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COMPUTER VISION

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HOMEWORK 1

1 - Perspective Projection

1 - We want to find the vanishing points of lines on a plane (P)

Let us define this plane by a point A and two non-colinear vectors $\vec{u}$ and $\vec{v}$ ($\vec{u} \notin \text{span}(\vec{v})$ and $\vec{v} \notin \text{span}(\vec{u})$).

Let us choose a line $\mathcal{L}$ on this plane.

$\mathcal{L}$ is defined by a point $B \in (P)$ and a vector $\vec{w} \in \text{span}(\vec{u}, \vec{v}) \setminus \{\vec{0}\}$

As A could have been defined as any point of (P), we can safely assume $A \equiv B$.

We have $\vec{w} \in \text{span}(\vec{u}, \vec{v}) \setminus \{\vec{0}\}$, so there exists $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{0,0\}$ such that:

$$\vec{w} = \alpha \vec{u} + \beta \vec{v}$$

Let us notice that $\forall t \in \mathbb{R}$, $t\vec{w} = \alpha t \vec{u} + \beta t \vec{v}$ still defines the same direction and thus does not change our line $\mathcal{L}$.

Let us further assume $\begin{cases} w_z \neq 0 \\ (u_z, v_z) \neq (0,0) \end{cases}$. Then we can scale back by w_z:

Let us define $\alpha' = \frac{\alpha}{w_z}$ and $\beta' = \frac{\beta}{w_z}$

Therefore, let us redefine $\vec{w}$ as $\vec{w} = \alpha'\vec{u} + \beta'\vec{v}$

Let us now take a point M_λ of our line $\mathcal{L}$: $M_\lambda = A + \lambda\vec{w}$.

Then we have the following coordinates of M_λ on the image:

$$x_\lambda = \frac{f(A_x + \lambda\alpha' u_x + \lambda\beta' v_x)}{A_z + \lambda\underbrace{(\alpha' u_z + \beta' v_z)}_{1}} \qquad y_\lambda = \frac{f(A_y + \lambda\alpha' u_y + \lambda\beta' v_y)}{A_z + \lambda\underbrace{(\alpha' u_z + \beta' v_z)}_{1}}$$

$$x_\infty = \lim_{\lambda\to+\infty} x_\lambda = f(\alpha' u_x + \beta' v_x) \qquad y_\infty = \lim_{\lambda\to+\infty} y_\lambda = f(\alpha' u_y + \beta' v_y)$$

Let us recall that
$$\begin{cases} u_x, u_y, u_z, v_x, v_x, v_y, v_z \text{ are fixed (i)} \\ \alpha' u_z + \beta' v_z = 1 \text{ (ii)} \\ w_z \neq 0 \text{ (iii)} \end{cases}$$

$w_z \neq 0$ implies that $(u_z, v_z) \neq (0,0)$

For example, let us assume $v_z \neq 0$ (the problem is symmetric in $u \leftrightarrow v$)

Then, using (ii): $\beta' = \frac{1 - \alpha' u_z}{v_z}$

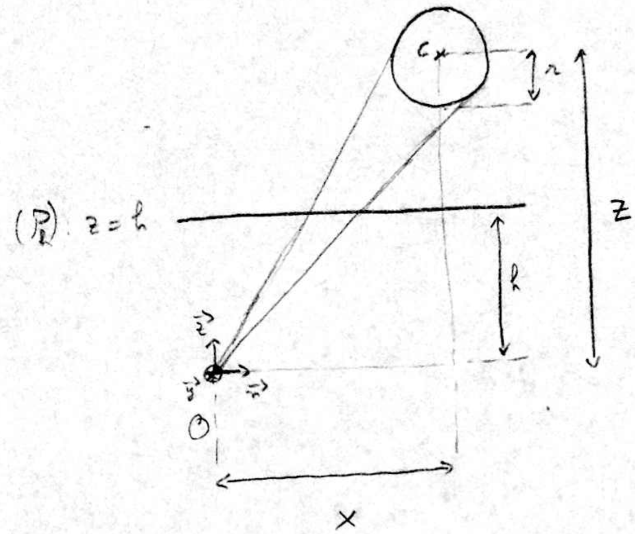
Therefore,
$$\begin{cases} x_\infty = f\left(\alpha' u_x + \frac{1-\alpha' u_z}{v_z} \times v_x\right) \\ y_\infty = f\left(\alpha' u_y + \frac{1-\alpha' u_z}{v_z} \times v_y\right) \end{cases}$$

Using (i), we finally notice that all entries above are fixed parameters of the problem, except α'.

