# Fundamentals of Computer Graphics

## Metric geometry

Emanuele Rodolà
rodola@di.uniroma1.it

SAPIENZA
UNIVERSITÀ DI ROMA

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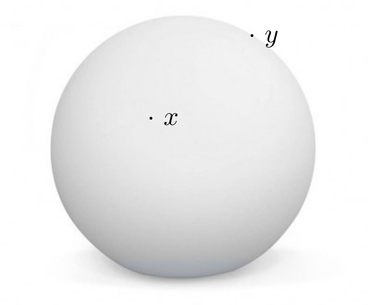
# Exercises

- Triangle mesh data structure
- Point cloud data structure

# Measuring distance

Working with curved surfaces rather than flat domains requires us to reconsider all the basic notions that we took for granted in high school geometry.

How do you measure distance between $x$ and $y$ in this picture?



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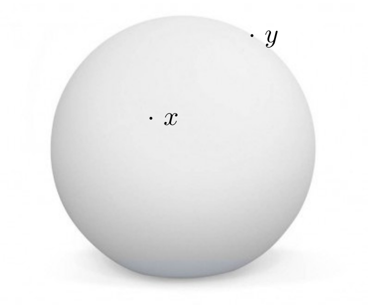
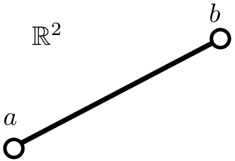



There is not a unique way!

- You can pass through the sphere with a straight line (Euclidean)
- You can walk on the surface in a "straight" path (non-Euclidean)

# Euclidean distance

The Euclidean distance measures the length of a straight line connecting two points:



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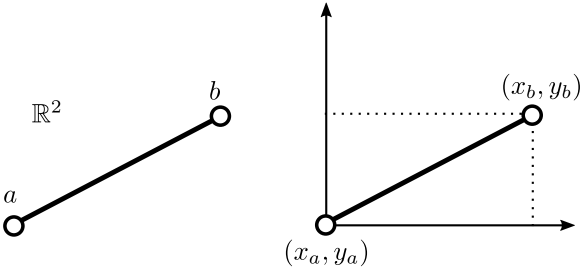

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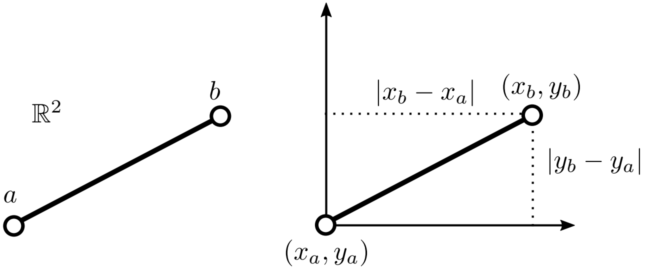

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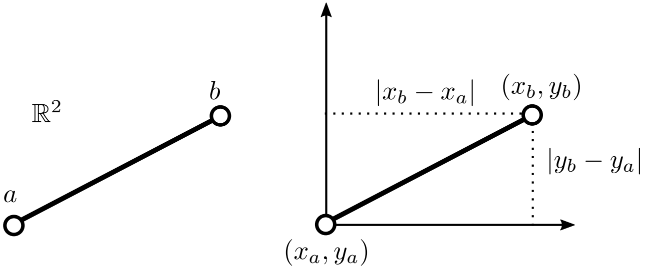


Apply Pythagoras' theorem: $d(a,b) = (|x_b - x_a|^2 + |y_b - y_a|^2)^{\frac{1}{2}}$

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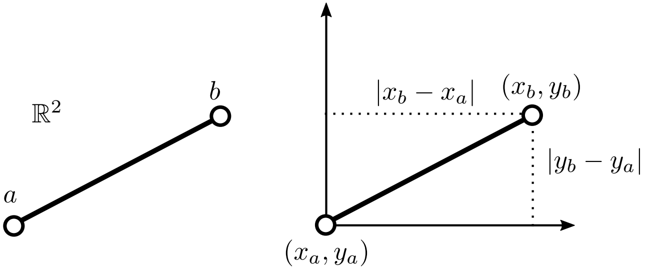


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In vector notation:

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|_2$$

where $\mathbf{a} = \begin{pmatrix} x_a \\ y_a \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_b \\ y_b \end{pmatrix}$

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One can generalize to different power coefficients $p \geq 1$:

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This definition gives us the $L_p$ distance between vectors in $\mathbb{R}^k$

