The long and the short of risk parity

This version: April 2020

Alexandre Rubesam[a,*]

[a]*Department of Finance, IESEG School of Management (LEM-CNRS 9221), Paris, France*

Abstract

I propose an approach to create risk parity portfolios combining multiple long-short strategies defined over a universe of assets. The proposed approach nests as special cases risk parity methods that have been proposed for specific trading strategies such as trend following and pairs trading, but it is more general. I apply the approach empirically to three types of quantitative investment strategies commonly used by practitioners: trend following, pairs trading and factor investing. In all cases, the proposed method achieves the objective of attaining equal risk contributions, while also yielding improvements in terms of risk-adjusted returns and significantly reducing drawdowns. The gains appear to be more relevant, statistically and economically, for trend following and factor investing.

Keywords: risk parity, equal risk contribution, long-short portfolios, factor investing, pairs trading

JEL: G11, G32.

*Corresponding author. IESEG School of Management, 1 Parvis de La Défense - F-92044 Paris La Défense, France. Tel: +33 (0)1435 882622.

Email address: a.rubesam@ieseg.fr (Alexandre Rubesam)

1. Introduction

Risk parity (henceforth, RP) has emerged as a popular risk-based approach to build portfolios, partly because it does not require knowledge of the expected returns of assets and thus overcomes one of the main issues with mean-variance portfolio optimization, i.e. the extreme sensitivity of mean-variance optimal portfolios to the forecasts of expected returns (see e.g. Merton, 1980; Michaud, 1989; Best and Grauer, 1991)). Additionally, the underlying idea of risk parity - creating portfolios with balanced risk contributions from different assets - is intuitive and appealing, and RP portfolios have been shown to historically outperform mean-variance tangency portfolios, as well as other popular heuristic portfolio construction methods (see eg Chaves et al., 2011).

Most of the literature on RP focuses on long-only portfolios, i.e., short positions on individual assets are ruled out. For long-only portfolios, it can be shown that the RP portfolio is the solution to a convex optimization problem and, under reasonable conditions, a unique solution exists (Maillard et al., 2010). If short positions are allowed, multiple risk parity solutions may exist, as shown by (Bai et al., 2016), who propose a non-convex least-squares optimization model to find optimal solutions in such cases. In many practical situations, however, the investor knows *a priori* which assets should be shorted. For example, many well-known quantitative trading strategies, such as pairs trading, trend-following, and long-short factor investing, rely on knowledge of which assets should be bought or sold. In such cases, a convex model still applies, and standard techniques to solve for long-only RP portfolios can be extended to obtain a solution.

Although methods to obtain RP portfolios that include short positions have been developed for specific purposes, such as pairs trading and trend-following, to my knowledge, there have been no papers exploring RP in the context of combinations of more general strategies involving long and/or short positions. In this paper, I propose a simple approach to build RP portfolios combining an arbitrary number of long, short or long-short strategies defined over a universe of assets. The objective is to build a portfolio such that the risk contribution of each strategy matches a given risk budget, such as for example equal risk contributions across strategies. The method relies on a straightforward representation of each strategy as a linear combination of the original assets. If there are m strategies defined over a universe of n assets, an $n \times m$ matrix W is defined, with each column of W representing a strategy in terms of weights in the individual assets. The overall portfolio is defined by an $m \times 1$ vector of strategy weights, where the weight of each strategy is positive, and the sum of all strategy weights equals one.

The method I propose nests as special cases (and extends) two previous risk parity approaches for specific long-short strategies, namely the method proposed by Baltas (2015) for trend following, and the analysis of risk parity for pairs trading discussed by Roncalli (2016, chapter 5.2). Baltas (2015) considers a model in which trend following signals (+1 for a long position or -1 for a short position) are used to create a portfolio with equal risk contributions across assets. This method is suitable for a portfolio that has a single long or short position in each asset at any point in time. Roncalli (2016, chapter 5.2) discusses an application of RP to a portfolio of pair trades where each asset in the portfolio is present in either the long or short leg of a single pair of assets. Mathematically, the approach involves partitioning the assets (and the corresponding covariance matrix) into those with long and short positions and therefore, each asset cannot be present in more than one pair; in particular, it is not possible to deal with pairs taking opposite positions in the same stock, which in practice happens in pairs trading. In comparison with these previous approaches, the method I propose can handle any combination of long, short, or long-short strategies, each one involving any number of assets, regardless of whether the strategies have positions in the same assets or in opposite directions. All that is required is a suitable specification of the matrix W. Since the method only requires a forecast of the covariance matrix of the original assets, which will be readily available to the portfolio manager, whenever there are any changes in the set of strategies, or in the positions in the original assets within one or more strategies, a new RP allocation can be readily obtained by making the relevant changes in W.

The approach I propose can be applied in a variety of scenarios involving combinations of strategies employing long and short positions, including trend following, pairs trading, statistical arbitrage in general, and factor investing. It is therefore especially relevant for quantitative portfolio managers. I test the approach empirically in three situations commonly faced by practitioners: trend following, pairs trading and factor investing. In all cases, the proposed approach produces the desired risk budgets, whereas a naive risk parity approach, i.e. using only the volatility of the assets, does not. Additionally, the RP approach yields improvements in terms of risk-adjusted returns. The gains appear to be more relevant, statistically and economically, for the trend following and factor investing applications.

The rest of this paper is organized as follows. Section 2 provides an overview of existing risk parity approaches for specific trading strategies involving long and short positions. Section 3 explains the more general approach proposed in this paper. Section 4 reports the results on empirical applications. Section 5 concludes.

2. The problem

The risk budgeting approach consists in constructing portfolios such that the risk contribution from each asset matches a desired risk budget. The specific case in which the risk budget specifies that all assets have the same risk contribution is commonly referred to as the risk parity or Equal Risk Contribution (ERC) portfolio. Different versions of this heuristic approach are widely used by practitioners, including multi-strategy hedge funds and

CTAs (Maillard et al., 2010). Among the cited advantages of this approach are the benefits of risk diversification, i.e. avoiding concentrations in terms of the risk contributions of positions to the overall portfolio; the fact that RP does not require investors to create forecasts of expected returns and are free from the impact of estimation errors in those forecasts, and the simplicity and transparency of the method, which may help mitigate the risk of behavioral biases influencing allocation decisions (Chaves et al., 2011).

I briefly review a standard approach to obtain a RP portfolio under a long-only restriction on the weights of the assets. Consider a portfolio $x = (x_1, \ldots, x_n)$ of n risky assets, whose covariance matrix is Σ. The portfolio risk is measure by the volatility, $\sigma_P(x) = \sqrt{x'\Sigma x}$.[1] The marginal risk contribution of asset i (MRC_i) is defined as

$$MRC_i = \partial_{x_i}\sigma_P(x) = \frac{\partial \sigma_P(x)}{\partial x_i} = \frac{(\Sigma x)_i}{\sqrt{x'\Sigma x}},$$

where $(\Sigma x)_i$ denotes the i-th element of Σx. The risk contribution of asset i to the portfolio is the product of the portfolio allocation to asset i and its marginal risk contribution: $\mathcal{R}C_i = x_i \partial_{x_i}\sigma_{P,t}$, and represents the proportion of total risk attributable to that asset. It follows that the risk of the portfolio can be decomposed as the sum of all risk contributions: $\sigma_P(x) = \sum_{i=1}^{n} \mathcal{R}C_i$. Given a risk budget $b = (b_1, \ldots, b_m)$ reflecting how much risk the portfolio manager desires to allocate to each asset, a risk budget portfolio can be found by solving the following system:

$$\begin{cases} \mathcal{R}C_i = b_i \\ x_i > 0 \\ \sum_i x_i = 1 \end{cases} . \tag{1}$$

In practice, problem (1) is usually transformed into the equivalent optimization problem:[2]

$$\begin{aligned} \min_x \quad & \sigma_P(x) \\ \text{subject to} \quad & \begin{cases} \sum_{i=1}^{n} b_i \ln x_i \geq c \\ x_i \geq 0, \ i = 1, \ldots, n \end{cases} , \end{aligned} \tag{2}$$

where c is an arbitrary constant. Once a solution is obtained, the vector of weights x is normalized to sum to one.

