On the Toeplitz Lemma, Convergence in Probability, and Mean Convergence

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Abstract Three examples are provided which demonstrate that "convergence in probability" versions of the Toeplitz lemma, the Cesàro mean convergence theorem, and the Kronecker lemma can fail. "Mean convergence" versions of the Toeplitz lemma, Cesàro mean convergence theorem, and the Kronecker lemma are presented and a general mean convergence theorem for a normed sum of independent random variables is established. Some additional problems are posed.
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1 Introduction

The Toeplitz lemma is a result in mathematical analysis which is a useful tool for proving a wide variety of probability limit theorems. It is stated as follows and its proof may be found in Loève (1997, p. 250).

Theorem 1.1 (Toeplitz lemma). *Let $\{a_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be a double array of real numbers such that $\lim_{n\to\infty} a_{nk} = 0$ for all $k \geq 1$ and $\sup_{n\geq 1} \sum_{k=1}^{k_n} |a_{nk}| < \infty$. Let $\{x_n, n \geq 1\}$ be a sequence of real numbers.*

(i) If $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} \sum_{k=1}^{k_n} a_{nk}x_k = 0$.

(ii) If $\lim_{n\to\infty} x_n = x$ finite and $\lim_{n\to\infty} \sum_{k=1}^{k_n} a_{nk} = 1$, then $\lim_{n\to\infty} \sum_{k=1}^{k_n} a_{nk}x_k = x$.

The Toeplitz lemma contains, as corollaries, the following well-known and important results.

Corollary 1.1 (Cesàro Mean Convergence Theorem). *Let $\{x_n, n \geq 1\}$ be a sequence of real numbers and let $\bar{x}_n = \sum_{k=1}^{n} x_k/n$, $n \geq 1$. If $\lim_{n\to\infty} x_n = x$ finite, then $\lim_{n\to\infty} \bar{x}_n = x$.*

Corollary 1.2 (Kronecker Lemma). *Let $\{x_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be a sequence of real numbers with $0 < b_n \uparrow \infty$. If the series $\sum_{k=1}^{n} x_k/b_k$ converges, then $\lim_{n\to\infty} \sum_{k=1}^{n} x_k/b_n = 0$.*

The proof of the Cesàro mean convergence theorem follows immediately from the Toeplitz lemma (ii) by taking $k_n = n$, $n \geq 1$ and $a_{nk} = n^{-1}$, $1 \leq k \leq n, n \geq 1$. See Loève (1977, p. 250) for a proof of the Kronecker lemma which also follows from the Toeplitz lemma (ii).

It is clear that the Toeplitz lemma and its corollaries are are valid when the numerical sequence $\{x_n, n \geq 1\}$ and real number x are replaced, respectively, by a sequence of random variables

$\{X_n, n \geq 1\}$ and random variable X provided the convergence statements involving the random variables are couched in terms of almost sure (a.s.) convergence.

It is natural to inquire as to whether or not the Toeplitz lemma and its corollaries hold when the mode of convergence is changed from a.s. convergence to convergence in probability or to mean convergence of some order. In Section 2, we demonstrate by three examples that both corollaries of the Toeplitz lemma fail when a.s convergence is replaced by convergence in probability, and that a variety of possible limiting behaviors can prevail. Some open problems are also posed. In Section 3 we present several "mean convergence" versions of the Toeplitz lemma, Cesàro mean convergence theorem, and the Kronecker lemma. A general mean convergence theorem for a normed sum of independent random variables is established in Section 4.

Dugué (1957) investigated the "convergence in probability" problem for the sequence of Cesàro means $\{\bar{X}_n = \sum_{k=1}^n X_k, n \geq 1\}$ where $\{X_n, n \geq 1\}$ is a sequence of independent random variables and proved that if $\bar{X}_n \xrightarrow{P} c$ for some constant c, then $\frac{1}{n}\min_{1\leq k\leq n} X_k \xrightarrow{P} 0$ and $\frac{1}{n}\max_{1\leq k\leq n} X_k \xrightarrow{P} 0$. Then, for a sequence of independent random variables $\{X_n, n \geq 1\}$ where X_n has distribution function

$$F_n(x) = \left(1 - \frac{1}{x+n}\right) I_{(0,\infty)}(x), x \in \mathbb{R}, n \geq 1,$$

Dugué showed $X_n \xrightarrow{P} 0$, $\frac{1}{n}\max_{1\leq k\leq n} X_k \overset{P}{\nrightarrow} 0$, and, consequently, $\bar{X}_n \overset{P}{\nrightarrow} 0$. However, for his example, Dugué did not actually characterize the weak limiting behavior of $\bar{X}_n$. In our Examples 2.1 and 2.2, we present sequences of independent random variables $\{X_n, n \geq 1\}$ wherein $X_n \xrightarrow{P} 0$ and $\bar{X}_n \overset{P}{\nrightarrow} 0$ and we characterize the weak limiting behavior of $\bar{X}_n$.

