Spiking Network
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Exercise 4: State Space of Deterministic System

Introduction

In a model where an excitatory population $E_1$and an inhibitory population E_2(I(t) in *Fig. 1*) interacts in a deterministic manner as shown in *Fig. 1*, we would like to study the dynamics of the activities of the two populations with different initial states and with different strength of the auto-excitation of E_1, namely w.

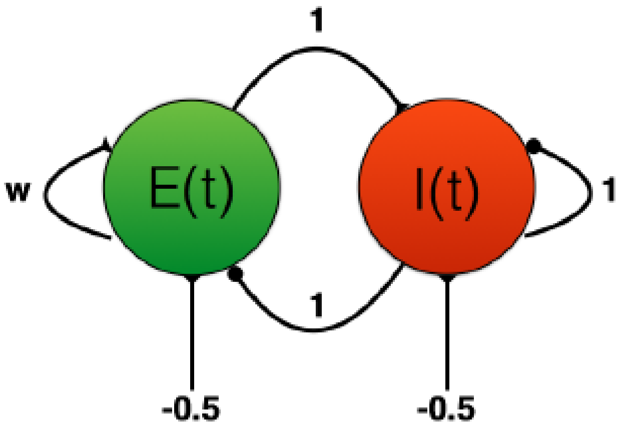


Fig. 1 Schematic diagram of a network of E_1 and E_2 (I(t) in the diagram). For both populations, they receive the same inhibition from an external source, together with auto-activity and interaction between the two populations. Notably, the strength of auto-excitation of $E_1$is determined by the variable w.

The dynamic equations used were as followed:

$$F_1 = \tau\frac{dE_1}{dt} = -E_1 + \Phi[wE_1 - E_2 - 0.5]$$

$$F_2 = \tau\frac{dE_2}{dt} = -E_2 + \Phi[E_1 - E_2 - 0.5]$$

$$\Phi(x) = \frac{x^2}{\kappa^2 + x^2}$$

In this assignment, the model described above was studied using MatLab. In particular, the isoclines, gradient, and the steady point of the system were found and plotted together with the time-development of a certain initial states.

Method

In this exercise, the main script *SN_Exercise_4_LeePoShing.m* was divided into two parts. In the first part, with a fixed value of w, the time-developments of a certain initial states were plotted on the state-space with the corresponding flow-field, isoclines and steady-points. In the second part, with different value of w(w in the code), the change in steady-points was studied.

For the first part, the time-development was calculated with the help of the function *timetrajectory.m*. In this function, the connectivity was set inside (A and b in the code), therefore with w set as 1 or 4, the time parameters (t, dt, and tau in the code), and non-linear value κ(kappa in the code) as the input, the state at each time point will be calculated iteratively and output as E in the code together with the final steady-state E_{ss} (E_ss in the code). For the properties of the state-space, including the gradient or flow-field F_1 and F_2 (F_1, and F_2 in the code), the isoclines for $\frac{dE_1}{dt} = 0$ and $\frac{dE_2}{dt} = 0$, and the steady-point(s) (steadypoint in the code) were calculated by the function *plotstatespace.m*. Firstly, the isoclines were calculated with the input w according to the following analytical calculation:

$$E_1 = \Phi[wE_1 - E_2 - 0.5] \qquad\qquad E_2 = \Phi[E_1 - E_2 - 0.5]$$

$$E_1 = 1 - \frac{\kappa^2}{\kappa^2 + (wE_1 - E_2 - 0.5)^2} \qquad\qquad E_2 = 1 - \frac{\kappa^2}{\kappa^2 + (E_1 - E_2 - 0.5)^2}$$

$$\frac{\kappa^2 + (wE_1 - E_2 - 0.5)^2}{\kappa^2} = \frac{1}{1 - E_1} \qquad\qquad \frac{\kappa^2 + (E_1 - E_2 - 0.5)^2}{\kappa^2} = \frac{1}{1 - E_2}$$

$$(wE_1 - E_2 - 0.5)^2 = \kappa^2(-1 + \frac{1}{1 - E_1}) \qquad (E_1 - E_2 - 0.5)^2 = \kappa^2(-1 + \frac{1}{1 - E_2})$$

$$wE_1 - E_2 - 0.5 = \pm\kappa\sqrt{-1 - \frac{1}{E_1 - 1}} \qquad E_1 - E_2 - 0.5 = \pm\kappa\sqrt{-1 - \frac{1}{E_2 - 1}}$$

$$E_2 = wE_1 - 0.5 \pm \kappa\sqrt{-\frac{E_1}{E_1 - 1}} \qquad E_1 = E_2 + 0.5 \pm \kappa\sqrt{-\frac{E_2}{E_2 - 1}}$$

