Límites - Teoría

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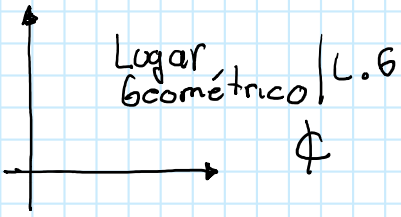
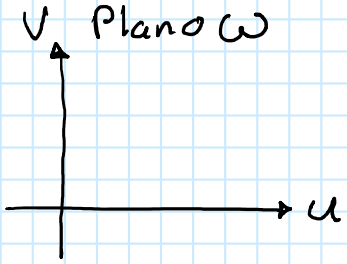
• Plano z





$F(z)$
- Trigonométrica
- Exponencial
- Logaritmica
- Polinomial

• $F(z) = u(x,y) + iv(x,y)$

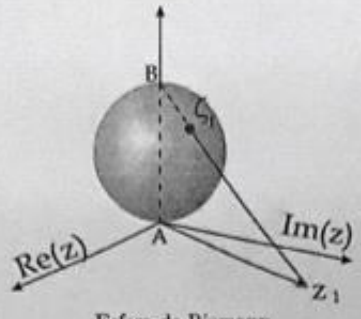
↳ $z = x + iy$

• $F(z) = u(r\theta) + iv(r\theta)$

↳ $z = r\,\text{cis}\,\theta$

• Región
• Función
• $D_{F(z)} \triangleq$ Dominio

Aspectos generales para cualquier función (Debemos definir)



Esfera de Riemann

El plano de Argand

Para poder estudiar el cálculo son necesarias las definiciones que a continuación se muestran:

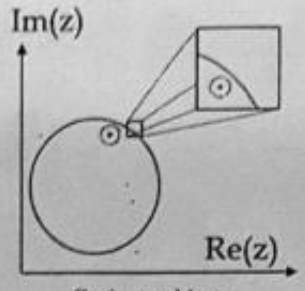
Punto: (.)

Conjunto: Una colección de puntos en el plano complejo.

Vecindad: Se llama vecindad (o entorno) de radio r, de un punto z_0, al conjunto de puntos situados en el interior de un círculo de radio r centrado en z_0, es decir la región $|z - z_0| < r$

Im(z)

Conjunto abierto: Un conjunto abierto es aquel en el que, para todo elemento, existe una vecindad cuyos puntos pertenecen al conjunto.

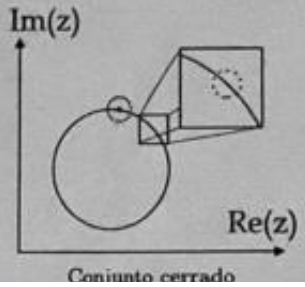


Conjunto abierto

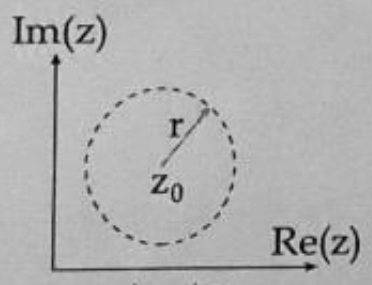
/* Cada elemento tiene su vecindad */

Ejemplo.- $|z| < 1$.

Conjunto cerrado: Un conjunto cerrado es aquel en el que, para al menos un elemento, no existe una vecindad cuyos puntos pertenecen todos al conjunto.

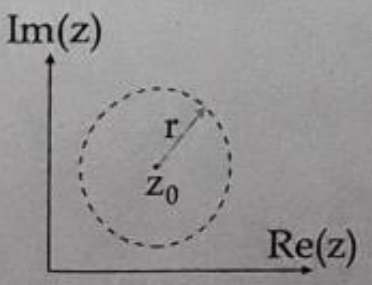


Conjunto cerrado



$|z - z_0| < r$

Vecindad punteada: Una vecindad punteada de z_0 es el conjunto de puntos tal que $0 < |z - z_0| < r$

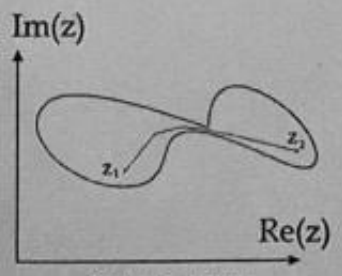


$0 < |z - z_0| < r$

Conjunto cerrado

Ejemplo .- $|z| \leq 1$.

