Fundamentals of Computer Graphics

Regular surfaces

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Differential geometry

The study of local properties of curves and surfaces

(Properties which depend on the behavior in the neighborhood of a point)

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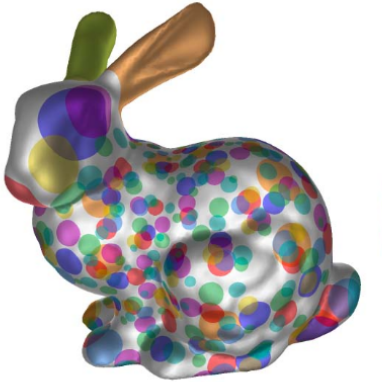
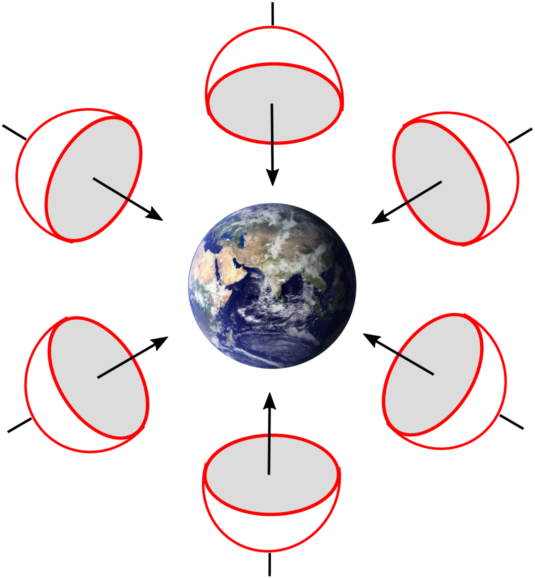


What constitutes a valid neighborhood?

Differential geometry

Differential geometry gives us powerful tools to compute lengths, areas, integrals, gradients, etc. on surfaces

A union of charts



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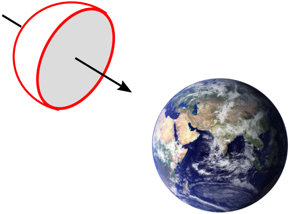

chart

Each chart can be seen as a mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^3$

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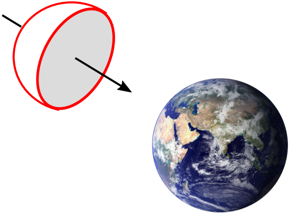

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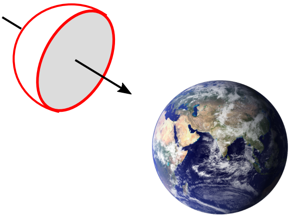

chart

Each chart can be seen as a mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^3$

- Cannot be an isometry!
- We require ϕ to be smooth and invertible
  (also called diffeomorphism)

Regular surfaces

Recipe for a regular surface in $\mathbb{R}^3$:

- Cut pieces of a plane

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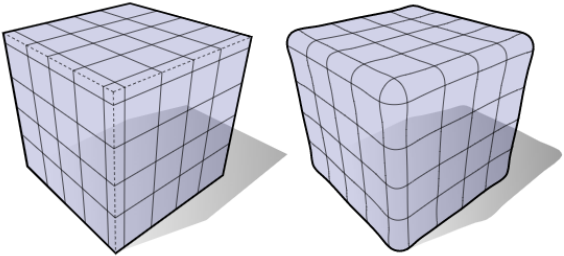
Regular surfaces

Regularity ensures that we can talk about tangent planes at each point



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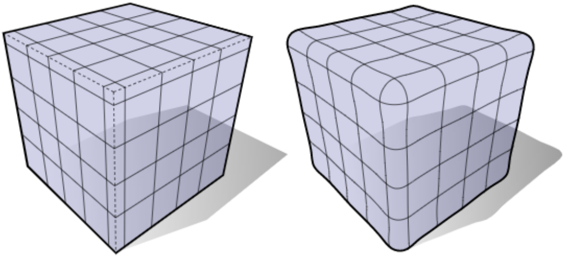