2 Three Counterexamples

We present three counterexamples in this section. In the first example, $X_n \xrightarrow{P} 0$ yet the corresponding sequence of Cesàro means $\bar{X}_n$ has a nondegenerate limiting distribution. The following two lemmas are used in the verification of Example 2.1.

Lemma 2.1. *For $x > -1$, $\frac{x}{x+1} \leq \log(1+x) \leq x$.*

Proof. This is well known; see Bartle (1976, p. 238). □

Lemma 2.2. *For all $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} \max_{1\leq k\leq n} \frac{|e^{kt/n} - 1|}{k} = 0. \tag{2.1}$$

Proof. For $t \geq 0$ and $2 \leq m \leq n$,

$$\begin{aligned} 0 &\leq \max_{1\leq k\leq n} \frac{|e^{kt/n} - 1|}{k} \\ &\leq \max_{1\leq k\leq m-1} \frac{|e^{kt/n} - 1|}{k} + \max_{m\leq k\leq n} \frac{|e^{kt/n} - 1|}{k} \\ &\leq e^{(m-1)t/n} - 1 + e^t/m \overset{n\to\infty}{\longrightarrow} e^t/m \overset{m\to\infty}{\longrightarrow} 0. \end{aligned}$$

A similar argument works for $t < 0$ with the right-hand side of the last inequality replaced by $1 - e^{(m-1)t/n} + m^{-1}$ and so (2.1) holds for all $t \in \mathbb{R}$. □

Example 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $P(X_n = 0) = 1 - n^{-1}$ and $P(X_n = n) = n^{-1}$, $n \geq 1$. It is clear that $X_n \xrightarrow{P} 0$. Set $\bar{X}_n = \sum_{k=1}^{n} X_k/n$, $n \geq 1$. For $k \geq 1$, the moment generating function of X_k is

$$m_{X_k}(t) = 1 + \frac{e^{kt} - 1}{k}, t \in \mathbb{R}$$

and so the cumulant generating function of $\bar{X}_n$ is

$$\kappa_n(t) = \sum_{k=1}^{n} \log\left(1 + \frac{e^{kt/n} - 1}{k}\right), t \in \mathbb{R}, n \geq 1.$$

Let $t \in \mathbb{R}$ and $n \geq 1$. By Lemma 2.1,

$$\kappa_n(t) \leq \sum_{k=1}^{n} \frac{e^{kt/n} - 1}{k} = \frac{1}{n}\sum_{k=1}^{n} \frac{e^{tk/n} - 1}{k/n} \xrightarrow{n\to\infty} \int_0^1 \frac{e^{tx} - 1}{x}\, dx,$$

where we have used the fact that the upper bound for $\kappa_n(t)$ is a Riemann sum. Hence,

$$\limsup_{n\to\infty} \kappa_n(t) \leq \int_0^1 \frac{e^{tx} - 1}{x}\, dx. \tag{2.2}$$

Again let $t \in \mathbb{R}$ and fix $0 < \epsilon < 1$. By Lemmas 2.1 and 2.2, for all large n,

$$\kappa_n(t) \geq \sum_{k=1}^{n} \frac{(e^{kt/n} - 1)/k}{1 + (e^{kt/n} - 1)/k} = \sum_{k=1}^{n} \frac{\frac{e^{tk/n}-1}{k/n}\frac{1}{n}}{1 + \frac{e^{kt/n}-1}{k}} \geq \frac{1}{1 \pm \epsilon}\frac{1}{n}\sum_{k=1}^{n} \frac{e^{tk/n} - 1}{k/n},$$

where $\pm$ is taken to be $+$ when $t \geq 0$ and $-$ when $t < 0$. Thus

$$\liminf_{n\to\infty} \kappa_n(t) \geq \frac{1}{1 \pm \epsilon}\int_0^1 \frac{e^{tx} - 1}{x}\, dx,$$

and since $0 < \epsilon < 1$ is arbitrary, we have

$$\liminf_{n\to\infty} \kappa_n(t) \geq \int_0^1 \frac{e^{tx} - 1}{x}\, dx. \tag{2.3}$$

