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Black-Litterman Model

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1 Introduction

The Black-Litterman model is an asset allocation model developed in 1990 by Fischer Black and Robert Litterman at Goldman Sachs. This model combines ideas from the Capital Asset Pricing Model (CAPM), Bayesian Statistics and the Markowitz's mean-variance optimization model to provide a tool for investors to calculate the optimal portfolio weights under specified parameters.

Prior to the Black-Litterman model, investors used to input expected returns of the assets into the Markowitz's model to generate portfolio weights. Because there is a complex mapping between them, and because there is no natural starting point for the expected return assumptions, users of the standard portfolio optimizers often find their specification of expected returns produces output portfolio weights which do not seem to make sense. In the Black-Litterman model the user inputs any number of views, which are statements about the expected returns of arbitrary portfolios, and the model combines the views with equilibrium, producing both the set of expected returns of assets as well as the optimal portfolio weights.

The investor starts by holding the scaled market equilibrium portfolio, reflecting his uncertainty on the equilibrium, then invests in portfolios representing his views. The Black-Litterman model computes the weight to put on the portfolio representing each view according to the strength of the view, the covariance between the view and the equilibrium, and the covariances among the views.

This paper will provide an overview of the process of using the Black-Litterman Model as having three distinct steps, which are represented as three sections. The first section is about the calculation of the informative prior estimate of returns. The prior is derived from the CAPM equilibrium portfolio. The second step is the specification of the investor's views, and the third one is the mixed estimation process used to blend the prior estimates of returns with the views to create the posterior estimates of the returns along with estimates of the uncertainty of the estimates.

2 Computing the equilibrium returns

This section describes in more detail how we find equilibrium returns. In the Black-Litterman framework, expected returns are viewed as a blend of equilibrium returns and an actual set of investor views. Equilibrium, according to Litterman, is an idealized state in which supply equals demand. The equilibrium returns can be interpreted as the long-run returns provided by the global capital markets. Under this interpretation, the equilibrium returns represent the information that is available through the capital markets. Investor views, by contrast, correspond to the interpretation of information that is unique to the individual investor. Thus, expected returns are a blend of the information available in the capital markets and information unique to a specific investor. As the mixture of sources of information changes, expected returns will also change.

The Black-Litterman model relies on General Equilibrium theory to state that if the aggregate portfolio is at equilibrium, each sub-portfolio must also be at equilibrium. It can be used with any utility function which makes it very flexible. In practice most practitioners use the Quadratic Utility function and assume a risk free asset, and thus the equilibrium model simplifies to the Capital Asset Pricing Model (CAPM). The neutral portfolio in this situation is the CAPM Market portfolio.

Here we will use the Quadratic Utility function, CAPM and unconsrained mean-variance because it is a well understood model.

2.1 CAPM

This theory, that has been separately developed by William Sharpe (1964), John Lintner (1965), Jan Mossin (1966) and Jack Treynor (1961), builds on the mean-variance analysis of Markowitz to develop a model that can compute the expected return of an asset if an equilibrium would exist in the market.

The model is based on the following assumptions. Suppose that every investor bases his investing decisions on mean-variance theory. Suppose also that the investors agree on the future performance of every asset in the investment universe, hence everyone assigns the same mean, variance and covariance to the assets. Assume there is a unique risk-free rate of borrowing or lending available to all investors and that there are no transaction costs.

Given our previous assumptions, the prior distribution for the Black-Litterman model is the estimated mean excess return from the CAPM market portfolio. The process of computing the CAPM equilibrium excess returns is presented here.

CAPM is based on the concept that there is a linear relationship between the rate of return of any security and the security's systematic risk called beta. Further, it requires returns to be normally distributed. This model is of the form

$$E(r_i) = r_f + \beta_i(E(r_m) - r_f), \tag{1}$$

where

- r_f the risk free rate;
- $E(r_m)$ the excess return of the market portfolio;
- β_i a regression coefficient computed as $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$ where σ_{im} represents the covariance between asset i and the market portfolio m and σ_m^2 represents the variance of the market portfolio.

In the CAPM world, all investors should hold the same risky portfolio, the CAPM market portfolio. Because all investors hold risky assets only in the market portfolio, at equilibrium the market capitalization of the various assets will determine their weights in the market portfolio. It is optimal in the sense that the CAPM market portfolio has the maximum Sharpe Ratio (excess return divided by excess risk) of any portfolio on the efficient frontier.

The investor can also invest in a risk free asset. This risk free asset has essentially a fixed positive return for the time period over which the investor is concerned. The Capital Market Line is a line through the risk free rate and the CAPM market portfolio. The Two Fund Separation Theorem, closely related to the CAPM, states that all investors should hold portfolios on the Capital Market Line. Any portfolio on the Capital Market Line dominates all portfolios on the Efficient Frontier, the CAPM market portfolio being the only point on the Efficient Frontier and on the Capital Market Line.