Moreover, (x_∞, y_∞) is uniquely and linearly parametrized by α'.

This α' parametrizes the vanishing line of the vanishing points of lines of plane (P). <u>QED</u>

2 -



Hypotheses: The "observer" is in $O\,(0,0,0)$

The plane projection is vertical : $(P_h): z = h$ with h a parameter (we will discuss later conditions on the value of h).

Let C be the center of the sphere

Then $\vec{OC} : \begin{pmatrix} X \\ 0 \\ Z \end{pmatrix}$

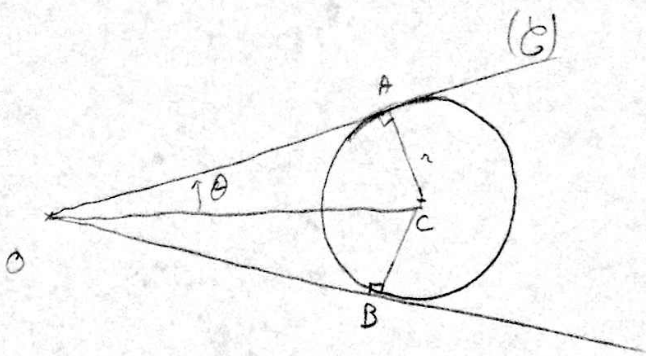
Let us define $\vec{u} = \dfrac{\vec{OC}}{\|\vec{OC}\|} = \begin{pmatrix} \dfrac{X}{\sqrt{X^2+Z^2}} \\ 0 \\ \dfrac{Z}{\sqrt{X^2+Z^2}} \end{pmatrix}$

Let θ be the angle of the cone $(\mathcal{C})$ going through O and tangent to the sphere:

we have:

$\sin\theta = \dfrac{r}{OC}$

$\sin\theta = \dfrac{r}{\sqrt{X^2+Z^2}}$



As O is the origin of $(\mathcal{C})$, we can either talk about vectors or points pertaining to $(\mathcal{C})$.

$\forall\ \vec{v} \in \mathbb{R}^3,\ \vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in (\mathcal{C}) \iff \vec{U} \cdot \frac{\vec{v}}{\|\vec{v}\|} = \cos\theta$ (remember that $\|\vec{U}\| = 1$).

$$\iff \frac{Xx + Zz}{\sqrt{X^2+Z^2}\sqrt{x^2+y^2+z^2}} = \cos\theta$$

$$\iff \frac{Xx + Zz}{\sqrt{X^2+Z^2}\sqrt{x^2+y^2+z^2}} = \sqrt{1-\sin^2\theta}$$ ($\cos\theta \geq 0$ because θ obviously $\in [0, \frac{\pi}{2}]$)

$$\iff \frac{Xx + Zz}{\sqrt{X^2+Z^2}\sqrt{x^2+y^2+z^2}} = \sqrt{1 - \frac{r^2}{X^2+Z^2}}$$

$$\iff \frac{(Xx + Zz)^2}{(X^2+Z^2)(x^2+y^2+z^2)} = 1 - \frac{r^2}{X^2+Z^2}$$ ($Xx + Zz \geq 0$ from the beginning because $\cos\theta \geq 0$)

$$\iff (Xx + Zz)^2 = (X^2+Z^2)(x^2+y^2+z^2) - r^2(x^2+y^2+z^2)$$

$$\iff x^2X^2 + Z^2z^2 + 2XxZz = (X^2+Z^2)(x^2+y^2+z^2) - r^2(x^2+y^2+z^2)$$

$$\iff x^2(Z^2 - r^2) + z^2(X^2 - r^2) + y^2(X^2 + Z^2 - r^2) - 2XxZz = 0$$

Therefore, $\forall\ \vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3,\ \vec{v} \in (\mathcal{C}) \cap (P_h)$

$$\iff \begin{cases} z = h \\ x^2(Z^2-r^2) + z^2(X^2-r^2) + y^2(X^2+Z^2-r^2) - 2XxZz = 0 \end{cases}$$

$$\iff \begin{cases} z = h \\ x^2(Z^2-r^2) + h^2(X^2-r^2) + y^2(X^2+Z^2-r^2) - 2XxZh = 0 \quad (*) \end{cases}$$

