Integración Compleja

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- Sea $f:[a,b]\subset \mathbb{R}\to\mathbb{C}$ una función compleja de variable real.
- Entonces $f(t)=u(t)+i\,v(t)$

$$\int_a^b f(t)\,dt=\int_a^b u(t)\,dt+i\int_a^b v(t)\,dt$$

solo si existen

▸ Calcular

$$\int_a^b f(t) \;\Big|\; f(t)=e^{t-it} \quad y \quad -\pi/2\le t\le \pi/2$$

$$\int f(t)\,dt=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{t-it}\,dt=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{t}e^{-it}=$$

$$=\int_{-\pi/2}^{\pi/2} e^{t}(\cos t-i\,\operatorname{sen} t)\,dt=$$

No olvidar

$$=\int_{-\pi/2}^{\pi/2} e^{t}\cos t\,dt-i\int_{-\pi/2}^{\pi/2} e^{t}\operatorname{sen} t\,dt=$$

$u=e^t \quad dv=\cos t$ | $u=e^t \quad dv=\operatorname{sen} t$

$du=e^t \quad v=\operatorname{sen} t$ | $du=e^t \quad v=-\cos t$

$$=\underbrace{e^t\operatorname{sen} t\Big|_{-\pi/2}^{\pi/2}-\int_{-\pi/2}^{\pi/2} e^t\operatorname{sen} t\,dt}_{\mathbb{R}}\;\Big|\;-i\left[\cos t\Big|_{-\pi/2}^{\pi/2}-\int -e^t\cos t\,dt\right]$$

Parte Real

$$I=\int_{-\pi/2}^{\pi/2} e^t\cos t\,dt$$

$$I=e^t\operatorname{sen} t\Big|_{-\pi/2}^{\pi/2}-\left(-e^t\cos t\Big|_{-\pi/2}^{\pi/2}+\int e^t\cos t\,dt\right)$$

$$2I=e^t\operatorname{sen} t\Big|_{-\pi/2}^{\pi/2}+e^t\cos t\Big|_{-\pi/2}^{\pi/2}\cancel{-I}$$

$$=e^t\left[\operatorname{sen} t+\cos t\right]\Big|_{-\pi/2}^{\pi/2}=e^{\pi/2}\operatorname{sen} t\Big|_{-\pi/2}^{\pi/2}+e^{\pi/2}\cos t\Big|_{-\pi/2}^{\pi/2}=$$

$\pi/2 \quad 1 \quad (-\pi/2) \quad -1$

$$= e^{\pi/2}\,\cancel{\operatorname{sen}(\pi/2)}^{\,1} - e^{(-\pi/2)}\,\cancel{\operatorname{sen}(-\pi/2)}^{\,-1}$$

$$I = \frac{e^{\pi/2} + e^{-\pi/2}}{2} = \cosh\left(\frac{\pi}{2}\right)$$

Parte Compleja

$$I_2 = \int_{-\pi/2}^{\pi/2} e^{t}\operatorname{sen} t\,dt$$

$$= -e^{t}\cos t\Big|_{-\pi/2}^{\pi/2} \cancel{+} \int_{-\pi/2}^{\pi/2} -e^{t}\cos t\,dt =$$

$$= -e^{t}\cos t\Big|_{-\pi/2}^{\pi/2} + e^{t}\operatorname{sen} t\Big|_{-\pi/2}^{\pi/2} - \underbrace{\int_{-\pi/2}^{\pi/2} e^{t}\operatorname{sen} t\,dt}_{I_2} =$$

$$\therefore\ 2I_2 = -e^{t}\cos t\Big|_{-\pi/2}^{\pi/2} + e^{t}\operatorname{sen} t\Big|_{-\pi/2}^{\pi/2}$$

$$= -e^{\pi/2}\,\cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0} + e^{-\pi/2}\,\cancel{\cos(-\pi/2)}^{\,0}$$

$$+ e^{\pi/2}\,\cancel{\operatorname{sen}\left(\frac{\pi}{2}\right)}^{\,1} - e^{-\pi/2}\,\cancel{\operatorname{sen}(-\pi/2)}^{\,-1} =$$

$$2I_2 = e^{\pi/2} + e^{-\pi/2}$$

$$I_2 = \frac{e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}}{2} = \cosh\left(\frac{\pi}{2}\right)$$

$$\therefore \int_a^b F(t)\,dt = \cosh\left(\frac{\pi}{2}\right) - i\cosh\left(\frac{\pi}{2}\right)$$

$$= \cosh(1-i)$$

Nota: Como $F(t)$ es analítica, aplican todos los teoremas de Variable Real

2° Forma

$$\int_{-\pi/2}^{\pi/2} e^{t} - e^{ti} = \int_{-\pi/2}^{\pi/2} e^{t(1-i)}\,dt =$$

$$\int_{-\pi/2}^{\pi/2} e^{t(1-i)} = \frac{1}{(1-i)}\,e^{t(1-i)}\Big|_{-\pi/2}^{\pi/2} =$$

$$// \left(\frac{1+i}{1+i}\right)\frac{1}{(1-i)} = \frac{1+i}{1+1} = \frac{1+i}{2}$$

$$= \frac{1+i}{2} e^{t(1-i)}\Big|_{-\pi/2}^{\pi/2} = \frac{1+i}{2} e^{t} e^{-it}\Big|_{-\pi/2}^{\pi/2} =$$

