Fundamentals of Computer Graphics

Euclidean embeddings

Emanuele Rodolà
rodola@di.uniroma1.it

SAPIENZA
UNIVERSITÀ DI ROMA

1st semester a.y. 2018/2019 · October 01, 2018

Exercises

- L_2 distance in $\mathbb{R}^3$
- Farthest point sampling
- Voronoi decomposition

Embeddings

We have seen isometric embeddings as maps f that preserve the metric:

$$d_{\mathcal{M}}(x, y) = d_{\mathcal{N}}(f(x), f(y))$$

And we know that the sphere cannot be embedded into $\mathbb{R}^k$ for any k

Embeddings

We have seen isometric embeddings as maps f that preserve the metric:

$$d_{\mathcal{M}}(x, y) = d_{\mathcal{N}}(f(x), f(y))$$

And we know that the sphere cannot be embedded into $\mathbb{R}^k$ for any k

Q1: Can we embed it anyway, with distortion?

Embeddings

We have seen isometric embeddings as maps f that preserve the metric:

$$d_{\mathcal{M}}(x, y) = d_{\mathcal{N}}(f(x), f(y))$$

And we know that the sphere cannot be embedded into $\mathbb{R}^k$ for any k

Q1: Can we embed it anyway, with distortion?

Further, imagine that for a given mesh, we accidentally lose its vertex coordinates; however, we still remember the length of all edges.

Embeddings

We have seen isometric embeddings as maps f that preserve the metric:

$$d_{\mathcal{M}}(x, y) = d_{\mathcal{N}}(f(x), f(y))$$

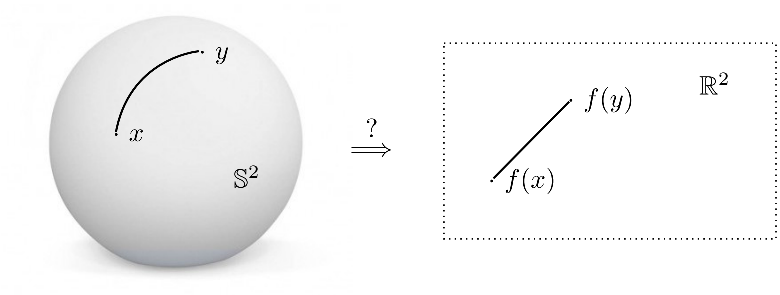
And we know that the sphere cannot be embedded into $\mathbb{R}^k$ for any k

Q1: Can we embed it anyway, with distortion?

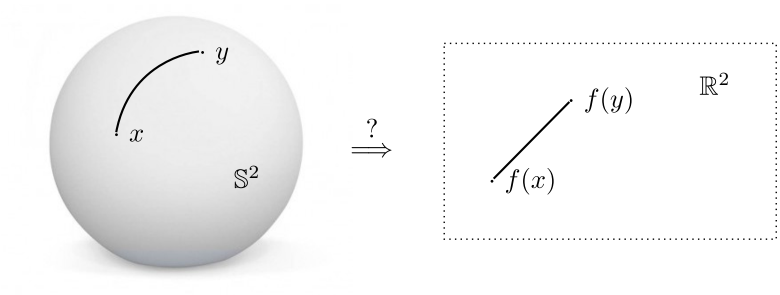
Further, imagine that for a given mesh, we accidentally lose its vertex coordinates; however, we still remember the length of all edges.

Q2: Can we recover the vertex coordinates?

Embeddings: Q1 example



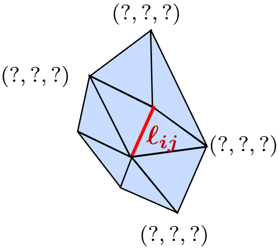
Embeddings: Q1 example



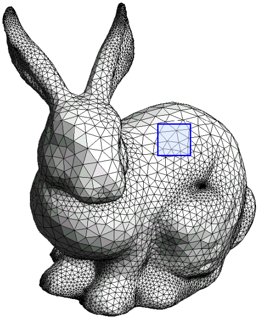
Given $d_g(x, y)$, find $f(x), f(y) \in \mathbb{R}^2$ such that :

$$d_g(x, y) \approx \|f(x) - f(y)\|_2 \text{ for all } x, y \in \mathbb{S}^2$$

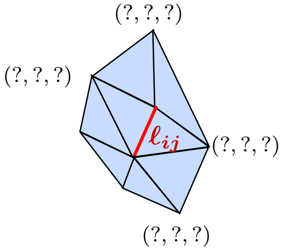
Embeddings: Q2 example



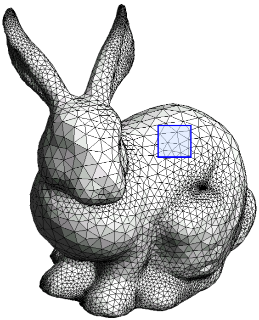
$\overset{?}{\Longrightarrow}$



Embeddings: Q2 example



$\overset{?}{\Longrightarrow}$



Given ℓ_{ij}, find $x_i, x_j \in \mathbb{R}^3$ such that:

$$\ell_{ij} \approx \|x_i - x_j\|_2 \text{ for all } i, j$$

Application: Shape modeling

Art directed nearly-isometric shape deformation: mimic the behavior of thin materials with high membrane stiffness

user-provided (initialization)

optimized (nearly-isometric to the uncrushed can)

Chern et al., "Shape from metric". TOG 37(4), 2018

Application: Shape modeling

Art directed nearly-isometric shape deformation: mimic the behavior of thin materials with high membrane stiffness

initialized with user-provided landmarks

Chern et al., "Shape from metric". TOG 37(4), 2018

Application: Visualization

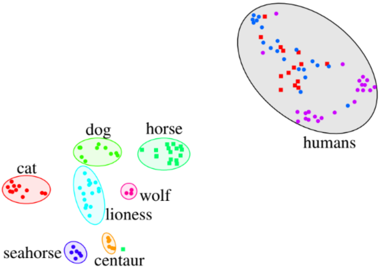
In computer vision and pattern recognition, very often we deal with high-dimensional vectors (e.g. representing images or more abstract concepts) together with some distance or similarity between them.

Application: Visualization

In computer vision and pattern recognition, very often we deal with high-dimensional vectors (e.g. representing images or more abstract concepts) together with some distance or similarity between them.