Examples:

- Euclidean ($L_2$) distance between 3D points
- Manhattan ($L_1$) distance between cities in a map

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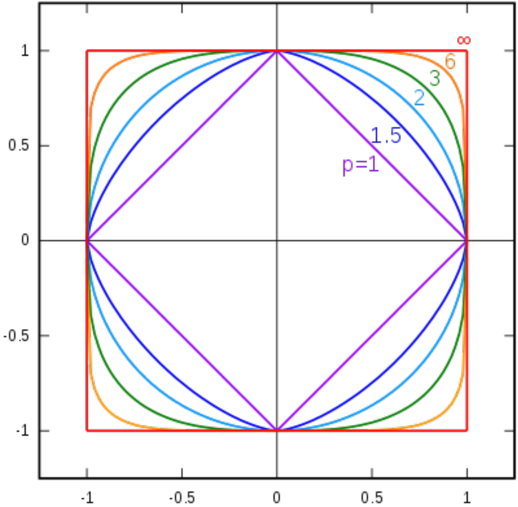
- Do all the above for points on the point cloud $P$

# $L_p$ unit balls



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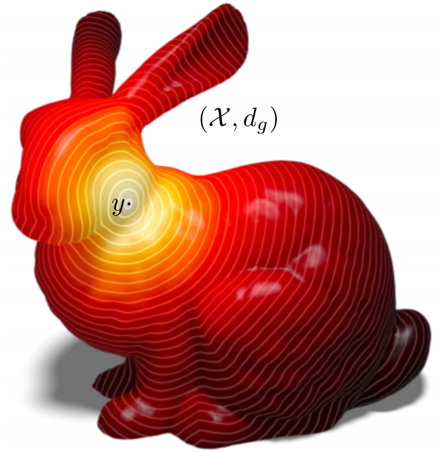
We will specify a metric space as the pair $(\mathcal{M}, d_{\mathcal{M}})$

Example:

- The sphere with Euclidean distance is $(\mathbb{S}^2, d_{L_2})$
- The sphere with geodesic distance is $(\mathbb{S}^2, d_g)$

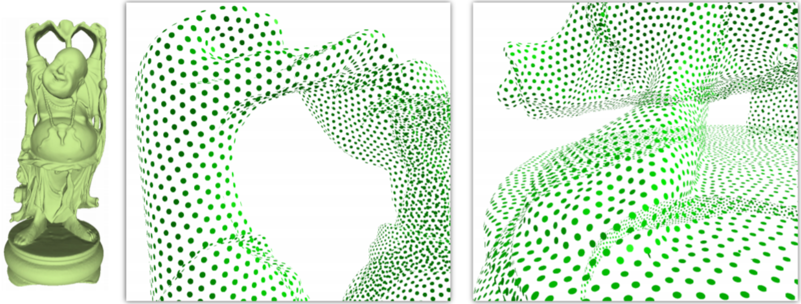
# Example: Geodesic isolines




Each isoline identifies a set of points $x \in \mathcal{X}$ at the same distance (according to $d_g$) from some reference $y \in \mathcal{X}$

# Exercise: Farthest point sampling

Implement a farthest point sampling (FPS) scheme using this algorithm:



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Implement a farthest point sampling (FPS) scheme using this algorithm:

- Fix $n$ and let $\mathcal{S}^{(0)} = \{y\}$ for some $y \in \mathcal{X}$
- Proceed recursively:
  - At step $k$, given $\mathcal{S}^{(k-1)}$, select $x \in (\mathcal{X}, d_{\mathcal{X}})$ such that

$$x = \arg\max_{x \in \mathcal{X}} d_{\mathcal{X}}(x, \mathcal{S}^{(k-1)})$$

  - Set $\mathcal{S}^{(k)} = \mathcal{S}^{(k-1)} \cup x$
  - Repeat until $k = n$
- Test with different starting points $y$
- Test with a fixed starting point and gradually increasing $n$

Use the Euclidean distance for the definition of $d_{\mathcal{X}}$.