Conjunto conexo: Un conjunto es conexo si dados dos puntos cualesquiera del conjunto, existe una trayectoria formada por segmentos de recta que los une, y cuyos puntos pertenecen al conjunto.



Conjunto conexo

Dominio: Llamamos dominio a un conjunto abierto conexo.

Dominio simplemente conexo: Un dominio sin agujeros.

Dominio multiplemente conexo: Un dominio con agujeros.

Límite

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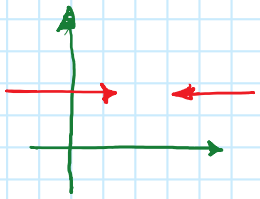
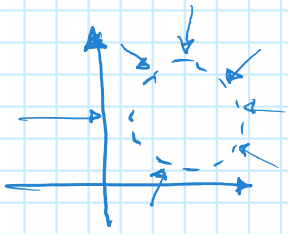
- Variable Real | • Limite por la derecha - izquierda



- Variable Compleja | • Limite por cualquier trayectoria



Caso Especial - Función Racional

- $w = \dfrac{az+b}{cz+d} \quad | \; cz+d \neq 0$ | • Función Lineal o Binomial

- Si no hay agujeros en la región
  ∴ La función es continua

1] Obtener el limite

- $F(z) = x^2 - y^2 + 2x - 1 + i(2xy + 2y)$

- $\lim\limits_{z \to z_0} f(z) \quad | \; z_0 = 1+i$

// Para determinar el tipo de función

1) Descomponer

$x = \dfrac{z+\bar{z}}{2}$ | $z = x+yi$

$y = \dfrac{z-\bar{z}}{2i}$

* $F(z) = z^2 + 2z + 1$

- La región es todo el espacio complejo

- $F(z) = z^n \in \mathbb{C}$

- $D_{F(z)} \in \mathbb{C}$ // No se indeterminca

∴ $\exists \lim\limits_{z \to z_0} F(z)$

// Otra forma

// Obteniendo $\lim_{z \to z_0 = 1+i} f(z)$

- $\lim_{z \to 1+i} F(z) = (x^2 - y^2 + 2x - 1) + i(2xy + 2y)$

$z_0 = 1+i \begin{cases} x = 1 \\ y = 1 \end{cases}$

// Sustituyendo

$= (\cancel{1-1}^{0} + \cancel{2-1}^{1}) + i(2+2) =$

$= 1 + 4i$

// Otra forma

$\lim_{z \to z_0} f(z) = z^2 - 2z + 1$

$= (1+i)^2 - 2(1+i) + 1 = 1 + 4i$

Propiedades Limites

- $\lim_{z \to z_0} [f(z) + g(z)] = w_1 + w_2$
- $\lim_{z \to z_0} [c f(z)] = c w_1$
- $\lim_{z \to z_0} \frac{g(z)}{f(z)} = \frac{w_2}{w_1}$ si $w_1 \neq 0$

Continuidad

- $w = f(z)$ es continua en $z = z_0$ si satisface

1) $f(z_0)$ está definido
2) $\lim_{z \to z_0} f(z) \exists$ y $\lim_{z \to z_0} f(z) = f(z_0)$

2) $\lim_{z \to z_0} f(z) \exists$ y $\lim_{z \to z_0} f(z) = f(z_0)$

$$F(z): \begin{cases} \to \text{Continua / Analítica / Entera / Holomorfa} \\ \to \text{No continua} \end{cases}$$

Derivada - Analítica

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Derivada Compleja

- $f'(z_0) = \left.\dfrac{df}{dz}\right|_{z_0}$

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

// Sólo si existe el limite

Forma Normal

1] $f(z) = z^2$ y $f(z) = z^3$

- si existe, es continua $F(z)$ y así aplicamos teoremas de variable real.

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^3 - (z)^3}{\Delta z} =$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^3} + 3\Delta z^2 z + 3\Delta z z^2 + \Delta z^3 - \cancel{z^3}}{\Delta z} =$$

$$= \lim_{\Delta z \to 0} \frac{3\Delta z^2 z + 3\Delta z z^2 + \Delta z^2}{\Delta z} =$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(3z\Delta z + 3z^2 + \Delta z)}{\cancel{\Delta z}} =$$

$$= \lim \quad \Delta z \to 0 \quad \Delta z \to 0$$

$$= \lim_{\Delta z \to 0} \cancel{3z\Delta z}^{\,0} + 3z^2 + \cancel{\Delta z}^{\,0} =$$

$$= \underline{3z^2}$$

$$f'(z) = \underline{2z}$$

Forma con Cartesianas

- $f'(z_0)$:

$$\frac{d f(z_0)}{dz} \quad o \quad \left.\frac{df}{dz}\right|_0$$

- Si la derivada existe los limites son iguales:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