[1] In this paper, I use the volatility as the risk measure, but other risk measures can be equally used.

[2] The equivalence of problems (1) and (2) is obtained by writing the Lagrangian in the optimization problem (2) and solving the first order conditions, see Bruder and Roncalli (2012). An analogous demonstration is provided in Appendix A for the long-short case.

This approach can be extended to allow for short positions. Assume that a signal for each asset is available in the vector $s = (s_1, \dots, s_n)'$, with $s_i = 1$ ($s_i = -1$) indicating that the position in asset i should be long (short). Therefore, we should have $x_i > 0$ ($x_i < 0$) whenever $s_i = 1$ ($s_i = -1$). Consider the following risk budget problem:

$$\begin{cases} \mathcal{R}C_i = b_i, \quad i = 1, \dots, n \\ x_i > 0 \quad \text{if} \quad s_i > 0 \\ x_i < 0 \quad \text{if} \quad s_i < 0 \\ \sum_i |x_i| = 1 \end{cases} . \tag{3}$$

The problem above specifies a risk budget portfolio where each of the weights x_i is restricted to have the same sign as the corresponding signal s_i. In addition, the gross leverage of the portfolio is restricted to one (an arbitrary value). This problem can be solved by considering a modification of the optimization problem (2):

$$\begin{aligned} \min_x \qquad & \sigma_P(x) \\ \text{subject to} \quad & \begin{cases} \sum_{i=1}^n b_i \ln |x_i| \geq c \\ x_i s_i > 0 \end{cases} . \end{aligned} \tag{4}$$

As shown in Appendix A, this formulation is equivalent to problem (3).[3] Notice that, other than restricting the sign of each weight x_i to match that of the signal s_i (through the constraint $x_i s_i > 0$), the optimal weights are unconstrained. After obtaining a solution, we can rescale x to the desired degree of gross leverage. For example, a portfolio with gross leverage equal to one is obtained by dividing the solution x^* by sum of the absolute values of each of its elements.

The method above can be used for a portfolio with a single long or short position in each of the assets, such as in a trend following model with multiple assets. In some cases, such as in pairs trading and statistical arbitrage in general, simultaneous long and short positions in two or more assets are required. Roncalli (2016, chapter 5.2) discusses an application of RP to an equity market neutral portfolio, described as a portfolio of pair trades where each asset in the portfolio is present in either the long or short leg of a single pair of assets. The objective is to attain equal risk contributions across all pairs of stocks. Mathematically, the approach involves partitioning the

[3]This formulation is equivalent to (in fact, the dual of) the optimization problem defined by Baltas (2015).

assets (and the corresponding covariance matrix) into those with long and short positions, and restricting the short weights to be equal to minus the corresponding positive weights. Their formulation can be described as follows. Assume there are m pair trades, and that the assets are ordered such that the first (last) m assets correspond to the long (short) positions. Let $x_+ = (x_1^+, \ldots, x_m^+)'$ and $x_- = (x_1^-, \ldots, x_m^-)'$ denote the vectors of weights of the long and short exposures. The covariance matrix can be partitioned as

$$\Sigma = \begin{pmatrix} \Sigma_{++} & \Sigma_{+-} \\ \Sigma_{-+} & \Sigma_{--} \end{pmatrix},$$

where Σ_{++} is the covariance matrix of the long assets, Σ_{--} is the covariance matrix of the short assets, and $\Sigma_{+-} = \Sigma_{-+}$ contains the covariances between the long and short assets. The variance of the portfolio is

$$\sigma_P = \sqrt{x_+' \Sigma_{++} x_+ + x_-' \Sigma_{--} x_- + 2 x_+' \Sigma_{+-} x_-'}$$

and therefore the risk contribution of the $i - th$ pair trade is

$$\mathcal{R}C_i = x_i^+ \frac{\partial \sigma_P}{\partial x_i^+} + x_i^- \frac{\partial \sigma_P}{\partial x_i^-}$$

The restriction $x^- = -x_j^+$ is imposed and the following problem is solved:

$$\begin{cases} \mathcal{R}C_i = \mathcal{R}C_{i'} \\ x_i^+ > 0 \\ x_i^+ + x_i^- = 0 \\ x_1^+ = c \end{cases},$$

where c is an arbitrary positive value. The portfolio is then rescaled to achieve a desired volatility.

This approach has a few limitations. First, it only works for portfolios of pairs where each asset is present in only one pair trade, in one direction (i.e. bought or sold). In practice, a certain asset can be present in more than one pair, possibly in different directions (see e.g. Do and Faff, 2010). In this case, this approach cannot be used, because it is not clear if the net position on an asset will be long or short. In addition, other types of statistical arbitrage or long-short strategies may require simultaneous positions in multiple assets. In these cases, the long and short positions do not correspond to pair trades in individual assets. For example, implementations of factor investing (Ang, 2014) frequently involve the creation of long-short portfolios where stocks are ranked according to some characteristic (e.g. past 12-month returns for a momentum factor), and a portfolio is formed by going long (correspondingly, short) the group of stocks with the highest (correspondingly, lowest) value of the characteristic. In this case, even though the number of long and short positions may be the same, each long position is not necessarily a pair of a short position, and different factor portfolios may take opposite positions in certain stocks. Moreover, filters or discretion applied in practice by portfolio managers may result in a portfolio with different numbers of long and short positions.

In what follows, I propose an approach to combine multiple, arbitrary types of long-short strategies into a portfolio, such that the risk contributions of the individual strategies matches a given risk budget. This method accomodates the two models described above as special cases.

3. A simple risk parity approach to combine strategies with long and short positions

I propose a general approach to solve risk budget problems involving multiple strategies, each of which may have only long, only short, or long and short positions. I assume there are m such strategies involving a total of n assets. The vector of weights for the $j-th$ long-short strategy is $\omega_j = (\omega_{j1}, \ldots, \omega_{jn})'$, and is assumed known. The specific values of ω_j will depend on the type of strategy. For example, for a cash neutral long-short portfolio, we could have $\sum_{i:\omega_{ji}>0} \omega_{ji} = 1$ and $\sum_{i:\omega_{ji}<0} \omega_{ji} = -1$. Let $W = [\omega_1 \cdots \omega_m]$ be the $n \times m$ matrix containing the weights defining all strategies. Denote by $r_t = (r_{1t}, \ldots r_{nt})'$ the vector of returns of all assets at time t, and by Σ_t its covariance matrix. The return of the $j-th$ strategy at time t is $r_{Strat_j,t} = \omega_j' r_t$. Finally, let $r_{Strats,t} = (r_{Strat_1,t}, \ldots, r_{Strat_m,t})'$ be the vector of returns of all strategies.

Consider a portfolio combining the m individual long-short strategies, with a vector of allocations $x = (x_1, \ldots, x_m)'$, where each $x_j > 0$ and $\sum_{j=1}^{m} x_j = 1$.[4] The return on the portfolio is

$$r_{P,t} = x' r_{Strats,t} = x' \begin{pmatrix} r_{Strat_1,t} \\ \vdots \\ r_{Strat_m,t} \end{pmatrix} = x' \begin{pmatrix} \omega_1' r_t \\ \vdots \\ \omega_m' r_t \end{pmatrix} = x' W' r_t.$$

[4]Notice that the allocations are now at the level of the strategies, not the individual assets.

The portfolio volatility is thus $\sigma_{P,t} = \sqrt{x'W'\Sigma_t Wx}$, and the risk contribution of the $j-th$ long-short strategy is

$$\mathcal{R}C_j = x_j \frac{\partial \sigma_{P,t}}{\partial x_j} = x_j \frac{(W'\Sigma_t Wx)_j}{\sigma_{P,t}},$$

where $(W'\Sigma_t Wx)_j$ denotes the $j-th$ element of $W'\Sigma_t Wx$. If we denote by $b = (b_1, \ldots, b_m)$ the risk budget vector, where b_j is the risk budget of the strategy j, we can then solve the problem below:

$$\begin{cases} \mathcal{R}C_j = b_j, \quad j = 1, \ldots, m \\ x_j > 0, \quad j = 1, \ldots, m \\ \sum_{j=1}^{m} x_j = 1 \end{cases} . \tag{5}$$

In practice, we can solve the equivalent optimization problem (2), where the portfolio volatility is calculated as $\sigma_{P,t} = \sqrt{x'W'\Sigma_t Wx}$, since all the weights to be chosen (those in the x vectors) are positive. Once we obtain a solution x^*, we can rescale the weights to sum to one. The final portfolio weights at the asset level can be calculated simply as Wx^*.