Combining (2.2) and (2.3) gives

$$\lim_{n\to\infty} \kappa_n(t) = \int_0^1 \frac{e^{tx} - 1}{x}\, dx, t \in \mathbb{R}.$$

Thus the sequence of moment generating functions $\{m_{\bar{X}_n}(\cdot), n \geq 1\}$ corresponding to $\{\bar{X}_n, n \geq 1\}$ satisfies

$$\lim_{n\to\infty} m_{\bar{X}_n}(t) = \lim_{n\to\infty} e^{\kappa_n(t)} = \exp\left(\int_0^1 \frac{e^{tx} - 1}{x}\right), t \in \mathbb{R}.$$

Then by the continuity theorem for moment generating functions (Curtis 1942, Theorem 3), the function

$$m(t) = \exp\left(\int_0^1 \frac{e^{tx} - 1}{x}\, dx\right), t \in \mathbb{R}$$

is the moment generating function of a random variable and the sequence of Cesàro means $\bar{X}_n$ has a limiting distribution with moment generating function $m(\cdot)$. It is clear that the limiting distribution of $\bar{X}_n$ is nondegenerate.

In the next example, $X_n \xrightarrow{P} 0$ yet the corresponding sequence of Cesàro means $\bar{X}_n$ approaches ∞ in probability.

Example 2.2. Let $\alpha > 1$ and let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $P(X_n = 0) = 1 - n^{-1}$ and $P(X_n = n^\alpha) = n^{-1}$, $n \geq 1$. It is clear that $X_n \xrightarrow{P} 0$. Set $\bar{X}_n = \sum_{k=1}^n X_k/n, n \geq 1$. Let $M \geq 1$ be arbitrary. Let k_n be the smallest integer greater than or equal to $(Mn)^{1/\alpha}$, $n \geq 1$. Then for all large n,

$$\begin{aligned} P(\bar{X}_n \geq M) &= P(\sum_{k=1}^n X_k \geq Mn) \geq P\left(\bigcup_{k=k_n}^n [X_k = k^\alpha]\right) \\ &= 1 - P\left(\bigcap_{k=k_n}^n [X_k \neq k^\alpha]\right) = 1 - \prod_{k=k_n}^n \frac{k-1}{k} \\ &= 1 - \frac{k_n - 1}{n} > 1 - \frac{(Mn)^{1/\alpha}}{n} \xrightarrow{n\to\infty} 1. \end{aligned}$$

Thus $\bar{X}_n \xrightarrow{P} \infty$ since $M \geq 1$ is arbitrary.

Remark 2.1. The previous two examples demonstrate the failure of the Toeplitz lemma (i) and (ii) when the mode of convergence is "convergence in probability", taking $a_{nk} = n^{-1}$, $1 \leq k \leq k_n = n, n \geq 1$.

The next example demonstrates that a convergence in probability version of the Kronecker lemma also fails; we note, however, that for the Kronecker lemma to fail we must have dependence among the X_k since otherwise convergence in probability of $\sum_{k=1}^n X_k/b_k$ to a random variable implies convergence a.s. (see, e.g., Theorem 3.3.1 of Chow and Teicher 1997, p. 72), in which case the traditional Kronecker lemma yields $\sum_{k=1}^n X_k/b_n \to 0$ a.s. and hence in probability.

Example 2.3. Let $\{Y_n, n \geq 1\}$ be a sequence of independent random variables with $P(Y_n = 0) = 1 - \frac{1}{2n-1}$ and $P(Y_n = 16^{n-1}) = \frac{1}{2^{n-1}}$, $n \geq 1$. For $n \geq 1$ define $X_{2n-1} = Y_n$ and $X_{2n} = -2Y_n$. Then

$$\frac{X_{2n-1}}{2^{2n-1}} + \frac{X_{2n}}{2^{2n}} = 0, n \geq 1$$

and so for $n \geq 1$

$$\sum_{k=1}^n \frac{X_k}{2^k} = \frac{X_n}{2^n} I(n \text{ is odd}).$$

Then, since it is clear that $X_n \xrightarrow{P} 0$, we have that $\sum_{k=1}^n \frac{X_k}{2^k} \xrightarrow{P} 0$ as well.

