Fundamentals of Computer Graphics

The Laplace-Beltrami operator

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Course quality questionnaire

The questionnaire is completely anonymous – your privacy is respected

Instructions:

- Disable pop-up block in your browser
- Go to https://www.uniroma1.it
- Click on **Studenti** and access **Infostud 2.0**
- Click on **Corsi di laurea**
- On the left menu, click on **Opinioni studenti**
- Enter the OPIS code and click on **Vai al questionario**

OPIS codes are:

- **LNM5IDE3**
- **CUP342HA** for the curriculum "Networks and security"

Motivation: Heat diffusion

For a subset $U \subset \mathbb{R}^2$ the diffusion of heat is described by the heat equation:

$$\frac{\partial}{\partial t} u(x,t) = \Delta u(x,t)$$
$$u(x,0) = u_0(x)$$

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where the Laplacian is defined as:

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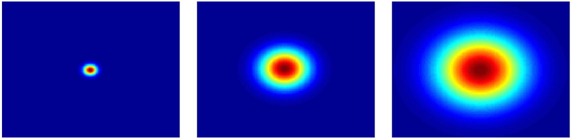


$t = 0$ $t = 1.3$ $t = 3.7$

Motivation: Heat diffusion on a surface

For a surface $\mathcal{X} \in \mathbb{R}^3$ the diffusion of heat is described by the heat equation:

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) &= \Delta u(x,t) \\ u(x,0) &= u_0(x) \end{aligned}$$

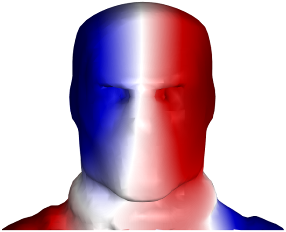
where the Laplacian is defined as:

$$\Delta u(x,t) = -\mathrm{div}(\nabla u) = ?$$

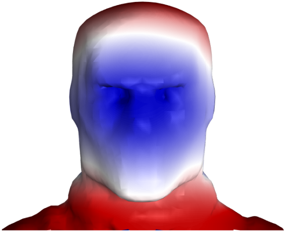
Inner product on a manifold

Given two scalar functions $f, g : \mathcal{X} \to \mathbb{R}$, we define their inner product as:

$$\langle f, g \rangle_{\mathcal{X}} = \int_{\mathcal{X}} f(x) g(x) dx$$



$f : \mathcal{X} \to \mathbb{R}$



$g : \mathcal{X} \to \mathbb{R}$

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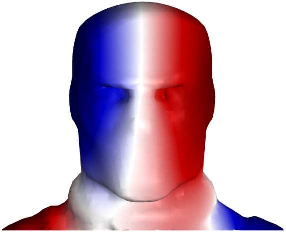
When discretized on a triangle mesh of n vertices, this boils down to:

$$\mathbf{f}^\top \mathrm{diag}(a_i) \mathbf{g}$$

where a_i with $i = 1, \ldots, n$ are the local area elements at each vertex



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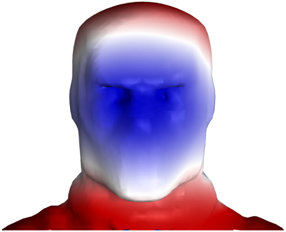
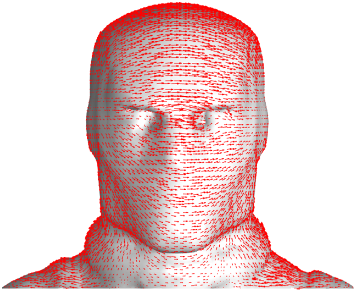


$g : \mathcal{X} \to \mathbb{R}$

Inner product on a manifold

Given two tangent vector fields $F, G : \mathcal{X} \to T\mathcal{X}$, we define their inner product as:

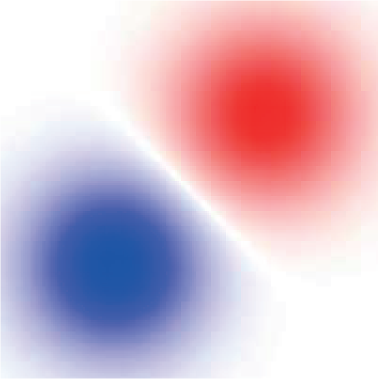
$$\langle F, G \rangle_{T\mathcal{X}} = \int_{\mathcal{X}} \langle F(x), G(x) \rangle_{T_x\mathcal{X}} dx$$



$F : \mathcal{X} \to T\mathcal{X}$

$G : \mathcal{X} \to T\mathcal{X}$

Geometric intuition

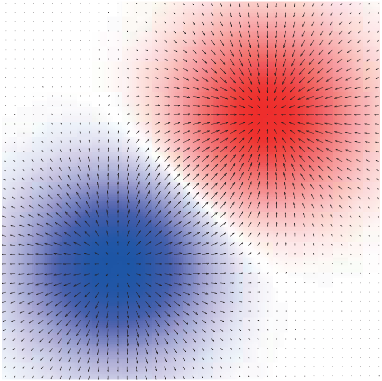


Smooth scalar field f

Geometric intuition

- Gradient $\nabla f(x)$

  'direction of the steepest increase of f at x'



Smooth vector field F

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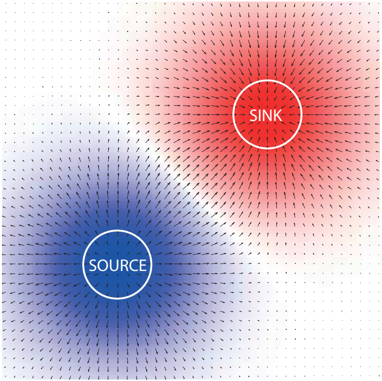
- Divergence $\mathrm{div}(F(x))$

  'scalar density of an outward flux of F from an infinitesimal volume around x'



Smooth vector field F

Geometric intuition

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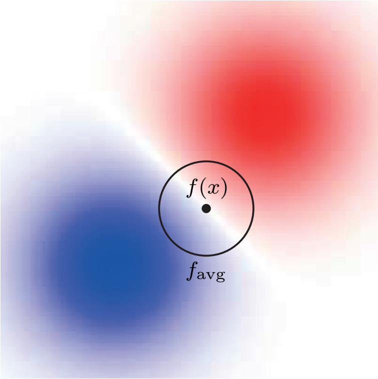
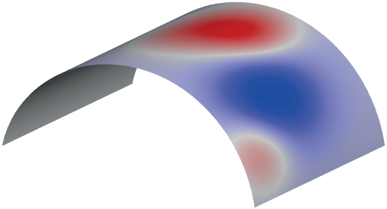
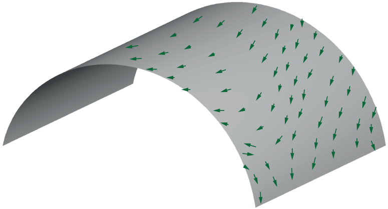
- Divergence $\mathrm{div}(F(x))$
  'scalar density of an outward flux of F from an infinitesimal volume around x'
- Laplacian $\Delta f(x) = -\mathrm{div}(\nabla f(x))$
  'scalar difference between $f(x)$ and the average of f on an infinitesimal sphere around x'



Adjointness





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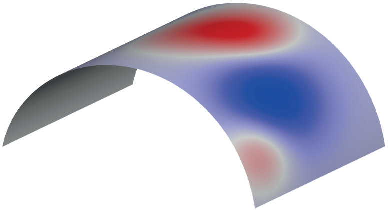
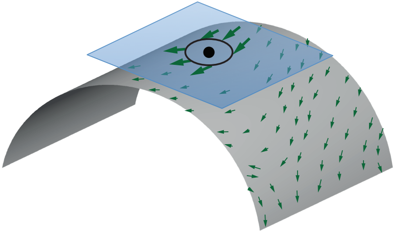