We can visualize these objects as points in a Euclidean space to get an idea of how the space looks like:

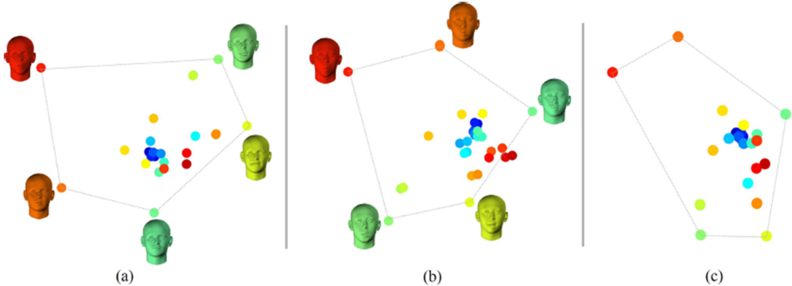
$$d_{\mathcal{GH}}(\mathcal{X}, \mathcal{Y}) \Rightarrow$$



This type of visualization is a key tool in vision and data science

Application: Alignment of shape collections

Low-dimensional embeddings are metric spaces themselves. We can then process them as we do with shapes, e.g., by looking for point-to-point alignment:



Embedding of collection $\mathcal{A}$ | Embedding of collection $\mathcal{B}$ | $\mathcal{A}$ aligned to $\mathcal{B}$

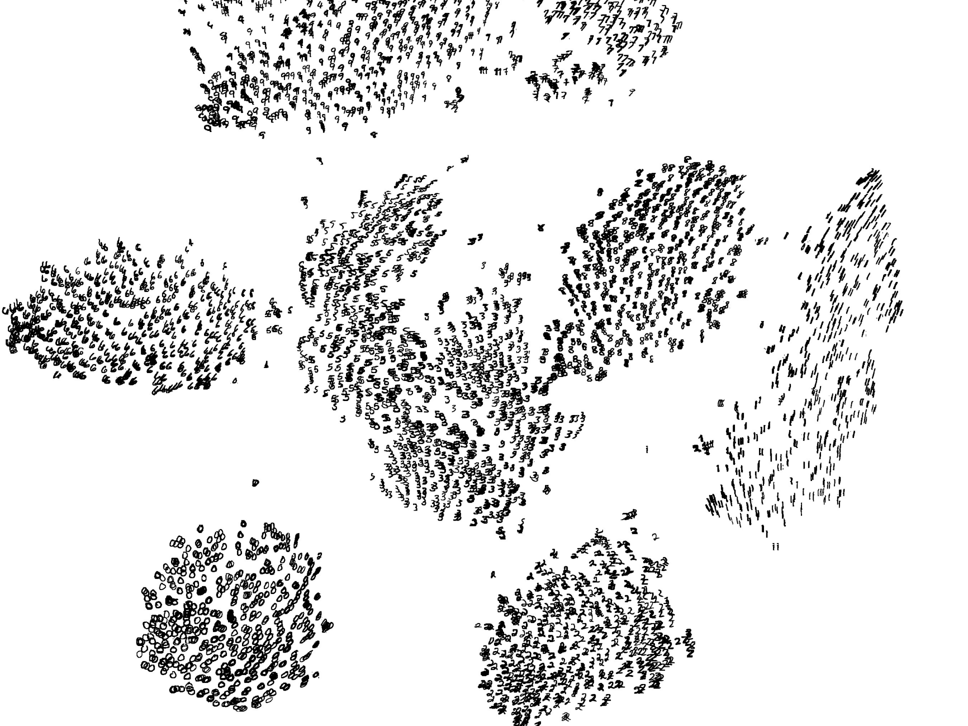
Shapira and Ben-Chen, "Cross-Collection Map Inference by Intrinsic Alignment of Shape Spaces". CGF 33(5), 2014

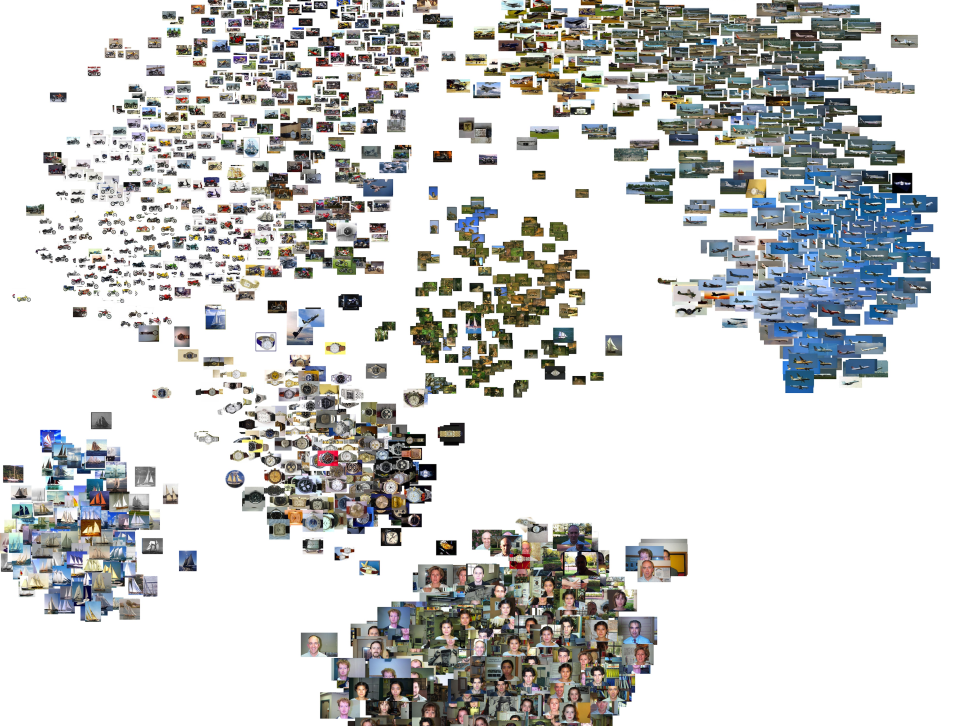
Application: Alignment of shape collections

Aligned shape collections

The shape-to-shape alignment done this way does not require computing a map between the individual shapes

Shapira and Ben-Chen, "Cross-Collection Map Inference by Intrinsic Alignment of Shape Spaces". CGF 33(5), 2014





Other applications

- Whenever we deal with data that we know or suspect having a geometric nature, and we want to visualize it

Other applications

- Whenever we deal with data that we know or suspect having a geometric nature, and we want to visualize it
- In shape modeling and shape optimization, we can design or prescribe certain properties of the desired surface in terms of distances. Optimizing for the geometry that realizes these properties amounts to solving a Euclidean embedding problem