# Exercise: Voronoi decomposition

For a given sampling $\mathcal{S}$, the associated Voronoi regions are defined as:

$$V_i(\mathcal{S}) = \{x \in \mathcal{X} \ : \ d_{\mathcal{X}}(x, x_i) < d_{\mathcal{X}}(x, x_j), x_{j \neq i} \in \mathcal{S}\}$$

- How do these regions look like?
- Implement Voronoi decomposition for meshes and point clouds using the Euclidean metric and using farthest point sampling for $\mathcal{S}$
- Visualize the Voronoi regions by assigning to each of them a random color

In order to color point clouds in Matlab, you can use the scatter3 () function

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- $\mathcal{X} = \mathcal{A} \times \mathcal{B}\,, \quad d_{\mathcal{X}}((a_1, b_1), (a_2, b_2)) = \sqrt{d_{\mathcal{A}}(a_1, a_2)^2 + d_{\mathcal{B}}(b_1, b_2)^2}$

# Ambient space and restriction

If $\mathcal{A}$ is a metric space and $\mathcal{X} \subset \mathcal{A}$, then $\mathcal{A}$ is called ambient space for $\mathcal{X}$.


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A metric on $\mathcal{X}$ can be obtained by the restriction $d_{\mathcal{X}} = d_{\mathcal{A}|\mathcal{X}}$, such that:

$$d_{\mathcal{X}}(x, y) = d_{\mathcal{A}}(x, y)$$

for all $x, y \in \mathcal{X}$

# Isometries

Let $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$ be two metric spaces.

A bijective map $f : \mathcal{M} \to \mathcal{N}$ is called an isometry if:

$$d_{\mathcal{M}}(x, y) = d_{\mathcal{N}}(f(x), f(y))$$

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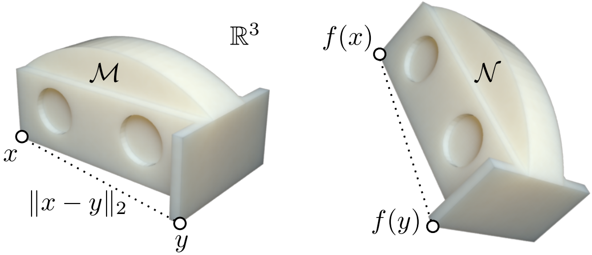

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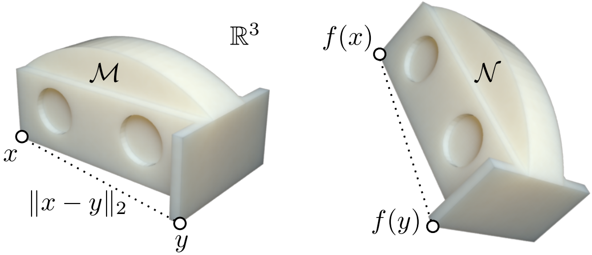
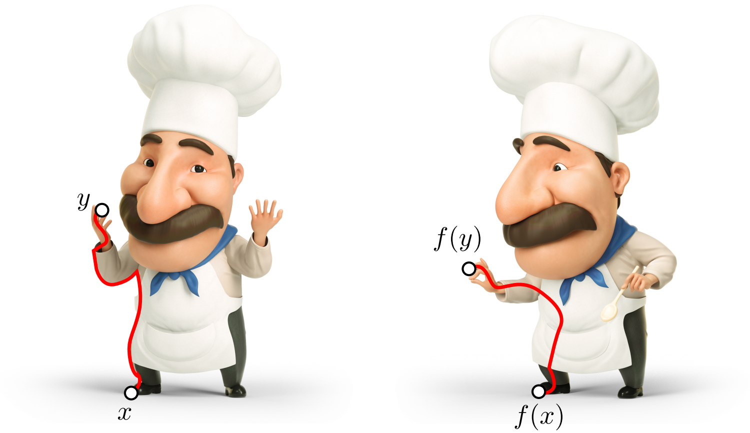



If $d_{\mathcal{M}} = \| \cdot \|_2$ and $d_{\mathcal{N}} = \| \cdot \|_2$ we say "rigid isometry"

# Example: Non-rigid “quasi”-isometries



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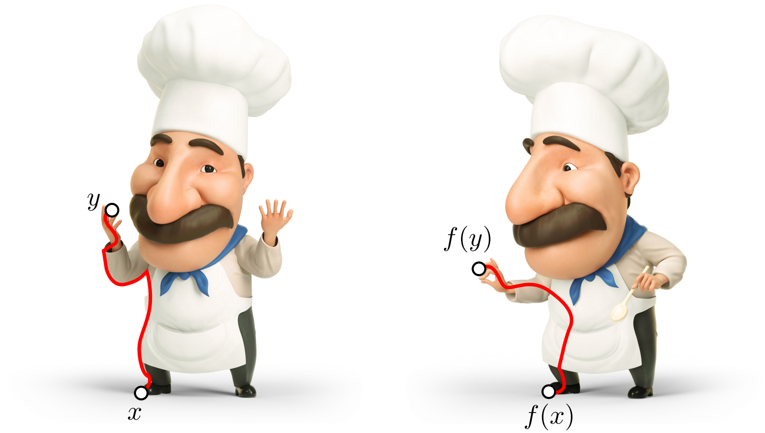