This approach is quite general; any type of long, short or long-short strategy can be represented in W, with different numbers of long and short positions, or opposite positions in assets across strategies. The risk budget approach proposed by Baltas (2015) for trend following corresponds to the choice $W = diag(s)$. That is, $n = m$ and each strategy is either long or short (only) its corresponding asset, depending on the sign of the signal s. The case where each long-short strategy is a pair trade can be represented by setting the elements corresponding to the long and short positions of the matrix W to 1 and -1, respectively, with each column of W representing a pair trade. When there are no trades involving the same assets, it yields the same results of the approach by Roncalli (2016), but it can also deal with more general cases, such as when some assets are present in more than one pair, in opposite directions, or when each trade is long or short multiple assets (e.g., cointegration-based statistical arbitrage strategies). Finally, when each strategy is a general long-short portfolio involving multiple assets, each column of W simply contains the asset weights that define the corresponding strategy.

4. Empirical applications

I consider three empirical applications of the RP approach with long and short positions. The first application is to a global, multi-asset-class trend following strategy. The second application concerns the pairs trading strategy of Gatev et al. (2006). Finally, I apply the RP approach to a portfolio of long-short strategies based on popular factors.

4.1. Application of the RP approach to trend following

Trend following is a very popular strategy, used extensively by Commodity Trader Advisories (CTAs). It is a rules-based approach to investing that uses indicators such as moving averages crossovers to define trading signals. When a positive trend is identified (e.g. if the current price is above some moving average of past prices), a long position is opened, whereas a short position (or a cash position) is implemented for a negative trend.[5] As argued by Clare et al. (2016) and others, the mechanistic approach of trend following can be an interesting device to help investors mitigate the effects of behavioral biases, as it forces investors to cut losses, and helps them to benefit from winning positions when a trend is present. The aim in this paper is not to do an extensive survey of different trend following strategies, but rather to illustrate the proposed RP method. For this reason, I consider a simple trend following model based on moving averages, similar to the one described by Faber (2007), but adding the possibility of short positions. Specifically, I define the trend following signal for a given asset over the next month as long (short), whenever the previous month closing price is higher than (lower than) its twelve-month average.

The data consists of country level (commodity-level in the case of commodities) indices representing five broad asset classes: developed economy equities, emerging market equities, developed economy government bonds, commodities, and real estate. All data for this application comes from Thomson Reuters. For the developed economy and emerging markets equity, I use the prices and total returns of MSCI country indices. The developed economy equity market indices refer to the following 24 countries: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Hong Kong, Ireland, Israel, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Singapore, Spain, Sweden, Switzerland, United Kingdom, and the United States. The emerging markets indices are for the following 16 countries: Brazil, Chile, China, Colombia, India, Indonesia, Korea, Malaysia, Mexico, Peru, Philippines, Poland, South Africa, Taiwan, Thailand, and Turkey.

Next, I collect prices and total returns for country level government bond indices produced by S&P. The indices are for the following 23 countries: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Hong Kong, Japan, Ireland, Israel, Italy, Netherlands, Norway, New Zealand, Portugal, Singapore, Spain, Sweden, Switzerland, United Kingdom, and the United States.

[5] Trend following strategies are discussed extensively in Ilmanen (2011). See also Hurst et al. (2017), Baltas (2015), Clare et al. (2016) and the references therein.

I obtain excess and total returns for the following 23 commodity indices produced by Bloomberg: Aluminium, Cocoa, Coffee, Copper, Corn, Cotton, Crude Oil (WTI), Gold, Heating Oil, Lead, Lean Hogs, Live Cattle, Natural Gas, Nickel, Platinum, Silver, Soybeans, Soybean Oil, Sugar, Tin, Unleaded Gas, Wheat, and Zinc.

Finally, for the real estate asset class, I collect country level REIT indices produced by FTSE/EPRA, referring to the following 13 countries: Australia, Belgium, France, Germany, Hong Kong, Italy, Japan, Netherlands, Singapore, Sweden, Switzerland, United Kingdom and United States.

In total, there are 99 individual indices. For each index, I obtain daily and monthly data for the period from January 1992 to December 2018, a total of 324 months. It is important to highlight that, within each asset class, data on different indices becomes available at different dates. Each month, all the available assets are considered.

Using the signals from the trend following rule previously described, I build four global, multi-asset trend following strategies. To explain the construction of these strategies, I introduce some notation. Let $s_{i,t}$ denote the trend following signal for asset i at time t (i.e. +1 for a long position and -1 for a short position), and let $\omega_{i,t}^{gross}$ denote the absolute or gross weight invested in asset i at time t. Also, let $n_{j,t}$ denote the number of available assets in asset class j at time t, and let $N_t = \sum_{j=1}^{5} n_{j,t}$ be the total number of available assets at time t. The strategies are normalized to have a gross exposure equal to one, that is, $\sum_{i=1}^{N_t} \omega_{i,t}^{gross} = 1$. The *net* weight on an asset i is simply $s_{i,t}\omega_{i,t}^{gross}$ and the return on the trend following strategy at time t can be calculated as the product of the net weights by the returns on the assets:

$$r_t^{TF} = \sum_{i=1}^{N_t} s_{i,t}\omega_{i,t}^{gross} r_{i,t},$$

where $r_{i,t}$ is the return on asset i at time t. Considering the trend following signals $s_{i,t}$ fixed, different choices for $\omega_{i,t}^{gross}$ will lead to different strategies.

The first strategy, which I call the *Equally Weighted* trend following strategy, consists in using equal gross weights for any asset, i.e., the gross weight of an asset i is given by $|s_{i,t}|/N_t$. The second strategy is a commonly used approach in CTAs and hedge funds and consists in using the inverse of the volatily of each asset in an attempt to balance the risk contributions of asset classes.[6] I refer to this as the *Naive Risk Parity* trend following strategy. The gross weight of asset i in this strategy is given by $(|s_{i,t}|/\hat{\sigma}_{i,t})/(\sum_{i=1}^{N_t} |s_{i,t}|/\hat{\sigma}_{i,t})$, where $\hat{\sigma}_{i,t}$ is the estimate of the volatility of asset i at time t. The third strategy, called the Asset Risk Parity trend-following strategy, solves the risk parity problem (5). The matrix W in this case is given by a diagonal matrix whose elements are the the trend-following signals, and the risk budgets for each asset at time t equal to $b_i = 1/N_t$ for all assets. Thus, this

[6] See Baltas (2015) and the references therein.

strategy seeks to have equal risk contributions across all available assets. Finally, the *Asset Class Risk Parity* trend-following strategy solves a similar risk parity problem, in which the risk budgets are now equal across asset classes. The risk budgets for individual assets within each asset class at each month are given by the risk budget of the asset class, divided by the number of available assets in the asset class. In all cases, I estimate the covariance matrix of the returns of the assets using two years of daily returns and an exponentially-weighted moving average estimator with a smoothing parameters equal to 0.94.[7] Because of this, the results are based on the period from January to December 2018.

Figure 1 shows the gross weights (graphs in the left column) and percentage risk contributions (right column) for each of the strategies. Although the Equally Weighted strategy is balanced in terms of gross weight allocations, the risk of the strategy comes mostly from the equity asset classes, reflecting their much higher risk compared to other asset classes. The Naive Risk Parity strategy shifts a significant portion of the gross weights to bonds, particularly after the mid-2000s. However, because this method ignores correlations, the risk contributions are far from balanced and vary significantly over time. The Asset Risk Parity strategy shifts asset allocations more dynamically, according to the changes in volatilities and correlations between the individual assets. For example, the gross allocation to equities is reduced quite dramatically during the 2008 crisis. The risk contributions are equal (at the asset level), but since the number of assets in each asset class varies over time, it is not balanced at the asset class level. Finally, the Asset Class Risk Parity achieves equal risk contributions per asset class, as it is intended to.