Other applications

- Whenever we deal with data that we know or suspect having a geometric nature, and we want to visualize it

- In shape modeling and shape optimization, we can design or prescribe certain properties of the desired surface in terms of distances. Optimizing for the geometry that realizes these properties amounts to solving a Euclidean embedding problem

  Example: shape interpolation and extrapolation

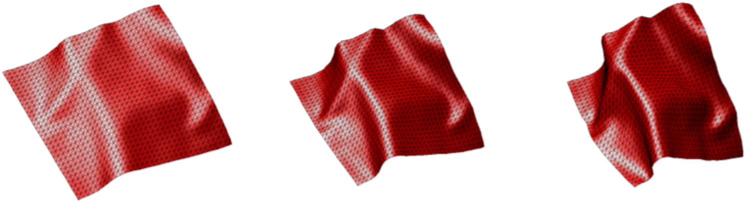


Figure: Corman et al., "Functional Characterization of Intrinsic and Extrinsic Geometry". TOG 36, 2017

Metric distortion

We need to specify a criterion for distortion that we want to minimize.

Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and $f : \mathcal{X} \to \mathcal{Y}$ an arbitrary (even not continuous) map.

Metric distortion

We need to specify a criterion for distortion that we want to minimize.

Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and $f : \mathcal{X} \to \mathcal{Y}$ an arbitrary (even not continuous) map.

The distortion induced by f on the metric can be quantified as follows:

- Measure the relative change between metrics ("dilation"):

$$\frac{d_{\mathcal{Y}}(f(x_1), f(x_2))}{d_{\mathcal{X}}(x_1, x_2)} \approx 1 \quad \text{for all } x_1, x_2 \in \mathcal{X}$$

Metric distortion

We need to specify a criterion for distortion that we want to minimize.

Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and $f : \mathcal{X} \to \mathcal{Y}$ an arbitrary (even not continuous) map.

The distortion induced by f on the metric can be quantified as follows:

- Measure the relative change between metrics ("dilation"):

$$\frac{d_{\mathcal{Y}}(f(x_1), f(x_2))}{d_{\mathcal{X}}(x_1, x_2)} \approx 1 \quad \text{for all } x_1, x_2 \in \mathcal{X}$$

- Measure the absolute distortion of the metric:

$$|d_{\mathcal{Y}}(f(x_1), f(x_2)) - d_{\mathcal{X}}(x_1, x_2)| \approx 0 \quad \text{for all } x_1, x_2 \in \mathcal{X}$$

Metric distortion

We need to specify a criterion for distortion that we want to minimize.

Let $(\mathcal{X}, d_\mathcal{X})$ and $(\mathcal{Y}, d_\mathcal{Y})$ be metric spaces and $f : \mathcal{X} \to \mathcal{Y}$ an arbitrary (even not continuous) map.

The distortion induced by f on the metric can be quantified as follows:

- Measure the relative change between metrics ("dilation"):

$$\frac{d_\mathcal{Y}(f(x_1), f(x_2))}{d_\mathcal{X}(x_1, x_2)} \approx 1 \quad \text{for all } x_1, x_2 \in \mathcal{X}$$

- Measure the absolute distortion of the metric:

$$|d_\mathcal{Y}(f(x_1), f(x_2)) - d_\mathcal{X}(x_1, x_2)| \approx 0 \quad \text{for all } x_1, x_2 \in \mathcal{X}$$

A minimum-distortion embedding is the f minimizing the measures above for all $x_1, x_2 \in \mathcal{X}$

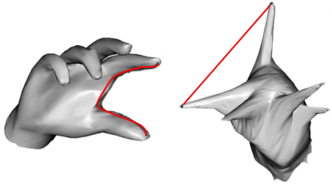
Canonical forms

If $d_{\mathcal{Y}} = \| \cdot \|_2$ we call the embedded shape (the image under f) the canonical form of $\mathcal{X}$

Canonical forms

If $d_{\mathcal{Y}} = \| \cdot \|_2$ we call the embedded shape (the image under f) the canonical form of $\mathcal{X}$

Example:

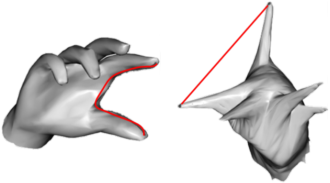


$(\mathcal{X}, d_{\mathcal{X}})$ $\quad$ $(f(\mathcal{X}), \| \cdot \|_2)$

Canonical forms

If $d_{\mathcal{Y}} = \| \cdot \|_2$ we call the embedded shape (the image under f) the canonical form of $\mathcal{X}$

Example:

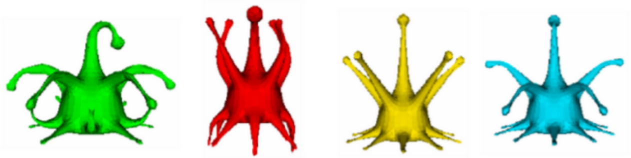


$(\mathcal{X}, d_{\mathcal{X}})$ $\quad$ $(f(\mathcal{X}), \| \cdot \|_2)$

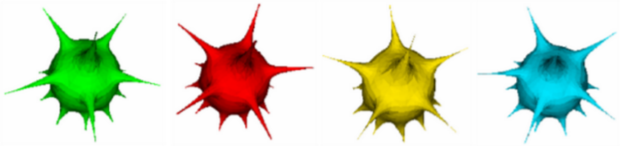
Note that in general the canonical form $f(\mathcal{X})$ is an approximation, since zero-distortion is not always achievable (Q: when is it achievable?)

Canonical forms

A canonical form defines an isometry class (equivalence class of shapes up to an isometry) in $\mathbb{R}^k$. These correspond to the rigid isometries (rotations, translations, reflections).



near-isometric deformations of a shape



canonical forms

We are reducing intrinsic isometries into extrinsic isometries

Stress minimization

As a global measure of distortion, we consider a quadratic stress

$$f = \arg \min_{f:\mathcal{X}\to\mathbb{R}^k} \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - \|f(x_i) - f(x_j)\|_2|^2$$

Stress minimization

As a global measure of distortion, we consider a quadratic stress

$$f = \arg \min_{f:\mathcal{X}\to\mathbb{R}^k} \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - \|f(x_i) - f(x_j)\|_2|^2$$

In matrix notation, define $\mathbf{z}_i = f(x_i)$ and arrange these vectors into a $n \times k$ matrix $\mathbf{Z}$.