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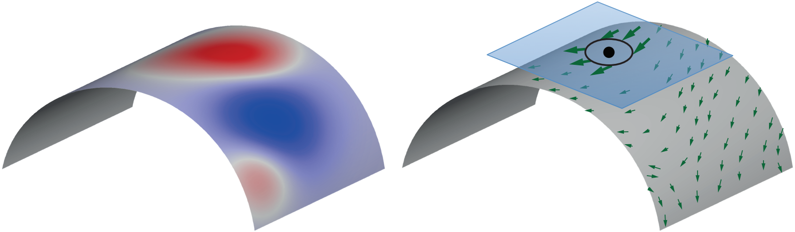

Gradient and divergence are (negative) adjoint operators:

$$\langle F, \nabla f \rangle_{T\mathcal{X}} = -\langle \text{div} F, f \rangle_{\mathcal{X}}$$

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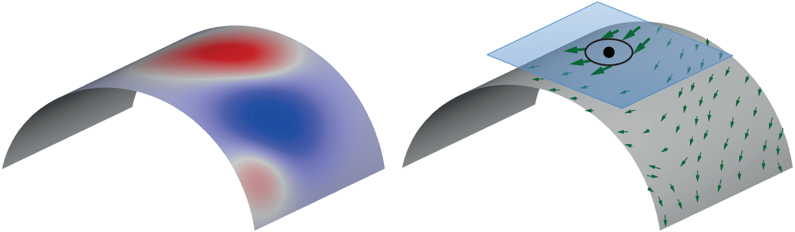

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Notice that the first inner product is among vector fields, while the second is among scalar functions

Finite elements method (FEM)

We will discretize the Laplace-Beltrami operator using a new approach

Overall idea:

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- Consider the equation

$$\Delta f(x) = g(x)$$

  We are interested in computing g, and then directly "solve for" Δ

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- To do so, we instead look at the weak formulation

  $$\langle \Delta f, h_j \rangle_{\mathcal{X}} = \langle g, h_j \rangle_{\mathcal{X}}$$

  where $h_j : \mathcal{X} \to \mathbb{R}$ are some test functions

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- To do so, we instead look at the weak formulation
$$\langle \Delta f, h_j \rangle_{\mathcal{X}} = \langle g, h_j \rangle_{\mathcal{X}}$$
where $h_j : \mathcal{X} \to \mathbb{R}$ are some test functions
- We obtain expressions for the left- and right-hand sides, from which we explicitly compute a matrix representation for Δ

Hat basis

Recall that on triangle meshes we approximate scalar functions as

$$f(x) \approx \sum_{i=1}^{n} f(v_i) h_i(x)$$

where h_i are hat basis functions defined at each vertex:

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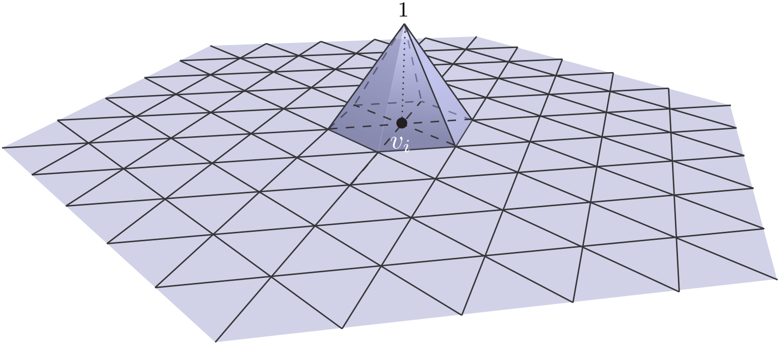

FEM discretization

We use the hat basis as test functions in the equation:

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&= \langle \nabla \sum_i f(v_i) h_i(x), \nabla h_j \rangle_{T\mathcal{X}}
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FEM discretization: Stiffness