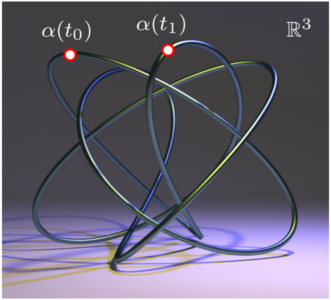
In the language of differential geometry:

"2-dimensional Riemannian sub-manifold"

Parametrized curves

A parametrized curve is a differentiable map $\alpha : (t_0, t_1) \to \mathbb{R}^3$

$$\alpha(t) = (x(t), y(t), z(t))$$



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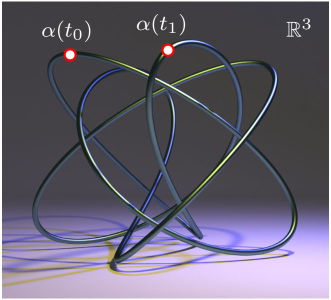

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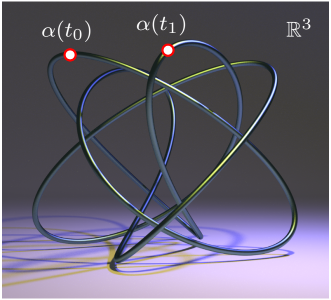

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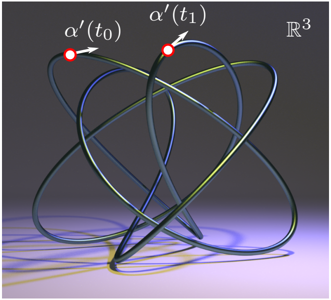
- t is called parameter
- $x(t), y(t), z(t)$ are differentiable
- The tangent vector at t is

$$\alpha'(t) = (x'(t), y'(t), z'(t))$$



Exercise: Parametrized curves

Draw the following parametrized curves:

$$\alpha(t) = (a\cos(t), a\sin(t), bt) \tag{1}$$

$$\alpha(t) = (t^3, t^2) \tag{2}$$

- Are these curves differentiable?
- What is $\alpha'(0)$?

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- What is $\alpha'(0)$?

Curve (2) is not regular, because $\alpha'(0) = (0, 0)$

Charts

A chart is an invertible and smooth map

$$\phi : U \to S \qquad \phi(u,v) = (x(u,v), y(u,v), z(u,v))$$

where $U \subset \mathbb{R}^2$ and $S \subset \mathbb{R}^3$

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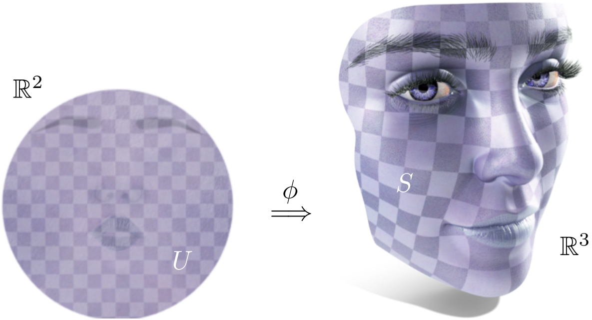

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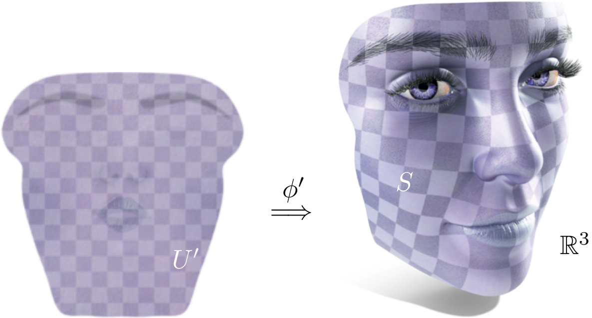

- The parametrization is not unique

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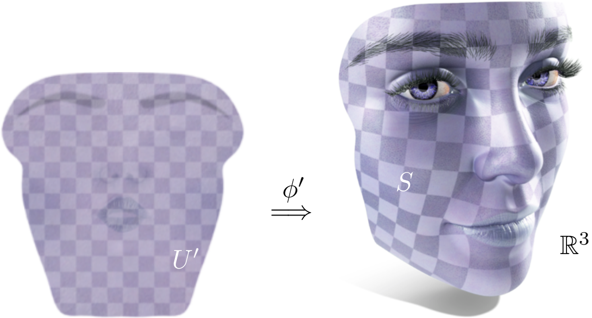


