Fundamentals of Computer Graphics

Lengths and areas

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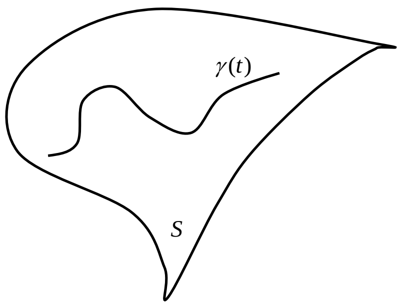
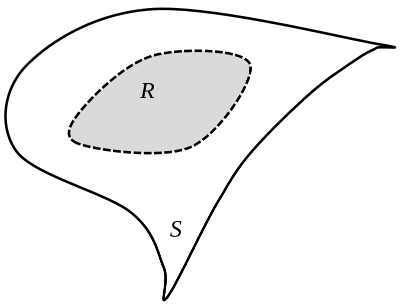
Measuring lengths and areas

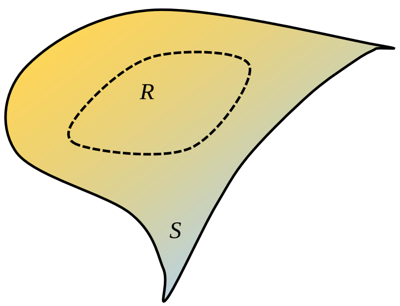
How to compute these?



Length of a curve



Area of a region



Integral of a function

$f : S \to \mathbf{R}$

First fundamental form

The quadratic form $I_p : T_p(S) \to \mathbf{R}$ given by

$$I_p(w) = \langle w, w \rangle_p = \|w\|^2$$

is called the first fundamental form of the regular surface *S* at *p*.

It is the key ingredient for computing lengths and areas on surfaces

First fundamental form

Let us denote by $\{\mathbf{x}_u, \mathbf{x}_v\}$ the basis spanning the tangent plane $T_p(S)$

Any vector $w \in T_p(S)$ is the tangent vector to a curve $\alpha(t) = \mathbf{x}(u(t), v(t))$ which lies on the surface, with $p = \alpha(0)$.

Then we can write:

chain rule

$$I_p(w) = I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle_p = \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle_p$$

$$= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p (u')^2 + 2 \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p u'v' + \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p (v')^2$$

$$= E(u')^2 + 2Fu'v' + G(v')^2$$

Metric tensor

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2$$

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p$$
$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p$$
$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

$$\Longrightarrow \quad I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

also called metric tensor

E(u,v), *F(u,v)*, and *G(u,v)* are the components of the first fundamental form

These play important roles in many intrinsic quantities of the surface

If *E*(*u*,*v*), *F*(*u*,*v*), *G*(*u*,*v*) are smooth, we have a Riemannian manifold

Metric tensor and Jacobian

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2$$

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p$$

$$F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p$$

$$G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

$$\Rightarrow \quad I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{\text{often called just } g} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

We have seen the Jacobian matrix:

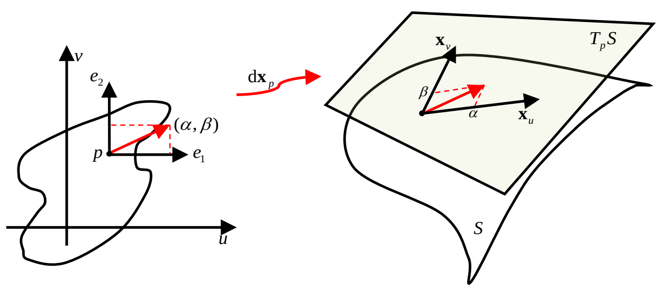
$$\mathrm{D}\mathbf{x} = \begin{pmatrix} \vdots & \vdots \\ \mathbf{x}_u & \mathbf{x}_v \\ \vdots & \vdots \end{pmatrix}$$

Then, it is easy to see that:

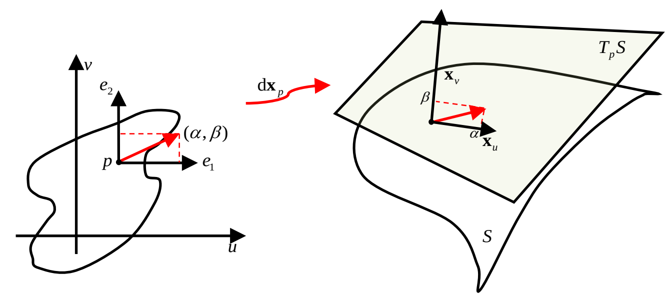
$$g = \mathrm{D}\mathbf{x}^{\mathrm{T}} \mathrm{D}\mathbf{x} = \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix}$$