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&= \sum_i f(v_i) \underbrace{\langle \nabla h_i, \nabla h_j \rangle_{T\mathcal{X}}}_{s_{ij} = s_{ji}}
\end{aligned}$$

where $\mathbf{S} = (s_{ij}) \in \mathbb{R}^{n \times n}$ is the symmetric stiffness matrix; so we have:

$$\langle \Delta f, h_j \rangle_{\mathcal{X}} = (\mathbf{S}\mathbf{f})_j$$

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FEM discretization: Mass

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where $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$ is the symmetric mass matrix; so we have:

$$\langle g, h_j \rangle_{\mathcal{X}} = (\mathbf{A}\mathbf{g})_j$$

Discrete Laplace-Beltrami operator

We get to:

$$\langle \Delta f, h_j \rangle_{\mathcal{X}} = \langle g, h_j \rangle_{\mathcal{X}}$$

Discrete Laplace-Beltrami operator

We get to:

$$(\mathbf{Sf})_j = (\mathbf{Ag})_j \qquad \text{for } j = 1, \ldots, n$$

Discrete Laplace-Beltrami operator

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$$\mathbf{Sf} = \mathbf{Ag}$$

Discrete Laplace-Beltrami operator

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Our discrete Laplacian is thus given by:

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- The matrices involved are sparse and symmetric

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- The matrices involved are sparse and symmetric
- They have the same structure as the vertex adjacency matrix
- The stiffness $\mathbf{S}$ is positive semi-definite
- They can be computed easily and efficiently for any given mesh

Mass integral

Assume $i \neq j$ (off-diagonal of $\mathbf{A}$):

$a_{ij} = \langle h_i, h_j \rangle_{\mathcal{X}}$

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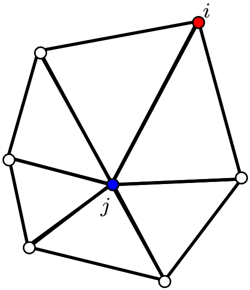

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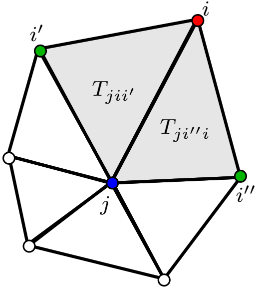

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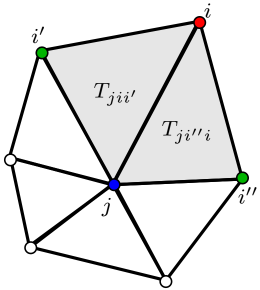

Functions h_i and h_j decrease linearly from 1 to 0

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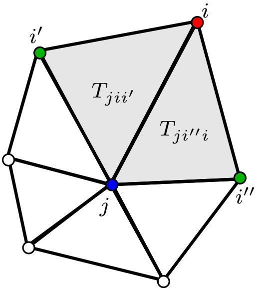

Functions h_i and h_j decrease linearly from 1 to 0

So, in parameter space, each of the two integrals can be written as:

$$
2A(T_\ell) \int_0^1 \int_0^{1-u} vu \; dvdu
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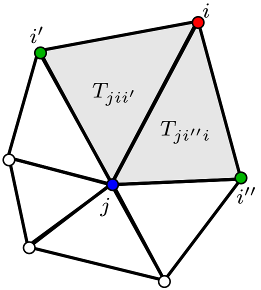

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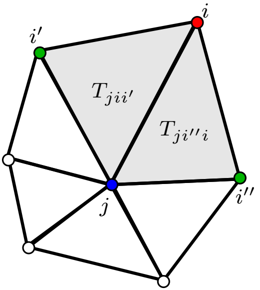

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$$2A(T_\ell) \int_0^1 \int_0^{1-u} vu \; dv du = 2A(T_\ell) \frac{1}{24} = \frac{1}{12} A(T_\ell)$$

