Fundamentals of Computer Graphics

Spectral geometry

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SAPIENZA
UNIVERSITÀ DI ROMA

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Course quality questionnaire

The questionnaire is completely anonymous – your privacy is respected

Instructions:

- Disable pop-up block in your browser
- Go to https://www.uniroma1.it
- Click on **Studenti** and access **Infostud 2.0**
- Click on **Corsi di laurea**
- On the left menu, click on **Opinioni studenti**
- Enter the OPIS code and click on **Vai al questionario**

OPIS codes are:

- **LNM5IDE3**
- **CUP342HA** for the curriculum "Networks and security"

Self-adjointness of Δ

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Hence, the eigenvalues of Δ are real

Laplacian eigenfunctions

$$\Delta\phi = \lambda\phi$$

Consider distinct eigenfunctions ϕ_i, ϕ_j with $\lambda_i \neq \lambda_j$. We have:

$$\langle \phi_i, \Delta\phi_j \rangle_{\mathcal{X}} = \langle \Delta\phi_i, \phi_j \rangle_{\mathcal{X}}$$

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Therefore, the eigenfunctions ϕ of Δ are orthogonal, and can be rescaled to be orthonormal:

$$\langle \phi_i, \phi_j \rangle_{\mathcal{X}} = \delta_{ij}$$

where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise

Spectral theorem

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Now recall that the Laplacian $\Delta : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X})$ operates on a vector space of functions $\mathcal{F}(\mathcal{X})$

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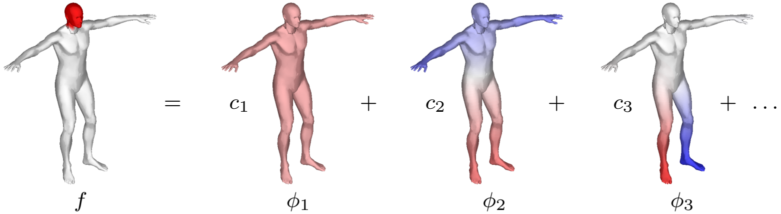

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And we consider the generalized eigenvalue problem:

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In other words, eigenfunctions have zero average

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Discrete spectrum

The eigenvalues of the Laplacian are countable and are canonically ordered non-decreasingly:

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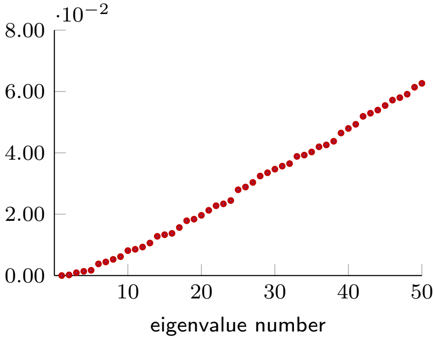

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The Laplacian eigenvalues follow Weyl's asymptotic law:

$$\lambda_i \approx \frac{\pi}{\int_{\mathcal{X}} da} i \qquad \text{for } i \to \infty$$

That is, they grow linearly with growth rate that is inversely proportional to surface area

Laplacian eigenvalues: Properties

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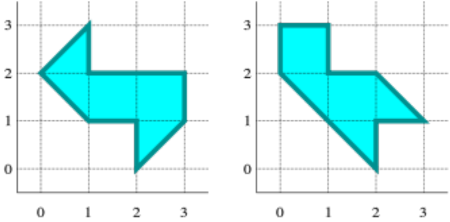
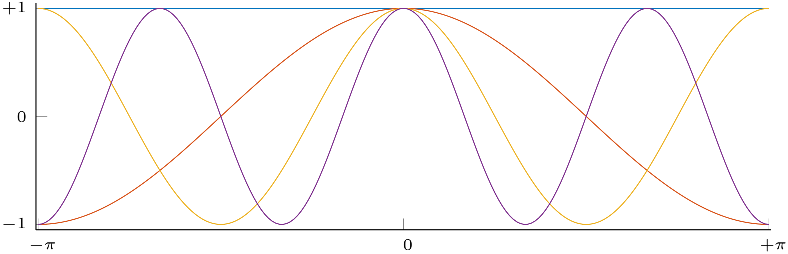
However, one "cannot hear the shape of the drum"