$$d_{\mathcal{M}}(x, y) \approx d_{\mathcal{N}}(f(x), f(y))$$

(here $d_{\mathcal{M}}, d_{\mathcal{N}}$ are geodesic distance functions)

# Isometry as equivalence

"Being isometric" is an equivalence relation, since it is:

- reflective ($a = a$)
- symmetric ($a = b \Rightarrow b = a$)
- transitive ($a = b \wedge b = c \Rightarrow a = c$)

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In this sense, we think of isometric shapes as being the same shape:

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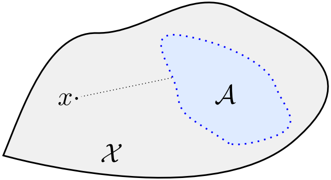
We start by defining the distance from a point $x$ to a set $\mathcal{A} \subseteq (\mathcal{X}, d_{\mathcal{X}})$:

$$\mathrm{dist}_{\mathcal{X}}(x, \mathcal{A}) = \min_{y \in \mathcal{A}} d_{\mathcal{X}}(x, y)$$



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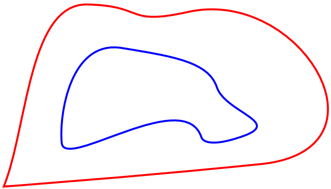

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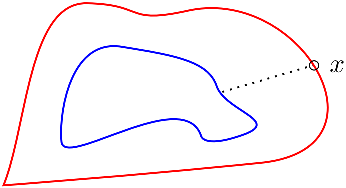

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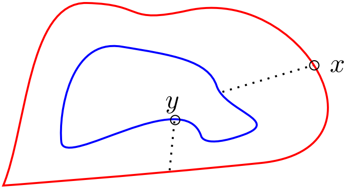

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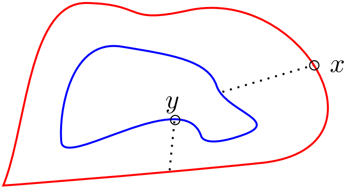

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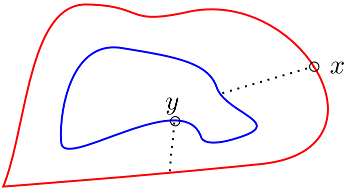

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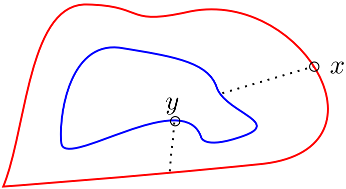
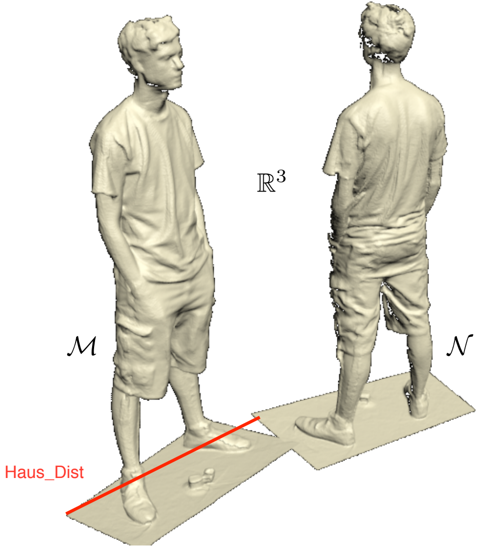