- $F(z) = u(x,y) + i\,v(x,y)$

$-(x,y)$

$-(x_0,y_0)$

$F(x+yi) = u(x,y) + i\,v(x,y)$

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{F(x+yi) - F(x_0+y_0 i)}{(x+yi) - (x_0+y_0 i)} =$$

$$= \lim_{z \to z_0} \frac{[u(x,y) + i v(x,y)] - [u(x_0,y_0) + i\,v(x_0,y_0)]}{(x+yi) - (x_0+y_0 i)} =$$

$$= \lim_{x+yi \to x_0+y_0 i} \frac{[u(x,y)-u(x_0,y_0)] + i[v(x,y)-v(x_0,y_0)]}{x-x_0+i(y-y_0)} =$$

1] • $f(z) = \dfrac{z^2-3}{z-5}$ es analítica en la región

$\to R = \{ z \in \mathbb{C} \mid \frac{1}{2}(1-i)z + \frac{1}{2}(1+i)\bar{z} - 3 < 0 \}$

// Analizando la región

• $z \neq 5$ | $z = x + iy \in \mathbb{C}$

$\mathbb{C}$ | $\frac{1}{2}(1-i)z + \frac{1}{2}(1+i)\bar{z} - 3 < 0$

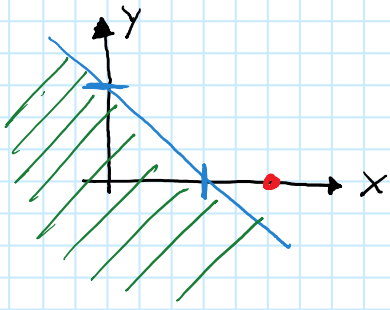
// Sustituyendo

$$\frac{1}{2}(1-i)(x+iy) + \frac{1}{2}(1+i)(x-yi) - 3 < 0$$

$$\left[\frac{1}{2}(x \cancel{-ix} \cancel{+iy} + y) + \frac{1}{2}(x \cancel{+ix} \cancel{-iy} + y) - 3 < 0\right] 2$$

$$2x + 2y - 6 < 0 \; ; \; x + y < 3$$

// Gráfica



if $z = 5$ /* No afecta a la región */

$\therefore$ La función es analítica

Ecuaciones de Cauchy-Riemann

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

- $F(z) = u(x,y) + iv(x,y)$

Ejemplo 1

$F(z) = z^2$; $F(z) = (x+iy)^2$

$F(z) = x^2 + 2xyi - y^2$

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

$\rightarrow \frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x$; $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✓

$\rightarrow \frac{\partial v}{\partial x} = 2y \qquad -\frac{\partial u}{\partial y} = -2y$; $2y = -(-2y)$ ✓

$\therefore F'(z) = 2x + 2yi = 2(x+yi)$ //

▸ Ecuaciones Cauchy-Riemann (Forma Polar)

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

2] Sea $f(z) = \frac{1}{z^2}$ con $z \neq 0$

a) Expresar en forma $f(z) = u(x,y) + iv(x,y)$

b) Validar que sea analítica

c) Demostrar que $f'(z) = \frac{-2}{z^3}$

c) Demostrar que $f'(z) = \dfrac{-2}{z^3}$

a) $z = x + iy \in \mathbb{C}$

$$f(z) = \frac{1}{(x+iy)^2} = \frac{1}{x^2 + 2xyi - y^2} =$$

$$= \frac{1}{x^2 - y^2 + 2xyi}\left(\frac{x^2 - y^2 - 2xyi}{x^2 - y^2 - 2xyi}\right) =$$

$$= \frac{x^2 - y^2 - 2xyi}{(x^2 - y^2)^2 + 4x^2y^2} = \frac{x^2 - y^2 - 2xyi}{x^4 - 2x^2y^2 + y^4 + 4x^2y^2} =$$

$$= \frac{}{x^4 + 2x^2y^2 + y^4} = \frac{x^2 - y^2 - 2xyi}{(x^2 + y^2)^2} =$$

$$\therefore u(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2} \qquad v(x,y) = \frac{-2xy}{(x^2+y^2)^2}$$

$$f(z) = \frac{1}{z^2}\left(\frac{\bar{z}^2}{\bar{z}^2}\right) = \frac{\bar{z}^2}{(z\bar{z})^2} = \frac{\bar{z}^2}{|z|^4}$$

$$f(z) = \frac{(x - yi)^2}{\left(\sqrt{x^2+y^2}\right)^4} = \frac{(x - yi)^2}{(x^2 + y^2)^2} = \frac{x^2 + 2xyi - y^2}{(x^2+y^2)^2} =$$

$$\therefore u(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2} \qquad v(x,y) = -\frac{2xy}{(x^2+y^2)^2}$$

b)

