Fundamentals of Computer Graphics

The gradient

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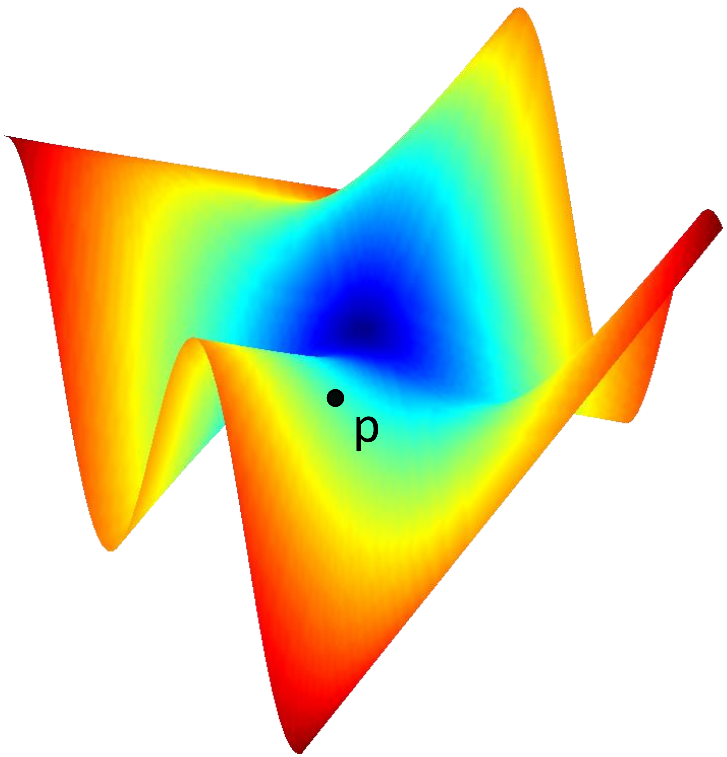
1st semester a.y. 2018/2019 – November 22, 2018

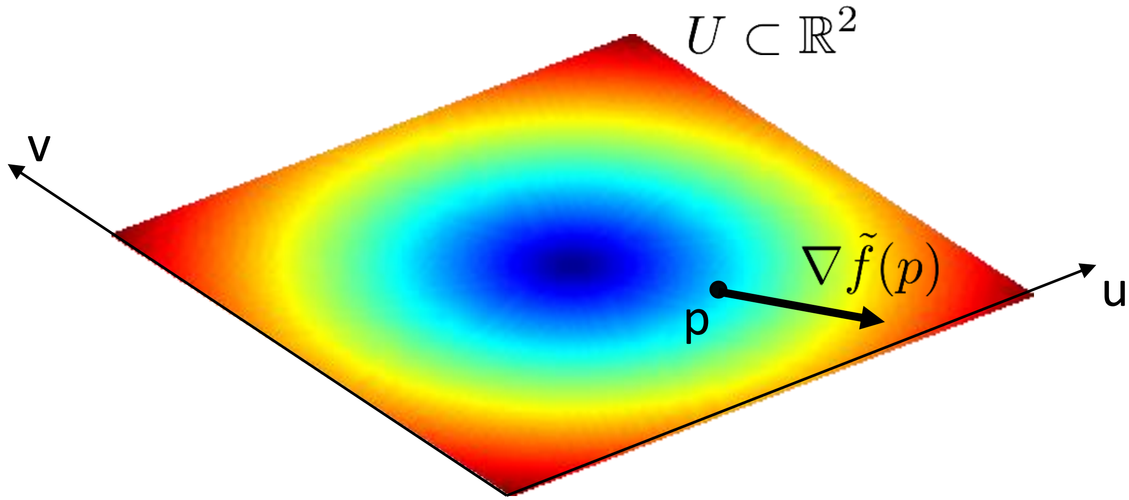
Gradient of a function

Consider a surface *S* with parametrization $\mathbf{x} : U \to S$ and a differentiable function $f : S \to \mathbb{R}$

We want to define the gradient $\nabla f(p)$ at a point $p \in S$



The gradient in R^2



The gradient of a differentiable function $\tilde{f} : U \to \mathbb{R}$ is the vector field

$$\nabla \tilde{f}(p) = \begin{pmatrix} \frac{\partial \tilde{f}}{\partial u}(p) \\ \frac{\partial \tilde{f}}{\partial v}(p) \end{pmatrix}$$