The diagram below illustrates the relationship between the Efficient Frontier and the Capital Market Line.

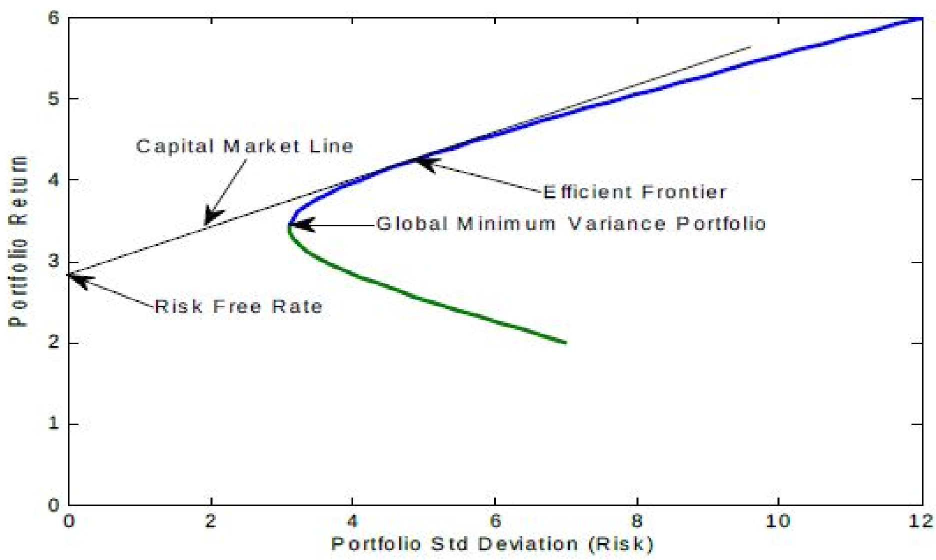

Figure 1: relationship between the Efficient Frontier and the Capital Market Line

2.2 The estimated distribution of expected returns

Since we are starting with the market portfolio, we will be starting with a set of weights which are all greater than zero and naturally sum to one. We will constrain the problem by asserting that the covariance matrix of the returns, Σ is known. Using the implied returns from CAPM as the prior and then adding the investors' views, a posterior distribution can be obtained.

As starting point for the Black Litterman approach we use equilibrium returns implied by reverse optimization method. Reverse optimization in contrast to classical mean-variance optimization uses portfolio weights as input and produces implied equilibrium returns. The main reason for reverse optimization being good way to forecast equilibrium returns and use weights as input is that the weights are sometimes easier to predict. Thus, reverse optimization produces more "healthy" expected returns than those in mean-variance optimization. Secondly, the weights in reverse optimization are easier interpreted by practitioners. Here we derive the equations for reverse optimization starting from the quadratic utility function:

$$U = w_m^T \Pi - (\frac{\delta}{2}) w_m^T \Sigma w_m \tag{2}$$

- U is the investors utility;
- w_m - vector of market portfolio weights (a benchmark or index portfolio is used as a proxy for the market weights, taking away the problem of estimating these weights);
- Π - vector of equilibrium excess returns for each asset (the expected return in the domestic currency minus the domestic risk free rate, $E(r) - r_f$);
- δ - risk aversion parameter;
- Σ - covariance matrix of the excess returns for the assets.

The Hessian matrix of U is negative definite ($-\delta\Sigma$) because the main property of a covariance matrix is its positive definiteness ($\Sigma > 0$), it means that U is a concave function, so it will have a single global maximum. If we maximize the utility without constraints, there is a closed form solution. We find the exact solution by taking the gradient of (2) with respect to the weights (w_m) and setting it to 0:

$$\nabla U = \Pi - \delta \Sigma w_m = 0.$$

The solution to this problem is

$$w_m = (\delta\Sigma)^{-1}\Pi.$$

In this case the weights are already known and we are interested in the vector of expected returns (Π), thus for a mean-variance efficient portfolio reverse optimization entails

$$\Pi = \delta \Sigma w_m. \tag{3}$$

In order to apply the equation (3) to solve for the CAPM market portfolio, we need to have a value for δ, the risk aversion coefficient of the market. We can find δ by deriving it from the CAPM formula:

$$E(r) - r_f = \beta(E(r) - r_f)$$

$$\Pi = \beta(E(r) - r_f) = \frac{cov(r, r'w_m)}{\sigma_m^2}(E(r) - r_f)$$

$$= \frac{cov(r, r')w_m}{\sigma_m^2}(E(r) - r_f) = \frac{(E(r) - r_f)}{\sigma_m^2} cov(r, r')w_m$$

$$= \delta \Sigma w_m, \text{ where } \delta = \frac{(E(r) - r_f)}{\sigma_m^2}.$$

The risk aversion coefficient is the rate that acts as a scaling factor for the reverse optimization estimate of access returns. It is thought to best describe human investment behavior. There is a specific field concerned with defining functions that can categorize preferences and formalizes the principle of risk aversion, this field is called utility theory.