- $u(x,y) = \dfrac{x^2-y^2}{(x^2+y^2)^2}$ • $v(x,y) = \dfrac{-2xy}{(x^2+y^2)^2}$

$$\frac{d}{dx}\frac{u}{v} = \frac{vu'-v'u}{v^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)^2(2x) - \overbrace{(x^2-y^2)}^{u}\left(4x(x^2+y^2)\right)}{\left[(x^2+y^2)^2\right]^2} =$$

$$= \frac{\cancel{(x^2+y^2)}\left[(x^2+y^2)(2x) - (4x)(x^2-y^2)\right]}{\cancel{(x^2+y^2)}\,(x^2+y^2)^3} =$$

$$= \frac{2x^3+2xy^2-4x^3+4xy^2}{(x^2+y^2)^3} =$$

$$= \frac{-2x^3+6xy^2}{(x^2+y^2)^3} \quad //$$

$$\frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$= \frac{-2x(x^2+y^2)^2 - \left[(-2xy)\left(2(x^2+y^2)(2y)\right)\right]}{(x^2+y^2)^4} =$$

$$= \frac{\cancel{(x^2+y^2)}\,(x^2+y^2)(-2x) - \left[(-2xy)(4y)\right]}{\cancel{(x^2+y^2)}\,(x^2+y^2)^3} =$$

$$= \frac{-2x^3 - 2xy^2 + 8xy^2}{(x^2+y^2)^3} = \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{-2y(x^2+y^2)^2 - \left[(-2xy)\left(2(x^2+y^2)(2x)\right)\right]}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(-2y) - (-2xy)(4x)}{(x^2+y^2)^2} =$$

$$= \frac{-2yx^2 - 2y^3 + 8x^2y}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$-\frac{\partial u}{\partial y} = \frac{-2y(x^2+y^2)^2 - \left[(x^2-y^2)(4y)(x^2+y^2)\right]}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2y) - (4y)(x^2+y^2)}{(x^2+y^2)^3} =$$

$$= \frac{-2yx^2 - 2y^3 - 4yx^2 + 4y^3}{(x^2+y^2)^3} =$$

$$= -\left[\frac{-6yx^2+2y^3}{(x^2+y^2)^3}\right] = \frac{6yx^2-2y^3}{(x^2+y^2)^3} \;//$$

$$\therefore \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \;//$$

c) Demostrar "$f'(z) = \frac{-2}{z^3}$"

① //Sabemos que es analítica

$$\therefore \frac{1}{z^2} = z^{-2} \;;\; f'(z) = -2z^{-3} = -\frac{2}{z^3}$$

② $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} =$$

$$= \frac{6xy^2-2x^3}{(x^2+y^2)^3} + i\,\frac{6x^2y-2y^3}{(x^2+y^2)^3} =$$

$$= \frac{6xy^2-2x^3+6x^2yi-2y^3i}{(x^2+y^2)^3} =$$

$$= \frac{-2(x^3-3xy^2-3x^2yi+y^3i)}{(x^2+y^2)^3} =$$

$$= \frac{-2(x-yi)^3}{(x^2+y^2)^3} = \frac{-2(\bar{z})^3}{|z|^6} = -\frac{2(\bar{z})^3}{(z\bar{z})^3} =$$

$$= -\frac{2}{z^3} \;//$$

$$= -\frac{2}{z^3}$$

Función Armónica

Thursday, February 28, 2019 8:23 PM

- Sea A una región en $\mathbb{R}^2$ y $\phi : A \to \mathbb{R}$
- Se dice que $\phi(x,y)$ es armónica en A si satisface:

$$\frac{\partial \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

- Ecuación de Laplace Bidimensional

NOTA | Si es Armónica es Analítica
| Si es Analítica puede* ser Armónica

$$\frac{\partial u}{\partial y} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3} \qquad \frac{\partial u}{\partial x} = \frac{6xy^2 - 2x^3}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{6xy^2 - 2x^3}{(x^2+y^2)^3} \qquad -\frac{\partial v}{\partial x} = \frac{6x^2y - 2x^3}{(x^2+y^2)^3}$$

$$\nabla^2 u = 0 \quad ; \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{6x^4 - 36x^2y^2 + 6y^4}{(x^2+y^2)^4}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-6x^4 + 36x^2y^2 - 6y^4}{(x^2+y^2)^4}$$

$$\therefore \frac{6x^4 - 36x^2y^2 + 6y^4}{(x^2+y^2)^4} + \frac{-6x^4 + 36x^2y^2 - 6y^4}{(x^2+y^2)^4} = 0$$