The Hausdorff distance is defined between subsets of a metric space

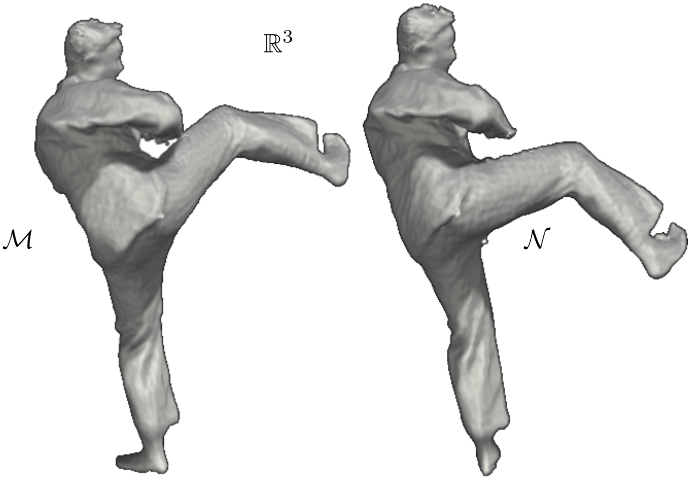
Note that perturbing one single point can make $d_{\mathcal{H}}^{\mathcal{Z}}$ arbitrarily large

# Example: Hausdorff distance, rigid case




What can we do to minimize $d_{\mathcal{H}}^{\mathbb{R}^3}(\mathcal{M}, \mathcal{N})$? overlap them

# Example: Hausdorff distance, non-rigid case

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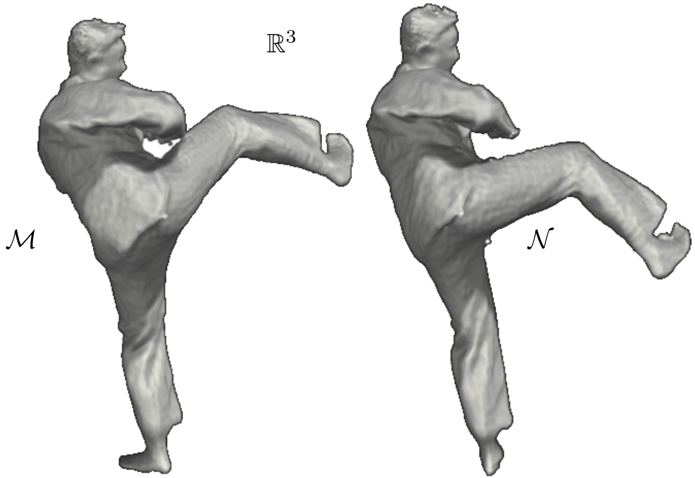


What can we do to minimize $d_{\mathcal{H}}^{\mathbb{R}^3}(\mathcal{M}, \mathcal{N})$?
The Hausdorff distance is better suited to compare rigid shapes

The Hausdorff distance can be used to compute the difference between meshes representing the same underlying surface (e.g. compare level-of-detail)

# Isometric embeddings

Hausdorff distance is between subsets of a common ambient space.

Can we extend it to general metric spaces, each with its own metric?

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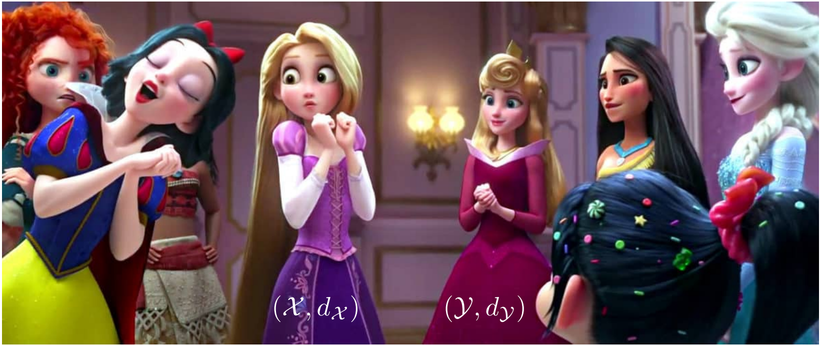
For example, we want to compare Disney princesses:



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An isometric embedding is a transformation $f : \mathcal{X} \to \mathcal{Z}$ which preserves the metric for all pairs $x, y \in \mathcal{X}$, i.e.