Table 1 reports return statistics for the four trend-following strategies. All strategies are profitable and deliver average monthly returns that range between 0.46% and 0.56%. The Equally Weighted strategy has the highest monthly standard deviation (2.82%), the highest maximum drawdown (31.41%), and the lowest Sharpe ratio (0.57). Thus, the simple combination of trend following strategies does not appear to be particularly attractive. The Naive Risk Parity strategy significantly reduces the risk, with a monthly standard deviation of 2.03%, while slightly increasing the average return to 0.50%. The resulting Sharpe ratio increases to 0.86, and the difference in Sharpe ratios is statistically significant at the 1% level.[8] The Naive Risk Parity also reduces the maximum drawdown to approximately 20%. The Asset Risk Parity strategy manages to reduce risk even further, while increasing the average return slightly in comparison with the Naive Risk Parity strategy. The monthly standard deviation is 1.03%, and the maximum drawdown is only 11.82%. The Sharpe ratio is more than double that of the Naive Risk Parity strategy, and the differences with respect to the Equally Weighted and the Naive Risk Parity strategies are

[7]Using other covariance matrix estimators produces similar results.

[8]The Ledoit and Wolf (2008) test is used to test for the difference between Sharpe ratios in this paper.

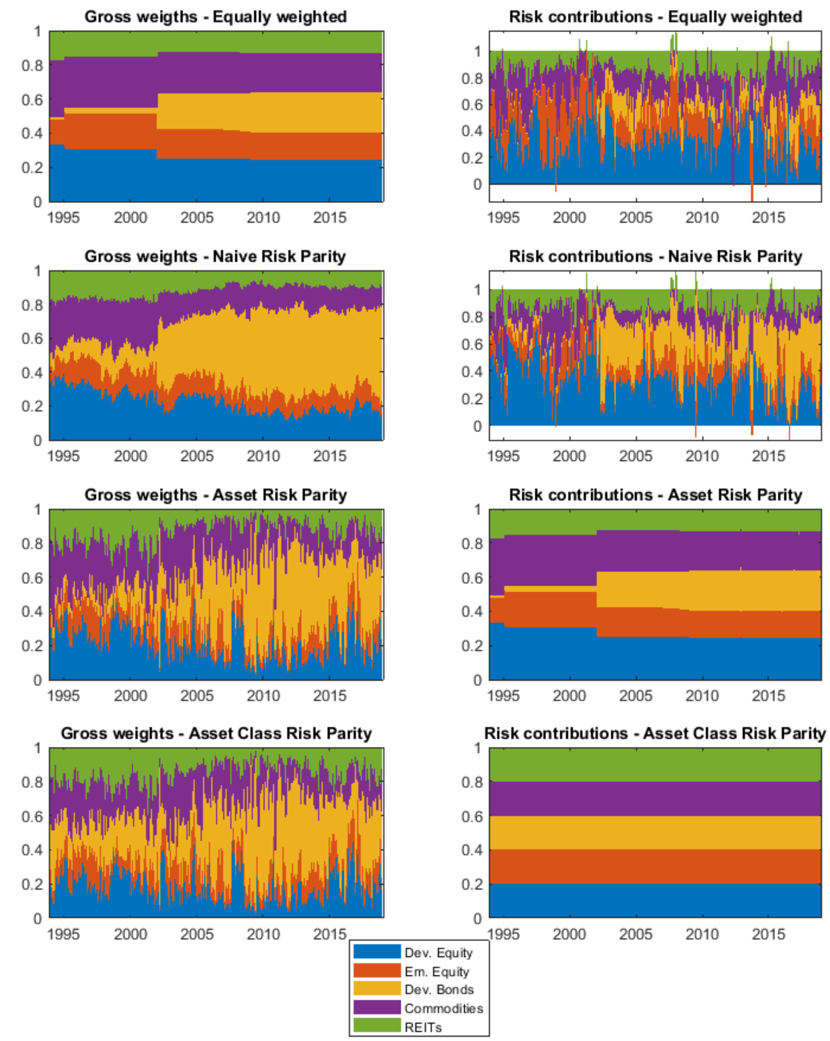

Figure 1: The graphs show the gross weights and marginal risk contributions (MRC), by asset class, for trend-following strategies. The asset classes are Developed Equity (Dev. Equity), Emerging Equity (Em. Equity), Developed Bonds (Dev. Bonds), Commodities, and Real Estate (REITs). All strategies have a total gross exposure equal to one. The Equally Weighted strategy invests an amount $s_{i,t}/N_t$ in each asset, where $s_{i,t} \in \{-1, 1\}$ is the trend following signal for asset i at time t, and N_t is the total number of available assets at time t. The Naive Risk Parity strategy invest in each asset according to the normalized inverse volatilities of the assets, i.e., the weight in asset i is $(s_{i,t}/\hat{\sigma}_{i,t})/(\sum_{i=1}^{N_t} s_{i,t}/\hat{\sigma}_{i,t})$, where $\hat{\sigma}_{i,t}$ is the estimate of the volatility of asset i at time t. The Risk Parity (asset) strategy is defined as the solution to a risk parity problem, where the risk budgets for all assets are equal. The Risk Parity (asset class) is defined as the solution to a risk parity problem, where the risk budgets for all asset classes are equal.

statistically significant at the 1% level. Finally, the Asset Class Risk Parity strategy performs similarly to the Asset Risk Parity strategy.

These results show that risk parity can significantly improve a trend-following strategy, and that properly accounting for the correlations between the assets brings substantial benefits. As argued by Hurst et al. (2017), despite

	Equally Weighted	Naive Risk Parity	Asset Risk Parity	Asset Clas Risk Parity
Average return	0.46%	0.50%	0.56%	0.55%
Std deviation	2.82%	2.03%	1.03%	1.04%
Standard Error	0.16%	0.12%	0.06%	0.06%
t-stat	2.83	4.29	9.43	9.08
Sharpe	0.57	0.86	1.89	1.82
Skewness	0.37	0.51	0.35	-0.13
Kurtosis	13.01	11.68	16.71	18.57
Max DD	31.41%	20.07%	11.82%	12.88%
Min Ret	-11.87%	-8.55%	-5.78%	-6.45%
Max Ret	19.96%	14.24%	7.45%	7.32%

Table 1: Return statistics for different global trend-following strategies. The sample period is from January 1994 to December 2018. All strategies have a total gross exposure equal to one. The Equally Weighted strategy invests an amount $s_{i,t}/N_t$ in each asset, where $s_{i,t} \in \{-1, 1\}$ is the trend following signal for asset i at time t, and N_t is the total number of available assets at time t. The Naive Risk Parity strategy invest in each asset according to the normalized inverse volatilities of the assets, i.e., the weight in asset i is $(s_{i,t}/\hat{\sigma}_{i,t})/(\sum_{i=1}^{N_t} s_{i,t}/\hat{\sigma}_{i,t})$, where $\hat{\sigma}_{i,t}$ is the estimate of the volatility of asset i at time t. The Asset Risk Parity strategy is defined as the solution to a risk parity problem, where the risk budgets for all assets are equal. The Asset Class Risk Parity is defined as the solution to a risk parity problem in which the risk budgets for all asset classes are equal, with the risk budgets for individual assets equal to the asset class risk budget, divided by the number of available assets.

the strong performance of trend following during the 2007-2009 Global Financial Crisis (GFC), the increased correlations among different asset classes since then have negatively impacted the performance of these strategies. Higher correlations imply fewer independent trend following signals, and lower risk adjusted returns. In my sample, the average absolute pairwise correlation in the period from 2007-2018 period (0.30) is double the one in the 1994-2006 period (0.15).

I investigate this further by summarizing the performance of the strategies during, before, and after the GFC. Panel A of Table 2 reports the performance of the strategies in the pre-GFC period, from January 1994 to October 2007. All trend-following strategies deliver higher average returns, compared to the full sample results shown in Table 1. During the GFC, the Equally Weighted and Naive Risk Parity trend-following strategies deliver much higher average returns (Panel B of Table 2), and have higher or similar Sharpe ratios compared to the pre-crisis period, whereas the Asset and Asset Class Risk Parity strategies show slightly higher returns in comparison with the pre-GFC period, but are also much less volatile than the other strategies. Nevertheless, none of the strategies' average returns are statistically significant. Finally, all trend-following strategies perform worse in the post-crisis period in terms of average returns or risk-adjusted returns (Panel C of Table 2). The Equally Weighted and Naive Risk Parity deliver negative average returns (which are not statistically significantly different from zero). In contrast, the Asset Risk Parity and Asset Class Risk Parity strategies continue to deliver positive and statistically significant returns. However, it can be noted that the average returns are about a third of their pre-GFC levels.