Notably, the isoclines for E_1&E_2have the domain ∈[0, 1). Since each set of E_1(E1 in the code) will generate two isoclines, the corresponding values of E_2 were stored as E_2_plus, and E_2_minus in the code; the same case applied for E2, E_1_plus, and E_1_minus. After that, the steady-point(s) were found by calculating the $\frac{dE_1}{dt}$(dE_1_E2 in the code) of the points of $\frac{dE_2}{dt} = 0$. They were firstly narrowed down by finding two consecutive values with one being negative and another positive. The positions were then stored in the variable idk. For each case, the isocline was further sampled again with an even smaller interval (E2_temp and E_1_temp in the code) and the narrowing-down process repeated again. The steady point (steadypoint in the code) was then set as the average value of the consecutive isocline points giving $\frac{dE_1}{dt}$ positive and negative. This method is better than direct calculation by mathematics as the non-linearity makes the solution very hard to compute; however, it cannot identify any saddle point of the state-space and is just a close approximation. Next, according to the logic value (plotting in the code), the gradient F_1 and F_2 were calculated based on the equations stated previously. At last, also depending on the same logic value, the properties listed above would be plotted on the same figure as the time-development.

In the second part, a vector of w was used as inputs in the *plotstatespace.m* in order to obtain the corresponding steady-points (steadypoint in the code). Since the logic value (plotting in the code) was set as zero, the function would just provide the steady-points coordinates without plotting the properties of the state-space. The corresponding value of w was spaced into w_temp in the code and subsequently plotted with the steady-points.

As an optional part, the properties of the three steady-points when $w = 4$ were studied. The Jacobian matrix of the following was firstly computed. Then the eigenvalues of the matrix were calculated and shown.

$$J_{E_1,E_2} = \begin{pmatrix} \frac{\partial F_1}{\partial E_1} & \frac{\partial F_1}{\partial E_2} \\ \frac{\partial F_2}{\partial E_1} & \frac{\partial F_2}{\partial E_2} \end{pmatrix}$$

$$\frac{\partial F_1}{\partial E_1} = \frac{\partial}{\partial E_1}(-E_1 + (1 - \frac{\kappa^2}{\kappa^2 + (wE_1 - E_2 - 0.5)^2})) \qquad \frac{\partial F_1}{\partial E_2} = \frac{\partial}{\partial E_2}(-E_1 + (1 - \frac{\kappa^2}{\kappa^2 + (wE_1 - E_2 - 0.5)^2}))$$

$$\frac{\partial F_1}{\partial E_1} = -1 + \frac{\partial}{\partial E_1}(-\frac{\kappa^2}{\kappa^2 + (wE_1 - E_2 - 0.5)^2})) \qquad \frac{\partial F_1}{\partial E_2} = \frac{\partial}{\partial E_2}(-\frac{\kappa^2}{\kappa^2 + (wE_1 - E_2 - 0.5)^2}))$$

$$\frac{\partial F_1}{\partial E_1} = -1 - \frac{-\kappa^2(0 + 2w^2E_1 - 2wE_2 - w)}{(\kappa^2 + (wE_1 - E_2 - 0.5)^2)^2} \qquad \frac{\partial F_1}{\partial E_2} = -\frac{-\kappa^2(0 + 2E_2 - 2wE_1 + 1)}{(\kappa^2 + (wE_1 - E_2 - 0.5)^2)^2}$$

$$\frac{\partial F_1}{\partial E_1} = -1 + \frac{2\kappa^2 w(wE_1 - E_2 - 0.5)}{(\kappa^2 + (wE_1 - E_2 - 0.5)^2)^2} \qquad \frac{\partial F_1}{\partial E_2} = \frac{-2\kappa^2(wE_1 - E_2 - 0.5)}{(\kappa^2 + (wE_1 - E_2 - 0.5)^2)^2}$$