(*) is the quadratic form of a conic, thus allowing to compute the eccentricity:

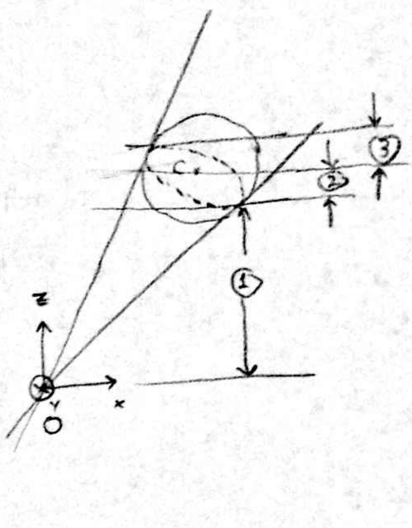
if $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,

we have $e = \sqrt{\dfrac{2\sqrt{(A-C)^2 + B^2}}{(A+C) + \sqrt{(A-C)^2 + B^2}}}$

Here we have $\begin{cases} A = z^2 - r^2 \\ B = 0 \\ C = x^2 + z^2 - r^2 \end{cases}$

By setting B to 0, e simplifies to: $e = \sqrt{\dfrac{|A-C|}{C}}$

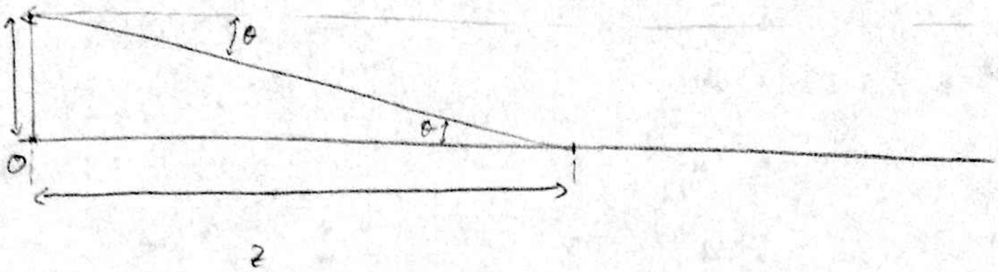
$$\boxed{e = \frac{x}{\sqrt{x^2+z^2-r^2}}}.$$



Yes, there are cases in which the planar projection is not an ellipse. Consider the 3 regions in the figure.

① Ellipse
② Parabola
③ Hyperbola.

3.



The observer is in O, has an eye at a height of h and is looking at an object on the ground in front of him at a depth $z \neq 0$

We have $\tan\theta = \dfrac{h}{z}$.

We want to study the impact of an error of $\delta\theta$ on the real value of θ on the estimated depth z':

$\tan(\theta + \delta\theta) = \dfrac{h}{z'}$, with $|\delta\theta| \ll 1$.

$$z' - z = h\left(\frac{1}{\tan\theta} - \frac{1}{\tan(\theta+\delta\theta)}\right)$$

$$\frac{1}{\tan(\theta+\delta\theta)} = \frac{\cos(\theta+\delta\theta)}{\sin(\theta+\delta\theta)} = \frac{\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta}{\sin\theta\cos\delta\theta + \cos\theta\sin\delta\theta}$$

$$= \frac{\cos\theta - \sin\theta \times \delta\theta + o(\delta\theta)}{\sin\theta + \cos\theta\,\delta\theta + o(\delta\theta)}$$

$$= \frac{\frac{1}{\tan\theta} - \delta\theta + o(\delta\theta)}{1 + \frac{\delta\theta}{\tan\theta} + o(\delta\theta)}$$

$$= \left(\frac{1}{\tan\theta} - \delta\theta + o(\delta\theta)\right)\left(1 - \frac{\delta\theta}{\tan\theta} + o(\delta\theta)\right)$$

$$= \frac{1}{\tan\theta} - \frac{\delta\theta}{\tan^2\theta} - \delta\theta + o(\delta\theta)$$

$\hookrightarrow$ $z - z' = h \times \delta\theta\left(1 + \dfrac{1}{\tan^2\theta}\right)$

$$\frac{z - z'}{z} = \delta\theta\left(\tan\theta + \frac{1}{\tan\theta}\right)$$

$$\frac{z' - z}{z} = \delta\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$\frac{z' - z}{z} = \frac{\delta\theta}{\sin\theta\cos\theta} = \frac{\delta\theta\,(z^2 + h^2)}{z h}$$

$$\boxed{z' - z = \frac{\delta\theta\,(z^2 + h^2)}{h}}$$

2- Rotations

First off, let us recall the following formula for cross-product:

$a \times (b \times c) = b(a.c) - c(a.b)$ with $\times$: cross-product
$\cdot$: dot-product.