Parametrizations

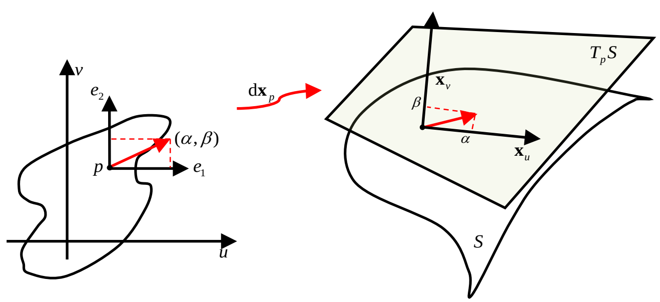
$$I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \qquad E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$



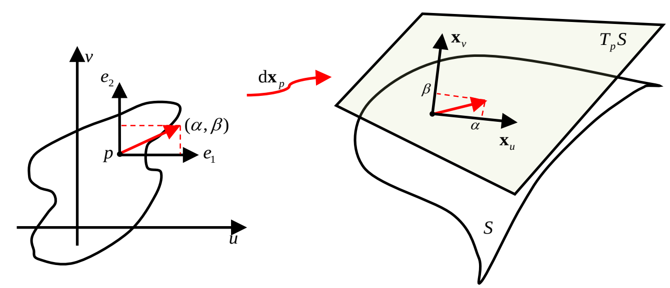
generic parametrization



orthogonal $F \equiv 0$



conformal $F \equiv 0\,, E = G$



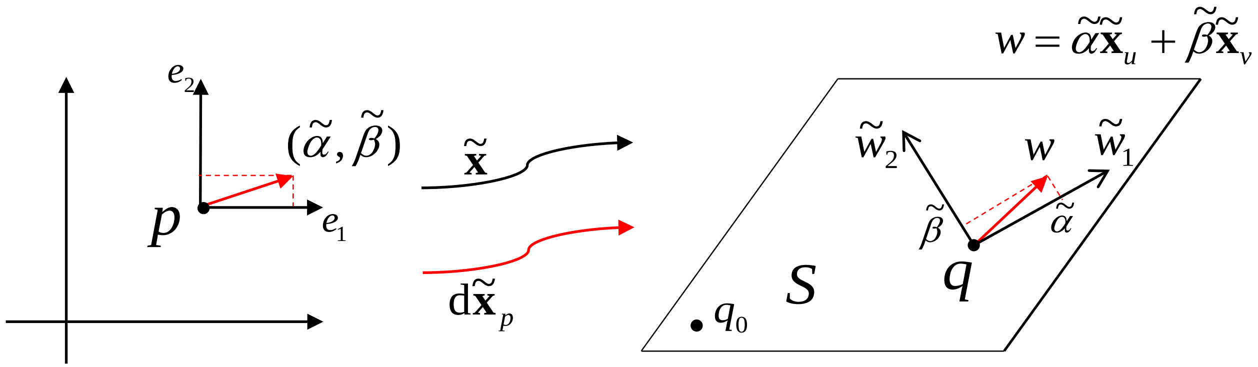
isometric $F \equiv 0\,, E = G = 1$

Example: The plane (1/3)

Consider a plane $S \subset \mathbf{R}^3$ passing through q_0 and containing the orthonormal vectors $\tilde{w}_1$ and $\tilde{w}_2$.

$$\tilde{\mathbf{x}}(u,v) = q_0 + u\tilde{w}_1 + v\tilde{w}_2 \quad \Rightarrow \quad \begin{aligned} \tilde{\mathbf{x}}_u &= \tilde{w}_1 \\ \tilde{\mathbf{x}}_v &= \tilde{w}_2 \end{aligned}$$

We want to compute the first fundamental form for an arbitrary point *q* in *S*.



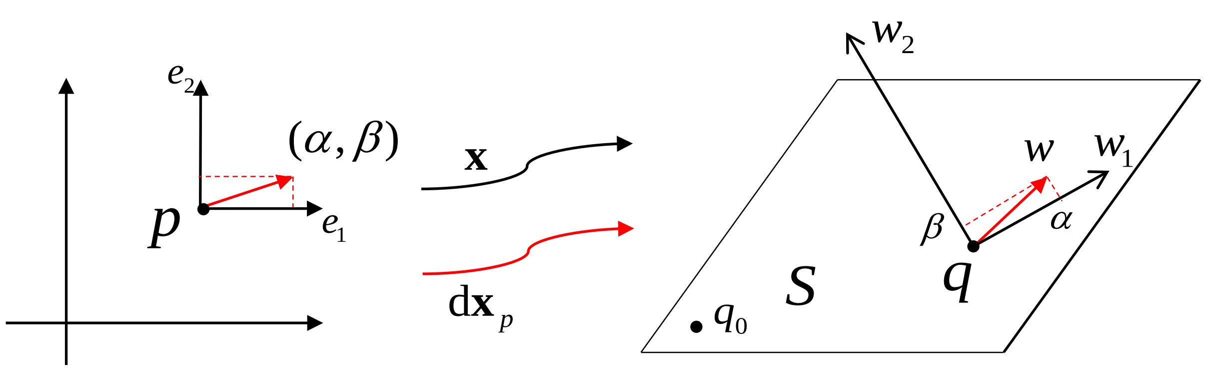
$$\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, the first fundamental form of *w* at *p* is $I_p((\tilde{\alpha}, \tilde{\beta})) = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