$\therefore$ u es armónica //

$\nabla^2 v = 0$

$$\frac{\partial^2 v}{\partial x^2} = \frac{-24x^3 + 24y^3}{(x^2+y^2)^4}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{24x^3 - 24y^3}{(x^2+y^2)^4}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ V es armónica //

$\therefore$ Como u y v son armónicas, entonces f(z) es armónica //

$$F(z) = u(x,y) + i\,v(x,y)$$

- Si la función analítica $\omega = F(z)$
- Expresada en coordenadas polares
  Demostrar que:

$$F'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

- $F(z) = r\,\text{cis}\,\theta = r[\cos\theta + i\,\text{sen}\,\theta] = r\cos\theta + ri\,\text{sen}\,\theta =$
  ↳ Analítica

- $u(r,\theta) = r\cos\theta$
- $v(r,\theta) = r\,\text{sen}\,\theta$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$F'(z) = e^{i\theta}\left(\frac{d\omega}{dr}\right)$$ // Sustituyendo

$$F'(z) = e^{i\theta}\left(\frac{dw}{dr}\right)$$

$$w = u(r,\theta) + i\,v(r,\theta)$$

$$\frac{\partial w}{\partial r} = \frac{\partial u}{\partial r} + \frac{\partial v}{\partial r}$$

// Sustituyendo

$$\therefore F'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Función Armónica 2

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- If $f(z)$ está dada en coordenadas polares

$$f(z) = \sqrt{r}\cos\frac{\theta}{2} + i\sqrt{r}\,\text{sen}\frac{\theta}{2}$$

- Donde el dominio está restringido a

$$r>0 \qquad -\pi<\theta<\pi$$

- Obtener $f'(z)$

$$f(z) = u(r,\theta) + iv(r,\theta)$$

$$u(r,\theta) = \sqrt{r}\cos\left(\frac{\theta}{2}\right) \qquad v(r,\theta) = \sqrt{r}\,\text{sen}\left(\frac{\theta}{2}\right)$$

- E-C-R $\left| \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \right. \vdots \left. \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \right.$

- $$\frac{\partial u}{\partial r} = \frac{1}{\sqrt{r}}\cos\left(\frac{\theta}{2}\right)$$

$$\frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right) = \frac{1}{r}\left(\frac{\sqrt{r}\cos\frac{\theta}{2}}{2}\right) = \frac{\sqrt{r}}{2r}\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{r}}\cos\left(\frac{\theta}{2}\right)$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right)$$

- $$\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\,\text{sen}\left(\frac{\theta}{2}\right)$$

$$-\frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) = \frac{1}{r}\left(-\frac{\sqrt{r}\,\text{sen}\frac{\theta}{2}}{2}\right) = \frac{1}{2\sqrt{r}}\,\text{sen}\left(\frac{\theta}{2}\right)$$

$$\therefore \frac{\partial v}{\partial r} = -\frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right)$$

$\therefore F(z)$ es analítica

$$\bullet F(z) = \sqrt{r}\cos\frac{\theta}{2} + i\sqrt{r}\,\text{sen}\frac{\theta}{2}$$

$$= \sqrt{r}\left(\cos\frac{\theta}{2} + i\,\text{sen}\frac{\theta}{2}\right) = \sqrt{r}\; e^{i\frac{\theta}{2}} =$$

$$= \sqrt{r\,e^{i\theta}} = \sqrt{z}$$

$$\therefore F'(z) = \frac{1}{2\sqrt{z}}$$

// Comprobar que sea armónica

Parte Real

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{2\sqrt{r}}\,\text{sen}\left(\frac{\theta}{2}\right)$$

$$\frac{\partial u}{\partial \theta} = -\frac{\sqrt{r}}{2}\,\text{sen}\left(\frac{\theta}{2}\right)$$