As mentioned by Baltas (2015), the profitability of a trend-following strategy depends not only on the existence of trends, but also on the allocation scheme used to combine different the assets. The higher correlations in the post-GFC period help explain why the Naive Risk Parity strategy performs worse than the Asset Risk Parity strategy. Since the former does not account for correlations, it fails to achieve balanced risk contributions, and therefore is adversely impacted when several correlated trend following signals result in losses. While the performance of these strategies in the future will depend on the existence of trends, investors can benefit from correctly accounting for correlations among the assets.

4.2. Application of the RP approach to pairs trading

I apply the RP approach to the simple pairs trading strategy investigated by Gatev et al. (2006).[9] The strategy selects pairs of stocks with minimum squared distance in the space of normalized, cumulative total returns with dividends reinvested, during a formation period of twelve months. The pairs are then traded over the next six months, at which point the formation and trading windows are rolled forward by six months. A trade is opened in a pair of stocks when the spread between the two stocks diverges by more than two standard deviations, as measured during the formation period. The trade is closed at the next crossing of prices, or when the trading period ends. Notice that a pair may open and close several times, or none at all, during the trading period. As in the original paper, I simulate the results of six overlapping portfolios of pairs, each starting one month after the previous one, and average their results. I focus on the top 20 pairs with minimum distance.

The calculation of excess returns on the portfolios of pairs also follows Gatev et al. (2006). The excess returns on individual pairs of stocks are calculated as the reinvested payoffs during the trading period on a daily basis. I calculate the returns on portfolios of pairs with the two methods used in the original paper, which are equally weighted across pairs. The *committed capital* portfolio allocates an equal capital of 1/20 to each pair, regardless of whether a position is opened. The *fully invested* portfolio considers, at each point in time, that the full capital is allocated to the currently open pairs. Returns are calculated daily and compounded to produce monthly returns. I use daily data from CRSP for the period January 1969 to December 2018, considering ordinary shares (share codes 10 and 11), and excluding stocks with no trading (zero volume) or with invalid returns in at least one day during the formation window.

[9]Although the returns of this strategy have deteriorated over time (Do and Faff, 2010), my purpose is merely to demonstrate the application of the methodology to a pairs trading strategy. The method can equally be applied to more sophisticated pairs trading and statistical arbitrage strategies (Huck, 2009, 2010; Krauss et al., 2017; Fischer and Krauss, 2018; Huck, 2019).

Panel A. Pre - Global Financial Crisis (Jan 1994 - Oct 2007)				
	Equally Weighted	Naive Risk Parity	Asset Risk Parity	Asset Class Risk Parity
Average return	0.67%	0.73%	0.74%	0.72%
Std deviation	2.10%	1.64%	0.90%	0.88%
t-stat	4.12	5.74	10.65	10.61
Sharpe	1.11	1.55	2.87	2.86
Max DD	9.79%	6.43%	3.22%	3.60%

Panel B. Global Financial Crisis (Nov 2007 - Feb 2009)				
	Equally Weighted	Naive Risk Parity	Asset Risk Parity	Asset Class Risk Parity
Average return	2.69%	2.04%	0.97%	0.93%
Std deviation	6.46%	4.68%	2.63%	2.74%
t-stat	1.72	1.80	1.52	1.40
Sharpe	1.44	1.51	1.28	1.17
Max DD	10.29%	8.55%	5.78%	6.45%

Panel C. Post - Global Financial Crisis (Mar 2009 - Dec 2018)				
	Equally Weighted	Naive Risk Parity	Asset Risk Parity	Asset Class Risk Parity
Average return	-0.16%	-0.04%	0.24%	0.24%
Std deviation	2.69%	1.76%	0.67%	0.72%
Sharpe	-0.20	-0.07	1.25	1.17
t-stat	-0.63	-0.22	3.91	3.67
Max DD	31.41%	18.37%	6.69%	7.37%

Table 2: Return statistics for different global trend-following strategies. All strategies have a total gross exposure equal to one. The Equally Weighted strategy invests an amount $s_{i,t}/N_t$ in each asset, where $s_{i,t} \in \{-1, 1\}$ is the trend following signal for asset i at time t, and N_t is the total number of available assets at time t. The Naive Risk Parity strategy invest in each asset according to the normalized inverse volatilities of the assets, i.e., the weight in asset i is $(s_{i,t}/\hat{\sigma}_{i,t})/(\sum_{i=1}^{N_t} s_{i,t}/\hat{\sigma}_{i,t})$, where $\hat{\sigma}_{i,t}$ is the estimate of the volatility of asset i at time t. The Asset Risk Parity strategy is defined as the solution to a risk parity problem, where the risk budgets for all assets are equal. The Asset Class Risk Parity is defined as the solution to a risk parity problem in which the risk budgets for all asset classes are equal, with the risk budgets for individual assets equal to the asset class risk budget, divided by the number of available assets.

I compare the Equally Weighted pairs trading strategies in Gatev et al. (2006) with Naive Risk Parity and Risk Parity pairs trading strategies. The Naive Risk Parity strategy weights each pair of stocks according to the inverse of its volatility. The implementation of the Risk Parity strategy is as follows. On each day, I take note of the pairs that are open, the corresponding stocks and the position (long or short) of each pair in the stock. With this information, I build the matrix W for that day. For example, on a certain day, if there are five open pairs involving four stocks, the matrix W will have dimension 4×5, and each column of W will have a 1 or -1 in the position corresponding to the long and short stocks, respectively. I estimate the covariance matrix of the stocks using the shrinkage estimator of Ledoit and Wolf (2004) and the last 252 daily returns of the stocks.[10] I then solve the RP problem (5) to obtain the weights x to be applied to each pair. For the committed capital RP portfolio, the allocation to each pair is calculated as the multiplication of the vector x by the total capital that would be available to all open pairs. For example, if five out of the 20 pairs are open, x will be multiplied by 0.25. For the fully invested RP portfolio, x is used directly. These weights are maintained until there is a change in the portfolio, i.e., if a new pair opens or an existing one closes. Similar results are obtained if the weights are updated on a daily basis.

Figure 2 reports the average number of open pairs and average number of stocks used in pairs in each month. If all pairs involved different stocks, the number of stocks would always be double the number of pairs. Clearly, some stocks are usually present in more than one pair trade at any given time. For example, on days when, on average, six pairs were open, the average number of stocks used ranged between six and ten.

The results of the pairs trading strategies are reported in Table 3. The average returns and standard deviations of the equally weighted pairs trading strategies are very close to those reported by (Gatev et al., 2006, Table 1).[11] Both the Naive Risk Parity as well as the Risk Parity approaches appear to improve the risk-adjusted returns of the pairs trading strategy, but overall, the gains seem modest. When the committed capital scheme is used, the average monthly return of the Equally Weighted pairs trading strategy is 0.74%, with a standard deviation of 1.12%, yielding a Sharpe ratio of 2.29. The Naive Risk Parity (Risk Parity) strategy yields a slightly lower return of 0.69% (0.65%), but with a lower standard deviation of 1.03% (0.95%) per month, improving the Sharpe ratio slightly to 2.32 (2.34), although the differences in Sharpe ratios are not statistically significant. The RP approach also reduces the maximum drawdown slightly. For the fully invested strategies, a similar pattern holds.

Although risk parity does not seem to improve the pairs trading strategies in terms of returns, its purpose

[10]Results using other covariance matrix estimators are not significantly different.

[11]I also document that the returns to pairs trading continued to decline over time, in line with the results of Do and Faff (2010). The average return of the committed capital and fully invested strategies from 2010 to 2018 are 0.15% and 0.27% (not tabulated), significantly lower than over the whole sample.