The need of covariance estimation along with expected return is fulfilled with the simplicity of assumption from Black Litterman specifying the structure of covariance matrix of estimation Σ_Π in proportion to to the covariance matrix of returns $\tau\Sigma$. The parameter τ is created as the constant of proportionality. It will be close to zero, because the uncertainty in the mean of the return is much smaller than the uncertainty in the return itself.

Then the prior distribution for Black-Litterman model is given by:

$$E(r) \sim N(\Pi, \tau\Sigma), r \sim N(E(r), \Sigma). \tag{4}$$

It represents the estimate of mean which is expressed as the distribution of actual unknown mean about our estimate.

The diagram below illustrates the Black-Litterman investor's portfolio with no views.

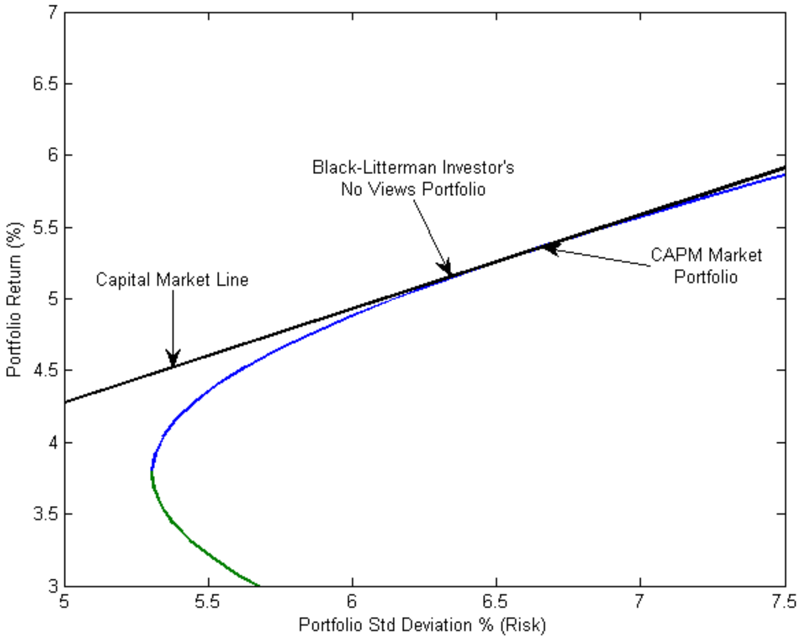

Figure 2: the Black-Litterman investor's portfolio with no views

3 Investor's views

This section will describe the process of specifying the investors views on the estimated mean excess returns. Most of the time, an investor has specific views regarding the expected returns of assets in portfolio. The mathematical representation of these views needs to meet a few characteristics. The views have to be specified relative to the vector of expected return $E(r)$, the views have to be specified relative to each other and it has to be possible to express a level certainty in the view.

The model allows the investor to express both absolute and relative views. An example of an absolute view is *"I expect that equities in country A will return X%"* an example of a relative view is *"I believe domestic bonds will outperform domestic equities by Y%"*. These k views on n assets can be represented using the following parameters:

- P, a $k \times n$ matrix of the assets weights within each view. For a relative view the sum of the weights will be 0, for an absolute view the sum of the weights will be 1.
- Q, a $k \times 1$ vector of the returns for each view.
- Ω, a $k \times k$ matrix of the covariance of the views. It is diagonal as the views are required to be independent and thus uncorrelated. Ω^{-1} is known as the confidence in the investor's views. The i-th diagonal element of Ω is represented as w_i.

These prerequisites lead to the following specification:

$$PE(r) = Q + \epsilon, \tag{5}$$

where $\epsilon \in \mathbb{R}^{k\times 1}$ is an error vector with mean zero and variance Ω ($\epsilon \sim N(0, \Omega)$).

That the mean is zero means that the investor does not have a standard bias against a certain set of assets. It is assumed that the views are mutually uncorrelated and therefore the covariance matrix Ω is diagonal. The vector $E(r)$ is the unknown expected return vector that needs to be estimated.

As an example of how these matrices are populated we examine four assets and two investor's views:

- a relative view in which the investor believes that Asset 1 will outperform Asset 3 by 2% with confidence w_{11};
- an absolute view in which the investor believes that Asset 2 will return 3% with confidence w_{22};
- that the investor has no view on Asset 4, and thus it's return should not be directly adjusted.

These views can be expressed in matrix form in the following way:

$$P = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};\ Q = \begin{pmatrix} 2 \\ 3 \end{pmatrix};\ \Omega = \begin{pmatrix} w_{11} & 0 \\ 0 & w_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} E(r) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \epsilon,\ \ \epsilon \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} w_{11} & 0 \\ 0 & w_{22} \end{pmatrix} \right)$$

3.1 Specifying Ω

Black and Litterman did not show any specific way of estimating Ω in their 1992 paper. It is up to the investor to compute the variance of the views Ω.