Armónica

$$\nabla^2 u = 0$$

$$\nabla^2 v = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\bullet \frac{\partial^2 u}{\partial r^2} = -\frac{1}{4}\, r^{-3/2}\cos\left(\frac{\theta}{2}\right)$$

$$\frac{\partial^2 u}{\partial \theta^2} = -\frac{\sqrt{r}}{4}\cos\frac{\theta}{2}$$

$\bullet$ Sustituyendo

$$// \quad -\frac{1}{4r\sqrt{r}}\cos\frac{\theta}{2} + \frac{1}{r}\left(\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2}\right) + \frac{1}{r^2}\left(-\frac{\sqrt{r}}{4}\cos\frac{\theta}{2}\right) = 0$$

$$\cos\left(\frac{\theta}{2}\right)\left[-\frac{1}{4r\sqrt{r}} + \frac{1}{2r\sqrt{r}} - \cancel{\frac{\sqrt{r}}{4r^2}}\; -\frac{1}{4r\sqrt{r}}\right] = 0$$

$\therefore F(z) \overset{\Delta}{=} \text{es armónica respecto a } \nabla^2 u$

Parte Imaginaria

$$\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}} \operatorname{sen} \frac{\theta}{2}$$

$$\frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2} \cos \frac{\theta}{2}$$

$$\frac{\partial^2 v}{\partial r^2} = \frac{1}{4\sqrt{r}} \operatorname{sen} \frac{\theta}{2} \qquad \frac{\partial^2 v}{\partial \theta^2} = -\frac{\sqrt{r}}{4} \operatorname{sen} \frac{\theta}{2}$$

//Sustituyendo

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

$$-\frac{1}{4r\sqrt{r}} \operatorname{sen} \frac{\theta}{2} + \frac{1}{r}\left(\frac{1}{2\sqrt{r}} \operatorname{sen} \frac{\theta}{2}\right) + \frac{1}{r^2}\left(-\frac{\sqrt{r}}{4} \operatorname{sen} \frac{\theta}{2}\right) = 0$$

$$\therefore \nabla^2 v = 0$$

Armónica Conjugada

Tuesday, March 5, 2019 7:55 PM

1) $f(z) = u(x,y) + i\,v(x,y)$ es analítica alrededor de $z = \pi$ | Obtener función

$$u(x,y) = \frac{x}{x^2+y^2} \quad y \quad f(\pi) = \frac{1}{\pi}$$

$$F(z) = u(x,y) + i\,v(x,y)$$

$$\rightarrow F(z) = \frac{x}{x^2+y^2} + i\,v(x,y)$$

Análitica

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y} \quad (1)$$

$$-\frac{\partial u}{\partial y} = -\left[\frac{-x(2y)}{(x^2+y^2)^2}\right] = \cancel{-}\left[\frac{-2xy}{(x^2+y^2)^2}\right] = \frac{2xy}{(x^2+y^2)^2} \quad (2)$$

$$\therefore \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \; ; \int \partial v = \int \frac{y^2-x^2}{(x^2+y^2)^2}\,\partial y$$

$$v = \int \frac{y^2-x^2}{(x^2+y^2)^2}\,\partial y = -\frac{y}{x^2+y^2} + c$$

$$\therefore \frac{\partial u}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \; ; \int \partial u = \int \frac{2xy}{(x^2+y^2)^2}\,\partial y$$

$$u = \int \frac{2xy}{(x^2+y^2)^2}\,\partial y = \; ¿?$$

$$C(x) = C(y) = 6$$

$$F(z) = \frac{x}{x^2+y^2}\left(-\frac{y}{x^2+y^2} + c\right) i$$

$$*F(\pi) = \frac{1}{\pi} + i\,c$$

$$\therefore \cancel{\frac{1}{\pi}} + i\,c = \cancel{\frac{1}{\pi}} \quad i\,c = 0$$

$$\therefore F(z) = \frac{x}{x^2+y^2} - \frac{y\,i}{x^2+y^2} = \frac{x - y\,i}{x^2+y^2} = \frac{\cancel{\bar{z}}}{\cancel{\bar{z}}\,z} =$$

$$F(z) = \frac{1}{z}$$

2] Encontrar la función $f(z)$ que cumple la función $f(0)=0$ cuya parte real es

$$u(x,y) = 3e^x \operatorname{sen} y$$

- $F(z) = 3e^x \operatorname{sen} y + i\,v(x,y)$

- Analítica;

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

- $\dfrac{\partial u}{\partial x} = 3e^x \operatorname{sen} y = \dfrac{\partial v}{\partial y}$