Example: The plane (2/3)

This time let $\|w_1\| = 1$ and $\|w_2\| = 2$.
We are changing the parametrization $\mathbf{x}$, but still we expect that the lengths of vectors in $T_p(S)$ do *not* change (as they are a property of the surface).

As before, we have $\mathbf{x}_u = w_1,\ \mathbf{x}_v = w_2$ and then $g = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$



previous example: $w = \tilde{\alpha}\tilde{\mathbf{x}}_u + \tilde{\beta}\tilde{\mathbf{x}}_v$

this example: $w = \alpha\mathbf{x}_u + \beta\mathbf{x}_v$

The two bases, and thus the coefficients for *w* are different in the two examples.

$$\boxed{\alpha\mathbf{x}_u + \beta\mathbf{x}_v \overset{!}{=} \tilde{\alpha}\tilde{\mathbf{x}}_u + \tilde{\beta}\tilde{\mathbf{x}}_v}$$

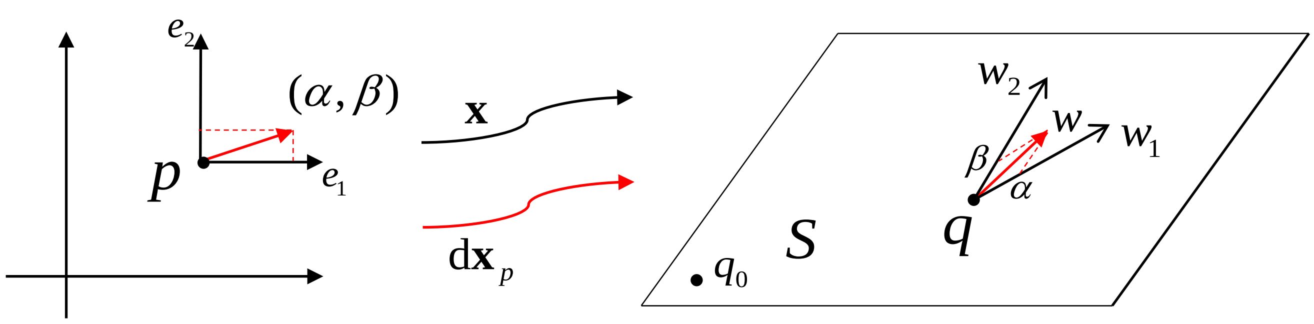
We can now compute $I_p((\alpha,\beta)) = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} & \frac{\tilde{\beta}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \frac{\tilde{\beta}}{2} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

Example: The plane (3/3)

Now let $\|w_1\| = 1, \|w_2\| = 1$, and $\langle w_1, w_2 \rangle = \dfrac{1}{\sqrt{2}}$.

We have $\mathbf{x}_u = w_1,\ \mathbf{x}_v = w_2$ and now $g = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix}$

Again we expect the first fundamental form to be the same as before.



$$\alpha w_1 + \beta w_2 \overset{!}{=} \tilde{\alpha}\tilde{w}_1 + \tilde{\beta}\tilde{w}_2$$

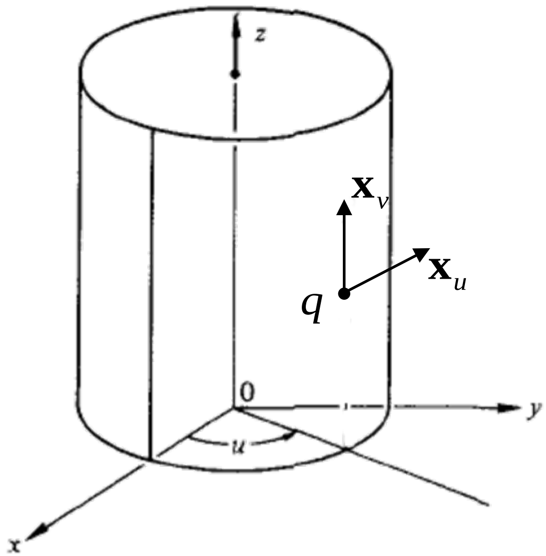
$$\Downarrow$$

$$w_1 = \tilde{w}_1,\quad w_2 = \frac{1}{\sqrt{2}}(\tilde{w}_1 + \tilde{w}_2)$$

$$\alpha = \tilde{\alpha} - \tilde{\beta},\quad \beta = \sqrt{2}\tilde{\beta}$$

So we get $I_p((\alpha,\beta)) = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} - \tilde{\beta} & \sqrt{2}\tilde{\beta} \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} - \tilde{\beta} \\ \sqrt{2}\tilde{\beta} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