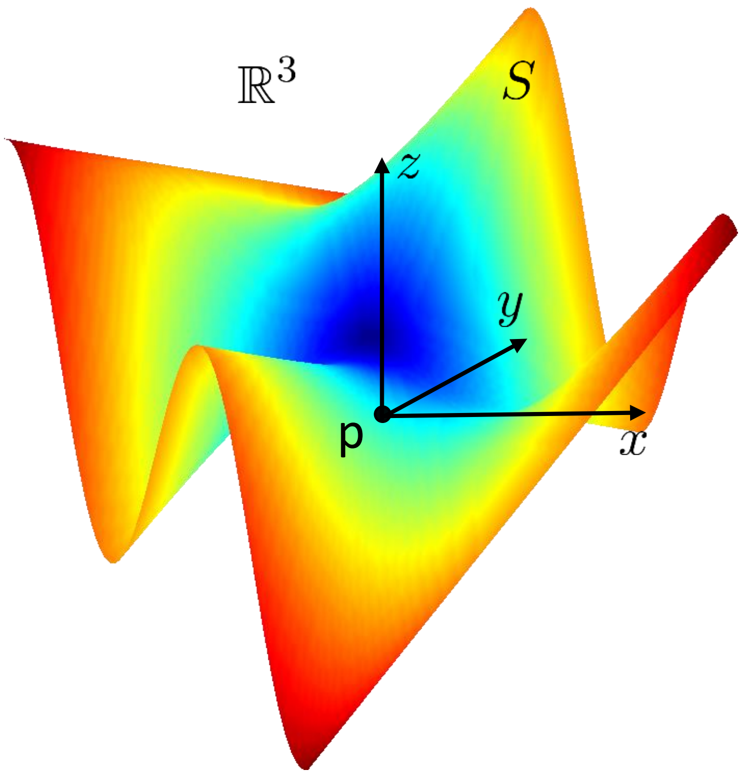
The gradient on a surface

Ideas how to define $\nabla f(p)$:

- Use the same formula as before, but in terms of *x*, *y*, *z*:

$$\nabla f(p) = \begin{pmatrix} \frac{\partial f}{\partial x}(p) \\ \frac{\partial f}{\partial y}(p) \\ \frac{\partial f}{\partial z}(p) \end{pmatrix}$$

Not a good choice, because we have no information about *f* outside of *S*!



The gradient on a surface

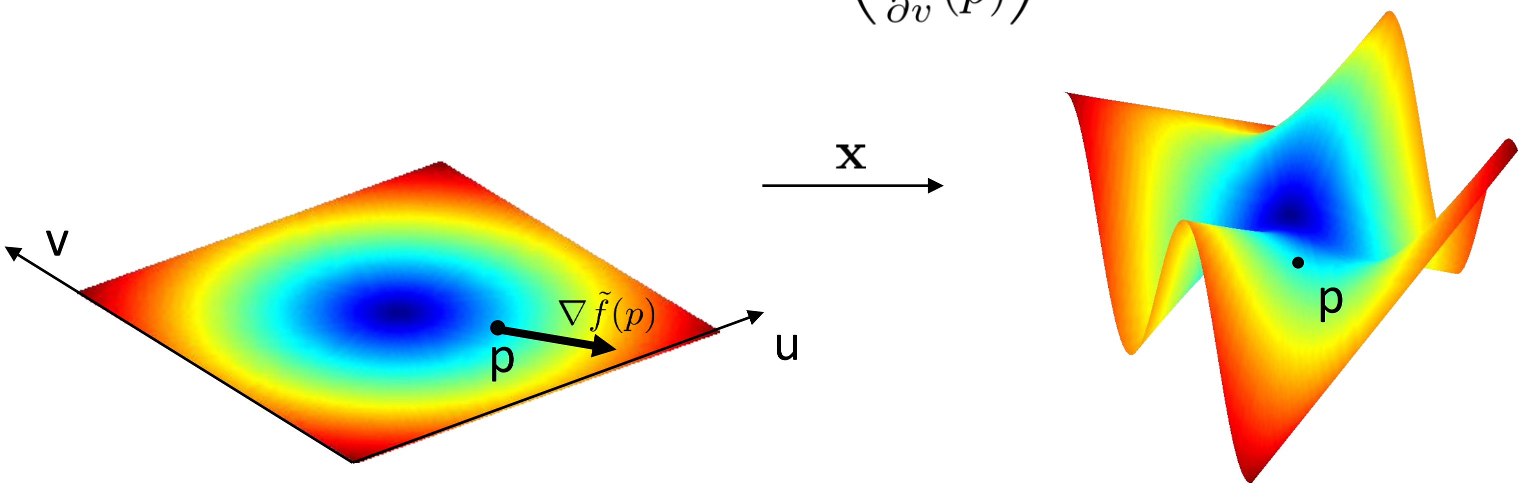
- Another possibility is to express our differentiable function f in terms of a parametrization **x**:

$$\tilde{f}(u,v) = f(\mathbf{x}(u,v))$$

and then set:

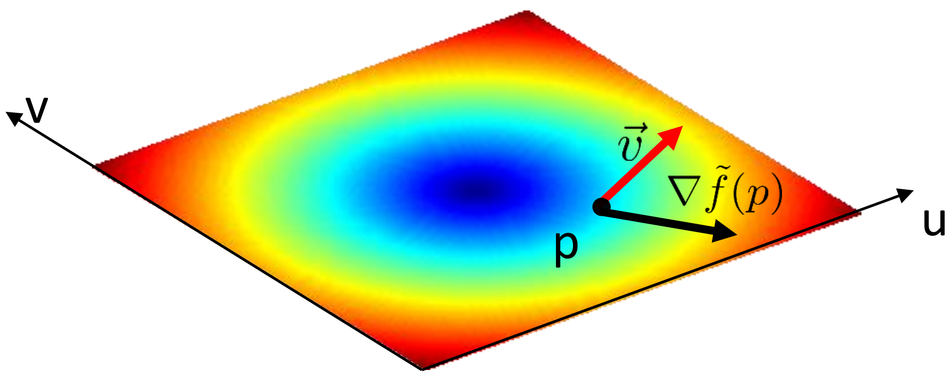
$$\nabla f(p) = \begin{pmatrix} \frac{\partial \tilde{f}}{\partial u}(p) \\ \frac{\partial \tilde{f}}{\partial v}(p) \end{pmatrix}$$

Depends on the choice of the parametrization!



The gradient on a surface

- Instead, let us try to interpret the geometric meaning of the gradient:
  - The vector that points in the direction of steepest increase of *f*
  - Its length measures the strength of increase
  - We have a relationship with the directional derivative:



$$
\begin{aligned}
d\tilde{f}_p(\vec{v}) &= \lim_{h \to 0} \frac{\tilde{f}(p + h\vec{v}) - \tilde{f}(p)}{h} \\
&= \frac{d}{dh} \tilde{f}(p + h\vec{v})|_{h=0} \\
&= \langle \nabla \tilde{f}, \vec{v} \rangle
\end{aligned}
$$

directional derivative of *f* at *p*, along direction *v*

Representation theorem

The gradient of any differentiable function f can be defined as the unique vector field ∇f such that the relationship holds:

$$\langle \nabla f, \vec{v} \rangle = df_p(\vec{v}) \qquad \text{where } df_p(\vec{v}) : T_pS \to \mathbb{R}$$

This is an application of the Riesz representation theorem:

$$H \qquad\qquad H^* = \{\phi : H \to \mathbb{R} \mid \phi \text{ continuous}\,, \text{linear}\} = \{df_p(\vec{v}) : T_pS \to \mathbb{R}\}$$

Inner product space

dual space of *H*

Then, every $\phi \in H^*$ can be written uniquely as an inner product:

$$\phi(y) = \langle y, x \rangle \quad \forall y \in H \quad \Rightarrow \quad df_p(\vec{v}) = \langle \vec{v}, x \rangle \quad \forall \vec{v} \in H$$

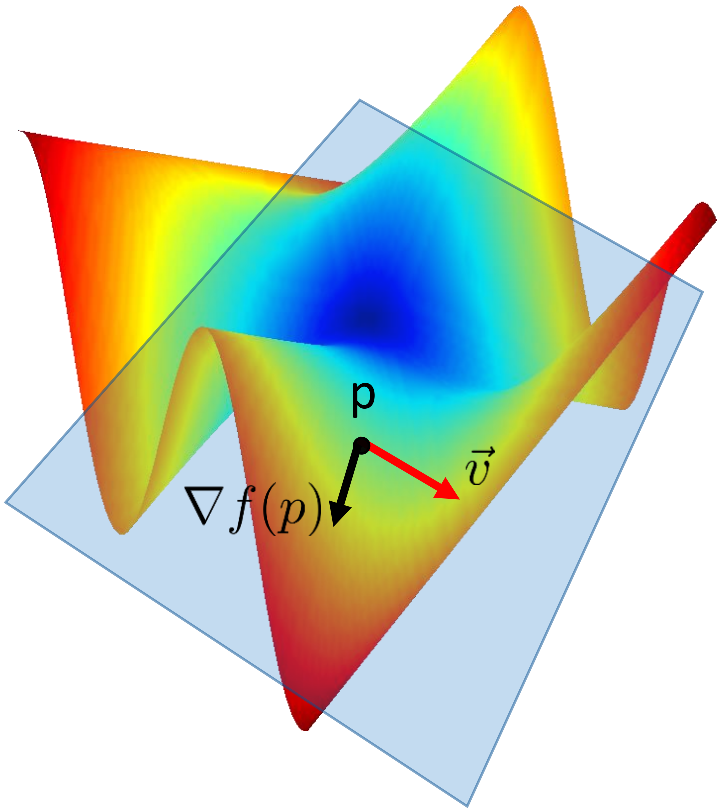
And one can then define the gradient as the unique $x \equiv \nabla f : S \to T_pS$

The gradient on a surface

We define the gradient $\nabla f(p) \in T_p S$ by means of the inner product:

$$I_p(\nabla f, \vec{v}) = df_p(\vec{v}) \quad \forall \vec{v} \in T_p S$$

Thus, if we are able to write down $df_p(\vec{v})$, we can "solve for" ∇f.