- $\dfrac{\partial u}{\partial y} = -\left[3e^x \cos y\right] = \dfrac{\partial v}{\partial x}$

$$v = \int 3e^x \operatorname{sen} y \, \partial y =$$

$$= -3e^x \cos y + c(y)$$

$$u = \int -3e^x \cos y \, d \quad =$$

$$= -3e^x \cos y + c(x)$$

$$c(x) = c(y) = c$$

$\therefore F(z) = 3e^x \operatorname{sen} y - i\, 3e^x \cos y + Ci$

$F(0) = 0 = 0 + i0$

$-3\cancel{e^0}^{1} \cancel{\cos(0)}^{1} i + Ci = 0$

$-3i + Ci = 0 \; ; \; -3i = -Ci \; ; \; \underline{C = 3}$

// Re-escribiendo

$F(z) = 3e^x \operatorname{sen} y - i\left[3e^x \cos y + 3\right] =$

$= 3e^x (\operatorname{sen} y - i \cos y) + 3i =$

$e^{iy} = \cos y + i \operatorname{sen} y$

$e^{-iy} = \cos y - i \operatorname{sen} y$

$= -3i\left[e^x (i \operatorname{sen} y + \cos y)\right] + 3i =$ $\quad -ii = 1$

$= -3i\left[\underbrace{e^x (e^{iy})}_{e^z}\right] + 3i$

$= -3i\, e^z + 3i$ //

3] Encontrar a $F(z)$ cuya parte imaginaria

$v(x,y) = x^3 - 3xy^2$ | Analítica

$* F(i) = 0$

$F(z) = u(x,y) + i\left[x^3 - 3xy^2\right]$

$$u(x,y) = \int -6xy\,dx = -3x^2y + c(x)$$

$$\cup$$

$$u(x,y) = \int -3x^2 + 3y^2\,dy = -3x^2y + y^3 + C(y)$$

$$u(x,y) = -3x^2y + y^3 + C(x) + c(y)$$

$$F(z) = -3x^2y + y^3 + C(x) + c(y) + i(x^3 - 3xy^2)$$

$$* \; F(i) = 0 + 0i \;\Big|\; \begin{array}{l} x = 0 \\ y = 1 \end{array}$$

$$= 0 + (i)^3 + \cancel{C(x)}^{\,0} + C(y) + i(0) =$$

$$= \cancel{i^2}^{\,-1} + C(y) = 0 \;;\quad \underline{C(y) = i}$$

// Pero es en la parte $\mathbb{R}$; $C(y) = 0$

$$\therefore F(z) = -3x^2y + y^3 + i(x^3 - 3xy^2)$$

Ejercicio

Thursday, March 7, 2019 7:22 PM

1] Comprobar que las partes Real e Imaginaria de la función $f(z) = z+2$ satisfacen las ecuaciones de Cauchy-Riemann en $\mathbb{R}^2$

- $f(z) = z + 2 \qquad z = x + iy$

$u(x,y) = x+2$

$v(x,y) = y$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial v}{\partial y} \end{aligned}\right\} \text{Cauchy Reimman}$$

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 1$

$\frac{\partial v}{\partial x} = 0 \qquad -\frac{\partial u}{\partial y} = 0$

$\therefore$ Si satisfacen

2] Obteniendo

$f(z) = (1+2i)\bar{z} - (1-2i) \quad ; \quad f(z) = u(x,y) + i\,v(x,y)$

$= (1+2i)(x-yi) - (1-2i) =$

$= x + 2ix - yi + 2y - 1 + 2i =$

- $u(x,y) = x + 2y - 1$
- $v(x,y) = 2\cancel{i}x - y\cancel{i} + 2\cancel{i}$

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$

$\frac{\partial v}{\partial x} = 2 \qquad -\frac{\partial u}{\partial y} = -2$

$\therefore f(z) \triangleq$ No es analítico

3] Dada la función $f(z) = \operatorname{sen}(z)$

a) Forma $f(z) = u(x,y) + iv(x,y)$

b) Demostrar, a partir del inciso anterior, que f es analítica en todo $\mathbb{C}$

c) Comprobar $f'(z) = \cos(z)$

a)

$$\operatorname{sen}(z) = \frac{e^{zi} - e^{-zi}}{2i} \qquad z = x + iy \in \mathbb{C}$$

$$\operatorname{sen}(z) = \frac{e^{(x+iy)i} - e^{-(x+iy)i}}{2i}$$

$$\operatorname{sen}(z) = \operatorname{sen}x \cosh y + i \cos x \operatorname{senh} y$$

b)

$$u(x,y) = \operatorname{sen}x \cosh y$$
$$v(x,y) = \cos x \operatorname{senh} y$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y \quad = \quad \frac{\partial v}{\partial y} = \cos x \operatorname{senh} y$$

$$\frac{\partial v}{\partial x} = -\operatorname{sen}x \operatorname{senh} y \quad = \quad -\frac{\partial u}{\partial y} = \operatorname{sen}x \operatorname{senh} y$$