Then we can consider the equivalent problem:

$$\mathbf{Z}^* = \arg \min_{\mathbf{Z}\in\mathbb{R}^{n\times k}} \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2$$

where $d_{ij}(\mathbf{Z}) = \|\mathbf{z}_i - \mathbf{z}_j\|_2$

Stress minimization

As a global measure of distortion, we consider a quadratic stress

$$f = \arg \min_{f:\mathcal{X}\to\mathbb{R}^k} \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - \|f(x_i) - f(x_j)\|_2|^2$$

In matrix notation, define $\mathbf{z}_i = f(x_i)$ and arrange these vectors into a $n \times k$ matrix $\mathbf{Z}$.

Then we can consider the equivalent problem:

$$\mathbf{Z}^* = \arg \min_{\mathbf{Z}\in\mathbb{R}^{n\times k}} \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2$$

where $d_{ij}(\mathbf{Z}) = \|\mathbf{z}_i - \mathbf{z}_j\|_2$

For any given configuration $\mathbf{Z}$, the stress measures how well that configuration matches the data. We look for the configuration of minimum stress.

Stress minimization

As a global measure of distortion, we consider a quadratic stress

$$f = \arg \min_{f:\mathcal{X}\to\mathbb{R}^k} \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - \|f(x_i) - f(x_j)\|_2|^2$$

In matrix notation, define $\mathbf{z}_i = f(x_i)$ and arrange these vectors into a $n \times k$ matrix $\mathbf{Z}$.

Then we can consider the equivalent problem:

$$\mathbf{Z}^* = \arg \min_{\mathbf{Z}\in\mathbb{R}^{n\times k}} \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2$$

where $d_{ij}(\mathbf{Z}) = \|\mathbf{z}_i - \mathbf{z}_j\|_2$

For any given configuration $\mathbf{Z}$, the stress measures how well that configuration matches the data. We look for the configuration of minimum stress.

This minimization problem does not have a unique solution.
In fact, applying any Euclidean isometry to the optimizer $\mathbf{Z}^*$ will not change the stress.

Multidimensional scaling

Why "stress"?

Problems of this sort started appearing in psychology several decades ago, and are usually called multidimensional scaling (MDS) problems.

> Empirical procedures of several diverse kinds have this in common: they start with a fixed set of entities and determine, for every pair of these, a number reflecting how closely the two entities are related psychologically. The nature of the psychological relation depends upon the nature of the entities. If the entities are all stimuli or all responses, we are inclined to think of the relation as one of similarity. A somewhat more objective (though less intuitive) characterization of such a relation, perhaps, is that of substitutability. The statement that stimulus A is more similar to B than to C, for example, could be interpreted to say that the psychological (or behavioral) consequences are greater when C, rather than B, is substituted for A. From this standpoint a natural procedure for determining similarities of stimuli or responses is by recording substitution errors during identification learning [2, 7, 12, 14, 17, 18]. In addition, though, disjunctive reaction time and sorting time have also been proposed as measures of psychological similarity [20]. Finally, of course, individuals have sometimes been instructed simply to rate each pair of stimuli, directly, on a scale of apparent similarity [1, 6]. The notion of similarity is not necessarily restricted to stimuli or responses (in the narrow sense of these words), however. Serviceable measures of similarity may also be found for concepts, attitudes, personality structures, or even social institutions, political systems, and the like.

Shepard, "The analysis of proximities: Multidimensional scaling with an unknown distance function". Psychometrika 27(2), 1962

Quadratic stress in matrix form

$$\sigma(\mathbf{Z}) = \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2$$

We want to rewrite $\sigma(\mathbf{Z})$ in matrix notation. This way, everything becomes easier to read and manipulate.

Quadratic stress in matrix form

$$\sigma(\mathbf{Z}) = \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2$$

This can be rewritten as:

$$\begin{aligned}\sigma(\mathbf{Z}) &= \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2 \\ &= \sum_{i>j} \underbrace{d_{ij}^2(\mathbf{Z})}_{\text{Term 1}} - \underbrace{2d_{ij}(\mathbf{Z})d_{\mathcal{X}}(x_i, x_j)}_{\text{Term 2}} + d_{\mathcal{X}}^2(x_i, x_j)\end{aligned}$$

We want to rewrite $\sigma(\mathbf{Z})$ in matrix notation. This way, everything becomes easier to read and manipulate.

Quadratic stress: Term 1

$$\sum_{i>j} d_{ij}^2(\mathbf{Z}) = \sum_{i>j} \|\mathbf{z}_i - \mathbf{z}_j\|_2^2$$

Quadratic stress: Term 1

$$\begin{aligned}\sum_{i>j} d_{ij}^2(\mathbf{Z}) &= \sum_{i>j} \|\mathbf{z}_i - \mathbf{z}_j\|_2^2 \\ &= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d - \mathbf{z}_j^d)^2\end{aligned}$$

Quadratic stress: Term 1

$$
\begin{aligned}
\sum_{i>j} d_{ij}^2(\mathbf{Z}) &= \sum_{i>j} \|\mathbf{z}_i - \mathbf{z}_j\|_2^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d - \mathbf{z}_j^d)^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d)^2 - 2\mathbf{z}_i^d \mathbf{z}_j^d + (\mathbf{z}_j^d)^2
\end{aligned}
$$

Quadratic stress: Term 1

$$\begin{aligned}
\sum_{i>j} d_{ij}^2(\mathbf{Z}) &= \sum_{i>j} \|\mathbf{z}_i - \mathbf{z}_j\|_2^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d - \mathbf{z}_j^d)^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d)^2 - 2\mathbf{z}_i^d \mathbf{z}_j^d + (\mathbf{z}_j^d)^2 \\
&= \sum_{i>j} \langle \mathbf{z}_j, \mathbf{z}_j \rangle + \langle \mathbf{z}_i, \mathbf{z}_i \rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j \rangle
\end{aligned}$$

Quadratic stress: Term 1

$$\begin{aligned}
\sum_{i>j} d_{ij}^2(\mathbf{Z}) &= \sum_{i>j} \|\mathbf{z}_i - \mathbf{z}_j\|_2^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d - \mathbf{z}_j^d)^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d)^2 - 2\mathbf{z}_i^d \mathbf{z}_j^d + (\mathbf{z}_j^d)^2 \\
&= \sum_{i>j} \langle \mathbf{z}_j, \mathbf{z}_j \rangle + \langle \mathbf{z}_i, \mathbf{z}_i \rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j \rangle \\
&= \sum_{i>j} \langle \mathbf{z}_j, \mathbf{z}_j \rangle + \langle \mathbf{z}_i, \mathbf{z}_i \rangle - 2\sum_{i>j} \langle \mathbf{z}_i, \mathbf{z}_j \rangle
\end{aligned}$$