$$= \left(\frac{1}{2}+\frac{i}{2}\right)\left[e^{t} e^{-it}\Big|_{-\pi/2}^{\pi/2}\right] = \left[\left(\frac{1}{2}+\frac{i}{2}\right)\left(e^{t}\right)\left(e^{-i\pi/2}\right)\right] + \left[\left(\frac{1}{2}+\frac{i}{2}\right)\left(e^{t}\right)\left(e^{+i\pi/2}\right)\right] =$$

$$= \left(\frac{1}{2}+\frac{i}{2}\right)\left(e^{\pi/2}\right)\left(\cancelto{0}{\cos \pi/2} - i\,\cancelto{1}{\operatorname{sen} \pi/2}\right) - \left[\left(\frac{1}{2}+\frac{i}{2}\right)\left(e^{-\pi/2}\right)\left(\cancelto{0}{\cos(-\pi/2)} - i\,\cancelto{-1}{\operatorname{sen}(-\pi/2)}\right)\right] =$$

$$= \left(\frac{1}{2}+\frac{i}{2}\right)\left(e^{\pi/2}\right) \cancel{-} \left(\frac{1}{2}+\frac{i}{2}\right)\left(e^{-\pi/2}\right)\cancel{(-1)} =$$

$$= \frac{e^{\pi/2}}{2} + \frac{i\,e^{\pi/2}}{2} + \frac{e^{-\pi/2}}{2} + \frac{i\,e^{-\pi/2}}{2} =$$

$$= \underbrace{\frac{e^{\pi/2}+e^{-\pi/2}}{2}}_{\cosh(\pi/2)} + i\,\underbrace{\frac{e^{\pi/2}+e^{-\pi/2}}{2}}_{\cosh \pi/2}$$

$$\int_a^b F(t) = \cosh(\pi/2) + i\cosh(\pi/2)$$

Integral de Línea

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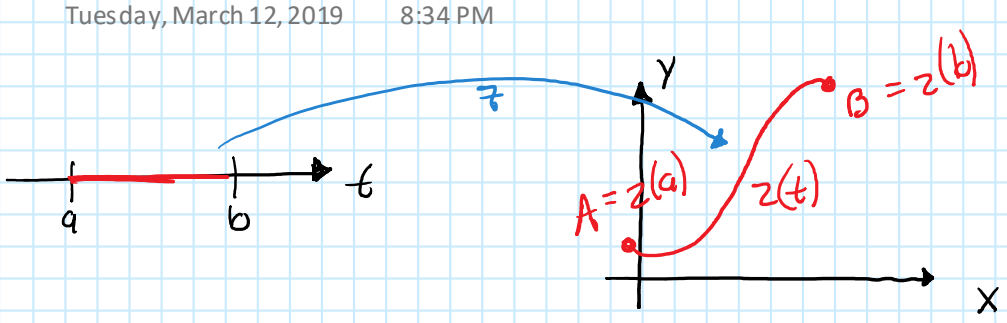


// Una función analítica no depende la curva
si no es analítica depende

Integral Compleja depende de C (curva)

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\, z'(t)\,dt$$

Ejercicio | Calcular integral

$\int_C (z^2-1)\,dz$ | a lo largo del segmento de parábola $y = x^2$ que se extiende del punto $(0,0)$ a $(2,4)$

// Es abierta

// Puede ser cerrada o abierta

$z(t) = x(t) + i\,y(t) \qquad a \le t \le b \qquad F(z) = z^2 - 1$

// Parametrización

$x = t$
$y = t^2$; $z(t) = t + i t^2 \qquad 0 \le t \le 2$

$z'(t) = 1 + 2ti$

$$\int_C F(z)\,dt = \int_C (z^2-1)\,dt = \int_0^2 (t+it^2)^2 - 1\,dt =$$

$$\int_C F(z)\,dz = \int_a^b F[z(t)]\,z'(t)\,dt$$

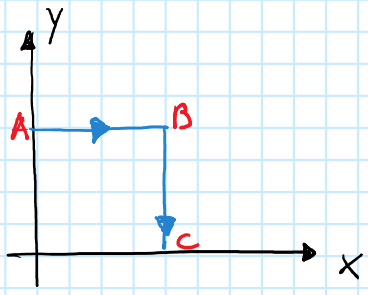
$$\int_0^2 (t+it^2)^2(1+2ti)\,dt =$$

$$= \int_0^2 -5t^4+t^2-1\,dt + i\int_0^2 -2t^5+4t^3-2t\,dt =$$

$$= \left(-t^5+\frac{t^3}{3}-t\right)\Big|_0^2 + i\left(\frac{t^6}{3}+t^4-t^2\right)\Big|_0^2 =$$

$$= -\frac{94}{3} - i\,\frac{28}{3}$$

Ejercicio 2



$A = i$
$B = 1+i$
$C = 1$

- Si μ es línea quebrada $\overline{AB} \cup \overline{BC}$
- Con sentido de A a C

$$\int_\mu \frac{1}{z}\,dz$$

$$z(t) = x(t) + i\,y(t)$$

$$\int_{\mu = C_T} F(z)\,dz = \int_{\overline{AB}} F(z)\,dz + \int_{\overline{BC}} F(z)\,dz$$

	x	y
A =	0	1
B =	1	1
C =	1	0

$$\int F(z)\,dz$$

$z(t) = t + i \qquad 0 \le t < 1$

$z'(t) = 1$

$$\int_{\overline{AB}} F(z)\,dz \quad \Bigg| \quad \begin{aligned} z(t) &= t + i \\ z'(t) &= 1 \end{aligned} \qquad 0 \le t < 1$$

$$\therefore \int_0^1 \left(\frac{1}{t+i}\right) dt = \int_0^1 \frac{t-i}{t^2+1}\,dt =$$

$$= \int_0^1 \frac{t}{t^2+1}\,dt - \int_0^1 \frac{i}{t^2+1}\,dt =$$

$$\int_{\overline{BC}} F(z)\,dz \quad \Bigg| \quad \begin{aligned} z(t) &= 1 + ti \\ z'\,t &= i \end{aligned}$$

