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Algebraic algorithm for maximum matching

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ela pode ser usada para uma dedicatória ou epígrafe.

Agradecimentos

Do. Or do not. There is no try.
— Mestre Yoda

Texto texto. Texto opcional.

Lista de abreviaturas

URL	Localizador Uniforme de Recursos (*Uniform Resource Locator*)
IME	Instituto de Matemática e Estatística
USP	Universidade de São Paulo

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Introduction

Something.

Chapter 1

Preliminaries

The purpose of this chapter is to introduce key concepts related to the maximum matching algorithm. The chapter covers important topics such as the definition of graph maximum matching, the Sherman-Morrison-Woodbury formula, and the Schur complement. These concepts are fundamental for understanding the algorithm's correctness and time complexity.

1.1 Graph theory

1.1.1 Graph: A **graph** G is a triple (V, E, φ) such that

(i) V is the **vertex set**;

(ii) E is the **edge set**;

(iii) $\varphi : E \to V \times V$ is a relation between each edge and a pair of vertices, called the **incidence function** of G.

Usually, it is used $V(G)$ or V_G to denote V and $E(G)$ or E_G to denote E. Also, if $e \in E(G)$ and $\varphi(e) = (u, v)$, then u and v are the **ends** of e; When the context is clear, (u, v) may be abbreviated to uv.

1.1.2 Matching: For a graph $G := (V, E, \varphi)$, a set $M \subseteq E$ is a **matching** of G if and only if no two edges in M share an end. A vertex $v \in V$ is M-covered if some edge of M incides in v, and it is said that M covers v; Otherwise, v is M-exposed. A matching M is:

- **maximal**, if there is no edge $e \in E \setminus M$ such that $M \cup \{e\}$ is a matching of G;
- **maximum**, if for every matching M' of G one has $|M| \geq |M'|$;
- **perfect**, if $2|V_G| = |M|$, i.e., every vertex of G is covered.

Now, the following problem can be introduced.

Maximum matching

Given a graph, find a maximum matching of this graph.

1.2 Linear algebra

1.2.3 Rings:

1.2.4 Fields:

Theorem 1.2.5 (Sherman-Morrison-Woodbury formula): *Let M be a $n \times n$ matrix, U and V be $n \times k$ matrices. Suppose that M is non-singular. Then*

(1) $M + UV^T$ is non-singular iff $I + V^T M^{-1} U$ is non-singular;

(2) If $M + UV^T$ is non-singular, then

$$(M + UV^T)^{-1} = M^{-1} - M^{-1}U(I + V^T M^{-1} U)^{-1} V^T M^{-1}.$$

Proof. For (1).

For (2), it suffices to verify that

$$(M + UV^T)(M^{-1} - M^{-1}U(I + V^T M^{-1} U)^{-1} V^T M^{-1}) = I.$$

Let $A := (I + V^T M^{-1} U)$, then

$$\begin{aligned}
&(M + UV^T)(M^{-1} - M^{-1}U(I + V^T M^{-1} U)^{-1} V^T M^{-1}) \\
=&(M + UV^T)(M^{-1} - M^{-1}UA^{-1}V^T M^{-1}) \\
=&(MM^{-1} - MM^{-1}UA^{-1}V^T M^{-1}) + UV^T(M^{-1} - M^{-1}UA^{-1}V^T M^{-1}) \\
=&(I - UA^{-1}V^T M^{-1}) + UV^T(M^{-1} - M^{-1}UA^{-1}V^T M^{-1}) \\
=&(I - UA^{-1}V^T M^{-1}) + (UV^T M^{-1} - UV^T M^{-1}UA^{-1}V^T M^{-1}) \\
=&(I + UV^T M^{-1}) - (UA^{-1}V^T M^{-1} + UV^T M^{-1}UA^{-1}V^T M^{-1}) \\
=&(I + UV^T M^{-1}) - U(A^{-1}V^T M^{-1} + V^T M^{-1}UA^{-1}V^T M^{-1}) \\
=&(I + UV^T M^{-1}) - U(I + V^T M^{-1}U)(A^{-1}V^T M^{-1}) \\
=&(I + UV^T M^{-1}) - UAA^{-1}V^T M^{-1} \\
=&(I + UV^T M^{-1}) - UIV^T M^{-1} = I + UV^T M^{-1} - UV^T M^{-1} = I.
\end{aligned}$$

□

Corollary 1.2.6 (): *Let M be a non-singular matrix and let N be its inverse, M' be a matrix which is identical to M except that $M'_{S,S} \neq M_{S,S}$ and $\Delta := M'_{S,S} - M_{S,S}$. If M' is non-singular, then*

$$M'^{-1} = N - N_{*,S}(I + \Delta N_{S,S})^{-1} \Delta N_{S,*}.$$

1.2.7 Schur complement: Let M be a square matrix of the form

$$M = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

where Z is a square matrix; Then, if Z is non-singular, the matrix

$$C = W - XZ^{-1}Y$$

is the *Schur complement* of Z in M.

Chapter 2

Simpler(?) algorithms

2.0.8 Indeterminates:

2.0.9 Tutte Matrix: Given a graph G and a function that maps f every edge of G to an indeterminate. Then, a tutte matrix is a matrix such that, for each $uv \in E(G)$, one has $T_{uv} = -T_{vu} = f(vu)$.

Fact 2.0.10 (Tutte matrix perfect matching condition): A graph G has a perfect matching iff T_G is non-singular.

Proof. This is direct from Pffafian. □

2.1 Basic algorithm

By fact 10, one can achieve a very direct algorithm. The idea is try to remove an edge e, if $G - e$ has a perfect matching, then this edge can be removed; Else, e belongs to a perfect matching of G. Repeat this step until only the perfect matching edges are left. Such approach achieves an $O(n^{\omega+2})$ time complexity.

2.2 Rank-2 update algorithm

One of the bottlenecks of the previous algorithm is the necessity to check after each edge if the matrix is non-singular, each of these checks is $O(n^{\omega})$. Thus, one can maintain the inverse through rank-2 updates. Achieving a time complexity of $O(n^4)$. For each edge, it suffices to check if $N_{ij} \neq -1/T_{ij}$. This condition is direct from corollary 6.

Chapter 3

Harvey’s algorithm

1. Brief of the idea;
2. Pseudo-algorithm;
3. Corretude;

Chapter 4

Extension to Maximum Matching

1. Theorem of Lovasz, the size of a maximum matching is the rank of the matrix;
2. Prove this theorem;
3. Extend graph to have a perfect matching and remove added vertices.

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