Quadratic stress: Term 1

$$
\begin{aligned}
\sum_{i>j} d_{ij}^2(\mathbf{Z}) &= \sum_{i>j} \|\mathbf{z}_i - \mathbf{z}_j\|_2^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d - \mathbf{z}_j^d)^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d)^2 - 2\mathbf{z}_i^d \mathbf{z}_j^d + (\mathbf{z}_j^d)^2 \\
&= \sum_{i>j} \langle \mathbf{z}_j, \mathbf{z}_j \rangle + \langle \mathbf{z}_i, \mathbf{z}_i \rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j \rangle \\
&= \sum_{i>j} \langle \mathbf{z}_j, \mathbf{z}_j \rangle + \langle \mathbf{z}_i, \mathbf{z}_i \rangle - 2 \sum_{i>j} \langle \mathbf{z}_i, \mathbf{z}_j \rangle \\
&= (n-1) \sum_{i=1}^{n} \langle \mathbf{z}_i, \mathbf{z}_i \rangle - \left(\sum_{i,j} \langle \mathbf{z}_i, \mathbf{z}_j \rangle - \sum_{i=1}^{n} \langle \mathbf{z}_i, \mathbf{z}_i \rangle \right)
\end{aligned}
$$

Quadratic stress: Term 1

$$
\begin{aligned}
\sum_{i>j} d_{ij}^2(\mathbf{Z}) &= \sum_{i>j} \|\mathbf{z}_i - \mathbf{z}_j\|_2^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d - \mathbf{z}_j^d)^2 \\
&= \sum_{i>j} \sum_{d=1}^{k} (\mathbf{z}_i^d)^2 - 2\mathbf{z}_i^d \mathbf{z}_j^d + (\mathbf{z}_j^d)^2 \\
&= \sum_{i>j} \langle \mathbf{z}_j, \mathbf{z}_j \rangle + \langle \mathbf{z}_i, \mathbf{z}_i \rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j \rangle \\
&= \sum_{i>j} \langle \mathbf{z}_j, \mathbf{z}_j \rangle + \langle \mathbf{z}_i, \mathbf{z}_i \rangle - 2\sum_{i>j} \langle \mathbf{z}_i, \mathbf{z}_j \rangle \\
&= (n-1)\sum_{i=1}^{n} \langle \mathbf{z}_i, \mathbf{z}_i \rangle - \left(\sum_{i,j} \langle \mathbf{z}_i, \mathbf{z}_j \rangle - \sum_{i=1}^{n} \langle \mathbf{z}_i, \mathbf{z}_i \rangle \right) \\
&= n\sum_{i=1}^{n} \langle \mathbf{z}_i, \mathbf{z}_i \rangle - \sum_{i,j} \langle \mathbf{z}_i, \mathbf{z}_j \rangle
\end{aligned}
$$

Quadratic stress: Term 1

$$\sum_{i>j} d_{ij}^2(\mathbf{Z}) = n\sum_{i=1}^{n}\langle \mathbf{z}_i, \mathbf{z}_i\rangle - \sum_{i,j}\langle \mathbf{z}_i, \mathbf{z}_j\rangle$$

Quadratic stress: Term 1

$$\begin{aligned}\sum_{i>j} d_{ij}^2(\mathbf{Z}) &= n\sum_{i=1}^{n}\langle \mathbf{z}_i, \mathbf{z}_i\rangle - \sum_{i,j}\langle \mathbf{z}_i, \mathbf{z}_j\rangle \\ &\overset{(a)}{=} n\operatorname{tr}(\mathbf{Z}\mathbf{Z}^\top) - \operatorname{tr}(\mathbf{1}\mathbf{1}^\top\mathbf{Z}\mathbf{Z}^\top)\end{aligned}$$

(a) See teaser exercise from Sep 27 lecture

Quadratic stress: Term 1

$$\sum_{i>j} d_{ij}^2(\mathbf{Z}) = n\sum_{i=1}^{n}\langle \mathbf{z}_i, \mathbf{z}_i\rangle - \sum_{i,j}\langle \mathbf{z}_i, \mathbf{z}_j\rangle$$
$$\overset{\text{(a)}}{=} n\mathrm{tr}(\mathbf{Z}\mathbf{Z}^\top) - \mathrm{tr}(\mathbf{1}\mathbf{1}^\top\mathbf{Z}\mathbf{Z}^\top)$$
$$\overset{\text{(b)}}{=} \mathrm{tr}(\mathbf{V}\mathbf{Z}\mathbf{Z}^\top)$$

where $\mathbf{V} = n\mathbf{I} - \mathbf{1}\mathbf{1}^\top$.

(a) See teaser exercise from Sep 27 lecture

(b) By linearity of the trace

Quadratic stress: Term 1

$$\begin{aligned}
\sum_{i>j} d_{ij}^2(\mathbf{Z}) &= n\sum_{i=1}^{n}\langle \mathbf{z}_i, \mathbf{z}_i\rangle - \sum_{i,j}\langle \mathbf{z}_i, \mathbf{z}_j\rangle \\
&\overset{\text{(a)}}{=} n\operatorname{tr}(\mathbf{Z}\mathbf{Z}^\top) - \operatorname{tr}(\mathbf{1}\mathbf{1}^\top\mathbf{Z}\mathbf{Z}^\top) \\
&\overset{\text{(b)}}{=} \operatorname{tr}(\mathbf{V}\mathbf{Z}\mathbf{Z}^\top) \\
&\overset{\text{(c)}}{=} \operatorname{tr}(\mathbf{Z}^\top\mathbf{V}\mathbf{Z})
\end{aligned}$$

where $\mathbf{V} = n\mathbf{I} - \mathbf{1}\mathbf{1}^\top$.