Example: The cylinder



$$\mathbf{x}(u,v) = (\cos u, \sin u, v)$$
$$U = \{(u,v) \in \mathbf{R}^2;\ 0 < u < 2\pi,\ -\infty < v < \infty\}$$

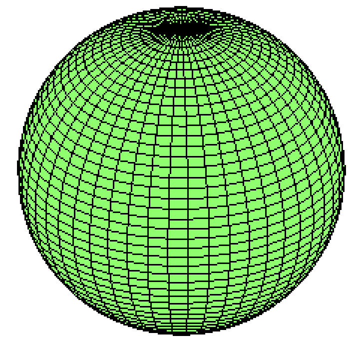
$$\mathbf{x}_u = (-\sin u, \cos u, 0),\ \ \mathbf{x}_v = (0,0,1)$$
$$E = \sin^2 u + \cos^2 u = 1$$
$$F = 0$$
$$G = 1$$
$$\Rightarrow \quad g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Plane and the cylinder behave locally in the same way, since their first fundamental forms are equal

We say that plane and cylinder are locally isometric

Example: The sphere

$$\mathbf{x} : (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3 \qquad \mathbf{x}(u,v) = \begin{pmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{pmatrix}$$



$$\mathrm{D}\mathbf{x} = \begin{pmatrix} \vdots & \vdots \\ \mathbf{x}_u & \mathbf{x}_v \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} -\sin(u)\cos(v) & -\cos(u)\sin(v) \\ \cos(u)\cos(v) & -\sin(u)\sin(v) \\ 0 & \cos(v) \end{pmatrix}$$

$$g = \mathrm{D}\mathbf{x}^{\mathrm{T}}\mathrm{D}\mathbf{x} = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix}$$

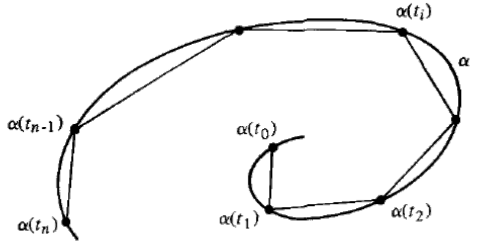
Here it is evident that *E*, *F*, *G* are indeed differentiable functions *E*(*u*,*v*), *F*(*u*,*v*), *G*(*u*,*v*).

Thus, if $w = \alpha \mathbf{x}_u + \beta \mathbf{x}_v$ is the tangent vector to the sphere at point **x**(*u*,*v*), then its squared length is given by $|w|^2 = I(w) = \alpha^2 \cos^2(v) + \beta^2$

Length of a curve

With the first fundamental form, we can treat metric questions on a regular surface without further references to the ambient space

arc-length of a curve $\alpha : (0,T) \to S$ $\quad s(t) = \int_0^t \|\alpha'(x)\| dx = \int_0^t \sqrt{I(\alpha'(x))}\, dx$



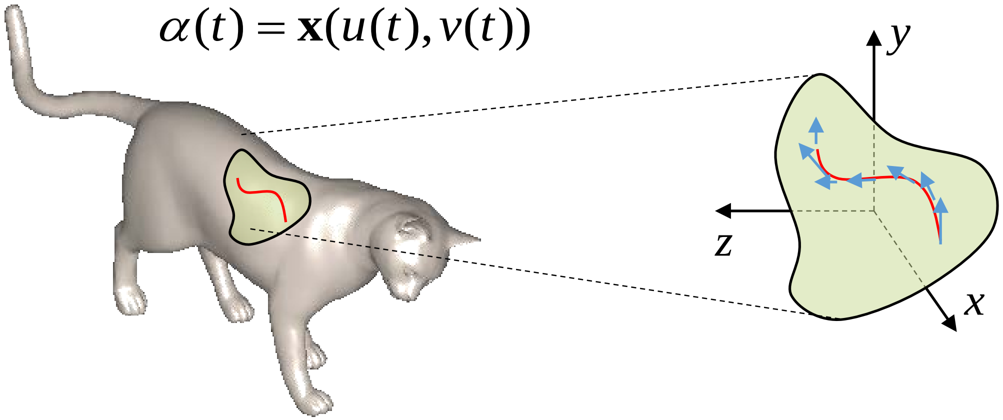
Remember that *E*, *F*, *G* are actually functions of (*u*,*v*), so in general they are changing along the curve.