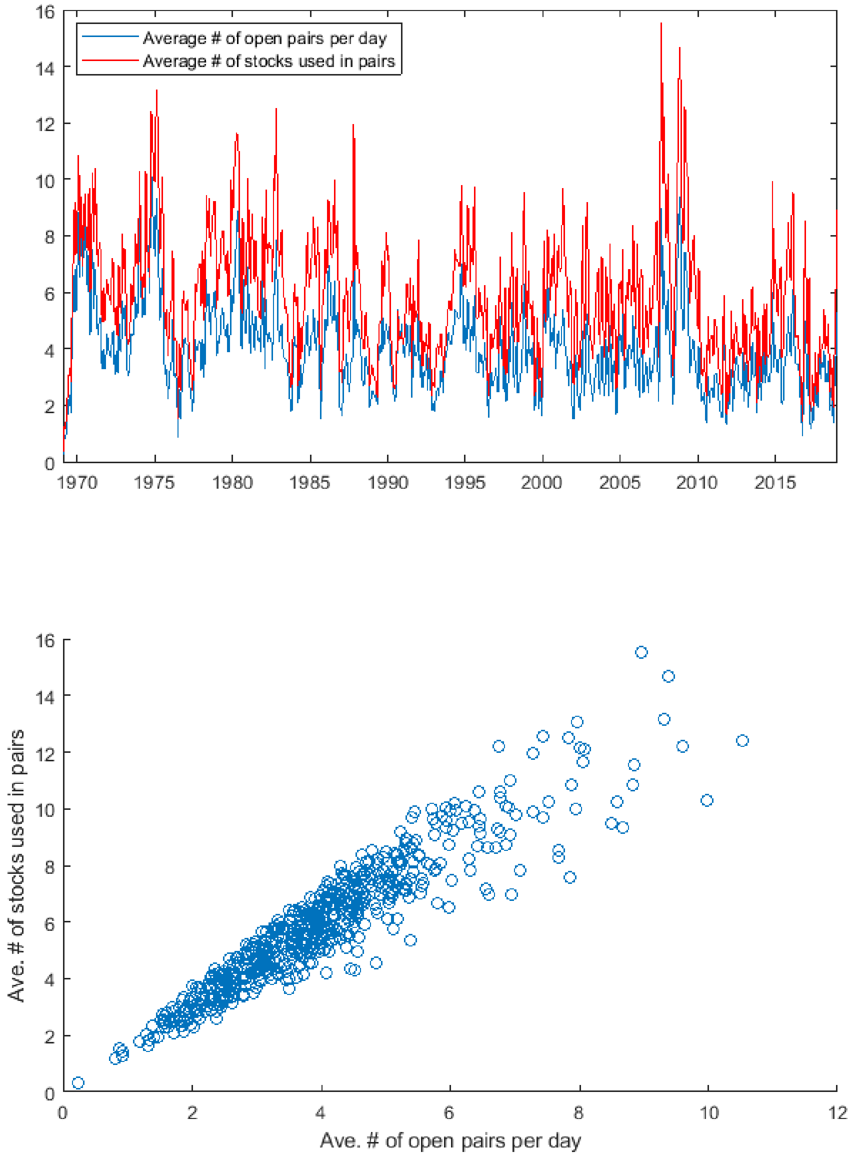

Figure 2: Average number of open pairs for the Gatev et al. (2006) pairs trading strategy. The top graph shows the average number of open pairs (blue line) and the average number of stocks that compose these pairs (red line) on each month. The scatter plot on the bottom shows the distribution of these quantities.

is to balance the risk contributions of the pairs to the overall portfolio. Since the number of open pairs varies significantly over time, I calculate, for each strategy, at each point in time, the average absolute distance between

	Commited capital			Fully invested		
	Equally Weighted	Naive Risk Parity	RiskParity	Equally Weighted	Naive Risk Parity	Risk Parity
Average return	0.74%	0.69%	0.65%	1.38%	1.23%	1.16%
Std deviation	1.12%	1.03%	0.95%	2.07%	1.72%	1.61%
Standard Error	0.05%	0.04%	0.04%	0.09%	0.07%	0.07%
t-stat	15.89	16.10	16.23	16.01	17.16	17.33
Sharpe	2.29	2.32	2.34	2.31	2.48	2.50
Skewness	1.23	1.13	1.10	1.87	1.14	1.05
Kurtosis	8.42	8.86	8.00	12.54	8.47	7.38
Max DD	6.09%	6.20%	5.79%	8.01%	8.57%	8.02%
Min Ret	-3.99%	-3.87%	-3.38%	-5.44%	-5.45%	-4.85%
Max Ret	7.46%	7.05%	6.32%	16.23%	12.17%	10.22%

Table 3: Return statistics for various pairs trading strategies. The table reports return statistics for the original (Equally Weighted) pairs trading strategy investigated by Gatev et al. (2006) using the top 20 pairs with minimum squared distance, a Naive Risk Parity strategy that weights pairs by the inverse of their volatility, and a Risk Parity version that obtains equal risk contributions for all pairs. The committed capital strategies allocate a fixed percentage of capital to each open pair, whereas the fully invested strategies allocate, at each point in time, all capital to the currently open pairs. All figures are monthly, with the exception of the Sharpe ratio, which is annualized.

the risk contributions of the open pairs and a vector of equal risk budgets. For example, if on a given date, there are 6 open pairs, I calculate the % risk contributions of each pair to the portfolio, and then calculate the average absolute distance between these and an equal risk budget of $1/6$ for each pair. The resulting distances are plotted in Figure 3. For a true risk parity strategy, the average distance should always equal zero, which is exactly what I obtain for the Risk Parity strategy (in orange). While the Naive Risk Parity pairs trading strategy (red line) reduces the distance from an equal risk budget in comparison with the Equally Weighted strategy (blue line), both are far away from zero. Overall, despite the fact that the Risk Parity strategy achieves its objective, the results from Table 3 suggest that differences in risk contributions are not particularly important for this specific pairs trading strategy.

4.3. Application of the RP approach to Factor investing

I apply the RP approach to form a portfolio combining individual long-short strategies based on the value, momentum, volatility and quality factors. These strategies are chosen due to their popularity in factor investing in practice, as well as their relevance in the asset pricing literature. Additionally, these strategies appear to yield significant returns in different markets and asset classes (Asness et al., 2013; Blitz and Vliet, 2007; Blitz et al.,

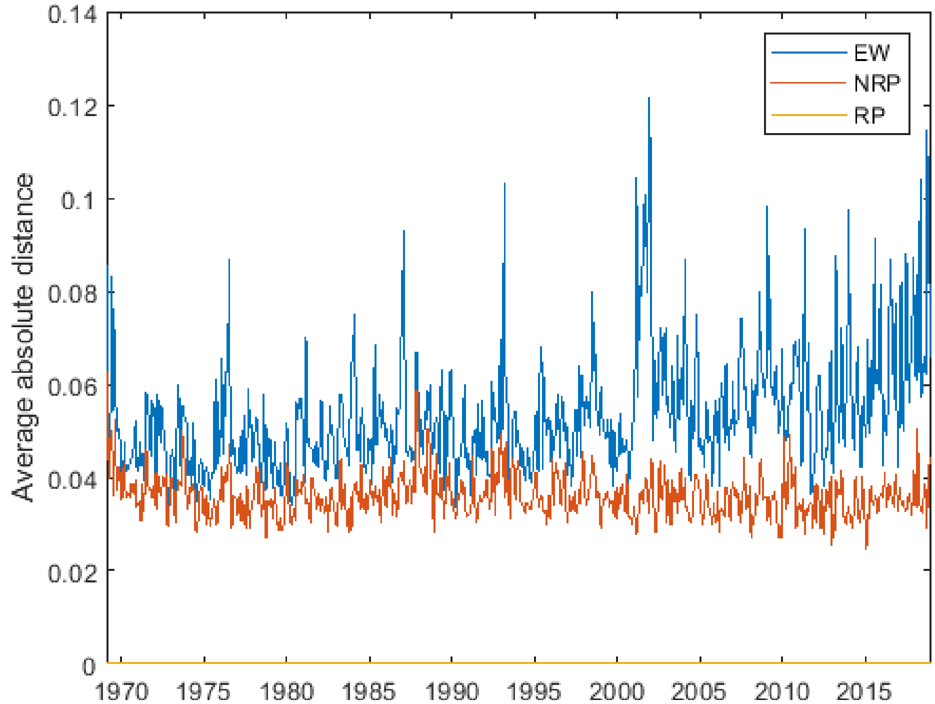

Figure 3: Average absolute distance between risk contributions and equal risk budget. For each date, the graph reports the distance between the actual risk allocations of different pairs trading strategies and a risk budget that allocates the same risk to all open pairs. The pairs trading strategies are either equally weighted (EW), inverse-volatility weighted or naive risk parity (NRP), and risk parity (RP).