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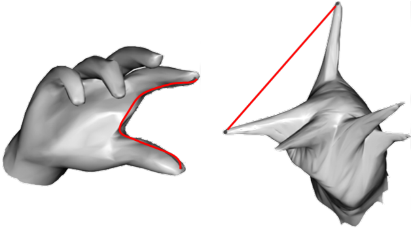
For example, take $d_{\mathcal{X}} = d_g$ and $d_{\mathcal{Z}} = \| \cdot \|_2$:

$(\mathcal{X}, d_g)$  $(\mathbb{R}^3, \| \cdot \|_2)$

# Gromov-Hausdorff distance

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- In which new metric space $(\mathcal{Z}, d_{\mathcal{Z}})$ should we embed?
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Choose the ones resulting in minimum Hausdorff distance:

$$d_{\mathcal{GH}}(\mathcal{X}, \mathcal{Y}) = \min_{\mathcal{Z}, f, g} d_{\mathcal{H}}^{\mathcal{Z}}(f(\mathcal{X}), g(\mathcal{Y}))$$

where $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are isometric embeddings

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# Gromov-Hausdorff distance

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$d_{\mathcal{GH}}($  $,$  $) = d_{\mathcal{GH}}($  $,$  $)$

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Question: What is the isometry class for the sphere $(\mathbb{S}^2, d_g)$?

# A cartographer's problem

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# A cartographer's problem

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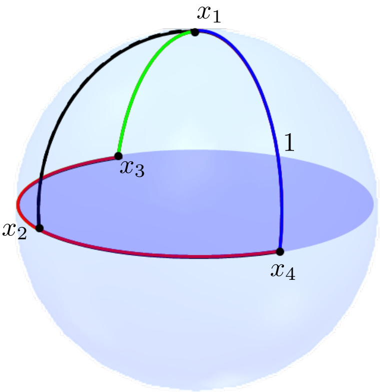
Consider the following:

An isometric embedding of $\mathbb{S}^2$ into $\mathbb{R}^2$ is not possible!

Any approximate solution introduces metric distortion

# Non-embeddability of the sphere

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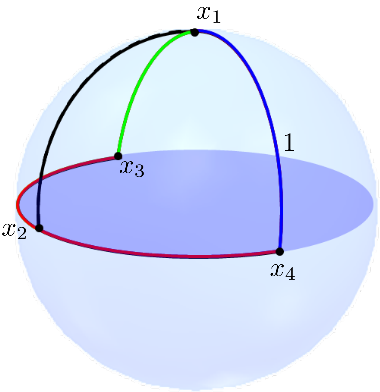


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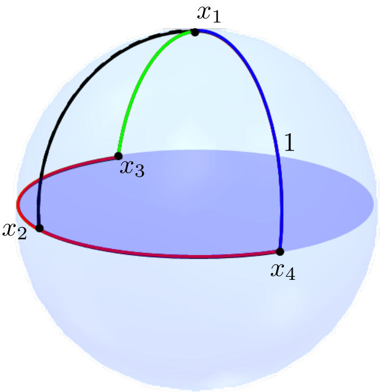


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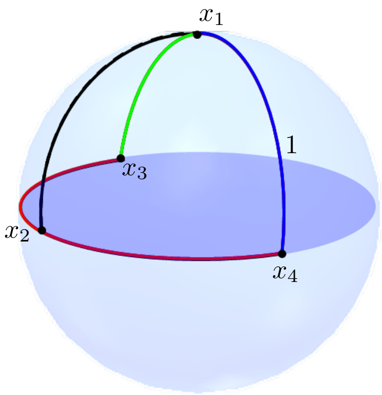


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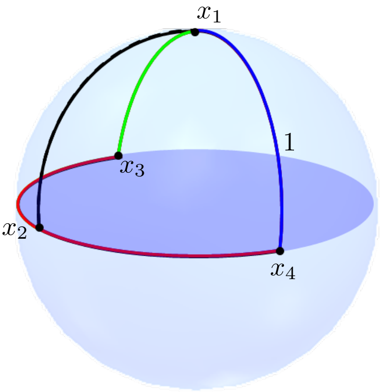


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- Consider the triangle $\Delta(x_2, x_3, x_4) \Rightarrow$ collinear!
- Then $x_1 = x_2$, which contradicts $d_g(x_1, x_2) = 1$
  $\Rightarrow$ This metric space cannot be embedded into $\mathbb{R}^k$ for any $k$

# A cartographer’s solution


# Teaser exercise: Matrix calculus

Let matrix $\mathbf{X} \in \mathbb{R}^{n \times 3}$ contain the 3D coordinates of points $x_i$ as its rows.

Consider the following expression:

$$n \sum_{i=1}^{n} \langle x_i, x_i \rangle - \sum_{i,j} \langle x_i, x_j \rangle$$

How do you write the expression above in matrix notation?

Tip: Use the trace operation, defined as $\operatorname{tr}(\mathbf{X}) = \sum_{i=1}^{n} x_{ii}$