There are several ways to calculate Ω :

- proportionality to the variance of the prior;
- use a confidence interval;
- use Idzorek's method to specify the confidence along the weight dimension.

3.1.1 Proportionality to the variance of the prior

This exact computational method was revealed by He and Litterman (1999). Employing this method, the elements of Ω are calculated in accordance with the following expression:

$$\Omega = diag(P(\tau\Sigma)P^T), \text{ where} \tag{6}$$

$$w_{i,j} = P(\tau\Sigma)P^T \ \forall i = j$$

$$w_{i,j} = 0 \ \forall i \neq j$$

Here it is assumed that the variance of the views is proportional to the variance of the asset returns, just as the variance of the prior distribution is.

3.1.2 Confidence Interval

Another way to specify the variance is using the confidence interval around the estimated mean return, because as defined before, Ω is the uncertainty in the estimate of the mean.

*Example:*an asset has an estimated 3% mean return. The expectation to be within the interval (2%,4%) is 68%. Knowing that 68% of the normal distribution falls within 1 standard deviation of the mean allows us to translate this into a variance for the view of $(1\%)^2$.

3.1.3 Idzorek's method

Idzorek's(2005) goal was to reduce the complexity of the Black-Litterman model for non-quantitative investors. He achieves this by allowing the investor to specify the investors confidence in the views as a percentage (0-100%) where the confidence measures the change in weight of the posterior from the prior estimate (0%) to the conditional estimate (100%). This linear relation is

$$\text{confidence} = \left(\frac{w - w_{mkt}}{w_{100} - w_{mkt}}\right),$$

where

- w is the weight of the asset under the specified view;

- w_{100} is the weight of the asset under 100% certainty in the view;
- w_{mkt} is the weight of the asset under no views.

These values are then combined to form Ω and the model is used to compute posterior estimates:

$$\Omega = \alpha P \Sigma P^T \tag{7}$$

α is the coefficient of uncertainty, is a scalar quantity in the interval $[\ 0, \infty\ ]$:

$$\alpha = \frac{1 - \text{confidence}}{\text{confidence}}. \tag{8}$$

When the investor is 100% confident in their views, then α will be 0, and when they are totally uncertain then α will be ∞. Using formulas (7) and (8) the investor can easily calculate the value of ω for each view and then roll them up into a single matrix Ω.
In the Black-Litterman model, the prior distribution is based on the equilibrium implied excess returns. The conditional distribution is based on the investor's views. The investor's views are specified as returns to portfolios of assets, and each view has an uncertainty which will impact the overall mixing process.

4 The Bayesian approach: Combining views with equilibrium

The algebraic expression from Bayes Theorem used to blend the prior distribution and the view distribution to form the posterior distribution of expected returns is known as *the Black-Litterman Formula.*

Bayes theorem.

$$P(A/B) = \frac{P(B/A)P(A)}{P(B)} \tag{9}$$

- $P(A/B)$ - the conditional probability of A given B. Known as **the posterior distribution**.
- $P(B/A)$ - the conditional probability of B given A. Known as **the sampling distribution.**
- $P(A)$ - the probability of A. Known as **the prior distribution**.
- $P(B) \neq 0$ - the probability of B. Known as **the normalizing constant.**

A general problem in using Bayes theory is to identify an intuitive and tractable prior distribution. One of the core assumptions of the Black-Litterman model is that asset returns are normally distributed with mean $E(r)$ and variance Σ. Furthermore, they assume that this mean itself is a random variable, is unobservable and stochastic. They choose a distribution for $E(r)$ heuristically assuming that the market is always moving to equilibrium, and is not necessarily in equilibrium. Therefore, the mean of the expected return should be equal to the expected returns that would hold if the market is in equilibrium, i.e. equal to the CAPM returns. The variance of $E(r)$ is assumed to be proportional to the variance of the returns r, proportional to Σ with proportionality constant τ.

It is assumed that the investor forms his views using knowledge of the equilibrium expected returns. Therefore, the equilibrium expected returns are considered the prior returns and these will be updated with the views of the investor. The posterior distribution combines both sources of information. For that reason we have the case of normally distributed conditional and prior distributions. Given that the inputs are normal distributions, then it follows that the posterior will also be normally distributed.

