variance, it will therefore be $E(t^2) - \left(E(t)\right)^2 = 2a^2 - \frac{\pi a^2}{2}$.

Analyse Simulated Process

From 10,000 uniformly distributed random numbers of survival function $u_i \in [0,1]$, the empirical intervals between spikes and thus the hazard $\rho(t)$, and the interval distribution $P(t)$ with 100 bins were calculated. Together with the analytical functions based on above calculations, they were plotted in *Fig. 1*. $\rho(t)$ followed the linear property of intervals while $P(t)$ displayed a Poisson distribution when they were plotted against t. $S(t)$, on the other hand, showed a sigmoidal pattern.

No major difference can be observed except a fluctuation in empirical $P(t)$ as in *Fig. 1(b)*. This is due to the random property of the number firstly generated and also the choice of number of bin.

The analytical and empirical mean values of the time interval were 25.0663 and 25.1030 *ms*; whereas those of variances were 171.6815 and 172.1102 *μs*^{2}. The values were very close, hinting that the calculations above can be correct.

(a) *(b)* *(c)*

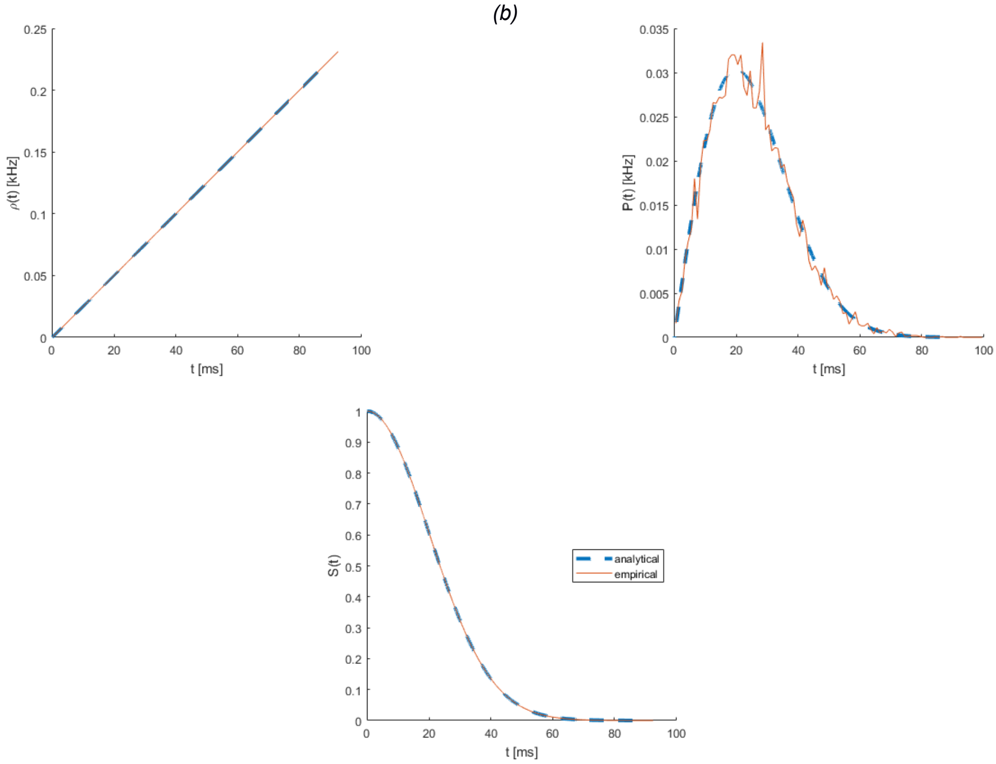

Fig. 1 Analytical and empirical values of (a) hazard, (b) interval density and (c) survivor fraction plotted again time interval. The dash lines showed the theoretical/analytical values; whereas the other ones the experimental/empirical values. The differences between experimental and theoretical values were not pronounced, except in interval density of the empirical calculation showed more fluctuations.