(a) See teaser exercise from Sep 27 lecture

(b) By linearity of the trace

(c) Because $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$

Exercise: Prove that $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$

Quadratic stress: Term 2

$$\begin{aligned}\sigma(\mathbf{Z}) &= \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2 \\ &= \sum_{i>j} \underbrace{d_{ij}^2(\mathbf{Z})}_{\mathrm{tr}(\mathbf{Z}^\top \mathbf{V} \mathbf{Z})} - \underbrace{2 d_{ij}(\mathbf{Z}) d_{\mathcal{X}}(x_i, x_j)}_{\text{Term 2}} + d_{\mathcal{X}}^2(x_i, x_j)\end{aligned}$$

Quadratic stress: Term 2

$$\begin{aligned}\sigma(\mathbf{Z}) &= \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2 \\ &= \sum_{i>j} \underbrace{d_{ij}^2(\mathbf{Z})}_{\mathrm{tr}(\mathbf{Z}^\top \mathbf{V}\mathbf{Z})} - \underbrace{2d_{ij}(\mathbf{Z})d_{\mathcal{X}}(x_i, x_j)}_{\text{Term 2}} + d_{\mathcal{X}}^2(x_i, x_j)\end{aligned}$$

$$\sum_{i>j} d_{ij}(\mathbf{Z})d_{\mathcal{X}}(x_i, x_j) = \sum_{i>j} d_{\mathcal{X}}(x_i, x_j) d_{ij}^{-1}(\mathbf{Z}) d_{ij}^2(\mathbf{Z})$$

Quadratic stress: Term 2

$$
\begin{aligned}
\sigma(\mathbf{Z}) &= \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2 \\
&= \sum_{i>j} \underbrace{d_{ij}^2(\mathbf{Z})}_{\mathrm{tr}(\mathbf{Z}^\top \mathbf{V}\mathbf{Z})} - \underbrace{2d_{ij}(\mathbf{Z})d_{\mathcal{X}}(x_i, x_j)}_{\text{Term 2}} + d_{\mathcal{X}}^2(x_i, x_j)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i>j} d_{ij}(\mathbf{Z})d_{\mathcal{X}}(x_i, x_j) &= \sum_{i>j} d_{\mathcal{X}}(x_i, x_j) d_{ij}^{-1}(\mathbf{Z}) d_{ij}^2(\mathbf{Z}) \\
&= \sum_{i>j} d_{\mathcal{X}}(x_i, x_j) d_{ij}^{-1}(\mathbf{Z}) \sum_{d=1}^{k} (\mathbf{z}_i^d - \mathbf{z}_j^d)^2
\end{aligned}
$$

Quadratic stress: Term 2

$$\sigma(\mathbf{Z}) = \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2$$
$$= \sum_{i>j} \underbrace{d_{ij}^2(\mathbf{Z})}_{\mathrm{tr}(\mathbf{Z}^\top \mathbf{V}\mathbf{Z})} - \underbrace{2 d_{ij}(\mathbf{Z}) d_{\mathcal{X}}(x_i, x_j)}_{\text{Term 2}} + d_{\mathcal{X}}^2(x_i, x_j)$$

$$\sum_{i>j} d_{ij}(\mathbf{Z}) d_{\mathcal{X}}(x_i, x_j) = \sum_{i>j} d_{\mathcal{X}}(x_i, x_j) d_{ij}^{-1}(\mathbf{Z}) d_{ij}^2(\mathbf{Z})$$
$$= \sum_{i>j} d_{\mathcal{X}}(x_i, x_j) d_{ij}^{-1}(\mathbf{Z}) \sum_{d=1}^{k} (\mathbf{z}_i^d - \mathbf{z}_j^d)^2$$
$$= \sum_{i>j} \underbrace{d_{\mathcal{X}}(x_i, x_j) d_{ij}^{-1}(\mathbf{Z})}_{a_{ij} = a_{ji}} (\langle \mathbf{z}_i, \mathbf{z}_i \rangle + \langle \mathbf{z}_j, \mathbf{z}_j \rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j \rangle)$$

Quadratic stress: Term 2

$$\sum_{i>j} d_{ij}(\mathbf{Z}) d_{\mathcal{X}}(x_i, x_j) = \sum_{i>j} a_{ij}(\langle \mathbf{z}_i, \mathbf{z}_i \rangle + \langle \mathbf{z}_j, \mathbf{z}_j \rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j \rangle)$$

Quadratic stress: Term 2

$$\begin{aligned}\sum_{i>j} d_{ij}(\mathbf{Z}) d_{\mathcal{X}}(x_i, x_j) &= \sum_{i>j} a_{ij}(\langle \mathbf{z}_i, \mathbf{z}_i \rangle + \langle \mathbf{z}_j, \mathbf{z}_j \rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j \rangle) \\ &= \sum_{i,j} a_{ij}(\langle \mathbf{z}_i, \mathbf{z}_i \rangle - \langle \mathbf{z}_i, \mathbf{z}_j \rangle)\end{aligned}$$

Quadratic stress: Term 2

$$\begin{aligned}
\sum_{i>j} d_{ij}(\mathbf{Z}) d_{\mathcal{X}}(x_i, x_j) &= \sum_{i>j} a_{ij}(\langle \mathbf{z}_i, \mathbf{z}_i \rangle + \langle \mathbf{z}_j, \mathbf{z}_j \rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j \rangle) \\
&= \sum_{i,j} a_{ij}(\langle \mathbf{z}_i, \mathbf{z}_i \rangle - \langle \mathbf{z}_i, \mathbf{z}_j \rangle) \\
&= \mathrm{tr}(\mathbf{B}\mathbf{Z}\mathbf{Z}^\top) \\
&= \mathrm{tr}(\mathbf{Z}^\top \mathbf{B}\mathbf{Z})
\end{aligned}$$

where $b_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j \\ -\sum_{\ell \neq i} b_{i\ell} & \text{if } i = j \end{cases}$

Quadratic stress: Term 2

$$
\begin{aligned}
\sum_{i>j} d_{ij}(\mathbf{Z}) d_{\mathcal{X}}(x_i, x_j) &= \sum_{i>j} a_{ij}(\langle \mathbf{z}_i, \mathbf{z}_i \rangle + \langle \mathbf{z}_j, \mathbf{z}_j \rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j \rangle) \\
&= \sum_{i,j} a_{ij}(\langle \mathbf{z}_i, \mathbf{z}_i \rangle - \langle \mathbf{z}_i, \mathbf{z}_j \rangle) \\
&= \mathrm{tr}(\mathbf{B}\mathbf{Z}\mathbf{Z}^\top) \\
&= \mathrm{tr}(\mathbf{Z}^\top \mathbf{B}\mathbf{Z})
\end{aligned}
$$

where $b_{ij} = \begin{cases} -d_{\mathcal{X}}(x_i, x_j) d_{ij}^{-1}(\mathbf{Z}) & \text{if } i \neq j \\ -\sum_{\ell \neq i} b_{i\ell} & \text{if } i = j \end{cases}$

