1 Intro

Manifold Mesh: Made of manifold triangles

- edges have at most two incident triangles
- edges with only one incident triangle form a mesh boundary
- faces incident to an edge form open/closed fan

Point Cloud Collection of points in the 3D space

- point cloud are interpreted as point-wise sampling of an underlying unknown surface
- Oriented point cloud has a norm for each point

Often come from depth sensors, can be very noisy.

2 Metric Geometry

Given a sphere, how do we compute distance from x to y?

- **Euclidean Distance:** straight walk from x to y
- **Geodesic Distance:** walk on the surface, from x to y

2.1 Distances

- Distances in R^2

  Euclidean Distance

  $$||a-b||_2 = ((a_x - b_x)^2 + (a_y - b_y)^2)^{\frac{1}{2}} \tag{1}$$

  L_p Distance

  $$||a-b||_p = ((a_x - b_x)^p + (a_y - b_y)^p)^{\frac{1}{p}} \tag{2}$$

- L_p distance between vectors in R^k

$$||a-b||_p = (\sum_{i=1}^{K}(a_i - b_i)^p)^{\frac{1}{p}}) \tag{3}$$

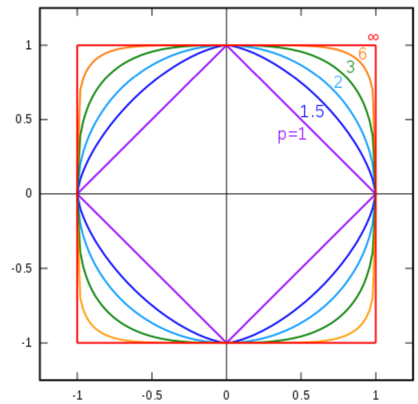


Figure 1: L_p unit balls

2.2 Metric Spaces

A *metric space* is a pair *(object, distance)*.
A set M is a metric space if for every pair of points $x, y \in M$ there is a *metric* (distance) function $d_M : M \times M : R_+$ such that:

- $d_M(x, y) = 0 \Leftrightarrow x = y$ (identity of indiscernibles)
- $d_M(x, y) = d_M(y, x)$ (symmetry)
- $d_M(x, y) \leq d_M(x, z) + d_M(z, y)$ for any $x, y, z \in M$ (triangle inequality)

In this course, a metric space is defined as the pair (M, d_M) E.g:

- Sphere with Euclidean distance is (S^2, d_{L_2})
- Sphere with Geodesic distance is (S^2, d_g)

Examples of metric spaces:

- $X = R, d_X(x, y) = |x - y|$
- $X = A \subset R^k, d_X(x, y) = ||x - y||_2$

- $X = R, d_X(x,y) = log(|x-y|+1)$
- $X = A \times B, d_X((a1,b1),(a2,b2)) = \sqrt{d_A(a1,a2)^2 + d_B(b1,b2)^2}$
- $X =$any set, $d_X(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

2.3 Geodesic Isolines

Identify a set of points $x \in X$ at the same distance (according to d_g) to some reference point $y \in X$.

2.4 Farthest Point Sampling

- fix n and $S^{(0)} = y$ for some $y \in X$.
- repeat until $k = n$:
  - At step k, given $S^{(k-1)}$, select $x \in X$ such that $x = \arg\max_{x \in X} d_X(x, S^{(k-1)})$
  - $S^(k) = S^{(k-1)} \cup x$

2.5 Voronoi Decomposition

For a given sampling S, associated *Voronoi regions* are defined as:

$$V_i(S) = \{x \in X : d_X(x,x_i) < d_X(x,x_j), x_{j \neq i} \in S\} \tag{4}$$

Voronoi regions can be implemented either for meshes and point clouds, using the Euclidean distance and using a farthest point sampling S.

2.6 Ambient Space and Restriciton

If A is a metric space and $X \subset A$ then A cis called *ambient space* for X. A metric on X can be obtained by the *restriction* $d_X = d_{A|X}$ such that:

$$d_X(x,y) = d_A(x,y) \tag{5}$$

for all $x, y \in X$

2.7 Isometries

Given two metric spaces (M, d_M) and (N, d_N), a bijective map $f : M \to N$ is an **isometry** iff:

$$d_M(x, y) = d_N(f(x), f(y)) \tag{6}$$

for any $x, y \in M$

if $d_M = ||\cdot||_2$ and $d_N = ||\cdot||_2$ it is a *rigid isometry*, meaning that we preserve the Euclidean distances, hence we're only rotating or translating the shape.