// Invertimos $\int_1^0 = -\int_0^1$

$$\therefore \int_0^1 \frac{1}{1+ti}\, i\,dt = -\int_0^1 \frac{i+t}{1+t^2}\,dt =$$

$$= -\int_0^1 \frac{t}{t^2+1}\,dt - i\int_0^1 \frac{1}{t^2+1}\,dt =$$

// Agrupando

$$\int_{C_T} F(z)\,dz = \cancel{\int_0^1 \frac{t}{t^2+1}\,dt} - \int_0^1 \frac{i}{t+1}\,dt$$

$$- \cancel{\int_0^1 \frac{t}{t^2+1}\,dt} - i\int_0^1 \frac{1}{t^2+1}\,dt =$$

$$= -2i \int_0^1 \frac{1}{t^2+1}\,dt =$$

$$= -2i \,\operatorname{angtan}(t)\Big|_0^1 = -2i\left(\frac{\pi}{4}\right) =$$

$$= -\frac{i\pi}{2} \quad //$$

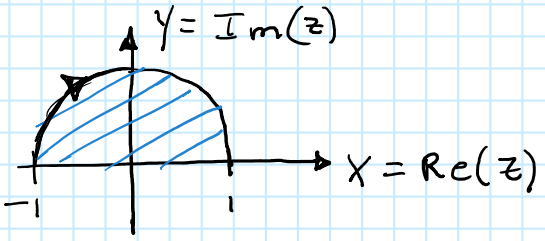
Ejercicio

$$\int_C \bar{z}\, dz \quad \text{desde } z=-1 \text{ hasta } z=1$$

- La curva es la semicircunferencia unitaria superior centrada en el origen, recorrida en sentido horario

NOTA: $\bar{z}$ no analítica

Nota: Si es analítica para c. cerrada entonces $\oint f(z)\,dz = 0$



$x^2+y^2=1$ $\qquad f(z)=\bar{z}$

// Parametrización de la función

$\rightarrow z(t) = x(t) + i\,y(t)$

$$\left| \begin{array}{l} y = \cos t \\ x = \operatorname{sen} t \end{array} \right. \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

$\rightarrow z(t) = \operatorname{sen} t + i\cos t$

$z'(t) = \cos t - i \operatorname{sen} t$

$$\int_C f(z)\,dz = \int_C \bar{z}\,dz$$

$$\therefore \int_C f(z)\,dt = \int_a^b f[z(t)]\, z'(t)\,dt$$

DUDA

$$= \int_{-\pi/2}^{\pi/2} (\operatorname{sen} t - i\cos t)(\cos t - i \operatorname{sen} t)\,dt =$$

$$= \int_{-\pi/2}^{\pi/2} \cancel{\operatorname{sen} t\cos t} - \underbrace{i\cos^2 t - i\operatorname{sen}^2 t}_{1} - \cancel{\operatorname{sen} t\cos t}\,dt$$

$$\int_{-\pi/2}$$

$$= \int_{-\pi/2}^{\pi/2} -i\,dt = -it\Big|_{-\pi/2}^{\pi/2} = -i\frac{\pi}{2} - \left(-i\frac{\pi}{2}\right) =$$

$$= -i\pi$$

Forma Alterna

$$z(t) = -i(\operatorname{sen} t + i\cos t) = -i\operatorname{sen} t + \cos t =$$

$$z(t) = \cos t - i\operatorname{sen} t \quad \Big| \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

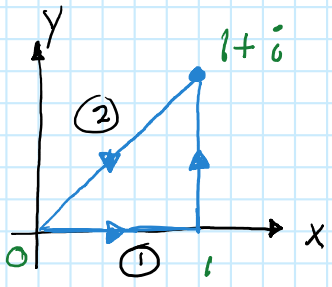
$$z(t) = e^{-ti} \qquad z'(t) = -i\,e^{-ti}\,dt$$

$$\int_C F(z)\,dz = \int_{-\pi/2}^{\pi/2} \underbrace{\left(e^{ti}\right)}_{\bar{z}}\left(-i\,e^{-ti}\right)dt =$$

$$= -i\,e^{ti-ti} = -i\,\cancel{e^{0}}^{\,1}$$

$$= \int_{-\pi/2}^{\pi/2} -i\,dt = -it\Big|_{-\pi/2}^{\pi/2} = -i\pi$$

Demostración | Demostrar que



$$\int_0^{1+i} \bar{z}\,dz =$$

Depende de la trayectoria

// Trayectoria 1

$z_1(0,0)$

$z_2(1,0)$

$$\int_{C_T} F(z)\,dz = \int F(z)\,dz + \int F(z)\,dz$$

$z_2(1,0)$

$z_3(1,1)$

R | I

$$\int_{C_T} F(z)\,dz = \int_{\overline{z_1 z_2}} F(z)\,dz + \int_{\overline{z_2 z_3}} F(z)\,dz$$