Quadratic stress: Term 2

$$\begin{aligned}\sum_{i>j} d_{ij}(\mathbf{Z}) d_{\mathcal{X}}(x_i, x_j) &= \sum_{i>j} a_{ij}(\langle \mathbf{z}_i, \mathbf{z}_i\rangle + \langle \mathbf{z}_j, \mathbf{z}_j\rangle - 2\langle \mathbf{z}_i, \mathbf{z}_j\rangle) \\ &= \sum_{i,j} a_{ij}(\langle \mathbf{z}_i, \mathbf{z}_i\rangle - \langle \mathbf{z}_i, \mathbf{z}_j\rangle) \\ &= \mathrm{tr}(\mathbf{B}\mathbf{Z}\mathbf{Z}^\top) \\ &= \mathrm{tr}(\mathbf{Z}^\top\mathbf{B}\mathbf{Z})\end{aligned}$$

where $\mathbf{B} = -\mathbf{D}_{\mathcal{X}} \oslash \mathbf{D}_{\mathbf{Z}} + \mathrm{diag}((\mathbf{D}_{\mathcal{X}} \oslash \mathbf{D}_{\mathbf{Z}})\mathbf{1})$ and $\oslash$ denotes element-wise division

Note that $\mathbf{B}$ directly depends on the unknown $\mathbf{Z}$, so we will write $\mathbf{B}(\mathbf{Z})$

Quadratic stress in matrix form

$$\sigma(\mathbf{Z}) = \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2$$

$$\Downarrow$$

$$\sigma(\mathbf{Z}) = \mathrm{tr}(\mathbf{Z}^\top \mathbf{V} \mathbf{Z}) - 2\mathrm{tr}(\mathbf{Z}^\top \mathbf{B}(\mathbf{Z}) \mathbf{Z}) + \sum_{i>j} d_{\mathcal{X}}^2(x_i, x_j)$$

where

$$\begin{aligned} \mathbf{V} &= n\mathbf{I} - \mathbf{1}\mathbf{1}^\top \\ \mathbf{B}(\mathbf{Z}) &= -\mathbf{D}_{\mathcal{X}} \oslash \mathbf{D}_{\mathbf{Z}} + \mathrm{diag}((\mathbf{D}_{\mathcal{X}} \oslash \mathbf{D}_{\mathbf{Z}})\mathbf{1}) \end{aligned}$$

Quadratic stress in matrix form

$$\sigma(\mathbf{Z}) = \sum_{i>j} |d_{\mathcal{X}}(x_i, x_j) - d_{ij}(\mathbf{Z})|^2$$

$$\Downarrow$$

$$\sigma(\mathbf{Z}) = \operatorname{tr}(\mathbf{Z}^\top \mathbf{V}\mathbf{Z}) - 2\operatorname{tr}(\mathbf{Z}^\top \mathbf{B}(\mathbf{Z})\mathbf{Z}) + \sum_{i>j} d^2_{\mathcal{X}}(x_i, x_j)$$

where

$$\begin{aligned} \mathbf{V} &= n\mathbf{I} - \mathbf{1}\mathbf{1}^\top \\ \mathbf{B}(\mathbf{Z}) &= -\mathbf{D}_{\mathcal{X}} \oslash \mathbf{D}_{\mathbf{Z}} + \operatorname{diag}((\mathbf{D}_{\mathcal{X}} \oslash \mathbf{D}_{\mathbf{Z}})\mathbf{1}) \end{aligned}$$

Our task is to solve the minimization problem:

$$\min_{\mathbf{Z} \in \mathbb{R}^{n \times k}} \sigma(\mathbf{Z})$$

We will use gradient descent.

Gradient descent

Consider the generic minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^k} f(\mathbf{x})$$

and assume that f is differentiable

Gradient descent

Consider the generic minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^k} f(\mathbf{x})$$

and assume that f is differentiable

The gradient descent algorithm proceeds as follows:

- Choose a starting point $\mathbf{x}^{(0)}$ (initialization)

Gradient descent

Consider the generic minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^k} f(\mathbf{x})$$

and assume that f is differentiable

The gradient descent algorithm proceeds as follows:

- Choose a starting point $\mathbf{x}^{(0)}$ (initialization)
- Iterate: $\mathbf{x}^{(t+1)} = \mathbf{x}^{(t)} - \alpha \nabla f(\mathbf{x}^{(t)})$

Gradient descent

Consider the generic minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^k} f(\mathbf{x})$$

and assume that f is differentiable

The gradient descent algorithm proceeds as follows:

- Choose a starting point $\mathbf{x}^{(0)}$ (initialization)
- Iterate: $\mathbf{x}^{(t+1)} = \mathbf{x}^{(t)} - \alpha\nabla f(\mathbf{x}^{(t)})$
- Stop when some convergence criterion is satisfied

Gradient descent

Consider the generic minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^k} f(\mathbf{x})$$

and assume that f is differentiable

The gradient descent algorithm proceeds as follows:

- Choose a starting point $\mathbf{x}^{(0)}$ (initialization)
- Iterate: $\mathbf{x}^{(t+1)} = \mathbf{x}^{(t)} - \alpha \nabla f(\mathbf{x}^{(t)})$
- Stop when some convergence criterion is satisfied

The recursive equation produces a non-increasing sequence:

$$f(\mathbf{x}^{(0)}) \geq f(\mathbf{x}^{(1)}) \geq f(\mathbf{x}^{(2)}) \cdots$$

Gradient descent for stress minimization

$$\min_{\mathbf{Z}\in\mathbb{R}^{n\times k}} \sigma(\mathbf{Z})$$

Gradient descent for stress minimization

$$\min_{\mathbf{Z}\in\mathbb{R}^{n\times k}} \sigma(\mathbf{Z})$$

Our gradient can be computed as:

$$\begin{aligned}\nabla\sigma(\mathbf{Z}) &= \nabla\left(\operatorname{tr}(\mathbf{Z}^\top\mathbf{V}\mathbf{Z}) - 2\operatorname{tr}(\mathbf{Z}^\top\mathbf{B}(\mathbf{Z})\mathbf{Z}) + \sum_{i>j} d_{\mathcal{X}}^2(x_i,x_j)\right)\\ &= 2\mathbf{V}\mathbf{Z} - 2\mathbf{B}(\mathbf{Z})\mathbf{Z}\end{aligned}$$