Using Bayes' formula this yields:

$$Pr(E(r)/PE(r)) = \frac{Pr(PE(r)/E(r))Pr(E(r))}{Pr(PE(r))}. \tag{10}$$

4.1 Black Litterman Formula

After combining the prior with the views in accordance with the original Black-Litterman framework, the posterior distribution is expressed as a normally distributed random vector with a mean equal to the vector $\mu*$ and a covariance matrix denoted M. Algebraically,

$$N(\mu*, M),$$

where

$$\mu* = ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}((\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q). \tag{11}$$

and

$$M = ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}. \tag{12}$$

The proof for these expressions is a straightforward application of Bayes' Theorem. The probability distributions can be deduced from the following assumptions:

Assumption 1:

The expected return $E(r)$ is a random variable, which is normally distributed with mean Π and variance $\tau\Sigma$:

$$E(r) \sim N(\Pi, \tau\Sigma).$$

Assumption 2:

$PE(r)/E(r)$ is normally distributed with mean Q and diagonal covariance matrix Ω :

$$PE(r)/E(r) \sim N(Q, \Omega).$$

The probability density functions of $E(r)$ and $PE(r)/E(r)$ are:

$$\frac{1}{\sqrt{(2\pi)^n \det(\tau\Sigma)}} \exp(-\frac{1}{2}(E(r)-\Pi)'(\tau\Sigma)^{-1}(E(r)-\Pi)) \tag{13}$$

and

$$\frac{1}{\sqrt{(2\pi)^k \det(\Omega)}} \exp(-\frac{1}{2}(PE(r)-Q)'\Omega^{-1}(PE(r)-Q)) \tag{14}$$

correspondingly.

These distributions can be substituted in equation (10) to obtain the posterior distribution. We will first concentrate on the numerator of the equation (10). Substituting the distributions into the formula gives

$$Pr(PE(r)/E(r))Pr(E(r)) =$$

$$\frac{1}{\sqrt{(2\pi)^n \det(\tau\Sigma)}} \exp(-\frac{1}{2}(E(r)-\Pi)'(\tau\Sigma)^{-1}(E(r)-\Pi))\times$$
$$\frac{1}{\sqrt{(2\pi)^k \det(\Omega)}} \exp(-\frac{1}{2}(PE(r)-Q)'\Omega^{-1}(PE(r)-Q)).$$

We leave out all the constants and are left with:

$$\exp(-\frac{1}{2}(E(r)-\Pi)'(\tau\Sigma)^{-1}(E(r)-\Pi) - \frac{1}{2}(PE(r)-Q)'\Omega^{-1}(PE(r)-Q)).$$

Expanding the brackets in the exponent and dropping the exponent and the factor a half in the exponent gives:

$$E(r)'P'\Omega^{-1}PE(r) - E(r)'P'\Omega^{-1}Q - Q'\Omega^{-1}PE(r) + Q'\Omega^{-1}Q+$$

$$E(r)'(\tau\Sigma)^{-1}E(r) - E(r)'(\tau\Sigma)^{-1}\Pi - \Pi'(\tau\Sigma)^{-1}E(r) + \Pi'(\tau\Sigma)^{-1}\Pi.$$

The term $Q'\Omega^{-1}PE(r)$ is equal to $E(r)'P'\Omega^{-1}Q$ due to the symmetric property of Ω, the same holds for $E(r)'(\tau\Sigma)^{-1}\Pi$ and $\Pi'(\tau\Sigma)^{-1}E(r)$, now due to the symmetry of Σ. The previous equation can thus be shortened to:

$$E(r)'P'\Omega^{-1}PE(r) - 2Q'\Omega^{-1}PE(r) + Q'\Omega^{-1}Q+$$

$$E(r)'(\tau\Sigma)^{-1}E(r) - 2\Pi'(\tau\Sigma)^{-1}E(r) + \Pi'(\tau\Sigma)^{-1}\Pi.$$

This can be expanded even further to:

$$E(r)'[P'\Omega^{-1}P + (\tau\Sigma)^{-1}]E(r) - 2[Q'\Omega^{-1}P + \Pi'(\tau\Sigma)^{-1}]E(r) + Q'\Omega^{-1}Q + \Pi'(\tau\Sigma)^{-1}\Pi.$$

The formula will be simplified by introducing three symbols C, H, A :

$$C = (\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q,$$

$$H = (\tau\Sigma)^{-1} + P'\Omega^{-1}P, \text{ where } H \text{ is symmetrical } H = H',$$

$$A = Q'\Omega^{-1}Q + \Pi'(\tau\Sigma)^{-1}\Pi.$$

With this shortened notation we can rewrite the exponent to:

$$E(r)'HE(r) - 2C'E(r) + A = E(r)'H'E(r) - 2C'E(r) + A.$$

We introduce the identity matrix $I = H^{-1}H$ into the equation which will be useful.