TRAYECTORIA 1

▸ $\overline{z_1 z_2}$

$z(t) = t \qquad 0 \le t \le 1$ // No afecta el $\bar{z}$

$z'(t) = 1$

$$\therefore \int_0^1 F(z)\,dz = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

▸ $\overline{z_2 z_3}$

$z(t) = 1 + ti \qquad 0 \le t \le 1$

$z'(t) = i$

$$\therefore \int_0^1 (1 - ti)(i)\,dt = \int_0^1 (i + t)\,dt =$$

$$= it + \frac{t^2}{2}\Big|_0^1 = i + \frac{1}{2}$$

TRAYECTORIA 2

$z(t) = t + ti \qquad 0 \le t \le 1$

$z'(t) = 1 + i$

$$\therefore \int_{\overline{z_1 z_3}} F(z)\,dz = \int_0^1 (t + ti)(1 + i)\,dt =$$

$$= \int_0^1 (t - ti + ti + t)\,dt =$$

$$= \int_0^1 2t\,dt = 2\,\frac{t^2}{2}\Big|_0^1 = 1$$

Conclusión $\int_{T_1} F(z)\,dz \neq \int_{T_2} F(z)\,dz$ //

Serie 3 – Ejercicio 1

- $F(z) = -(1-i)z + (2+i)$
- Curva $= (z+\bar{z})^2 + 8i(z-\bar{z}) = 0$
- Rango $=$ $z=0$ hasta $z = 2+i$

→ Validar $F(z)$ analítica

$$z = x + iy$$

$$F(z) = (-1+i)(x+iy) + (2+i) =$$

$$= -x + ix - iy - y + 2 + i =$$

$$\therefore \begin{cases} u(x,y) = -x - y + 2 \\ v(x,y) = x - y + \cancel{i}\,1 \end{cases}$$

// E.C.R

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$-1 = -1 \checkmark \qquad 1 = -(-1) \checkmark$$

$\therefore F(z)$ es cerrada → S_1 es cerrada | $\oint F(z)\,dz = 0$

→ S_1 es abierta

→ Validar si es abierta o cerrada

// Obteniendo C

$$(z+\bar{z})^2 + 8i(z-\bar{z}) = 0$$

$$\begin{cases} z = x+yi \\ \bar{z} = x-yi \end{cases} \in \mathbb{C}$$

$$[x+yi + x-yi]^2 + 8i[x+yi - x+yi] = 0$$

$$(2x)^2 + 8i[2yi] = 0$$

$$4x^2 - 16y = 0 \quad ; \quad x^2 - 4y = 0$$

∴ Es abierta

CASO 2

→ Analítica $\begin{cases} \text{Cerrada} \rightarrow \oint F(z)\,dz = 0 \\ \text{Abierta} \rightarrow \text{Integración directa} \end{cases}$

→ No analítica $\begin{cases} \text{Cerrada} \rightarrow \text{Definición} \\ \text{Abierta} \rightarrow \text{Definición} \end{cases}$

Definición $\int_{C_T} F(z)\,dz = \int_a^b F[z(t)]\, z'(t)\,dt$

// Sustituyendo

$$F(z) = (-1+i)z + (2+i) = -z + iz + (2+i)$$

$$\int_0^{2+i} -z + iz + (2+i)\,dz = -\frac{z^2}{2} + i\frac{z^2}{2} + 2z + iz \Big|_0^{2+i} =$$

$$\int_0 -z + iz + (2+i)\,dz = -\frac{z^2}{2} + i\frac{z^2}{2} + 2z + iz \Big|_0 =$$

$$= -\frac{1}{2} + \frac{7}{2}i$$

Integrales

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- Comprobar que parte real e imaginario de la función $f(z) = z e^{2z}$ $z = x + yi$

$$\oint_C z e^{2z} dz \qquad C \triangleq \text{Circunferencia } |z| = 1$$

- Analítica / E.C.R

$$F(z) = u(x,y) + i\, v(x,y)$$

$$F(z) = z e^{2z}$$

$$= (x+yi) e^{2(x+yi)} =$$

$$= (x+yi) e^{2x} e^{2yi} =$$

$$= (x+yi) e^{2x} [\cos(2y) + i \operatorname{sen}(2y)] =$$

$$= (x e^{2x} + y i e^{2x})(\cos 2y + i \operatorname{sen} 2y) =$$

$$= x e^{2x} \cos 2y + y i e^{2x} \cos 2x$$

$$x e^{2x} i \operatorname{sen} 2y - y e^{2x} \operatorname{sen} 2y$$

- $u(x,y) = x e^{2x} \cos 2y - y e^{2x} \operatorname{sen} 2y$
- $v(x,y) = y e^{2x} \cos 2y + x e^{2x} \operatorname{sen} 2y$

$$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = (2x e^{2x} + e^{2x}) \cos 2y - 2y e^{2x} \operatorname{sen} 2y$$

$$\frac{\partial v}{\partial y} = \left(2x e^{2x} + e^{2x}\right)\cos 2y - 2y e^{2x} \operatorname{sen} 2y$$

$$\frac{\partial v}{\partial x} = e^{2x}\left(-2x \operatorname{sen} 2y - 2y \cos 2y - \operatorname{sen} 2y\right)$$

$$-\frac{\partial u}{\partial y} = -e^{2x}\left(2x \operatorname{sen} 2y - 2y \cos 2y - \operatorname{sen} 2y\right)$$