To show that $\sum_{k=1}^n X_k/2^n \overset{P}{\not\to} 0$, we show that the convergence fails along a subsequence, namely that

$$\frac{\sum_{k=1}^{4n} X_k}{2^{4n}} \overset{P}{\not\to} 0. \tag{2.4}$$

Note that for all odd positive integers k,

$$X_k + X_{k+1} = X_k - 2X_k = -X_k$$

and consequently for all $n \geq 1$,

$$\frac{\sum_{k=1}^{2n} X_k}{2^{2n}} = \frac{-\sum_{k=1}^{n} X_{2k-1}}{2^{2n}}.$$

Thus (2.4) will hold if we can show that

$$\frac{\sum_{k=1}^{2n} X_{2k-1}}{16^n} \stackrel{P}{\not\to} 0. \tag{2.5}$$

To this end,

$$\begin{aligned}
P\left(\frac{|\sum_{k=1}^{2n} X_{2k-1}|}{16^n} \geq 1\right) &\geq P\left(\sum_{k=n+1}^{2n} X_{2k-1} \geq 16^n\right) \\
&\geq P\left(\bigcup_{k=n+1}^{2n} [X_{2k-1} \neq 0]\right) = 1 - \prod_{k=n+1}^{2n} P(X_{2k-1} = 0) \\
&\geq 1 - \prod_{k=n+1}^{2n} \left(1 - \frac{1}{2k}\right) \geq 1 - \left(1 - \frac{1}{4n}\right)^n \\
&\to 1 - e^{-1/4} > 0,
\end{aligned}$$

proving (2.5).

Remark 2.2. In view of Examples 2.1-2.3, it is natural to pose the following problems. Find conditions under which "convergence in probability" versions of the Toeplitz lemma, the Cesàro mean convergence theorem, and the Kronecker lemma hold for a sequence of random variables. A sufficient condition for Cesàro means to converge in probability is given in Remark 3.1 below, but it is not clear whether that result is optimal. Apropos of the Kronecker lemma, independence of the sequence $\{X_n, n \geq 1\}$ is sufficient to ensure that a "convergence in probability" version of the Kronecker lemma holds (see the discussion prior to Example 2.3). Thus, we seek conditions under which a "convergence in probability" version of the Kronecker lemma will hold without the independence assumption being imposed on the $\{X_n, n \geq 1\}$.

3 Mean Convergence Versions of the Toeplitz Lemma, the Cesàro Mean Convergence Theorem, and the Kronecker Lemma

In this section, we present "mean convergence" versions of the Toeplitz lemma (Theorem 3.1), the Cesàro mean convergence theorem (Corollary 3.1), and the Kronecker lemma (Theorems 3.2 and 3.3). In Theorems 3.1–3.3 and Corollary 3.1, no independence conditions are imposed on the random variables $\{X_n, n \geq 1\}$.

For $p \geq 1$, the space $\mathscr{L}_p$ of the absolute pth power integrable random variables is a Banach space with norm $\|X\|_p = (E|X|^p)^{1/p}$, $X \in \mathscr{L}_p$. Now the proof of the Toeplitz lemma in Loève (1997, p. 250) carries over to a Banach space setting, *mutatis mutandis*. We thus immediately obtain the following "mean convergence" version of the Toeplitz lemma.

Theorem 3.1. *Let $\{a_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be a double array of real numbers such that $\lim_{n\to\infty} a_{nk} = 0$ for all $k \geq 1$ and $\sup_{n\geq 1} \sum_{k=1}^{k_n} |a_{nk}| < \infty$. Let $\{X_n, n \geq 1\}$ be a sequence of $\mathscr{L}_p$ random variables for some $p \geq 1$.*

(i) If $X_n \xrightarrow{\mathscr{L}_p} 0$*, then* $\sum_{k=1}^{k_n} a_{nk} X_k \xrightarrow{\mathscr{L}_p} 0$.

(ii) If there exists a random variable X *such that* $X_n \xrightarrow{\mathscr{L}_p} X$ *and* $\lim_{n\to\infty} \sum_{k=1}^{k_n} a_{nk} = 1$*, then* $\sum_{k=1}^{k_n} a_{nk} X_k \xrightarrow{\mathscr{L}_p} X$.

Letting $k_n = n, n \geq 1$ and $a_{nk} = n^{-1}$, $1 \leq k \leq k_n$, $n \geq 1$ in Theorem 3.1 (ii) yields the following "mean convergence" version of the Cesàro mean convergence theorem.