$$E(r)'H'E(r) - 2C'E(r) + A =$$

$$(HE(r))'H^{-1}HE(r) - 2C'H^{-1}HE(r) + A =$$

$$(HE(r) - C)'H^{-1}(HE(r) - C) + A - C'H^{-1}C =$$

$$(E(r) - H^{-1}C)'H(E(r) - H^{-1}C) + A - C'H^{-1}C.$$

The terms in $A - C'H^{-1}C$ do not depend on $PE(r)$ and therefore disappear into the constant of integration. Thus, reintroducing the exponent and the factor half in the exponent leaves us with:

$$\exp(-\frac{1}{2}(E(r) - H^{-1}C)'H(E(r) - H^{-1}C)).$$

Hence, $E(r)/PE(r)$ has mean

$$((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}((\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q)$$

and variance

$$((\tau\Sigma)^{-1} + P'\Omega^{-1}P)^{-1}.$$

When deriving $\mu*$ the uncertainty associated with the prior distribution (as measured by τ) and the uncertainty associated with the view distribution (as measured by Ω) are taken into account.

An alternate representation of the same formula for the mean returns $\widehat{\mu}$ is

$$\widehat{\mu} = \Pi + \tau\Sigma P^T (P\tau\Sigma P^T + \Omega)^{-1}(Q - P\Pi) \tag{15}$$

The derivation of formula (15) is shown in Appendix (A).

Letting $\Omega \to 0$ by showing the 100% certainty of the views, the return can be defined as follows:

$$\widehat{\mu} = \Pi + \tau\Sigma P^T (P\tau\Sigma P^T)^{-1}(Q - P\Pi)$$

Furthermore, if P is invertible which means that a view on every asset has been offered, then

$$\widehat{\mu} = P^{-1}Q$$

If the investor is not sure about his views ($\Omega \to \infty$), then the formula (15) reduces to

$$\widehat{\mu} = \Pi.$$

Since expected returns themselves are assumed to be random variables in the Black-Litterman framework, the covariance matrix M can only be associated with the expected returns, i.e. the expected values of future returns. For an investor seeking to estimate the optimal portfolio weights, a crucial input needed to accomplish this is the covariances of the actual returns that are believed to prevail over the holding period. If the investor was to use the prior estimate of the covariance matrix of returns (i.e. Σ), he would fail to take into account that the expected returns themselves are not constants but random variables. The appropriate estimate of the posterior covariance matrix associated with the return distribution (denoted $\Sigma*$) was shown by He and Litterman (2002) as the expression

$$\Sigma* = \Sigma + M. \tag{16}$$

Substituting the posterior variance from (12) we get

$$\Sigma* = \Sigma + ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}.$$

The posterior covariance matrix takes into account that since the mean itself (μ) is a random variable, there is an added source of uncertainty that needs to be considered by the investor when estimating the covariances of future market returns. The distribution of returns can be described in accordance with the following equation:

$$r \sim N(\mu*, \Sigma*). \tag{17}$$

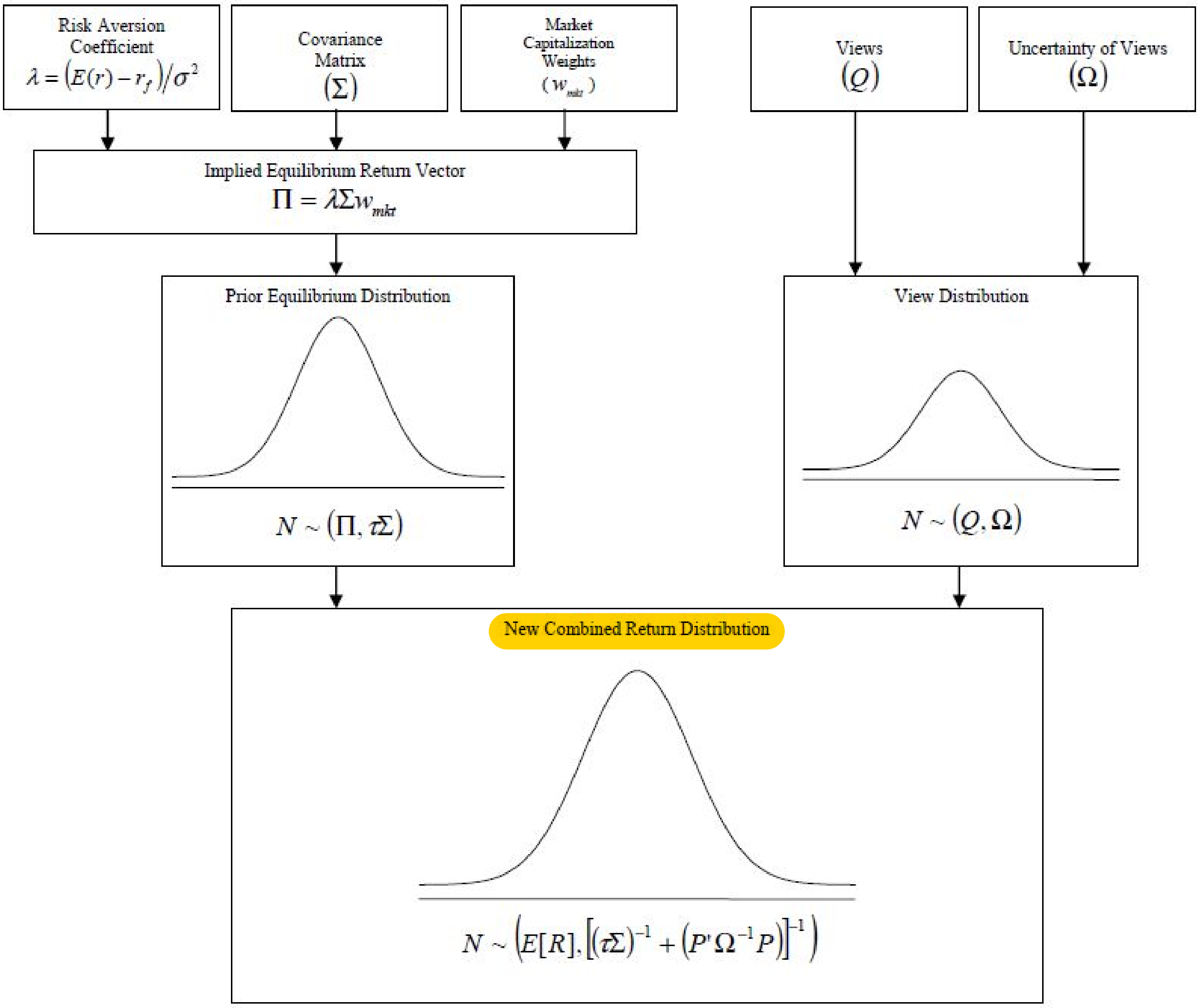

Figure 3: Deriving the New Combined Return Vector

4.2 The impact of τ

The meaning and impact of the parameter τ causes a great deal of confusion for many users of the Black-Litterman model. This section considers a richer understanding of τ and when it should be used, and if used how it should be calibrated.

He and Litterman (1999) propose considering τ as the ratio of the sampling variance to the distribution variance, and thus it is $1/t$. They use a value of τ of 0.05 which they describe as

> *"...corresponds to using 20 years of data to estimate the CAPM equilibrium returns."*

As described previously, τ is the constant of proportionality. This section describes three ways in which we might select the value for τ :

- use confidence intervals;
- examine the investors' uncertainty as expressed in their prior portfolio;

- basic statistics

It is important to remember that τ is a measure of the investor's confidence in the prior estimates, and as such it is largely a subjective factor.

1. Use of confidence intervals

The approach to establishing a reasonable value for τ is to use confidence intervals. The illustration of this concept is shown on a simple example of a plausible scenario with a prior estimate of 8% as the excess return and 15% as the known standard deviation of returns about the mean of a single asset. The table below shows the 95% and 99% confidence intervals for this scenario and various values of τ.

τ	95% Confidence	99% Confidence
0.0167	$\mu \in$ (4.13%, 11.87%)	$\mu \in$ (2.19%, 13.81%)
0.0250	$\mu \in$ (3.26%, 12.74%)	$\mu \in$ (0.88%, 15.12%)
0.0500	$\mu \in$ (1.30%, 14.71%)	$\mu \in$ (-2.06%, 18.06%)
0.2000	$\mu \in$ (-5.42%, 21.42%)	$\mu \in$ (-12.12%, 28.13%)
1.0000	$\mu \in$ (-22.00%, 38.00%)	$\mu \in$ (-37.00%, 53.00%)

Figure 4: confidence intervals for various values of τ

It is clear that setting τ too high makes a very weak statement for the prior estimate of the mean. For example where $\tau = 0.20$ is selected results in the estimate at the 99% confidence level is about 8% +/- 20% which is not a very precise estimate for the mean of a distribution.

2. The investors' uncertainty as expressed in their prior portfolio

Second, we can consider τ from the point of view of a Bayesian investor. Here τ is calibrated to the amount invested in the risk free asset given the prior distribution. The the portfolio invested in risky assets given the prior views will be

$$w = \Pi(\delta(1+\tau)\Sigma)^{-1}.$$

Thus the weights allocated to the assets are smaller by $\frac{1}{1+\tau}$ than the CAPM market weights. This is because the Bayesian investors is uncertain in their estimate of the prior, and they do not want to be 100% invested in risky assets. A value of $\tau = 1$ would lead to the investor being only 50% invested based on the prior estimates. A value of $\tau = 0.25$ would lead to the investor being 80% invested based on the prior estimates.A plausible example of this case would be a prior asset allocation of 90-95% which results in a value of τ of between 0.053 and 0.11.

3. Basic statistics

The first method to calibrate τ relies on falling back to basic statistics. When estimating the mean of a distribution, the uncertainty (variance) of the mean estimate will be proportional to the inverse of the number of samples. Given the covariance matrix from historical data, then

$$\tau = \frac{1}{T}$$

The maximum likelihood estimator
with T- the number of samples.