$\therefore$ $F(z)$ es analítica | La curva es cerrada

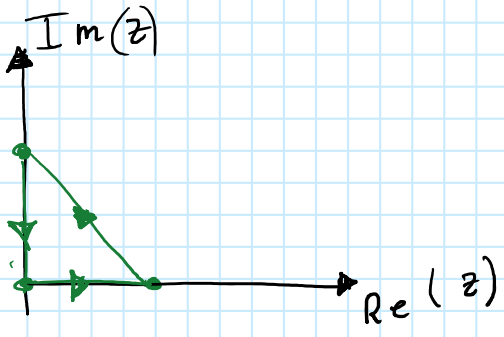
$$\therefore \oint_C F(z)\,dz = 0$$

- Verificar que se cumple el teorema Integral de Cauchy

$$\oint z^2\,dz$$

Efectuando el cálculo directamente, donde C es el triángulo con vertices en $(0,0)$, $(1,0)$, $(0,1)$

// Recorrido antihorario



$$C_{Total} = C_1 + C_2 + C_3$$

$$\int_C F(z)\,dz = \int_a^b F(z(t))\, z'(t)\,dt$$

▶ Para C_1

$0,0$ | $z(t) = t$ $\quad 0 \le t \le 1$

$$\begin{matrix} 0,0 \\ 1,0 \end{matrix} \quad \left| \begin{array}{l} z(t) = t \\ z'(t) = 1 \end{array} \right. \qquad 0 \le t \le 1$$

$$\therefore \int_0^1 t^2 (1)\, dt = \int_0^1 t^2\, dt = \frac{t}{3}\Big|_0^1 = \underline{1/3}$$

▶ Para C_2

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \quad \left| \begin{array}{l} z(t) = t + (1-t)\, i \\ z'(t) = 1 - i \end{array} \right.$$

Paso Mortal

$$0 \le t \le 1$$

$$-\int_{C_2} F(z)\, dz = -\int_0^1 \left[t + (1-t)\, i\right]^2 (1-i)\, dt =$$

$$= \int_0^1 \left[t^2 + 2ti - 2t^2 i - (1 - 2t + t^2)\right](1-i)\, dt =$$

$$= \int_0^1 \begin{array}{l} t^2 + 2ti - 2t^2 i - 1 + 2t - t^2 \\ \underline{-it^2 - 2t + 2t^2 + i - 2ti - it^2} \end{array}$$

$$= -\int_0^1 \left[-2t^2 i + 4t - 2t^2 - 1 + i\right] dt =$$

$$= -\left[-\frac{2t^3}{3} i + \frac{4t^2}{2} - \frac{2}{3} t^3 - t + it\right]\Big|_0^1 =$$

$$= -\frac{1}{3} - \underline{\frac{1}{3} i}$$

▶ Para C_3

$$-\int_{C_3} F(z)\, dz = \left| \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right. \qquad \begin{array}{l} z(t) = ti \\ z'(t) = i \end{array} \qquad 0 \le t \le 1$$

$$-\int_{C_3} F(z)\,dz = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \qquad z'(t) = i$$

$$= -\int_0^1 (ti)^2 (i)\,dt = \int_0^1 t^2 i = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$= \left(\frac{1}{3}\right)\left(-\frac{1}{3} - \frac{1}{3}i\right) + \left(\frac{1}{3}i\right) = 0$$

Teorema de Cauchy

$$\oint_C (z - z_0)^n\,dz = \begin{cases} 0 & \text{si } n \neq -1 \\ 1 & \text{si } n \end{cases}$$

Integrales

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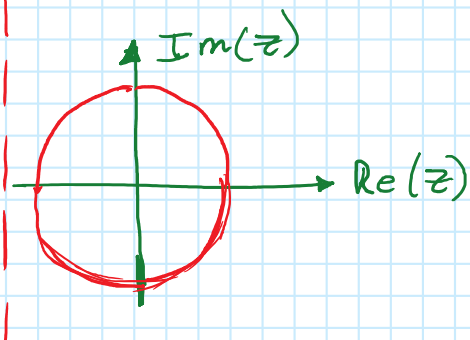
$$\oint \frac{z-\bar{z}}{\bar{z}}\,dz$$

donde la curva C está dada por $|z|=1$, recorrida una vez en sentido positivo



$$F(z) = \frac{z}{\bar{z}} - 1$$

$$\therefore \oint F(z)\,dz = \underbrace{\underbrace{\oint_{C:|z|=1} \frac{z}{\bar{z}}\,dz}_{\text{No analítica}} - \underbrace{\cancel{\oint_{C:|z|=1} 1\,dz}}_{\text{Cerrada y Analítica}}}_{\text{No análitica}} = 0$$