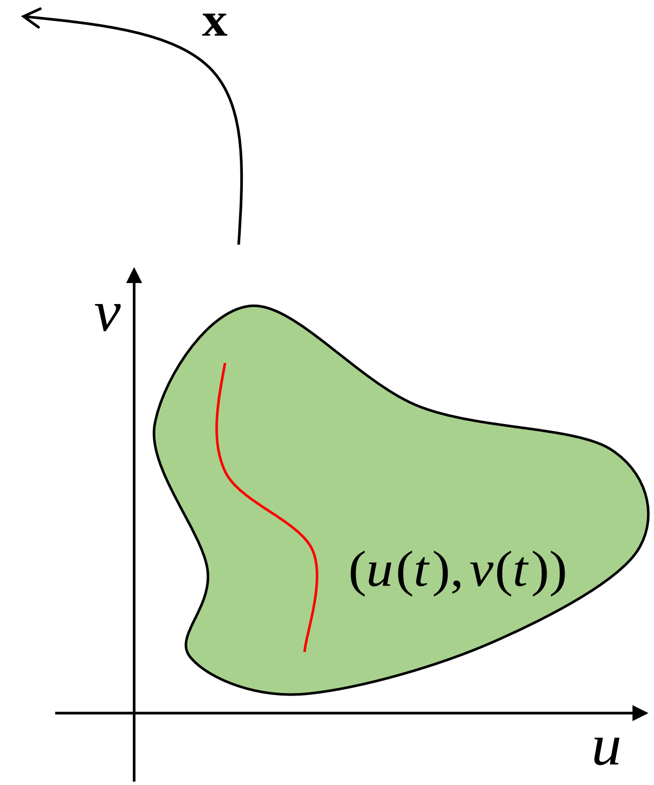
Thus, if $\alpha(t) = \mathbf{x}(u(t), v(t))$ is contained in a surface element parametrized by **x**(*u*,*v*), we can compute the length as:

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt$$

Length of a curve

$$\alpha(t) = \mathbf{x}(u(t), v(t))$$





length $$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\, dt$$

Arc length element

Length $s(t)=\int_0^t \|\alpha'(x)\| dx$

The first fundamental theorem of calculus gives us:

$$\frac{ds}{dt}=\|\alpha'(t)\|$$

$$\Downarrow$$

$$ds=\|\alpha'(t)\| dt$$

We get to the compact notation:

$$\mathrm{length}(\alpha)=\int_\alpha ds$$

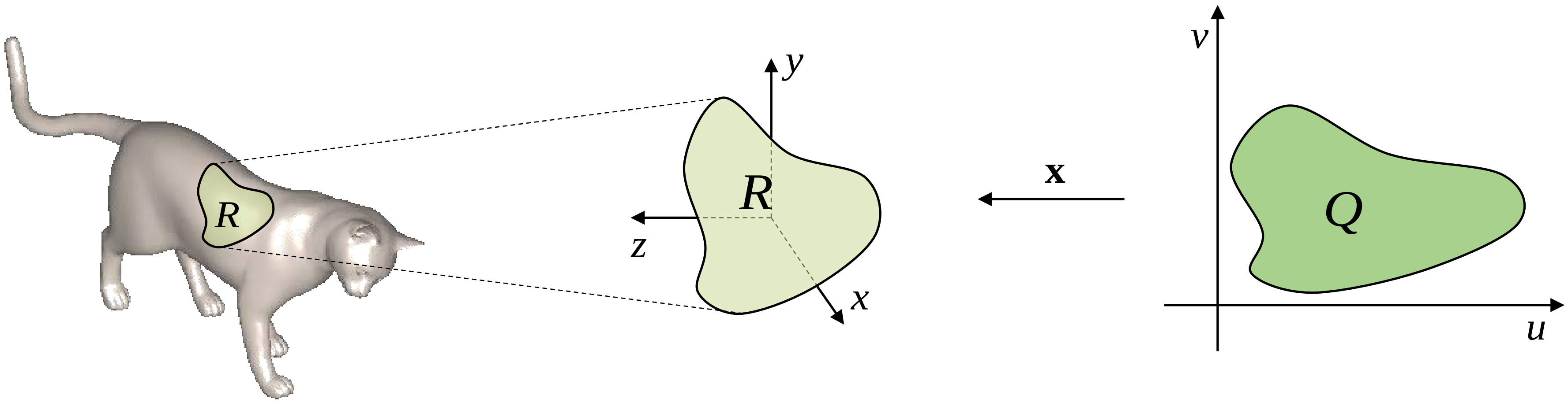
In terms of the metric tensor, the arc length element *ds* is given by:

$$ds=\sqrt{Edu^2+2Fdudv+Gdv^2}\,dt$$

Area of a region

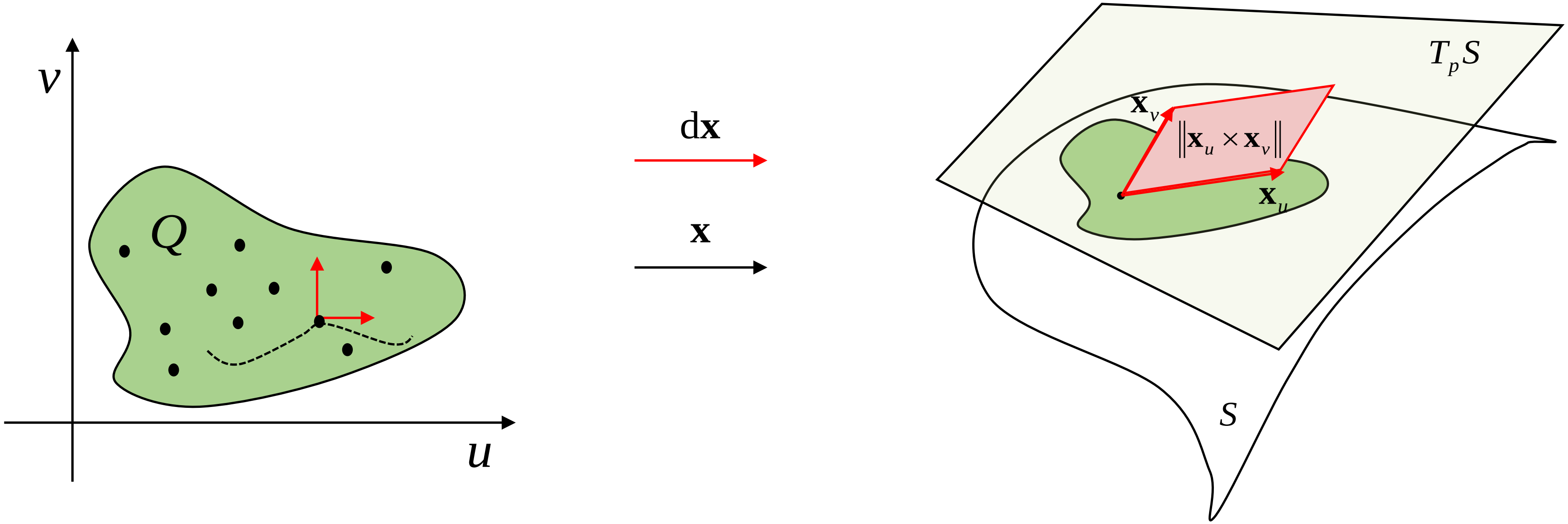
If $R \subset S$ is contained in the image of the parametrization $\mathbf{x}: U \subset \mathbf{R}^2 \to S$, the area of R is defined by

$$A(R) = \iint_Q \|\mathbf{x}_u \times \mathbf{x}_v\| dudv, \qquad Q = \mathbf{x}^{-1}(R)$$



Area of a region

$$A(R) = \iint_Q \|\mathbf{x}_u \times \mathbf{x}_v\| du dv, \qquad Q = \mathbf{x}^{-1}(R)$$



The area of a region on the surface is defined as the sum of the areas of parallelograms tangent to that surface region.

Area of a region

$$A(R) = \iint_Q \left\| \mathbf{x}_u \times \mathbf{x}_v \right\| du dv, \qquad Q = \mathbf{x}^{-1}(R) \qquad g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Observe that:

$$\left\| \mathbf{x}_u \times \mathbf{x}_v \right\|^2 = \left\| \mathbf{x}_u \right\|^2 \left\| \mathbf{x}_v \right\|^2 \sin^2 \omega = \left\| \mathbf{x}_u \right\|^2 \left\| \mathbf{x}_v \right\|^2 (1 - \cos^2 \omega) = \left\| \mathbf{x}_u \right\|^2 \left\| \mathbf{x}_v \right\|^2 - \left\langle \mathbf{x}_u, \mathbf{x}_v \right\rangle^2$$

We can then rewrite:

$$\left\| \mathbf{x}_u \times \mathbf{x}_v \right\| = \sqrt{\left\| \mathbf{x}_u \right\|^2 \left\| \mathbf{x}_v \right\|^2 - \left\langle \mathbf{x}_u, \mathbf{x}_v \right\rangle^2} = \sqrt{EG - F^2} = \sqrt{\det g}$$