Corollary 3.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of* $\mathscr{L}_p$ *random variables where* $p \geq 1$ *and let* $\bar{X}_n = \sum_{k=1}^{n} X_k/n$, $n \geq 1$*. If there exists a random variable* X *such that* $X_n \xrightarrow{\mathscr{L}_p} X$*, then* $\bar{X}_n \xrightarrow{\mathscr{L}_p} X$.

Remark 3.1. It follows from the $\mathscr{L}_p$-convergence theorem (see, e.g., Loève 1977, p. 165) and Corollary 3.1 that if $X_n \xrightarrow{P} X$ and if, for some $p \geq 1$, $E|X_n|^p \to E|X|^p < \infty$ or $\{|X_n|^p, n \geq 1\}$ is uniformly integrable, then $\bar{X}_n \xrightarrow{\mathscr{L}_p} X$ and, *a fortiori*, $\bar{X}_n \xrightarrow{P} X$.

By employing the Banach space version of the traditional Kronecker lemma (see, e.g., Taylor 1978, p. 101), the following "mean convergence" version of the Kronecker lemma is immediate.

Theorem 3.2. *Let* $\{X_n, n \geq 1\}$ *be a sequence of* $\mathscr{L}_p$ *random variables for some* $p \geq 1$ *and let* $\{b_n, n \geq 1\}$ *be a sequence of real numbers with* $0 < b_n \uparrow \infty$*. If there exists a random variable* S *such that*

$$\sum_{k=1}^{n} \frac{X_k}{b_k} \xrightarrow{\mathscr{L}_p} S, \tag{3.1}$$

then

$$\frac{\sum_{k=1}^{n} X_k}{b_n} \xrightarrow{\mathscr{L}_p} 0. \tag{3.2}$$

Remark 3.2. For a sequence of independent mean 0 random variables $\{X_n, n \geq 1\}$ and a sequence of real numbers $\{b_n, n \geq 1\}$ with $0 < b_n \uparrow \infty$, a sufficient condition for the existence of a random variable S satisfying (3.1) with $p \in [1, 2]$ is that

$$\sum_{n=1}^{\infty} \frac{E|X_n|^p}{b_n^p} < \infty. \tag{3.3}$$

This follows readily from Theorem 2 of von Bahr and Esseen (1965) and the Cauchy convergence criterion (see, e.g., Chow and Teicher 1997, p. 99). However, for $p \geq 2$, a necessary *and* sufficient condition for the existence of a random variable S satisfying (3.1) is that

$$\sum_{n=1}^{\infty} \frac{E|X_n|^p}{b_n^p} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{EX_n^2}{b_n^2} < \infty.$$

This follows readily from Theorem 2 of Rosenthal (1970) and the Cauchy convergence criterion.

It is easy to construct an example showing that when $p \in [1, 2)$, the condition (3.3) is not necessary for the existence of a random variable S satisfying (3.1); that is, (3.1) can hold when (3.3) fails.

We now establish a version of Theorem 3.2 for a sequence of *nonnegative* random variables $\{X_n, n \geq 1\}$. In view of Theorem 3.2, Theorem 3.3 is of interest only when $0 < p < 1$.

Theorem 3.3. *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative $\mathscr{L}_p$ random variables for some $0 < p < \infty$ and let $\{b_n, n \geq 1\}$ be a sequence with $0 < b_n \uparrow \infty$. If there exists a random variable S such that (3.1) holds, then (3.2) holds as well.*

Proof. The $\mathscr{L}_p$ convergence in (3.1) implies convergence in probability which, in turn, implies a.s. convergence since $\sum_{k=1}^n X_k/b_k$ is nondecreasing. Then by the traditional Kronecker lemma, $\sum_{k=1}^n X_k/b_n \to 0$ a.s. On the other hand, $\mathscr{L}_p$ convergence of $\sum_{k=1}^n \frac{X_k}{b_k}$ implies that the sequence $\{|\sum_{k=1}^n X_k/b_k|^p, n \geq 1\}$ is uniformly integrable by the mean convergence criterion (see, e.g., Chow and Teicher 1997, p. 99). By nonnegativity,

$$0 \leq \frac{\sum_{k=1}^n X_k}{b_n} \leq \sum_{k=1}^n \frac{X_k}{b_k}$$

and so the sequence $\{|\sum_{k=1}^n X_k/b_n|^p, n \geq 1\}$ is uniformly integrable as well. Combining this with $\sum_{k=1}^n X_k/b_n \to 0$ a.s. and employing the mean convergence criterion gives (3.2). $\square$