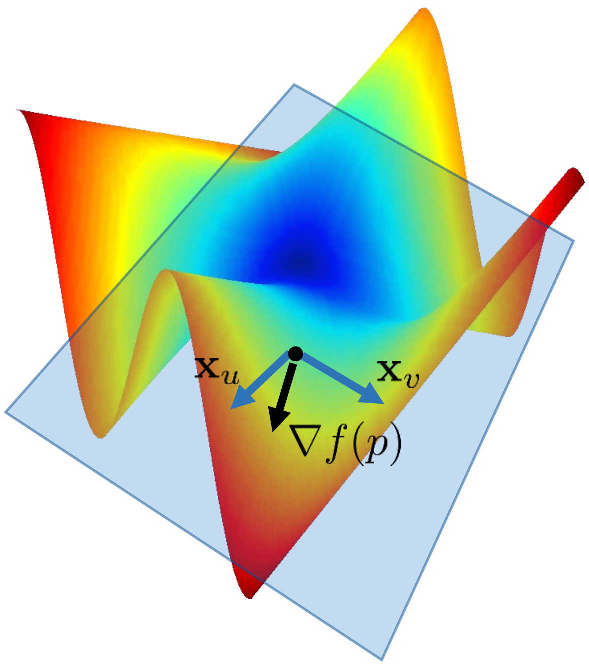


The gradient in local coordinates

Since the gradient is a member of T_pS , we can express it in the local basis:

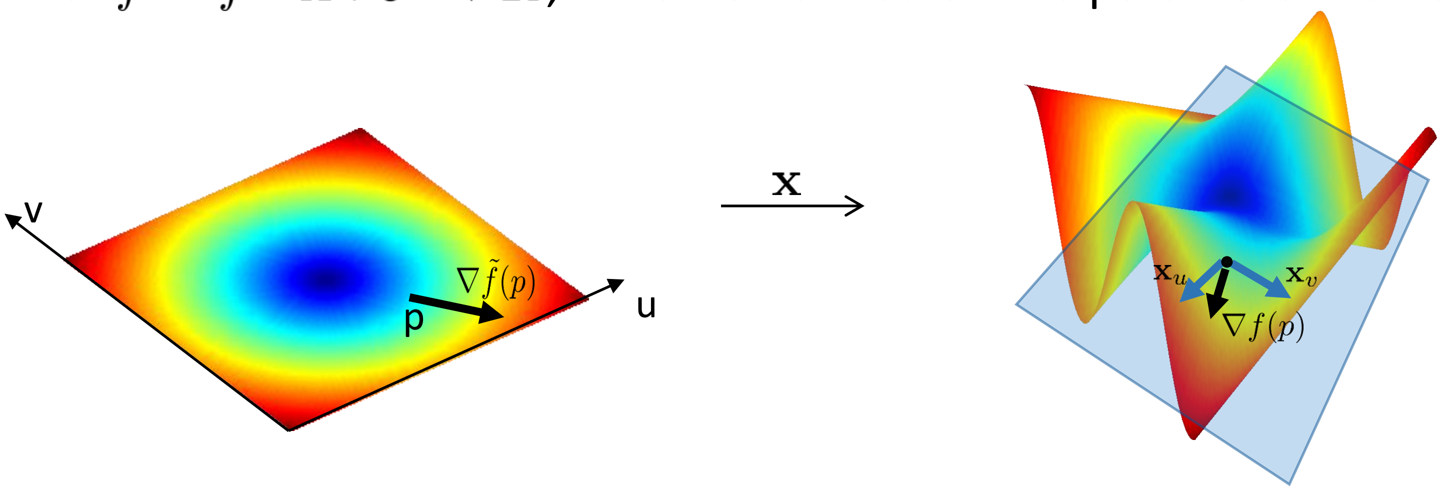
$$\nabla f(p) = f_1 \mathbf{x}_u + f_2 \mathbf{x}_v = \mathrm{D}\mathbf{x} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Hence, knowledge of the coefficients f_1 , f_2 corresponds to knowledge of $\nabla f(p)$.



The gradient in local coordinates

It turns out that the coefficients f_1 , f_2 can be obtained by considering the gradient of $\tilde{f} = f \circ \mathbf{x} : U \to \mathbb{R}$, which is defined on the parameter domain.



Since we have $f(\mathbf{x}(u,v)) = \tilde{f}(u,v)$, we also have that the change of f in the direction $\mathbf{x}_u/\mathbf{x}_v$ corresponds to the change of $\tilde{f}$ in the direction u/v:

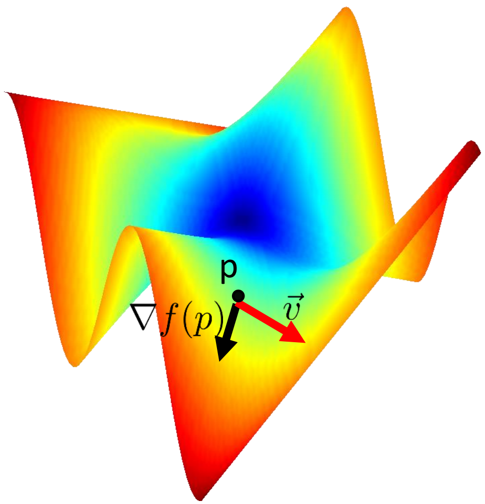
$$\frac{\partial \tilde{f}}{\partial u} = \frac{\partial (f \circ \mathbf{x})}{\partial u} \underset{\substack{\uparrow \\ \text{chain} \\ \text{rule}}}{=} \frac{\partial f}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial u} \underset{\substack{\uparrow \\ \text{Riesz}}}{=} df_p(\mathbf{x}_u)$$