There are other estimators, but usually, the definition above is commonly used.

5 Conclusion and further research

The purpose of this paper has been to investigate the popular Black-Litterman model for asset allocation. It details the process of developing the inputs for the Black-Litterman model, which enables investors to combine their unique views with the implied equilibrium return vector to form a new combined return vector. The new combined return vector leads to intuitive, well-diversified portfolios. The two parameters of the Black-Litterman model that control the relative importance placed on the equilibrium returns vs. the view returns, the scalar τ and the uncertainty in the views Ω, are very difficult to specify. It could be a major improvement if the model could be adapted such that there is a method to compute these parameters.

A major drawback of the model is that it assumes that the returns have a normal distribution, which is often not the case. It would be an improvement to develop the Black-Litterman formula under the assumption of other distributions that better reflect the return distribution.

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Appendices

A Alternative formula for the posterior expected return

This appendix presents a derivation of the alternate formulation of the Black-Litterman master formula for the posterior expected return. Here it is shown in details how the formula (15) is derived from (11).

$$\mu* = ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}((\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q) =$$
$$((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}(\tau\Sigma)^{-1}\Pi + ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}P^T\Omega^{-1}Q =$$
$$(\tau\Sigma - \tau\Sigma P^T(P\tau\Sigma P^T + \Omega)^{-1}P\tau\Sigma)(\tau\Sigma)^{-1}\Pi + ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}P^T\Omega^{-1}Q =$$
$$(\Pi - \tau\Sigma P^T(P\tau\Sigma P^T + \Omega)^{-1}P\Pi) + ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}P^T\Omega^{-1}Q =$$
$$(\Pi - \tau\Sigma P^T(P\tau\Sigma P^T + \Omega)^{-1}P\Pi) + (\tau\Sigma)(\tau\Sigma)^{-1}((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}P^T\Omega^{-1}Q =$$
$$(\Pi - \tau\Sigma P^T(P\tau\Sigma P^T + \Omega)^{-1}P\Pi) + (\tau\Sigma)(I + P^T\Omega^{-1}P\tau\Sigma)^{-1}P^T\Omega^{-1}Q =$$
$$(\Pi - \tau\Sigma P^T(P\tau\Sigma P^T + \Omega)^{-1}P\Pi) + (\tau\Sigma)(\Omega(P^T)^{-1} + P\tau\Sigma)^{-1}Q =$$
$$(\Pi - \tau\Sigma P^T(P\tau\Sigma P^T + \Omega)^{-1}P\Pi) + (\tau\Sigma P^T(P^T)^{-1})(\Omega(P^T)^{-1} + P\tau\Sigma)^{-1}Q =$$
$$(\Pi - \tau\Sigma P^T(P\tau\Sigma P^T + \Omega)^{-1}P\Pi) + (\tau\Sigma P^T)(\Omega + P\tau\Sigma P^T)^{-1}Q =$$
$$\Pi + (\tau\Sigma P^T(P\tau\Sigma P^T + \Omega)^{-1})(Q - P\Pi),$$

which is the alternative form of the Black-Litterman formula for expected return.

B Steps for implementing the Black-Litterman Model

This section of the document summarizes the steps required to implement the Black-Litterman model.

Given the following inputs

- w_m - equilibrium weights for each asset class. Derived from capitalization weighted CAPM Market portfolio;
- Σ matrix of covariances between the asset classes that can be computed from historical data;
- r_f risk free rate for base currency;
- δ the risk aversion coefficient of the market portfolio, this can be assumed, or can be computed if one knows the return and standard deviation of the market portfolio;
- τ a measure of uncertainty of the equilibrium variance, usually set to a small number of the order of 0.025 – 0.050.

First the reverse optimization is used to compute the vector of equilibrium returns Π :

$$\Pi = \delta \Sigma w_m.$$

Then the investors formulates his views and specifies P, Ω, and Q. Given k views and n assets, then P is a $k \times n$ matrix where each row sums to 0 (relative view) or 1 (absolute view). Q is a $k \times 1$ vector of the excess returns for each view. Ω is a diagonal $k \times k$ matrix of the variance of the views, or the confidence in the views.

Next assuming the investor is uncertain in all the views, the the Black-Litterman 'master formula' can be applied to compute the posterior estimate of the returns

$$\mu* = \Pi + (\tau \Sigma P^T (P \tau \Sigma P^T + \Omega)^{-1})(Q - P\Pi).$$

Then the posterior variance is computed and closely followed by the computation of the sample variance from formula (16).

At the end the portfolio weights for the optimal portfolio on the unconstrained efficient frontier can be computed:

$$w = \Pi(\delta\Sigma)^{-1}, w = \Pi(\delta\Sigma*)^{-1}.$$