$$\frac{\partial F_2}{\partial E_1} = \frac{\partial}{\partial E_1}(-E_2 + (1 - \frac{\kappa^2}{\kappa^2 + (E_1 - E_2 - 0.5)^2}))$$

$$\frac{\partial F_2}{\partial E_2} = \frac{\partial}{\partial E_2}(-E_2 + (1 - \frac{\kappa^2}{\kappa^2 + (E_1 - E_2 - 0.5)^2}))$$

$$\frac{\partial F_2}{\partial E_1} = \frac{\partial}{\partial E_1}(-\frac{\kappa^2}{\kappa^2 + (E_1 - E_2 - 0.5)^2}))$$

$$\frac{\partial F_2}{\partial E_2} = -1 + \frac{\partial}{\partial E_1}(-\frac{\kappa^2}{\kappa^2 + (E_1 - E_2 - 0.5)^2}))$$

$$\frac{\partial F_2}{\partial E_1} = -\frac{-\kappa^2(0 + 2E_1 - 2E_2 - 1)}{(\kappa^2 + (E_1 - E_2 - 0.5)^2)^2}$$

$$\frac{\partial F_2}{\partial E_2} = -1 - \frac{-\kappa^2(0 + 2E_2 - 2E_1 + 1)}{(\kappa^2 + (E_1 - E_2 - 0.5)^2)^2}$$

$$\frac{\partial F_2}{\partial E_1} = \frac{2\kappa^2(E_1 - E_2 - 0.5)}{(\kappa^2 + (E_1 - E_2 - 0.5)^2)^2}$$

$$\frac{\partial F_2}{\partial E_2} = -1 - \frac{2\kappa^2(E_1 - E_2 - 0.5)}{(\kappa^2 + (E_1 - E_2 - 0.5)^2)^2}$$

Results & Discussion

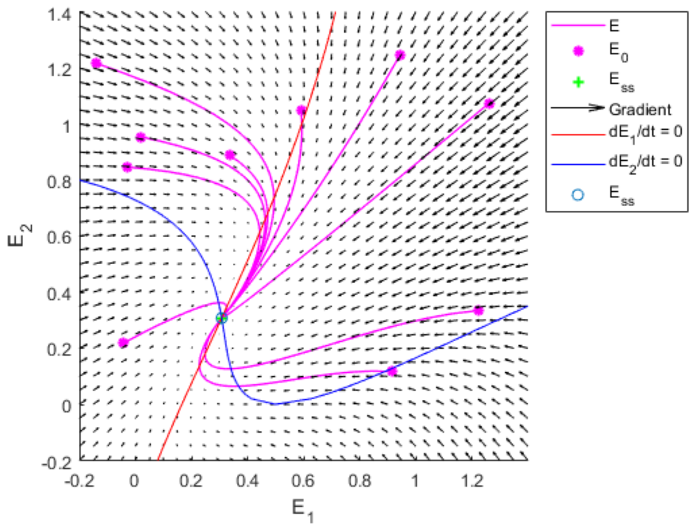
Time-development of random initial states stabilized at steady-points defined by the state-space

In *Fig. 2*, for some randomly generated initial states, they progressed along time and stabilized at one of the steady-points defined by the state-space. As one can easily see, the time-development for each of them followed the gradient/flow-field of the state-space and stopped in the end at one of the steady-points of the state-space, indicated as the intersections of the two isoclines. Notably, with an increasing in w, the number of steady-point started from 1 to 3, meaning that there are more stable activity patterns when the auto-excitation increases. However, as shown in the *Fig. 2(b)*, there is a steady-point where no initial state fell on that. This is because that is an unstable steady-point as if a peak of the energy profile. In the following section, the property of each steady-point was studied.