Remark 3.3. For $\{X_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ as above and $0 < p \leq 1$, a sufficient condition for verifying (3.1) is that

$$\sum_{n=1}^\infty \frac{EX_n^p}{b_n^p} < \infty$$

in view of

$$E\left(\sum_{k=n+1}^m \frac{X_k}{b_k}\right)^p \leq E\left(\sum_{k=n+1}^m \frac{X_k^p}{b_k^p}\right) = \sum_{k=n+1}^m \frac{EX_k^p}{b_k^p}, \quad m > n \geq 1$$

and the Cauchy convergence criterion.

4 A General Mean Convergence Theorem

We close by establishing in Theorem 4.1 a general mean convergence theorem for a normed sum of independent random variables. Its proof uses three results which will now be stated for the convenience of the reader.

The following result is the famous de La Vallée Poussin criterion for uniform integrability; a proof of it may be found in Meyer (1996, p. 19).

Proposition 4.1. *A sequence of random variables $\{U_n, n \geq 1\}$ is uniformly integrable if and only if there exists a nondecreasing convex function G defined on $[0, \infty)$ with*

$$G(0) = 0, \lim_{u \to \infty} \frac{G(u)}{u} = \infty, \text{ and } \sup_{n \geq 1} E(G(|U_n|)) < \infty. \tag{4.1}$$

The next result is a so-called "contraction principle" and is due to Jain and Marcus (1975).

Proposition 4.2. *Let φ be a nonnegative nondecreasing convex function defined on $[0, \infty)$, let $n \geq 1$, let $\{\lambda_1, ..., \lambda_n\} \subset \mathbb{R}$, and let $Y_1, ..., Y_n$ be independent symmetric random variables. Then*

$$E\left(\varphi\left(\left|\sum_{k=1}^n \lambda_k Y_k\right|\right)\right) \leq E\left(\varphi\left((\max_{1 \leq k \leq n} |\lambda_k|)\left|\sum_{k=1}^n Y_k\right|\right)\right).$$

The next result is referred to as a "symmetrization moment inequality" and its proof may be found in Loève (1997, p. 258).

Proposition 4.3. *Let U^* be a symmetrized version of a random variable U and let m be any median of U. Then for all $p > 0$,*

$$E|U - m|^p \leq 2E|U^*|^p.$$

Theorem 4.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent $\mathscr{L}_p$ random variables for some $p \geq 1$ and let $\{b_n, n \geq 1\}$ and $\{B_n, n \geq 1\}$ be a sequence of real numbers with $0 < B_n \uparrow \infty$. Suppose that the sequence*

$$\left\{ \left| \sum_{k=1}^{n} \frac{X_k}{b_k} \right|^p, n \geq 1 \right\} \quad \textit{is uniformly integrable.} \tag{4.2}$$

(i) If $b_n = O(B_n)$ and

$$\frac{\sum_{k=1}^{n} X_k}{B_n} \xrightarrow{P} 0, \tag{4.3}$$

then

$$\frac{\sum_{k=1}^{n} X_k}{B_n} \xrightarrow{\mathscr{L}_p} 0. \tag{4.4}$$

(ii) If $b_n = o(B_n)$ and $m_n = o(B_n)$ where m_n is any median of $\sum_{k=1}^{n} X_k, n \geq 1$, then

$$\frac{\sum_{k=1}^{n} X_k}{B_n} \xrightarrow{\mathscr{L}_p} 0. \tag{4.5}$$

Proof. Let $\{X'_n, n \geq 1\}$ be an independent copy of $\{X_n, n \geq 1\}$ and consider the sequence of symmetrized random variables $\{X^*_n = X_n - X'_n, n \geq 1\}$. Now (4.2) holds with X_k replaced by X'_k, $k \geq 1$ and since

$$\left| \sum_{k=1}^{n} \frac{X^*_k}{b_k} \right|^p \leq 2^{p-1} \left(\left| \sum_{k=1}^{n} \frac{X_k}{b_k} \right|^p + \left| \sum_{k=1}^{n} \frac{X'_k}{b_k} \right|^p \right), n \geq 1,$$

the sequence

$$\left\{ \left| \sum_{k=1}^{n} \frac{X^*_k}{b_k} \right|^p, n \geq 1 \right\} \quad \text{is uniformly integrable.} \tag{4.6}$$