Quasi Isometries: non rigid isometries, where

$$d_M(x, y) \approx d_N(f(x), f(y)) \tag{7}$$

(d_M and d_N are geodesic distance functions)

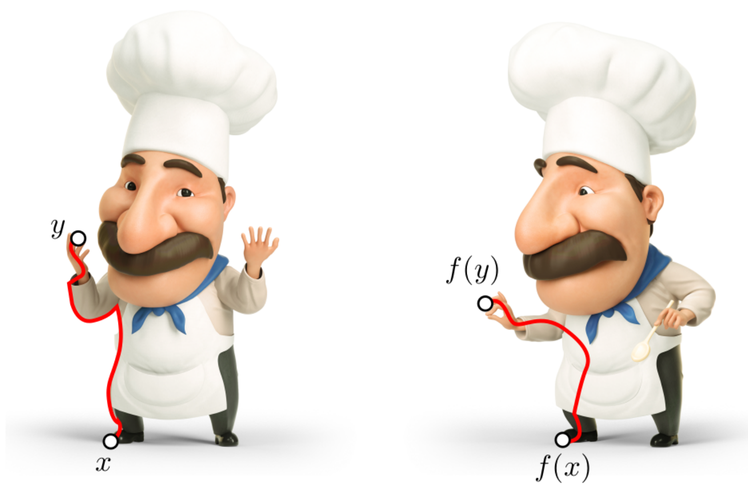


Figure 2: non-rigid isometry

Isometry can be seen as an **equivalence** relation, since it is:

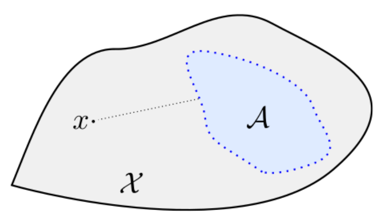
- **reflective:** $a = a$
- **symmetric:** $a = b \Rightarrow b = a$
- **transitive:** $a = b \wedge b = c \Rightarrow a = c$

meaning that we can think of **isometric shapes** as being the **same shape**.

2.8 Distance

There are many notions of **distance between shapes**. **Distance from point x to set $A \subseteq (X, d_X)$:**

$$dist_X(x, A) = \min_{y \in A} d_X(x, y) \tag{8}$$



2.9 Hausdorff Distance

The **Hausdorff Distance** is defined between *subsets of a metric space* Given two subsets $X, Y \subset (Z, d_Z)$

$$d_H^Z = \max\{\max_{x \in X} dist_Z(x, Y), \max_{y \in Y} dist_Z(y, X)\} \tag{9}$$

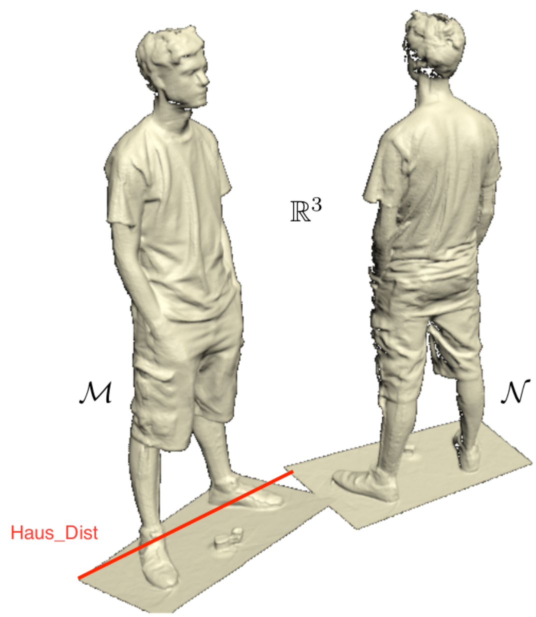


Figure 3: Hausdorff distance, rigid case

To minimize the Hausdorff distance between these two shapes, we can overlap them.