NOTA

$\bar{z} \overset{\Delta}{=}$ No analítica

// Definición

$$\int_C F(z)\,dz = \int_a^b F(z(t))\,(z'(t))\,dt$$

// Parametrización

$$\begin{aligned} x &= \cos t \\ y &= \operatorname{sen} t \end{aligned} \quad 0 \le t \le 2\pi$$

$$\therefore z(t) = \cos t + i \operatorname{sen} t = e^{ti}$$

$$z'(t) = i e^{ti}$$

$$\oint_{C:|z|=1} \frac{z}{\bar{z}}\,dz = \int_0^{2\pi} \left(\frac{e^{ti}}{e^{-ti}}\right)\left(i e^{ti}\right) dt =$$

$$= \int^{2\pi} e^{ti} \cdot e^{ti} e^{ti}\,dt = \int^{2\pi} i e^{3ti}\,dt =$$

$$= \int_0^{2\pi} e^{ti}\, i\, e^{ti} e^{ti}\, dt = \int_0^{2\pi} i\, e^{3ti}\, dt =$$

$$= \frac{\cancel{i}}{3\cancel{i}} e^{3ti} \Big|_0^{2\pi} = \frac{1}{3}\Big[\underbrace{\cancel{e^{3(2\pi)i}}}^{1} - \cancel{e^{0}}^{1}\Big] =$$

$$= 0$$

Recuerda Propiedad

$$e^{2\pi n i w} = e^{6\pi i} = 1$$

$$\oint \frac{dz}{z^3 - 4z^2 + 8z}$$

$C: |z - 2i| = 4$

recorrido en sentido positivo

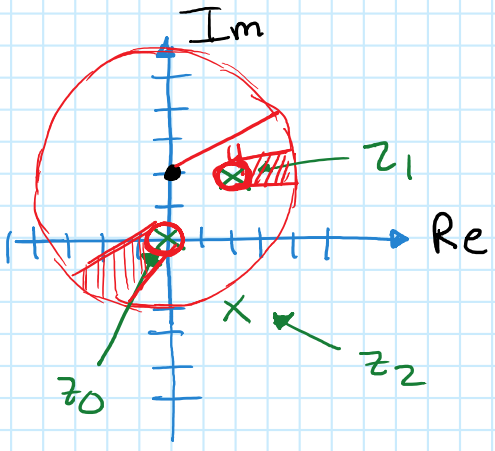
$$z(z^2 - 4z + 8)$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4(1)(8)}}{2(1)} =$$

$$= \frac{4 \pm \sqrt{-16}}{2} = \frac{4 \pm -4i}{2} = 2 \pm 2i$$

Raíces | $z = 0$ $\quad z_1 = 2 + 2i \quad z_2 = 2 - 2i$

Analizando a $\mathbb{C}$



$|z - 2i| = 4$

Arriba

/// Omitimos los puntos (singularidades)

La función tiene 2 indeterminaciones

La función tiene 2 indeterminaciones
∴ No es analítica

// Expresando a $F(z)$ utilizando fracciones parciales

$$F(z) = \frac{1}{z^3 - 4z^2 + 8z} = \frac{A}{z} + \frac{B}{z-(2+2i)} + \frac{C}{z-(2-2i)} =$$

$$= \frac{A}{z} + \frac{B}{z-(2+2i)} + \frac{C}{z-(2-2i)} =$$

// Analisis previo

$$\oint_{C:|z-2i|=4} F z\, dz = \oint \frac{A}{z} dz + \oint \frac{B}{z-2+2i} dz + \underbrace{\cancelto{0}{\oint \frac{C}{z-(2-2i)} dz}}_{F(z) \text{ es analítica}}$$

$$\frac{1}{z\underbrace{(z-2-2i)(z-2+2i)}_{z^2-4z+8}} = \frac{A}{z} + \frac{B}{z-2+2i}$$

→ Fracciones Parciales

$$1 = A(z^2-4z+8) + B(z^2-2(2-2i))$$

$$A = 1/8 \qquad B = \frac{1}{8-8i}$$

// Integrales

$$\oint_{C:|z-2i|=4} F(z)\, dz = \oint \frac{\frac{1}{8}}{z} dz + \oint \frac{\frac{1}{8-8i}}{z-2+2i} + \cancelto{0}{\oint}$$

$$= \frac{1}{8}\oint \frac{1}{z} dz + \frac{1}{8-8i}\oint \frac{1}{z-2+2i} + 0 =$$

$$= \frac{1}{8}\oint \frac{1}{z}\,dz + \frac{1}{8-8i}\oint \frac{1}{z-2+2i} + 0 =$$

$$= \frac{1}{8}\left[2\pi i\right] + \frac{1}{8-8i}\left[2\pi i\right] =$$

NOTA $\left| \oint_C (z-z_0)^n dz \right|$ 0 si $n \neq -1$; $2\pi i$ si $n = -1$

$$= \frac{2\pi}{8} i + \left(\frac{2\pi i}{8-8i}\right)\left(\frac{8+8i}{8+8i}\right) =$$

$$= \frac{2\pi i}{8} - \frac{\pi}{8} + \frac{\pi}{8} i = -\frac{\pi}{8} + \frac{3\pi}{8} i$$