2013; Asness et al., 2019).[12]

I form long-short portfolios for each strategy using data on all U.S. common stocks (excluding financials) for the period 1968 to 2018 from CRSP and Compustat. All strategies are value-weighted, rebalanced on a monthly basis, and consider only non-microcap stocks, defined as those with a market capitalization above the 20th percentile of NYSE stocks. The value strategy is based on going long (short) stocks in the highest (lowest) decile of book-to-market ratio. The momentum strategy goes long (short) stocks in the highest (lowest) deciles of past 12-month returns (skipping one month). The volatility strategy goes long (short) stocks in the lowest (highest) decile of

[12]The value effect was first identified by Rosenberg et al. (1985), who documented the outperformance of value stocks (those with high book-to-market ratios) relative to growth stocks (those with low book-to-market ratios). The seminal paper on momentum is Jegadeesh and Titman (1993), who documented that stocks with high returns over the previous 6 to 12 months outperform those with low returns in the future. The low-volatility effect, that stocks with low volatility tend to ouperform those with higher volatility, dates back to Haugen and Heins (1975), see also Blitz and Vliet (2007) and Ang et al. (2006). Asness et al. (2019) study the impact of quality on the price of stocks and finds that a factor that goes long high-quality stocks and shorts low-quality stocks earns positive and significant risk-adjusted returns in the U.S. and a number of countries.

realized volatility. Finally, the quality strategy is similar to the one in Asness et al. (2019) and is based on the following characteristics: the fraction of earnings composed of cash (i.e., minus accruals), return on equity, return on assets, low leverage, and low earnings volatility. Each month, I calculate the z-score of the rank transformation of each variable. I build a quality score as the z-score of the sum of z-scores for these measures, and create a portfolio that goes long (short) the stocks in the highest (lowest) quality score decile.

I build three different portfolios that invest in the four factor strategies. All of these portfolios are normalized such that the sum of the positive (negative) weights equals one (minus one). The first portfolio is an Equally Weighted combination of the four factor strategies. The second is a Naive Risk Parity portfolio, which invests according to the inverse volatility of each factor strategy. Finally, I apply the RP approach to form a portfolio such that each of the four long-short factor strategies has the same equal risk contribution. To form this portfolio, each month I gather daily return data for the last two years for all stocks that are present in any of the strategies, and estimate their covariance matrix using the shrinkage method of Ledoit and Wolf (2004). I also build the matrix W, with each column representing the weights of each stock in each strategy.[13] I then solve the RP optimization problem (5) in order to obtain the 4×1 vector of RP allocations x. The individual stock weights for the combined portfolio is then calculated as Wx, which is re-normalized so that the portfolio has a gross leverage ratio of 2.

Table 4 shows return statistics for the individual long-short factor strategies and the portfolios of factor strategies. The individual factor strategies yield positive monthly returns and, with the exception of the volatility strategy, these are statistically significant. Nevertheless, the strategies typically have high monthly standard deviations, yielding relatively low annualized Sharpe ratios. They also suffer large maximum drawdowns of between 38% (quality strategy) to 68% (momentum strategy). The Equally Weighted portfolio improves substantially on the individual factor strategies in terms of risk-adjusted returns. It yields a monthly return of 0.70%, a volatility of 3.90%, and a Sharpe ratio of 0.62, much higher than those of the individual strategies. However, the equally weighted portfolio still has a high maximum drawdown of 47%. The Naive Risk Parity strategy produces a slightly lower average return, of 0.65% per month, but reduces the standard deviation to 3.20%, improving the Sharpe ratio to 0.71, while also reducing the maximum drawdown to 37%. Finally, the RP strategy achieves a slightly lower monthly return of 0.63%, but with an even lower volatility of 2.53%, which is 35% lower compared to the Equally Weighted portfolio. Its Sharpe ratio of 0.87 is 40% higher than that of the equally weighted portfolio, and the difference is statistically significant at the 5% significance level. The maximum drawdown is approximately 25%,

[13] I use market-capitalization weights. Similar results are obtained for equally-weighted factor strategies.

almost half that of the Equally Weighted portfolio.

	Panel A. Original factor strategies				Panel B. Portfolios of factor strategies		
	Value	Momentum	Volatility	Quality	Equally weighted	Naive Risk Parity	Risk Parity
Average return	0.40%	0.89%	0.47%	0.39%	0.70%	0.65%	0.63%
Std deviation	4.37%	7.21%	7.35%	3.73%	3.90%	3.20%	2.53%
Standard Error	0.19%	0.31%	0.32%	0.16%	0.17%	0.14%	0.11%
t-stat	2.14	2.86	1.50	2.45	4.19	4.74	5.81
Sharpe	0.32	0.43	0.22	0.36	0.62	0.71	0.87
Skewness	0.49	-0.57	0.11	0.24	-0.68	-0.62	-0.47
Kurtosis	5.27	10.98	7.41	6.11	10.70	8.62	6.34
Max DD	57.94%	67.55%	76.03%	38.28%	47.11%	36.82%	25.46%
Min Ret	-14.58%	-43.98%	-37.60%	-16.13%	-24.00%	-18.72%	-12.99%
Max Ret	21.17%	45.95%	33.48%	22.14%	20.40%	14.45%	9.04%

Table 4: Return statistics of individual long-short factor strategies and portfolios of factor strategies. All strategies are monthly-rebalanced and market-capitalization weighted. The value strategy goes long (short) stocks in the top (bottom) decile of book-to-market ratio. The momentum strategy goes long (short) stocks in the top (bottom) decile of past 12-month returns. The volatility strategy goes long (short) stocks in the bottom (top) decile of realized volatility. The quality strategy goes long (short) stocks in the top (bottom) decile of a quality score based on accruals, leverage, return on equity, return on assets, and earnings volatility. All portfolios of strategies in Panel B are normalized such that the sum of all the long (short) weights at the stock level equals one (minus one). The equally-weighted portfolio is defined by averaging the weights of the four long-short factor strategies. The Naive Risk Parity portfolio allocates weights to individual long-short factor strategies in proportion to the inverse volatility of each strategy. The Risk Parity strategy allocates weights to each factor strategy such that the risk contributions of each individual strategy to the risk of the overall portfolio is the same. All figures are monthly, with the exception of the Sharpe ratio, which is annualized.

Figure 4 shows the allocations to each of the four factor strategies under each portfolio, and the resulting risk contributions. The risk contributions of the factor strategies in the Equally Weighted portfolio vary substantially over time, showing that such an approach leads to portfolios whose risk can, at times, be dominated by a particular factor. The Naive Risk Parity portfolio does not fare much better; it is not able to obtain balanced risk contributions, because it ignores the correlations between the factors. In contrast, the Risk Parity portfolio achieves equal risk contributions for all factors, dynamically adjusting the factor allocations according to the changes in volatilities as well as correlations among the strategies.

5. Conclusions

In this paper, I have proposed a risk parity approach to construct portfolios combining multiple long-short strategies. The proposed method nests nests as special cases previous risk parity approaches that were proposed for trend following and pairs trading strategies, but it is more general. The approach is tested empirically in three situations involving long-short strategies: trend following, pairs trading, and factor investing. The results show that the approach achieves its objective in all three cases, namely delivering equal risk contributions in each case. I also show that naive risk parity approaches, which ignore correlations among assets, do not. Additionally, in all cases, risk parity improves risk-adjusted returns relative to equally weighted or naive risk parity strategies. The improvement is particularly relevant for trend following and factor investing, and less significant for the pairs trading strategy we tested.