We get the compact expression:

$$A(R) = \iint_Q \sqrt{\det g}\, du dv$$

Area element

We define the area element *da* as:

$$da = \sqrt{\det g}\, du dv$$

Leading to:

$$A(R) = \int_R da$$

The area element is also called (Riemannian) volume form

In the case of 2-dimensional manifolds (our case), volume corresponds to area

Wrap-up

We have two alternative expressions for measuring lengths and areas

One in parameter space, the other directly on the surface

Parameter space

$$\text{length}(\alpha) = \int_0^T \|\alpha'(t)\| dt = \int_0^T \sqrt{Edu^2 + 2Fdudv + Gdv^2}\, dt$$

Surface

$$\text{length}(\alpha) = \int_\alpha ds \qquad ds = \sqrt{Edu^2 + 2Fdudv + Gdv^2}\, dt$$

Parameter space

$$A(R) = \iint_Q \|\mathbf{x}_u \times \mathbf{x}_v\| dudv, \qquad Q = \mathbf{x}^{-1}(R)$$

Surface

$$A(R) = \int_R da \qquad da = \sqrt{\det g}\, dudv$$

Integral of a function

$$\int_R f(x)dx$$

We have the definition:

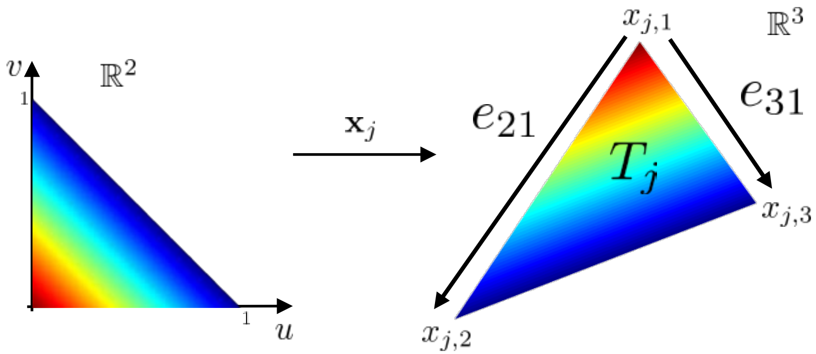
$$\int_R f(x)dx = \iint_Q f(\mathbf{x}(u,v))\sqrt{\det g}\,du\,dv, \qquad Q = \mathbf{x}^{-1}(R)$$

$$\int_{\phi(U)} f(\mathbf{v})\,d\mathbf{v} = \int_U f(\phi(\mathbf{u}))\,|\det(\mathrm{D}\phi)(\mathbf{u})|\,d\mathbf{u}.$$

Generalizes the substitution rule in classical multivariate calculus

Discretization: Metric tensor

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



We simply have:

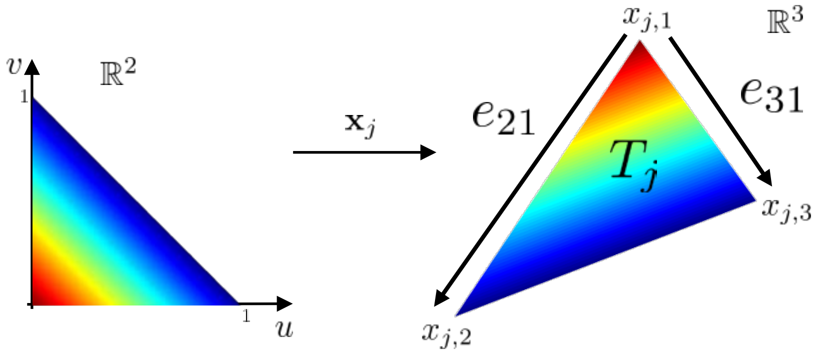
$$\mathbf{x}_u = x_{j,2} - x_{j,1} = e_{21}$$
$$\mathbf{x}_v = x_{j,3} - x_{j,1} = e_{31}$$

The coefficients for the metric tensor are thus given by:

$$g_j = \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix} = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31} \rangle \\ \langle e_{21}, e_{31} \rangle & \|e_{31}\|^2 \end{pmatrix}$$