For u, v two vectors, let us define the sequence:

$$\begin{cases} u_1 = u \times v \\ \forall n \geq 1, \ u_{n+1} = u \times u_n \end{cases}$$

Let us find an explicit relationship between u_n, u and v.

More explicitely, let us prove that:

$\forall k \in \mathbb{N}^*$, $u_{2k-1} = (-1)^{k+1} \|u\|^{2k-2} u \times v$

$u_{2k} = (-1)^{k+1} \|u\|^{2k-2} ((u.v)u - \|u\|^2 v)$

For $k = 1$: $\begin{cases} u_1 = u \times v \quad \checkmark \\ u_2 = u \times u_1 = u \times (u \times v) = (u.v)u - \|u\|^2 v \quad \checkmark \end{cases}$

Let $k \in \mathbb{N}^*$.

Let us assume $\begin{cases} u_{2k-1} = (-1)^{k+1} \|u\|^{2k-2} u \times v \\ u_{2k} = (-1)^{k+1} \|u\|^{2k-2} ((u.v)u - \|u\|^2 v) \end{cases}$

* $u_{2k+1} = u \times u_{2k}$

$= (-1)^{k+1} \|u\|^{2k-2} \left((u.v) \underbrace{u \times u}_{0} - \|u\|^2 u \times v \right)$

$u_{2k+1} = (-1)^{k+2} \|u\|^{2k} u \times v \quad \checkmark$

* $u_{2k+2} = u \times u_{2k+1}$

$= (-1)^{k+2} \|u\|^{2k} u \times (u \times v)$

$= (-1)^{k} \|u\|^{2k} ((u.v)u - \|u\|^2 v) \quad \checkmark$

We have therefore proven that $\forall\, k \in \mathbb{N}^*$: $\begin{cases} u_{2k-1} = (-1)^{k+1} \|u\|^{2k-2} \; u \times v \\ u_{2k} = (-1)^{k+1} \|u\|^{2k-2} \left((u.v)\, u - \|u\|^2 v \right) \end{cases}$

Let us now go back to our problem:

we have $\begin{cases} \hat{s} \text{ a unit vector} \\ \theta \text{ a scalar angle} \end{cases}$ and $s = \theta\, \hat{s}$. (we assume $\theta \geq 0$ (otherwise $\begin{matrix} \theta \leftarrow -\theta \\ s \leftarrow -s \end{matrix}$))

Let S be the matrix associated with the cross-product with s

Then $\exp(S) = \sum_{k=0}^{+\infty} \frac{1}{k!} S^k$

For any vector v, we have $\exp(S)\, v = \sum_{k=0}^{+\infty} \frac{1}{k!} S^k v$

$$= v + \sum_{k=1}^{+\infty} \left(\frac{1}{(2k-1)!} S^{2k-1} v + \frac{1}{(2k)!} S^{2k} v \right)$$

Using the previous result, this yields:

$$\exp(S)\, v = v + \sum_{k=1}^{+\infty} \left(\frac{1}{(2k-1)!} (-1)^{k+1} \|s\|^{2k-2} s \times v + \frac{1}{(2k)!} (-1)^{k+1} \|s\|^{2k-2} \left((s.v)s - \|s\|^2 v \right) \right)$$

$$= v + \sum_{k=1}^{+\infty} \left(\frac{1}{(2k-1)!} (-1)^{k+1} \theta^{2k-1} \hat{s} \times v + \frac{1}{(2k)!} (-1)^{k+1} \theta^{2k} \left((\hat{s}.v)\hat{s} - v \right) \right)$$

$\exp(S)\, v = v + \sin\theta\; \hat{s} \times v - (\cos\theta - 1)\left((\hat{s}.v)\hat{s} - v \right)$ by identifying the series development of $\cos\theta$ and $\sin\theta$.

$\exp(S)\, v = \cos\theta\, v + \sin\theta\; \hat{s} \times v + (1 - \cos\theta)(\hat{s}.v)\hat{s}$.