Gradient descent for stress minimization

$$\min_{\mathbf{Z} \in \mathbb{R}^{n \times k}} \sigma(\mathbf{Z})$$

Our gradient can be computed as:

$$\begin{aligned}
\nabla \sigma(\mathbf{Z}) &= \nabla \left(\operatorname{tr}(\mathbf{Z}^\top \mathbf{V} \mathbf{Z}) - 2\operatorname{tr}(\mathbf{Z}^\top \mathbf{B}(\mathbf{Z}) \mathbf{Z}) + \sum_{i>j} d_{\mathcal{X}}^2(x_i, x_j) \right) \\
&= 2\mathbf{V}\mathbf{Z} - 2\mathbf{B}(\mathbf{Z})\mathbf{Z}
\end{aligned}$$

Then we can follow the algorithm:

- Start with a random configuration of points $\mathbf{Z}^{(0)}$

Gradient descent for stress minimization

$$\min_{\mathbf{Z}\in\mathbb{R}^{n\times k}} \sigma(\mathbf{Z})$$

Our gradient can be computed as:

$$\begin{aligned}\nabla\sigma(\mathbf{Z}) &= \nabla\left(\operatorname{tr}(\mathbf{Z}^\top\mathbf{V}\mathbf{Z}) - 2\operatorname{tr}(\mathbf{Z}^\top\mathbf{B}(\mathbf{Z})\mathbf{Z}) + \sum_{i>j} d_{\mathcal{X}}^2(x_i, x_j)\right)\\ &= 2\mathbf{V}\mathbf{Z} - 2\mathbf{B}(\mathbf{Z})\mathbf{Z}\end{aligned}$$

Then we can follow the algorithm:

- Start with a random configuration of points $\mathbf{Z}^{(0)}$
- Iterate
  $\mathbf{Z}^{(t+1)} = \mathbf{Z}^{(t)} - \alpha\nabla\sigma(\mathbf{Z}^{(t)}) = \mathbf{Z}^{(t)} - 2\alpha(\mathbf{V}\mathbf{Z}^{(t)} - \mathbf{B}(\mathbf{Z}^{(t)})\mathbf{Z}^{(t)})$

Gradient descent for stress minimization

$$\min_{\mathbf{Z}\in\mathbb{R}^{n\times k}} \sigma(\mathbf{Z})$$

Our gradient can be computed as:

$$\begin{aligned}\nabla\sigma(\mathbf{Z}) &= \nabla\left(\mathrm{tr}(\mathbf{Z}^\top\mathbf{V}\mathbf{Z}) - 2\mathrm{tr}(\mathbf{Z}^\top\mathbf{B}(\mathbf{Z})\mathbf{Z}) + \sum_{i>j} d^2_{\mathcal{X}}(x_i, x_j)\right)\\ &= 2\mathbf{V}\mathbf{Z} - 2\mathbf{B}(\mathbf{Z})\mathbf{Z}\end{aligned}$$

Then we can follow the algorithm:

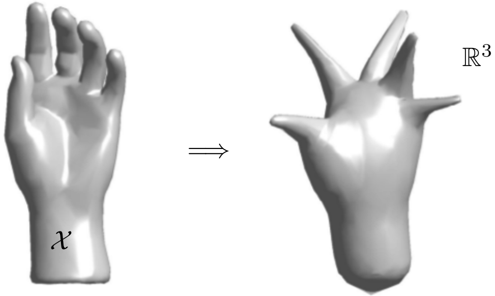
- Start with a random configuration of points $\mathbf{Z}^{(0)}$
- Iterate $\mathbf{Z}^{(t+1)} = \mathbf{Z}^{(t)} - \alpha\nabla\sigma(\mathbf{Z}^{(t)}) = \mathbf{Z}^{(t)} - 2\alpha(\mathbf{V}\mathbf{Z}^{(t)} - \mathbf{B}(\mathbf{Z}^{(t)})\mathbf{Z}^{(t)})$
- Terminate when $|\sigma(\mathbf{Z}^{(t+1)}) - \sigma(\mathbf{Z}^{(t)})| < \epsilon$

Example: Canonical forms

We can now apply this algorithm to solve Euclidean embedding problems.

- Example 1: Given a mesh representing some shape $(\mathcal{X}, d_{\mathcal{X}})$ with the geodesic metric, embed it into $\mathbb{R}^3$

  This is a straightforward application of the algorithm where we use the complete distance matrix $\mathbf{D}_{\mathcal{X}}$ as input data

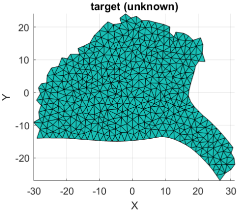


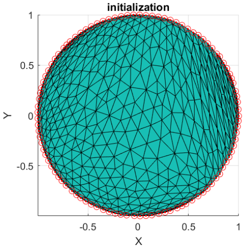
Example: Mesh from edge lengths

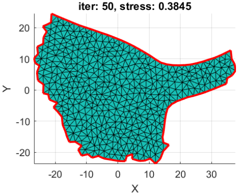
We can now apply this algorithm to solve Euclidean embedding problems.

- Example 2: Given the mesh connectivity together with edge lengths ℓ_{ij}, recover the vertex coordinates in $\mathbb{R}^2$

  This is a simple variant of the algorithm where the input data is an incomplete set of pairwise distances. This is realized simply by inputting a sparse distance matrix $\mathbf{D}_{\mathcal{X}}$, with $d_{ij} = 0$ whenever x_i and x_j are not connected by an edge







Exercise: Symmetry detection

Write an algorithm that finds the left-right labeling for the human mesh tr_reg_000 . off (download from course page).

- Compute a canonical form $f(M)$ for the mesh M
- Center $f(M)$ around $(0, 0, 0)$
- Align $f(M)$ to the x, y, z axes via PCA; in Matlab, you can use Z∗pca(Z) where Z is the $n \times 3$ matrix encoding $f(M)$
- Find the symmetry plane by selecting one of the (x, y), (x, z), (y, z) planes
- Visualize the resulting labels L and R on the original mesh M by interpreting them as scalar functions $L, R : M \to \{0, 1\}$

To speed-up computation, run the algorithm on a farthest point sampling of the mesh, and then expand the solution to all remaining points via Voronoi decomposition