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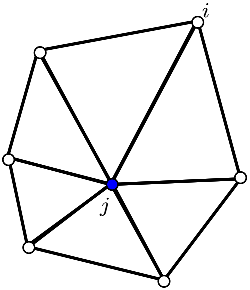

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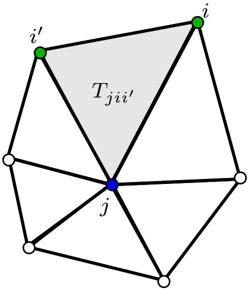
&= \sum_{\ell} \int_{T_\ell} h_j(x) h_j(x) dx \\
&= \sum_{i \in \mathcal{N}(j)} \int_{T_{jii'}} h_j(x) h_j(x) dx
\end{aligned}
$$



Mass integral

Now assume $i = j$ (diagonal of $\mathbf{A}$):

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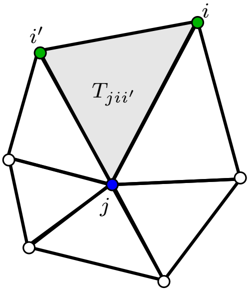

Like before, each integral in parameter space looks like:

$$2A(T_\ell) \int_0^1 \int_0^{1-u} u^2 \; dv du$$

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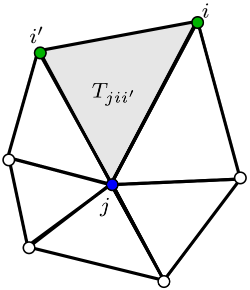

Like before, each integral in parameter space looks like:

$$2A(T_\ell) \int_0^1 \int_0^{1-u} u^2 \; dv du = \frac{1}{6} A(T_\ell)$$

Stiffness integral

Assume $i \neq j$ (off-diagonal of $\mathbf{S}$):

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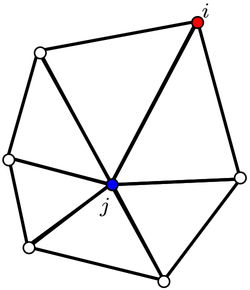

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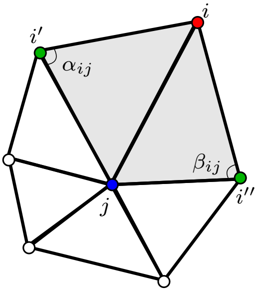

The integrals are non-zero in the same positions as the mass matrix

The formulas involve internal angles (also known as "cotangent Laplacian")

Stiffness and mass matrices

The discrete (FEM) Laplace-Beltrami operator is the $n \times n$ matrix:

$$\mathbf{L} = \mathbf{A}^{-1}\mathbf{S}$$

where

$$s_{ij} = \begin{cases} -\frac{1}{2}(\cot\alpha_{ij} + \cot\beta_{ij}) & \text{if } e_{ij} \in E \\ -\sum_{k \neq i} s_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$a_{ij} = \begin{cases} \frac{1}{12}(A(T_{jii'}) + A(T_{ji''i})) & \text{if } e_{ij} \in E \\ \frac{1}{6}\sum_{k \in \mathcal{N}(i)} A(T_k) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

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A lumped mass matrix (easier to invert) is obtained as:

$$\tilde{a}_{ij} = \begin{cases} \sum_{k=1}^{n} a_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

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$$\tilde{a}_{ij} = \begin{cases} \frac{1}{3}\sum_{k \in \mathcal{N}(i)} A(T_k) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Exercise: Discrete Laplacian

Implement the FEM Laplacian

Test it similarly to what we did with the graph Laplacian to:

- Denoise a scalar function on a surface
- Skeletonize a mesh
- Implement least-squares meshes

Suggested reading

- S. Axler, "Linear algebra done right – 3rd edition". Springer, 2015
  Section $7.A$

- Reuter et al., "Discrete Laplace-Beltrami operators for shape analysis and segmentation". CAG 33, 2009.
  Sections 1 *to* $2.1.2$