3 - Geometry

The transformation E can be defined as followed:

$$E_{\theta, t_1, t_2} : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Let us now define our loss function:

$$J(\theta, t_1, t_2) = \sum_{j=1}^{4} \| E_{\theta, t_1, t_2}(u_j) - v_j \|^2$$

$$= \sum_{j=1}^{4} \left\| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_j^1 \\ u_j^2 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} - \begin{pmatrix} v_j^1 \\ v_j^2 \end{pmatrix} \right\|^2$$

$$= \sum_{j=1}^{4} \left\| \begin{pmatrix} \cos\theta\, u_j^1 - \sin\theta\, u_j^2 + t_1 - v_j^1 \\ \sin\theta\, u_j^1 + \cos\theta\, u_j^2 + t_2 - v_j^2 \end{pmatrix} \right\|^2$$

$$= \sum_{j=1}^{4} (\cos\theta\, u_j^1 - \sin\theta\, u_j^2 + t_1 - v_j^1)^2 + (\sin\theta\, u_j^1 + \cos\theta\, u_j^2 + t_2 - v_j^2)^2$$

With the values of our problem, we obtain the following:

$$J(\theta, t_1, t_2) = (-3\cos\theta + t_1)^2 + (-3\sin\theta + t_2 - 3)^2 + (\cos\theta - \sin\theta + t_1 - 1)^2 + (\sin\theta + \cos\theta + t_2)^2$$
$$+ (\cos\theta + t_1)^2 + (\sin\theta + t_2)^2 + (\cos\theta + \sin\theta + t_1 + 1)^2 + (\sin\theta - \cos\theta + t_2)^2$$

To find the optimal values of our parameters, we are going to solve:

$$\nabla J(\theta^*, t_1^*, t_2^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, let us compute the derivatives.

$$\frac{\partial J}{\partial \theta}(\theta, t_1, t_2) = 2\Big(3\sin\theta(-3\cos\theta + t_1) - 3\cos\theta(-3\sin\theta + t_2 - 3) - (\cos\theta + \sin\theta)(\cos\theta - \sin\theta + t_1 - 1) + (\cos\theta - \sin\theta)(\sin\theta + \cos\theta + t_2) - \sin\theta(\cos\theta + t_1) + \cos\theta(\sin\theta + t_2) + (-\sin\theta + \cos\theta)(\cos\theta + \sin\theta + t_1 + 1) + (\cos\theta + \sin\theta)(\sin\theta - \cos\theta + t_2)\Big) \quad (1)$$

$$\frac{\partial J}{\partial t_1}(\theta, t_1, t_2) = 2\Big(-3\cos\theta + t_1 + \cancel{\cos\theta} - \cancel{\sin\theta} + t_1 - 1 + \cancel{\cos\theta} + t_1 + \cancel{\cos\theta} + \cancel{\sin\theta} + t_1 + 1\Big) = 8t_1$$

$$\frac{\partial J}{\partial t_2}(\theta, t_1, t_2) = 2\Big(-3\sin\theta + t_2 - 3 + \cancel{\sin\theta} + \cancel{\cos\theta} + t_2 + \cancel{\sin\theta} + t_2 + \sin\theta - \cos\theta + t_2\Big) = 8t_2 - 6$$

Setting $\frac{\partial J}{\partial t_1}(\theta^*, t_1^*, t_2^*)$ and $\frac{\partial J}{\partial t_2}(\theta^*, t_1^*, t_2^*)$ to zero yields: $\begin{cases} t_1 = 0 \\ t_2 = \frac{3}{4} \end{cases}$

Then injecting these results in $\frac{\partial J}{\partial \theta}(\theta^*, t_1^*, t_2^*) = 0$ gives the equation:

$$\cancel{-9\sin\theta^*\cos\theta^*} + \cancel{9\sin\theta^*\cos\theta^*} - 3\cos\theta^*\underbrace{\left(\tfrac{3}{4} - 3\right)}_{-\frac{9}{4}} - \cancel{(\cos^2\theta^* - \sin^2\theta^*)} + \cos\theta^* + \sin\theta^*$$
$$+ \cancel{\cos^2\theta^* - \sin^2\theta^*} + \frac{3}{4}(\cos\theta^* - \sin\theta^*) - \cancel{\cos\theta^*\sin\theta^*} + \cancel{\cos\theta^*\sin\theta^*} + \frac{3}{4}\cos\theta^*$$
$$+ \cancel{(\cos^2\theta^* - \sin^2\theta^*)} - \sin\theta^* + \cos\theta^* + \cancel{\sin^2\theta^* - \cos^2\theta^*} + \frac{3}{4}\cos\theta^* + \frac{3}{4}\sin\theta^* = 0$$