By (4.6) and Proposition 4.1, there exists a nondecreasing convex function G defined on $[0, \infty)$ satisfying (4.1) with $U_n = |\sum_{k=1}^{n} X^*_k / b_k|^p$, $n \geq 1$. Set $b^*_n = \max_{1 \leq k \leq n} |b_k|, n \geq 1$. Now the function $\varphi(u) = G(u^p)$, $u \geq 0$ is a nonnegative nondecreasing convex function and so by Proposition 4.2

$$\begin{aligned}
\sup_{n \geq 1} E\left(G\left(\left| \sum_{k=1}^{n} \frac{X^*_k}{b^*_n} \right|^p \right) \right) &= \sup_{n \geq 1} E\left(\varphi\left(\left| \sum_{k=1}^{n} \frac{b_k}{b^*_n} \frac{X^*_k}{b_k} \right| \right) \right) \\
&\leq \sup_{n \geq 1} E\left(\varphi\left(\left(\max_{1 \leq k \leq n} \left| \frac{b_k}{b^*_n} \right| \right) \left| \sum_{k=1}^{n} \frac{X^*_k}{b_k} \right| \right) \right) \\
&= \sup_{n \geq 1} E\left(\varphi\left(\left| \sum_{k=1}^{n} \frac{X^*_k}{b_k} \right| \right) \right) \\
&= \sup_{n \geq 1} E\left(G\left(\left| \sum_{k=1}^{n} \frac{X^*_k}{b_k} \right|^p \right) \right) < \infty
\end{aligned}$$

recalling (4.1). Then since $G(0) = 0$ and $\lim_{u\to\infty} \frac{G(u)}{u} = \infty$, again by applying Proposition 4.1 with the same function G we get that the sequence

$$\left\{ \left| \frac{\sum_{k=1}^n X_k^*}{b_n^*} \right|^p, n \geq 1 \right\} \quad \text{is uniformly integrable.} \tag{4.7}$$

We first prove part (i). It follows from $b_n = O(B_n)$ and $B_n \uparrow \infty$ that $b_n^* = O(B_n)$ and so by (4.7) the sequence

$$\left\{ \left| \frac{\sum_{k=1}^n X_k^*}{B_n} \right|^p, n \geq 1 \right\} \quad \text{is uniformly integrable.} \tag{4.8}$$

Now (4.3) also holds with X_k replaced by X_k', $k \geq 1$ and so

$$\frac{\sum_{k=1}^n X_k^*}{B_n} = \frac{\sum_{k=1}^n X_k}{B_n} - \frac{\sum_{k=1}^n X_k'}{B_n} \xrightarrow{P} 0. \tag{4.9}$$

Then by (4.8), (4.9), and the mean convergence criterion,

$$\frac{\sum_{k=1}^n X_k^*}{B_n} \xrightarrow{\mathscr{L}_p} 0. \tag{4.10}$$

Let m_n be any median of $\sum_{k=1}^n X_k$, $n \geq 1$. It follows from (4.3) that

$$m_n = o(B_n). \tag{4.11}$$

Then by Proposition 4.3, (4.10), and (4.11),

$$\begin{aligned} E\left| \frac{\sum_{k=1}^n X_k}{B_n} \right|^p &\leq 2^{p-1} \left(\frac{E\left|\sum_{k=1}^n X_k - m_n\right|^p}{B_n^p} + \frac{|m_n|^p}{B_n^p} \right) \\ &\leq 2^{p-1} \left(\frac{2E\left|\sum_{k=1}^n X_k^*\right|^p}{B_n^p} + \frac{|m_n|^p}{B_n^p} \right) \\ &\to 0 \end{aligned}$$

proving (4.4) thereby completing the proof of part (i).

Next, we prove part (ii). It follows from $b_n = o(B_n)$ and $B_n \uparrow \infty$ that $b_n^* = o(B_n)$. Then

$$E\left| \frac{\sum_{k=1}^n X_k^*}{B_n} \right|^p = \frac{b_n^{*p}}{B_n^p} E\left| \frac{\sum_{k=1}^n X_k^*}{b_n^*} \right|^p = o(1)O(1) = o(1)$$

recalling (4.7). Since $m_n = o(B_n)$ by hypothesis, the conclusion (4.5) follows by the same argument used to complete the proof of part (i). □

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