Hence, we can compute the directional derivative df_p directly in parameter space

The gradient in local coordinates

Now let $\vec{v} = v_1\mathbf{x}_u + v_2\mathbf{x}_v$, and apply df_p on both sides (note that df_p is linear):

$$\begin{aligned} df_p(\vec{v}) &= v_1 df_p(\mathbf{x}_u) + v_2 df_p(\mathbf{x}_v) \\ &= v_1 \frac{\partial \tilde{f}}{\partial u} + v_2 \frac{\partial \tilde{f}}{\partial v} \\ &= (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{aligned}$$



On the other hand, we can also write:

$$df_p(\vec{v}) = I_p(\nabla f, \vec{v}) = \begin{pmatrix} f_1 & f_2 \end{pmatrix} g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

This means $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g^{-1} \nabla \tilde{f}$

Wrap-up

Thus, according to our definition of the gradient, we have to find the unique ∇f such that:

$$\langle \nabla f, \vec{v} \rangle = df_p(\vec{v})$$

We can compute the directional derivative directly in *U*, as:

$$df_p(\vec{v}) = (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \text{where} \quad \tilde{f} = f \circ \mathbf{x}$$

defined on the surface *S*

defined on the parameter domain *U*

We can thus write $\langle \nabla f, \vec{v} \rangle = (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Wrap-up

Using the bilinear definition of first fundamental form, we can also write

$$\langle \nabla f, \vec{v} \rangle = I_p(\nabla f, \vec{v}) = \begin{pmatrix} f_1 & f_2 \end{pmatrix} g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Together with the last equation from the previous slide, we have

$$(\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \end{pmatrix} g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

And thus we finally obtain:

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g^{-1} \nabla \tilde{f}$$

Expression in local coordinates

The expression

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g^{-1} \nabla \tilde{f}$$

is giving the gradient coefficients w.r.t. a basis in T_pS, hence it is said to be given in local coordinates.

To obtain a vector $\nabla f \in \mathbb{R}^3$ (i.e. in global coordinates), we simply have to write:

$$\nabla f = f_1 \mathbf{x}_u + f_2 \mathbf{x}_v = \mathrm{D}\mathbf{x} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathrm{D}\mathbf{x}\ g^{-1} \nabla \tilde{f}$$

Gradient norm

In some applications we need the norm of the gradient. Our final expression for the gradient is given by:

$$\nabla f = \mathrm{D}\mathbf{x}\; g^{-1} \nabla \tilde{f}$$

Computing its (squared) norm is straightforward:

$$\begin{aligned}
\|\nabla f\|^2 = \nabla f^\top \nabla f &= \left(\mathrm{D}\mathbf{x}\; g^{-1} \nabla \tilde{f}\right)^\top \left(\mathrm{D}\mathbf{x}\; g^{-1} \nabla \tilde{f}\right) \\
&= \nabla \tilde{f}^\top g^{-1} \underbrace{\mathrm{D}\mathbf{x}^\top \mathrm{D}\mathbf{x}}_{g}\; g^{-1} \nabla \tilde{f} \\
&= \nabla \tilde{f}^\top g^{-1} \nabla \tilde{f}
\end{aligned}$$

Example: The sphere

Consider the function $f : \mathbb{S}^2 \setminus \{n\} \to \mathbb{R}$ that assigns to each point its distance to the north pole *n*:

$$f(p) = d_{\mathbb{S}^2}(n, p)$$

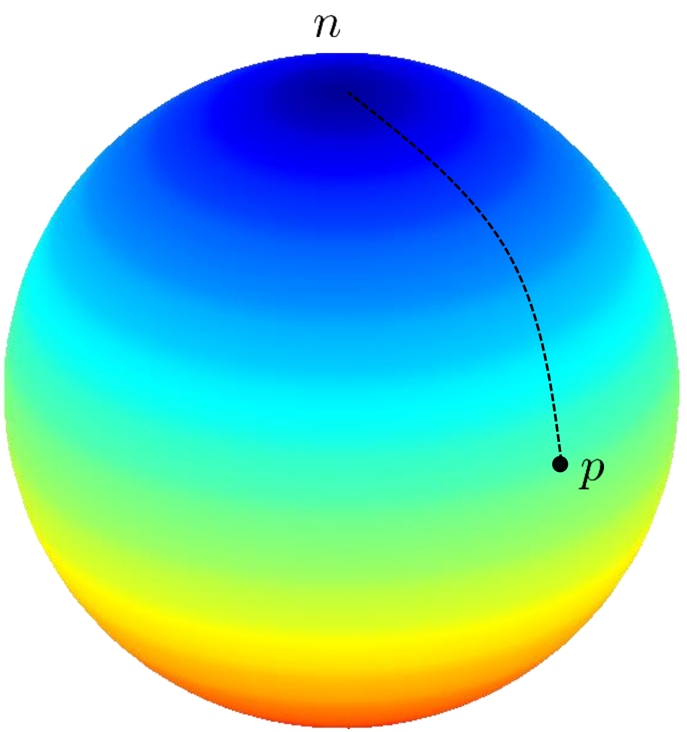
We want to compute its gradient:

$$\nabla f = \mathrm{D}\mathbf{x}\ g^{-1} \nabla \tilde{f}$$

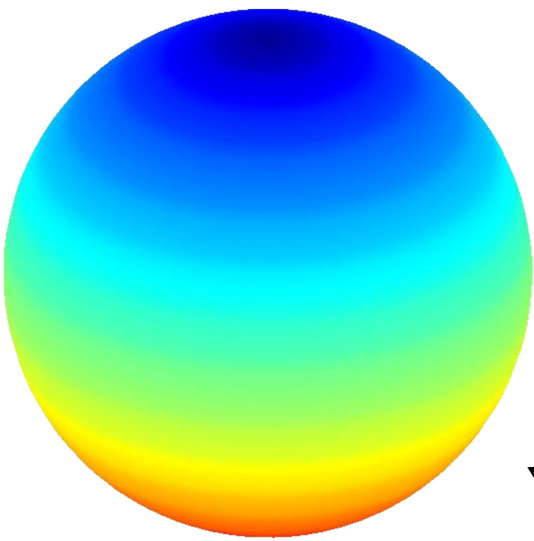
We consider the parametrization:

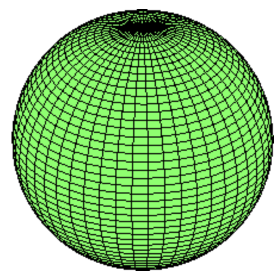
$$\mathbf{x} : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3$$

$$\mathbf{x}(u, v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$



Example: The sphere





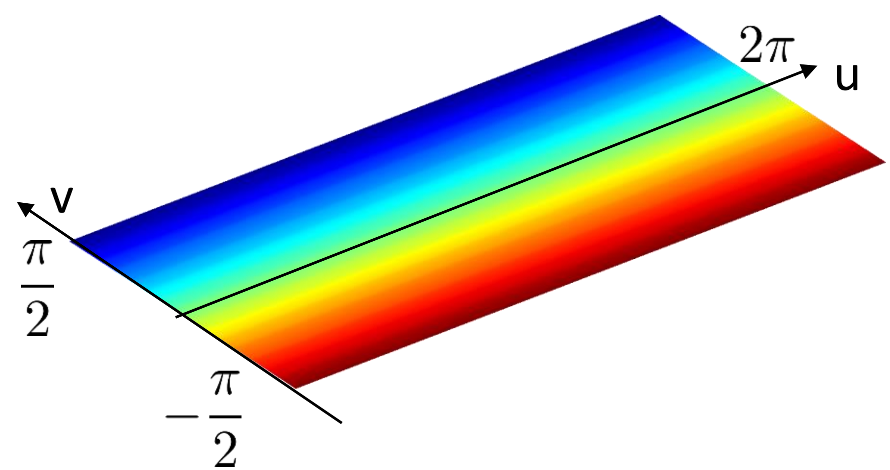
$$\mathbf{x} : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3$$