$$\Leftrightarrow \cos\theta^*\left(\frac{27}{4} + 1 + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4}\right) + \sin^*\left(1 - \frac{3}{4} - 1 + \frac{3}{4}\right) = 0$$

$$\Leftrightarrow \cos\theta^* = 0$$

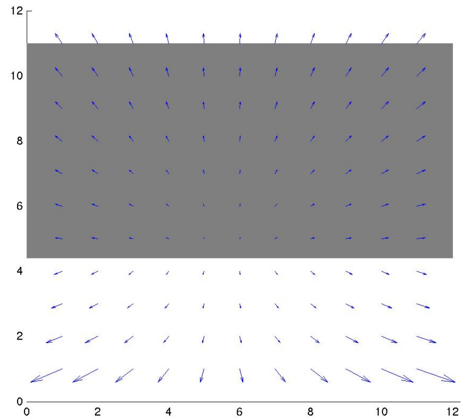
$$\Leftrightarrow \theta^* = \pm\frac{\pi}{2}.$$

CS280 HW

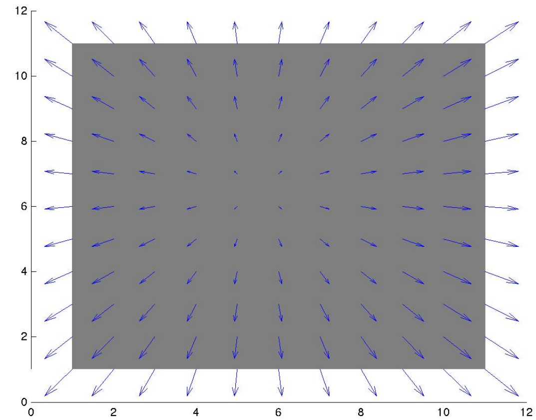
4. OPTICAL FLOW

Results:

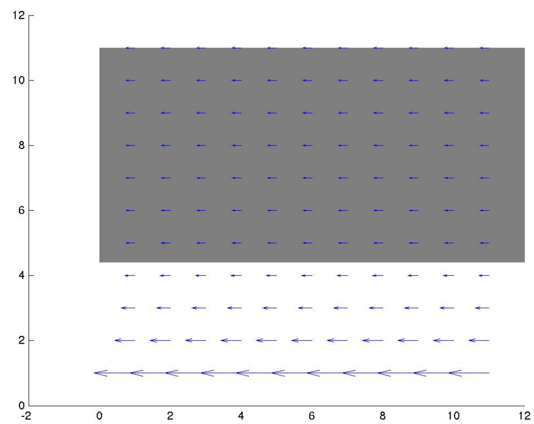
Car (on a road, moving to wall)



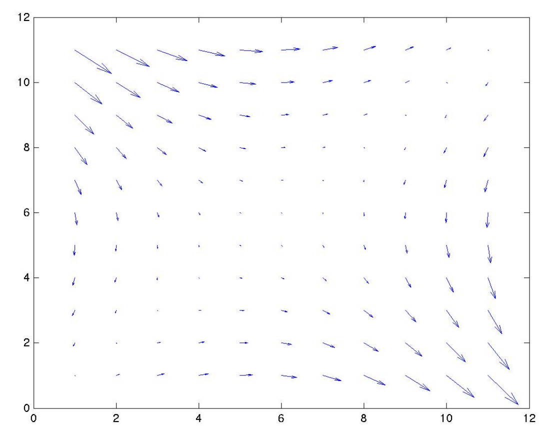
Plane moving towards ground



Train (ground outside, wall in distance)



Rotation



Code Snippet:

```
% TRAIN
u_train = (-train_vel./train_scene());

% CAR
for i = 1:M
    for j = 1:M
            v_car(i,j) = (car_vel*(i-M*k_road)/train_scene(i,j));
            u_car(i,j) = (car_vel*(j-M/2)/train_scene(i,j));
    end
end

% PLANE
for i = 1:M
```