- The parametrization is not unique
- The chart is regular if there exists a tangent plane at all points of S

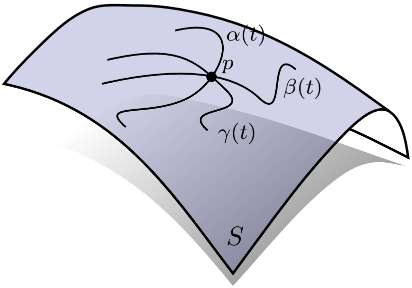
Example: Non-regular surface

The tip of the cone is a singular point

This is in the same sense as we had with regular curves: a tangent plane cannot be defined at p

Tangent plane

The set of tangent vectors to all possible parametrized curves on S passing though p



The curves are parametrized so that $p = \alpha(0) = \beta(0) = \cdots$

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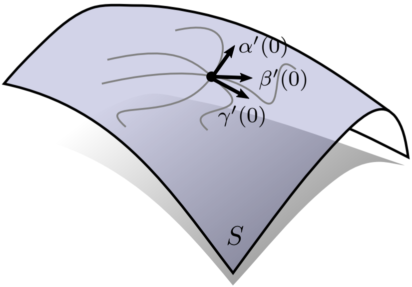

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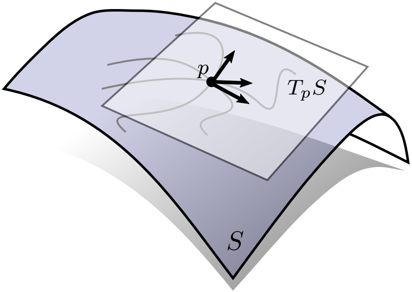

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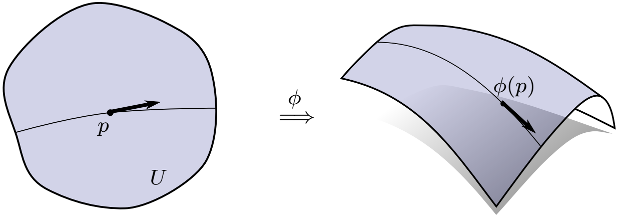
We denote the tangent plane at p by T_pS

Differential of a map



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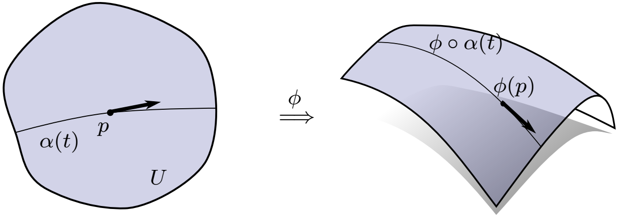

$$\alpha(t) = (u(t), v(t)) \in \mathbb{R}^2\,, \qquad (\phi \circ \alpha)(t) = \phi(u(t), v(t)) \in \mathbb{R}^3$$

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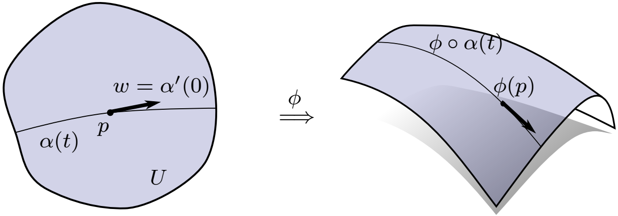

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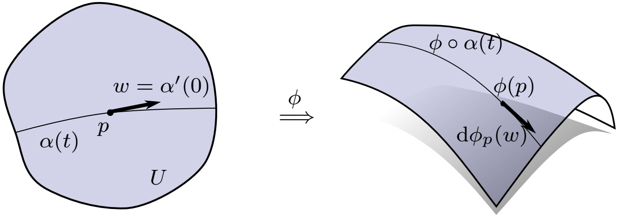