$$\mathbf{x}(u, v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$

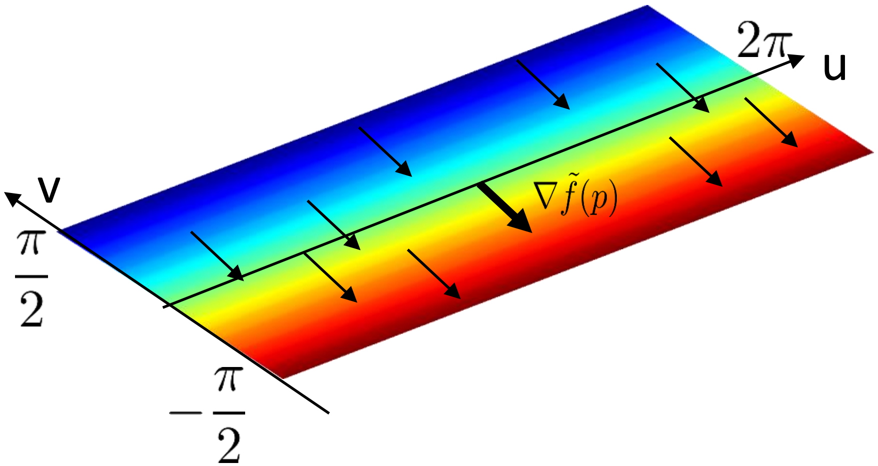
$\mathbf{x}$

$$f(p) = d_{\mathbb{S}^2}(n, p)$$

$$\tilde{f}(u, v) = \frac{\pi}{2} - v$$

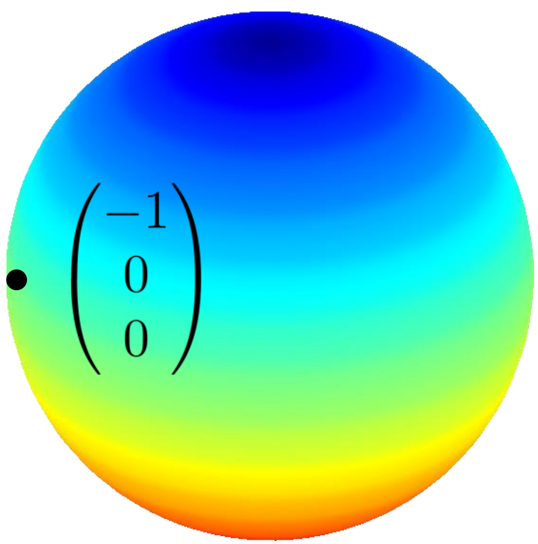


Example: gradients in R^2



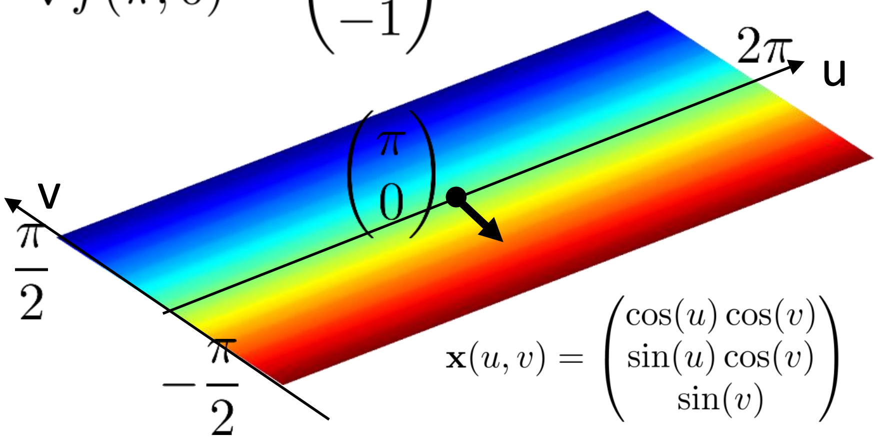
$$\tilde{f}(u,v) = \frac{\pi}{2} - v$$

$$\nabla \tilde{f}(u,v) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

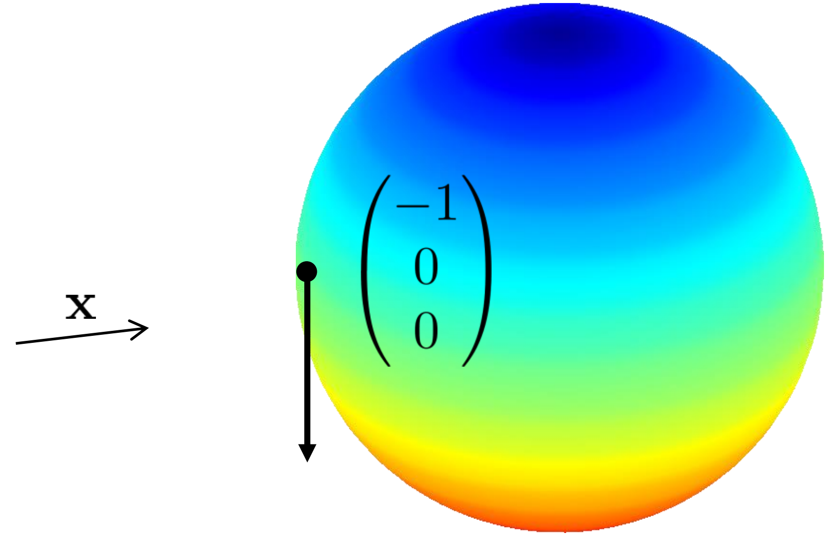


$$\nabla f(-1,0,0) = ?$$

$$\nabla \tilde{f}(\pi, 0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$



$$\mathbf{x}(u, v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$



$$\nabla f(-1, 0, 0) = \mathrm{D}\mathbf{x}\; g_{\mathbf{x}}^{-1} \nabla \tilde{f}(\pi, 0)$$