II – Serie 3

$\oint F(z)\,dz$ | $F(z) = \dfrac{iz+2}{z}$

Curvas | a) $z\bar{z} - 2i(z-\bar{z}) + 3 = 0$ | Sentido positivo

| b) $z\bar{z} + i(z-\bar{z}) - 3 = 0$

$z\bar{z} = x+y$

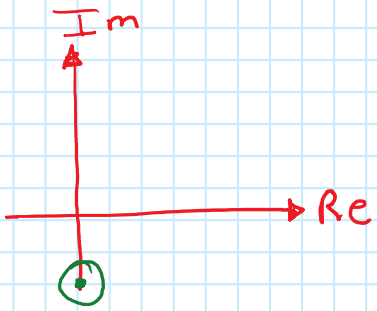
$z\bar{z} - 2i(z-\bar{z}) + 3 = 0$

$x^2+y^2 - 2i(\cancel{x}+yi - \cancel{x}+yi) + 3 = 0$

$x^2+y^2 - 2i(2yi) + 3 = 0$; $x^2+y^2+4y+3 = 0$

$x^2 + (y+2)^2 = 1$ // Circunferencia radio y con $C(0,-2)$

con $C(0,-2)$



$$F(z) = \frac{iz+2}{z}$$

$$\oint \frac{iz+z}{z}\,dz$$

$$C: x^2+(y+2)^2=1$$

Se indetermina en $z=0$; $(0,0)$

// No afecta el punto

$$\therefore \oint \frac{iz+2}{z}\,dz = 0$$

Porque es analítica y cerrada

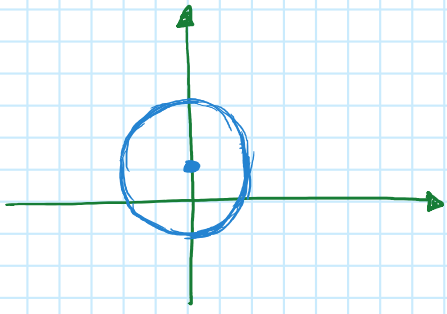
$$z\bar{z} + i(z-\bar{z}) - 3 = 0$$

$$x^2+y^2+i(\cancel{x}+yi-\cancel{x}+yi)-3=0$$

$$x^2+y^2-2y-3=0$$

$$x^2+(y-1)^2=4$$

Deformación de la trayectoria



- Centro $(0,1)$
- Radio $= 2$

$$\oint \frac{i\cancel{z}^{1}}{\cancel{z}} + \frac{2}{z}\,dz =$$

$$\oint \cancel{i\,dz}^{\,0} + \oint \frac{2}{z}$$

$C: x^2+(y-1)^2=4$ $\quad$ $C: x^2+(y-1)^2=4$

Cerrada y analítica | Se indetermina en $z=0$

CAMINO 1

TEOREMAS

CAMINO 2

Cerrada y analítica | se indeterminа en $z=0$

CAMINO 2

$$\oint_a^b F[z(t)]\, z'(t)\, dt$$

$$\therefore = 0 + 2\oint \frac{1}{z}\, dz$$

$$C: x^2 + (y-1)^2 = 4$$

$x = \cos t$ | $0 \le t \le 2\pi$

$y = \operatorname{sen} t + 1$ | $z(t) = \cos t + i(\operatorname{sen} t + 1)$

$$\therefore\ 2\oint \frac{1}{z}\, dz = 2\int_0^{2\pi} \frac{1}{\cos t + i(\operatorname{sen} t + 1)} = 2\left[2\pi i\right]$$

$$= 4\pi i$$ // ¿? Checar //

Desviación. Integrales Línea

Tuesday, March 19, 2019 8:41 PM

$$\oint \frac{1}{z^2+8z}\,dz \qquad C: |z|=1 \quad \text{sentido negativo}$$

$$z(z+8) \;\Big|\; \begin{array}{l} z_1 = 0 \\ z_2 = -8 \end{array}$$

$$\underset{C:|z|=1}{\oint} F(z)\,dz = \underset{C:|z|=1}{\oint} \frac{A}{z}\,dz + \underbrace{\underset{C:|z|=1}{\oint} \frac{B}{z+8}\,dz}$$

Aquí no se indetermina en la curva

$$= \underset{C:|z|=1}{\oint} \frac{A}{z}\,dz \qquad \oint_C F(z)\,dz = \int_a^b F[z(t)]\,z'(t)\,dt$$

$$x = \cos t \qquad y = \operatorname{sen} t$$

$$z(t) = \cos t + i \operatorname{sen} t \qquad 0 \le t \le 2\pi$$

$$z(t) = e^{-it} \qquad 0 \le t \le 2\pi$$

$$z' t = -i e^{-it}$$

Teorema

$$\int_0^{2\pi} (\cancel{e^{-it}})(-i\cancel{e^{-it}}) - i\,dt = -it\Big|_0^{2\pi} =$$

$$= -2\pi i$$

$$= A(-2\pi i) = \frac{1}{8}(-2\pi i) = -\frac{\pi}{4}\,i$$

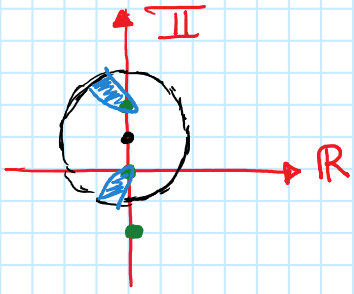
$$\frac{1}{x(x+8)} = \frac{A}{x} + \frac{B}{x+8} \;;\; 1 = A(x+8) + B(x)$$

if $x=0$

$1 = 8A \; ; \; A = 1/8$

$$\oint \frac{z^2+1}{z^3+4z}\,dz \qquad C: |z-i| = 2$$

$C(0,i)$



- Función Racional

$z^3+4z=0$

$z(z^2+4)=0$

$z_1=0$

$z_{2,3} = \pm 2i \begin{cases} z_2 = 2i \\ z_3 = -2i \end{cases}$

- Punto del denominador (Donde se indetermina)