The differential of the map ϕ at point p is defined as:

$$\mathrm{d}\phi_p(w) = (\phi \circ \alpha)'(0)$$

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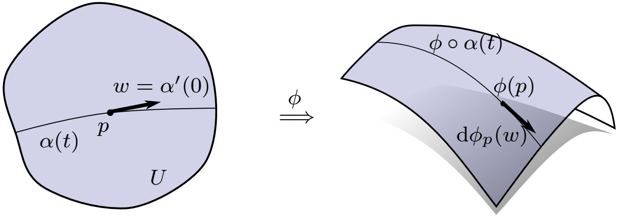

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It is a linear map that takes tangent vectors to tangent vectors:

$$\mathrm{d}\phi_p : \mathbb{R}^2 \to \mathbb{R}^3$$

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Recall, for a curve $\alpha(t)$ in $\mathbb{R}^2$:

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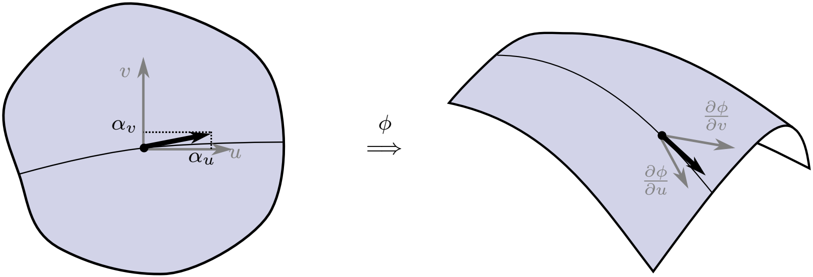

The vectors $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$ span the tangent plane at $\phi(p)$

Differential of a map

$$\mathrm{d}\phi_p \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u}\alpha_u + \frac{\partial x}{\partial v}\alpha_v \\ \frac{\partial y}{\partial u}\alpha_u + \frac{\partial y}{\partial v}\alpha_v \\ \frac{\partial z}{\partial u}\alpha_u + \frac{\partial z}{\partial v}\alpha_v \end{pmatrix} = \underbrace{\begin{pmatrix} | & | \\ \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \\ | & | \end{pmatrix}}_{\text{Jacobian}} \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix}$$

The Jacobian:

- is a 3×2 matrix
- is the matrix representation of the map differential
- has rank 2

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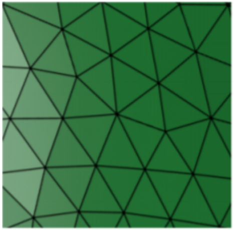
- is the matrix representation of the map differential
- has rank 2

For non-regular surfaces, the rank is < 2 at the singular points

This means that $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$ do not span a (tangent) plane

Mesh parametrization

We deal with discrete surfaces (triangle meshes):



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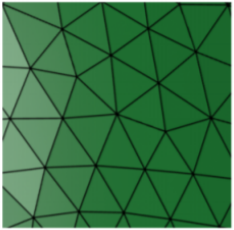


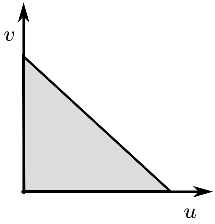
A chart for each triangle

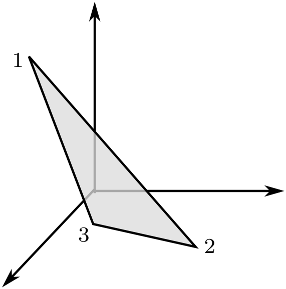
Mesh parametrization

For a mesh with m triangles, we have m charts $\phi_j : \mathbb{R}^2 \to \mathbb{R}^3$ with $j = 1, \ldots, m$:

$$\phi_j(u,v) = \begin{pmatrix} x_1^j \\ y_1^j \\ z_1^j \end{pmatrix} + u \left(\begin{pmatrix} x_2^j \\ y_2^j \\ z_2^j \end{pmatrix} - \begin{pmatrix} x_1^j \\ y_1^j \\ z_1^j \end{pmatrix} \right) + v \left(\begin{pmatrix} x_3^j \\ y_3^j \\ z_3^j \end{pmatrix} - \begin{pmatrix} x_1^j \\ y_1^j \\ z_1^j \end{pmatrix} \right)$$

with $u \in [0,1], v \in [0, 1-u]$





Suggested reading

See chapters 2.1 – 2.4 and Appendix 2.B of:

M. Do Carmo, "Differential geometry of curves and surfaces". Prentice-Hall, 1976