$$= \mathrm{D}\mathbf{x} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

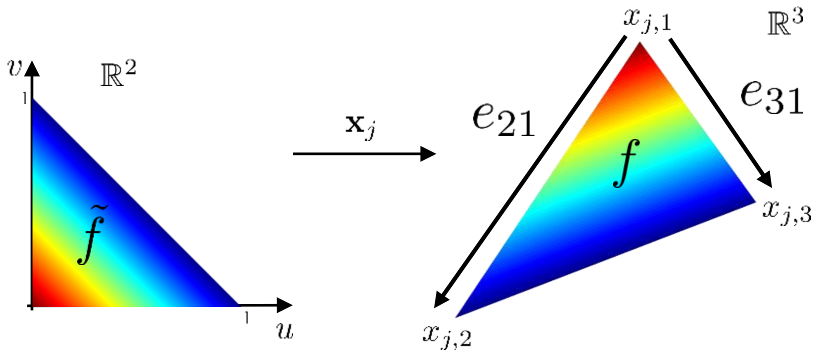
$$= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathrm{D}\mathbf{x}(\pi, 0) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g_{\mathbf{x}}(\pi, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (g_{\mathbf{x}}(\pi, 0))^{-1}$$

Discretization: The gradient

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



$$\mathrm{D}\mathbf{x} = (\mathbf{x}_u, \mathbf{x}_v) = (e_{21}, e_{31})$$

$$g_j = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31} \rangle \\ \langle e_{21}, e_{31} \rangle & \|e_{31}\|^2 \end{pmatrix}$$

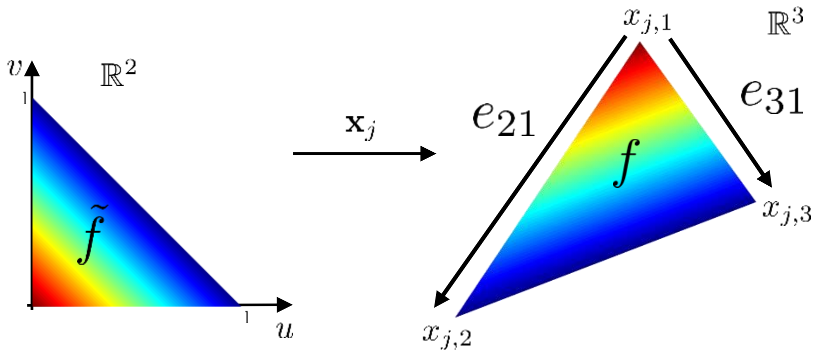
Recall that we are assuming functions to be linear within each triangle.

Our partial derivatives on the reference triangle can be simply approximated by the finite differences:

$$\frac{\partial \tilde{f}}{\partial u} = f(x_{j,2}) - f(x_{j,1}) \qquad \frac{\partial \tilde{f}}{\partial v} = f(x_{j,3}) - f(x_{j,1})$$

Discretization: The gradient

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



$$\mathrm{D}\mathbf{x} = (\mathbf{x}_u, \mathbf{x}_v) = (e_{21}, e_{31})$$

$$g_j = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31}\rangle \\ \langle e_{21}, e_{31}\rangle & \|e_{31}\|^2 \end{pmatrix}$$

The discrete gradient is then given by:

$$\nabla f = \mathrm{D}\mathbf{x}\ g^{-1}\nabla\tilde{f} = (e_{21}, e_{31}) \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}^{-1} \begin{pmatrix} f(x_{j,2}) - f(x_{j,1}) \\ f(x_{j,3}) - f(x_{j,1}) \end{pmatrix}$$

Also observe that: $\begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}^{-1} = \begin{pmatrix} G_j & -F_j \\ -F_j & E_j \end{pmatrix} \dfrac{1}{\det g_j}$

Discretization: Gradient norm

$$\|\nabla f\| = \sqrt{\nabla \tilde{f}^\top g^{-1} \nabla \tilde{f}}$$

We simply obtain:

$$\|\nabla f\| = \sqrt{(f_u, f_v) \begin{pmatrix} G_j & -F_j \\ -F_j & E_j \end{pmatrix} \begin{pmatrix} f_u \\ f_v \end{pmatrix} \frac{1}{\sqrt{\det g_j}}}$$

$$= \sqrt{\frac{f_u^2 G_j - 2 f_u f_v F_j + f_v^2 E_j}{\det g_j}}$$

Note that, since we take f to be linear, the gradient ∇f is constant within each triangle.

Discretization: Total variation

The total variation of a function $f : S \to \mathbb{R}$ is:

$$TV_S(f) = \sum_j \int_{T_j} \|\nabla f(x)\| da$$

We know how to compute integrals of functions on triangulated meshes. We simply get, for each triangle:

$$\begin{aligned}
\int_{T_j} \|\nabla f(x)\| da &= \int_0^1 \int_0^{1-u} \sqrt{\frac{f_u^2 - 2 f_u f_v F_j + f_v^2 E_j}{\det g_j}} \sqrt{\det g_j}\, du dv \\
&= \int_0^1 \int_0^{1-u} \sqrt{f_u^2 - 2 f_u f_v F_j + f_v^2 E_j}\, du dv \\
&= \frac{1}{2} \sqrt{f_u^2 - 2 f_u f_v F_j + f_v^2 E_j}
\end{aligned}$$