$$\frac{z^2+1}{z(z+2i)(z-2i)}$$

$$\oint_{C_T} F(z)\,dz = \oint_{C_1 \to z=2i} F z\,dz + \oint_{C_2 \to z=0} F z\,dz$$

$$\oint_{C_1 \to z=2i} \frac{z^2+1}{z(z+2i)(z-2i)}\,dz =$$

$$\oint \frac{\frac{z^2+1}{(z)(z+2i)}}{(z-2i)}\,dz = 2\pi i \left.\frac{z^2+1}{(z)(z+2i)}\right|_{z=2i} =$$

$$= (2\pi i)\left(\frac{(2i)^2+1}{(2i)(2i+2i)}\right) = (2\pi i)\left(\frac{3}{8}\right)$$

$$= \frac{6}{8}\pi i$$

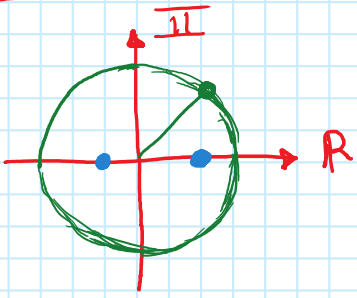
$$\oint \frac{\frac{z^2+1}{(z+2i)(z-2i)}}{z}\, dz = 2\pi i \left(\frac{z^2+1}{z^2+4}\right)\Bigg|_{z=0} =$$

$$= 2\pi i\left(\frac{1}{4}\right) = \frac{\pi}{2} i$$

$$\therefore \oint F(z)\, dz = \frac{3}{4}\pi i + \frac{2\pi}{4} i = \frac{5\pi}{4} i$$

$$\oint \frac{9z}{(z+1)(z-2)^2}\, dz$$

$C: |z| = 3$



→ Raíces

$z_1 = -1 \quad z_{2,3} = 2$

$$\oint_{C_T \to |z|=3} F_z\, dz = \oint_{C_1 \to z=-1} F_z\, dz + \oint_{C_2 \to z=2} F_z\, dz$$

// Sólo 2 deformaciones

$C \to z=-1$

$$\oint \frac{\frac{9z}{z-2}}{(z+1)}\, dz = 2\pi i \left.\frac{9z}{z-2}\right| =$$

$$\oint \frac{\frac{z-2}{(z+1)}}{} dz = 2\pi i \left.\frac{9z}{z-2}\right|_{z=-1} =$$

$$= -2\pi i$$

$C \to z=2$

$n+1=2;\ n=1$

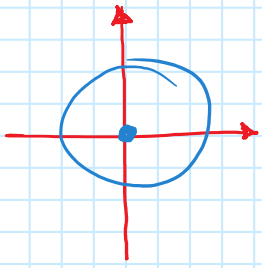
$$\oint \frac{\frac{9z}{z+1}}{(z-2)^{2}} \quad \frac{2\pi i}{1!} \frac{d}{dz}\left(\left.\frac{9z}{z+1}\right|_{z=0}\right) = 2\pi i$$

$2\pi i$

$$\oint_{C_T} F z\, dz = -2\pi i + 2\pi i = 0$$

13

$$F(z) = \frac{iz+i}{z^2}$$

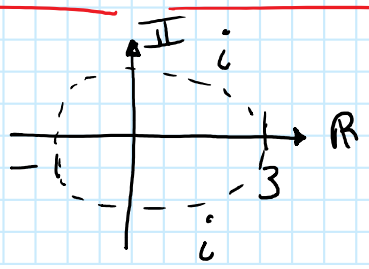


$C: |z+1| = 2$

$$\oint_{C_T:\ |z+1|=2} F(z)dz = \oint_{C_1 \to z=0} F(z)\,dz = \frac{2\pi i}{1!} F'(0) =$$

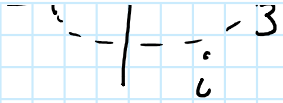
$$= \frac{2\pi i}{1!} \frac{d}{dz}(iz+i) = -2\pi$$

i



$$F(z) = \frac{1}{z^4+z^2}$$

$$z^2(z^2+1) \quad z_{1,2} = 0$$



$$z^2(z^2+1) \quad \begin{cases} z_{1,2} = 0 \\ z_{3,4} = \pm i \end{cases}$$

$$\oint_{C_F} F z\, dz = \oint_C \frac{dz}{z^4+z^2}$$

$$= \oint_{C_1 \to z=i} F(z)\,dz + \oint_{C_2 \to z=-i} F(z)\,dz + \oint_{C_3 \to z=0} F(z)\,dz =$$

$C_1 \to z = i$

$$\oint \frac{\frac{1}{(z^2)(z+i)}}{(z-i)}\,dz = 2\pi i \left(\frac{1}{(z^2)(z+i)}\right)\Bigg|_{z=i} = -\pi$$

$C_2 \to z = -i$

$$\oint \frac{\frac{1}{(z^2)(z-i)}}{(z+i)}\,dz = 2\pi i \left(\frac{1}{(z^2)(z-i)}\right)\Bigg|_{z=-i} = \pi$$

$C_3 \to z = 0$

$$\oint \frac{\frac{1}{z^2+1}}{z^2}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\left(\frac{1}{z^2+1}\right)\Bigg|_{z=0} = 0$$

$$\oint_C F(z)\,dz = 0$$