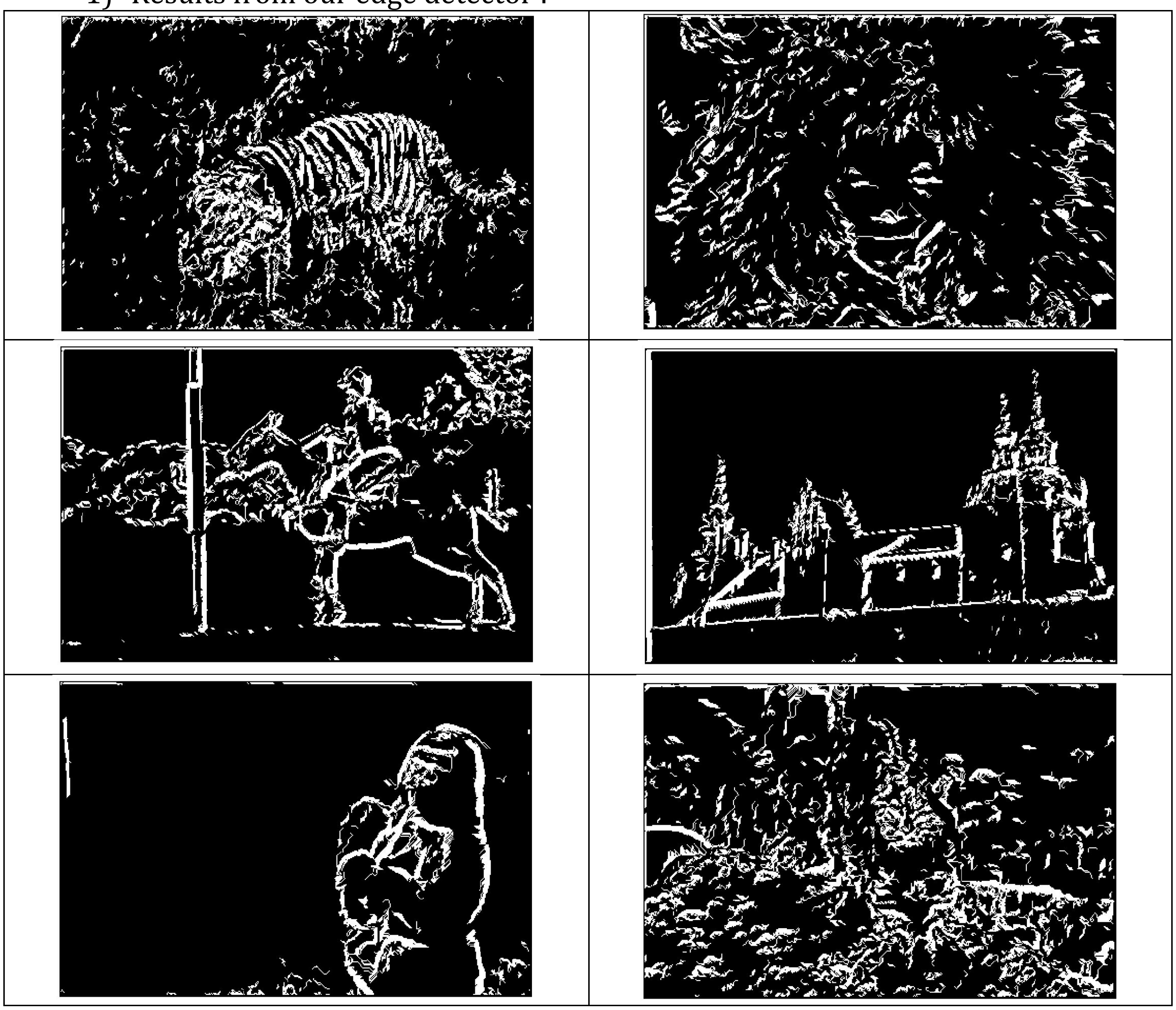
```
    for j = 1:M
            v_plane(i,j) = (-plane_y_vel+plane_z_vel*(i-
M*0.5))/plane_scene(i,j);
            u_plane(i,j) = (plane_z_vel*(j-M*0.5)/plane_scene(i,j));
    end
end

w = 10; %wx = wy

% ROTATION
for i = 1:M
    for j = 1:M
            v_rot(i,j) = -(1+(j-M/2)^2)*w*0.5 + (i-M/2)*(j-M/2)*w*0.5;
            u_rot(i,j) = (1+(i-M/2)^2)*w*0.5 - (i-M/2)*(j-M/2)*w*0.5;
    end
end
```

5. EDGE DETECTION

1) Results from our edge detector :



2) The canny edge detector gives thinner edges than our result, and also does a better job at connecting edges.