$\therefore$ La función es analítica

c) • $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ | Expresiones

- $\operatorname{sen}(iz) = i \operatorname{senh} z$
- $\cos(iz) = \cosh z$

$$\frac{\partial v}{\partial x} = \operatorname{sen}x \operatorname{senh} y \qquad \frac{\partial u}{\partial y} = -\operatorname{sen}x \operatorname{senh} y$$

$$f'(z) = \cos x \cosh y - i \operatorname{sen} x \operatorname{senh} y =$$

$$= \cos x \cosh(iy) - \operatorname{sen} x \operatorname{sen} h(iy) =$$

$$= \cos(x+iy) = \cos(z)$$

Obtener una armónica conjugada de la función donde (r,θ) son coordenadas polares

$$u(r,\theta) = \left(r + \frac{1}{r}\right)\cos\theta$$

Armónica $\longrightarrow$ Analítica

$\therefore$ cumple ecuaciones de Cauchy Reiman

$$\nabla^2 u = 0 \longleftarrow \text{Armónica Conjugada}$$

$\rightarrow$ E.C.R

$$\blacktriangleright \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \blacktriangleright \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\bullet \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} = \left(1 - \frac{1}{r^2}\right)\cos\theta =$$

$$\therefore \frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta$$

$$v = \int \left(r - \frac{1}{r}\right)\cos\theta \, dr =$$

$$\bullet -r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} = \left(r + \frac{1}{r}\right)\operatorname{sen}\theta = r\frac{\partial v}{\partial r}$$

$$\therefore \frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right)\operatorname{sen}\theta$$

$$v = \int 1 + \frac{1}{r^2} \operatorname{sen}\theta \, dr =$$

$\therefore C(r) = C(\theta) = C$

→ Ecuación

$$v(r,\theta) = \left(r - \frac{1}{r}\right)\operatorname{sen}\theta + C$$

$C = 0$ Armónica Conjugada

// Sustituyendo

$$F(z) = \left(r + \frac{1}{r}\right)\cos\theta + \left[\left(r - \frac{1}{r}\right)\operatorname{sen}\theta\right] i =$$

$$= r\cos\theta + \frac{1}{r}\cos\theta + i\,r\operatorname{sen}\theta - \frac{1}{r} i \operatorname{sen}\theta =$$

$$= \underbrace{r\left[\cos\theta + i\operatorname{sen}\theta\right]}_{= r\,\mathrm{cis}\,\theta} + \underbrace{\frac{1}{r}\left[\cos\theta - i\operatorname{sen}\theta\right]}_{} =$$

$\cos(-\theta) = \cos\theta$

$\operatorname{sen}(-\theta) = -\operatorname{sen}\theta$

$\therefore \cos(-\theta) + i\operatorname{sen}(\theta)$

$$+ \frac{1}{r}\left[\cos(-\theta) + i\operatorname{sen}(-\theta)\right]$$

$$\therefore F(z) = r\,\mathrm{cis}\,\theta + \frac{1}{r}\,\mathrm{cis}(-\theta) =$$

$$= r\,\mathrm{cis}\,\theta + \frac{1}{r\,\mathrm{cis}(\theta)} = z + \frac{1}{z}$$

Final $\Big|\; F(z) = z + \frac{1}{z} \;\Big|\; z \neq 0$

$$r\,\mathrm{cis}\,\theta = e^{i\theta} = z$$

$$r\,\mathrm{cis}(-\theta) = e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{z}$$

Conformalidad

Tuesday, March 12, 2019 7:09 PM

- Una transformación o mapeo $w=f(z)$ es conforme en z_0 si preserva ángulos entre curvas orientadas tanto en magnitud como sentido

FALTA

Regla de L'Hopital

- Sean f y g funciones analíticas en z_0
  Si $f(z_0)=0$, $g(z_0)=0$ y $g'(z_0)\neq 0$

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{f'(z)}{g'(z)}$$

- Solo aplica si son ambas analíticas //