Number of steady-points increased with the strength of auto-excitation

As shown in *Fig. 2*, when $w < 2.4$, there is only one steady-point; however, it becomes 2 then 3 with increasing w. This can be understood with reference to *Fig. 1*. By comparing *(a)* and *(b)*, there is an extra isocline in *(b)*. This red isocline grows upwards from $E_2 < 0.2$ (i.e. out of the frame) and then crosses the blue isoclines, when w increases. Therefore, at a certain value of w around 2.4, new steady-points start to emerge, forming the bifurcation diagram as shown in *Fig. 2*.

(a)



(b)

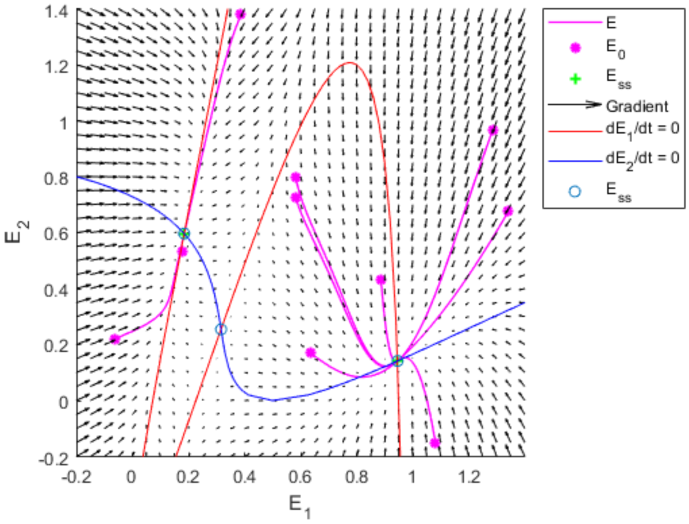


Fig. 2 Time-developments of random initial states became stabilized at steady-points of the state-space. With randomly generated initial states (magenta dots), the time-development for each of them (magenta lines) followed the gradient/flow-field of the state-space (black arrows) and stabilized in the end (green crosses) at some of the steady-points of the state-space (blue circles). The gradient on the isocline for E_1 (red lines) pointed only vertically; whereas that of E_2 (blue lines) horizontally. Notably, with an increase in w, the number of steady-points increased. (a) $w = 1$, (b) $w = 4$.

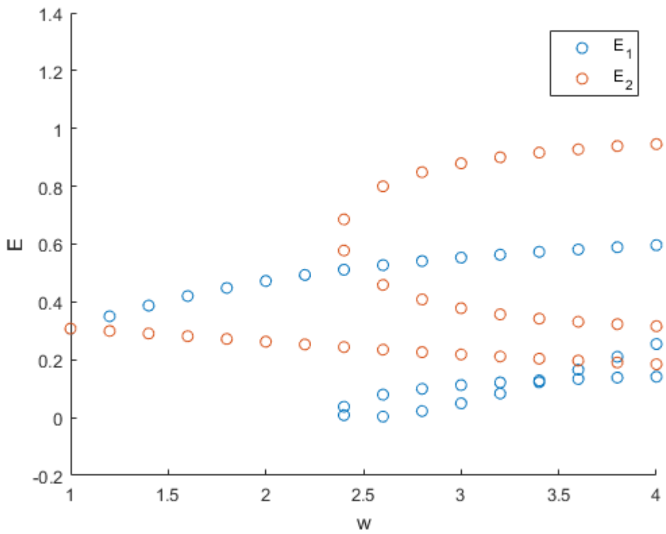


Fig. 3 Number of steady-point increased with w. When $w < 2.4$, there is only one steady-point; however, it becomes 2 then 3 with increasing w.

Eigenvectors of the Jacobian matrix at different steady points reveal the nature of the steady points

In *Fig. 1(a)*, there are three steady-points, from right to left, with coordinates (0.9460, 0.1415), (0.3163, 0.2545), and (0.1846, 0.5965). By putting the number in the Jacobian matrix, the eigenvalues are -0.8990 and -1.7691, 2.6537 and -0.3973, and -4.2598 and -0.5898. Since the real parts of the eigenvalues of first and third points are negative, while the complex part doesn't exist, the two points are predicted to be stable fixed points without spiralling. On the other hand, since the real part of one of the eigenvalues of the second steady-point is positive, it is predicted that the point is an unstable fixed point. The above prediction is correct when one compares it with *Fig. 1(b)*.