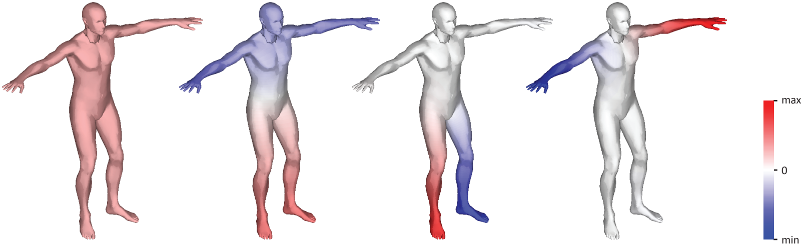


Laplacian eigenfunctions: Euclidean



First eigenfunctions of 1D Euclidean Laplacian = standard Fourier basis

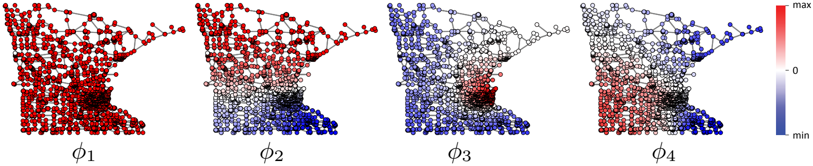
Laplacian eigenfunctions: manifold



ϕ_1 ϕ_2 ϕ_3 ϕ_4

First eigenfunctions of a manifold Laplacian

Laplacian eigenfunctions: graph

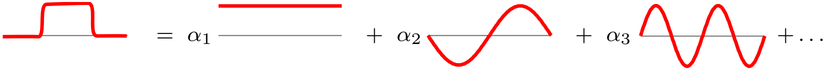


First eigenfunctions of a graph Laplacian

Fourier analysis: Euclidean space

A function $f : [-\pi, \pi] \to \mathbb{R}$ can be written as Fourier series

$$f(x) = \sum_{k \geq 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') e^{-ikx'} dx' \, e^{ikx}$$



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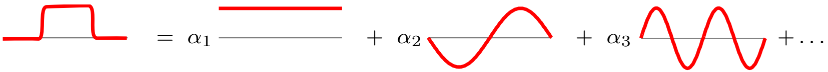

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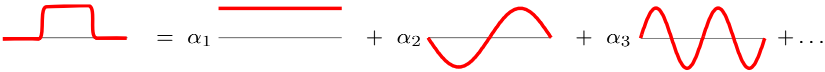
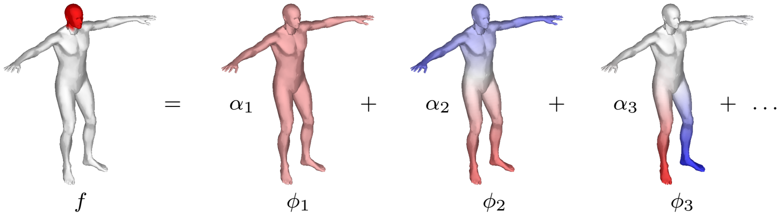


Fourier basis = Laplacian eigenfunctions: $\frac{d^2}{dx^2} e^{ikx} = k^2 e^{ikx}$

Fourier analysis: non-Euclidean space

A function $f : \mathcal{X} \to \mathbb{R}$ can be written as Fourier series

$$f(x) = \sum_{k \geq 1} \underbrace{\int_{\mathcal{X}} f(x') \phi_k(x') dx'}_{\hat{f}_k = \langle f, \phi_k \rangle_{L^2(\mathcal{X})}} \; \phi_k(x)$$



Fourier basis = Laplacian eigenfunctions: $\Delta \phi_k(x) = \lambda_k \phi_k(x)$

Teaser exercise: Maps in the Laplacian eigenbasis

Consider two isometric shapes $\mathcal{X}, \mathcal{Y}$ with Laplacian eigenfunctions $\{\phi_i\}, \{\psi_j\}$ respectively spanning the functional spaces $\mathcal{F}(\mathcal{X}), \mathcal{F}(\mathcal{Y})$.

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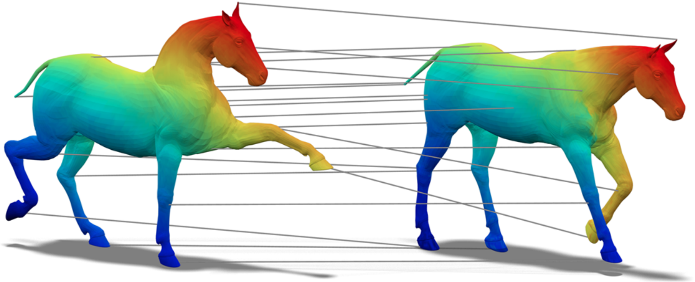
Now let $T : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y})$ be the ground-truth map.



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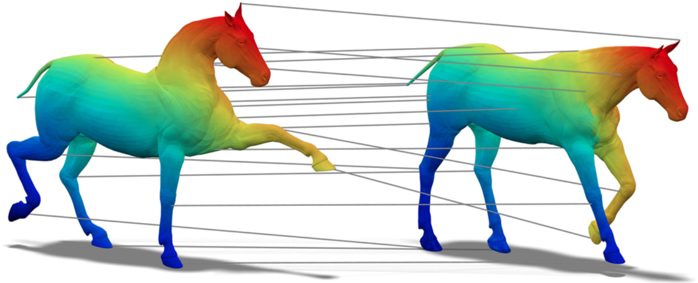

How does the matrix representation of T look like, in the Laplacian eigenbases?