Discretization: Area element

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



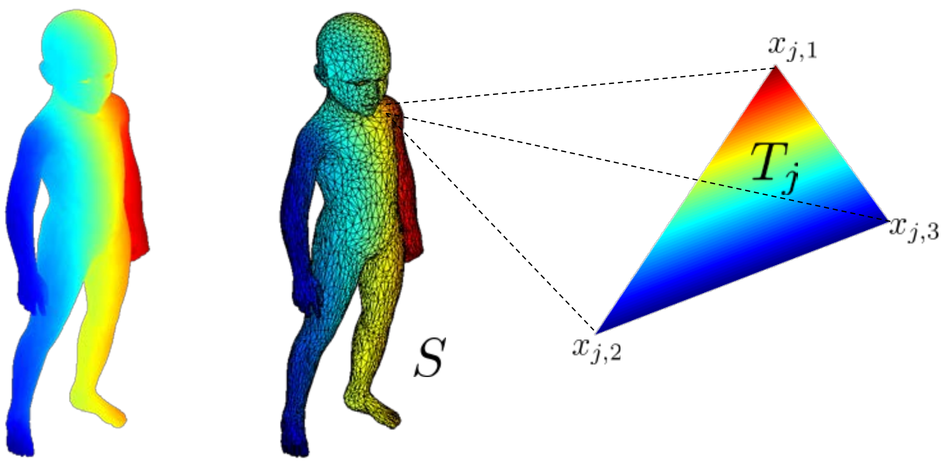
$$g_j = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31}\rangle \\ \langle e_{21}, e_{31}\rangle & \|e_{31}\|^2 \end{pmatrix}$$

The area of the triangle is the area of a region:

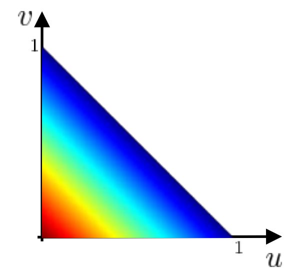
$$\int_{T_j} da = \int_0^1 \int_0^{1-u} \sqrt{\det g_j}\, du dv = 2A(T_j) \int_0^1 \int_0^{1-u} du dv = 2A(T_j)\frac{1}{2} = A(T_j)$$

Discretization: Integral of a function

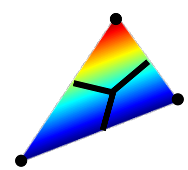
$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



$f : S \to \mathbb{R}$ behaves linearly within each triangle and it is uniquely determined by its values at the vertices of the triangle.

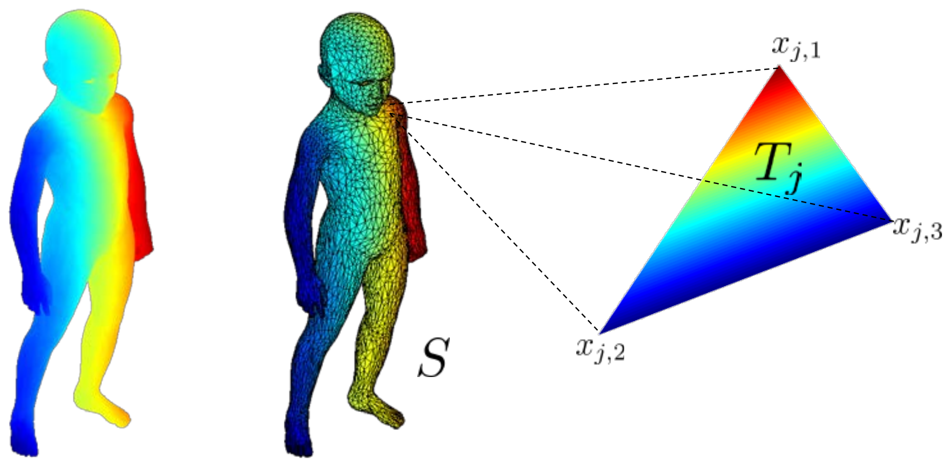


$$\begin{aligned}
\int_{T_j} f \; da &= \int_0^1 \int_0^{1-u} f(\mathbf{x}(u,v)) \sqrt{\det g_j} du dv \\
&= \int_0^1 \int_0^{1-u} f(x_{j,1})(1-u-v) + f(x_{j,2})u + f(x_{j,3})v \sqrt{\det g_j} du dv \\
&= \frac{1}{6}(f(x_{j,1}) + f(x_{j,2}) + f(x_{j,3})) 2A(T_j) \\
&= \frac{1}{3}(f(x_{j,1}) + f(x_{j,2}) + f(x_{j,3})) A(T_j)
\end{aligned}$$



Discretization: Integral of a function

$$\mathbf{x}_j(u,v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



$f : S \to \mathbb{R}$ behaves linearly within each triangle and it is uniquely determined by its values at the vertices of the triangle.

The integral of *f* over a region $R \subseteq S$ is just the sum:

$$\int_R f \; da = \sum_{j=1}^{|R|} \int_{T_j} f \; da$$

Exercise: Integral of a function

Write the code to compute the integral of a function on a triangle mesh

Test it by computing the integral of the constant function $f(x)=1$, and check if it returns the total surface area

Suggested reading

- *Differential geometry of curves and surfaces*. Do Carmo – Chapters 2.5, Appendix 2.B