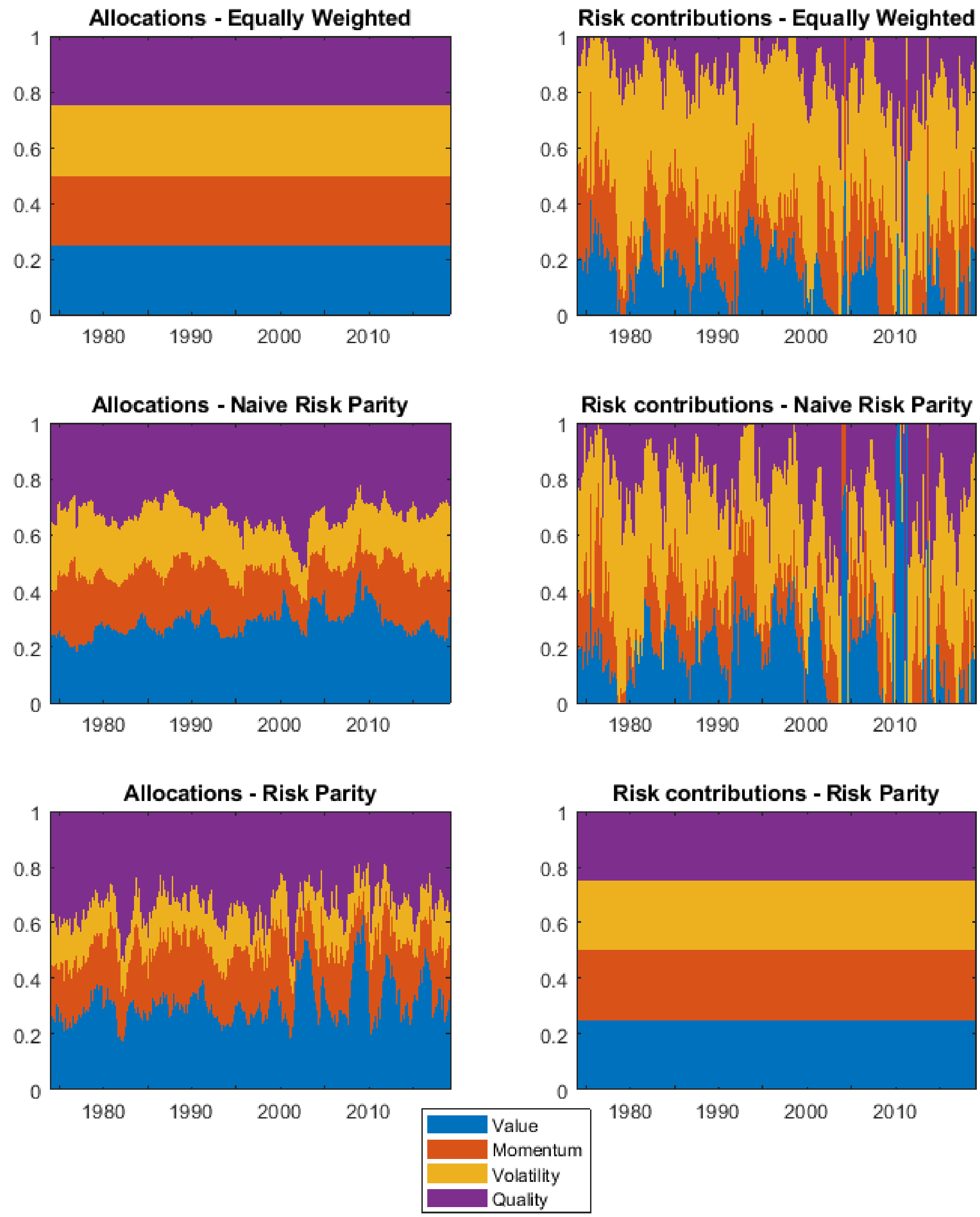

Figure 4: Allocations and risk contributions of portfolios of factor strategies. The graph shows the weight allocations to value, momentum, volatility, and quality factor strategies, and the resulting risk contributions, for three portfolios of factor strategies. The Equally Weighted portfolio invests an equall amount in each of the four factor strategies. The Naive Risk Parity approach invests in each factor strategy according to the inverse of its volatility. The Risk Parity strategy solves a risk parity problem such that the risk contributions of each factor strategy to the overall portfolio are equal.

References

Ang, A. (2014), *Asset management: A systematic approach to factor investing*, Oxford University Press.

Ang, A., Hodrick, R. J., Xing, Y. and Zhang, X. (2006), 'The cross-section of volatility and expected returns', *Journal of Finance* **61**, 259–299.

Asness, C. S., Frazzini, A. and Pedersen, L. H. (2019), 'Quality minus junk', *Review of Accounting Studies* **24**(1), 34–112.

Asness, C. S., Moskowitz, T. J. and Pedersen, L. H. (2013), 'Value and momentum everywhere', *The Journal of Finance* **68**(3), 929–985.

Bai, X., Scheinberg, K. and Tutuncu, R. (2016), 'Least-squares approach to risk parity in portfolio selection', *Quantitative Finance* **16**(3), 357–376.

Baltas, N. (2015), Trend-following, risk-parity and the influence of correlations, *in* 'Risk-Based and Factor Investing', Elsevier, pp. 65–95.

Best, M. J. and Grauer, R. R. (1991), 'On the sensitivity of mean-variance efficient portfolios to changes in asset means: Some analytical and computational results', *Review of Financial Studies* **4**, 315–342.

Blitz, D., Pang, J. and Van Vliet, P. (2013), 'The volatility effect in emerging markets', *Emerging Markets Review* **16**, 31–45.

Blitz, D. and Vliet, P. V. (2007), 'The Volatility Effect', *Journal of Portfolio Management* (Fall), 102–113.

Bruder, B. and Roncalli, T. (2012), Managing risk exposures using the risk budgeting approach.
URL: *https://ssrn.com/abstract=2009778*

Chaves, D., Hsu, J., Li, F. and Shakernia, O. (2011), 'Risk parity portfolio vs. other asset allocation heuristic portfolios', *The Journal of Investing* **20**(1), 108–118.

Clare, A., Seaton, J., Smith, P. N. and Thomas, S. (2016), 'The trend is our friend: Risk parity, momentum and trend following in global asset allocation', *Journal of Behavioral and Experimental Finance* **9**, 63–80.

Do, B. and Faff, R. (2010), 'Does simple pairs trading still work?', *Financial Analysts Journal* **66**(4), 83–95.

Faber, M. T. (2007), 'A quantitative approach to tactical asset allocation', *The Journal of Wealth Management* **9**(4), 69–79.

Fischer, T. and Krauss, C. (2018), 'Deep learning with long short-term memory networks for financial market predictions', *European Journal of Operational Research* **270**(2), 654–669.

Gatev, E., Goetzmann, W. N. and Rowenhorst, K. G. (2006), 'Pairs Trading: Performance of a Relative-Value Arbitrage Rule', *Review of Financial Studies* **19**, 797–827.

Haugen, R. A. and Heins, A. J. (1975), 'Risk and the rate of return on financial assets: Some old wine in new bottles', *Journal of Financial and Quantitative Analysis* **10**(5), 775–784.

Huck, N. (2009), 'Pairs selection and outranking: An application to the s&p 100 index', *European Journal of Operational Research* **196**(2), 819–825.

Huck, N. (2010), 'Pairs trading and outranking: The multi-step-ahead forecasting case', *European Journal of Operational Research* **207**(3), 1702–1716.

Huck, N. (2019), 'Large data sets and machine learning: Applications to statistical arbitrage', *European Journal of Operational Research* **278**(1), 330–342.

Hurst, B., Ooi, Y. H. and Pedersen, L. H. (2017), 'A century of evidence on trend-following investing', *The Journal of Portfolio Management* **44**(1), 15–29.

Ilmanen, A. (2011), *Expected returns: An investor's guide to harvesting market rewards*, John Wiley & Sons.

Jegadeesh, N. and Titman, S. (1993), 'Returns to buying winners and selling losers: Implications for stock market efficiency', *The Journal of finance* **48**(1), 65–91.

Krauss, C., Do, X. A. and Huck, N. (2017), 'Deep neural networks, gradient-boosted trees, random forests: Statistical arbitrage on the s&p 500', *European Journal of Operational Research* **259**(2), 689–702.

Ledoit, O. and Wolf, M. (2004), 'Honey, I Shrunk the Sample Covariance Matrix', *Journal of Portfolio Management* **30**, 110–119.

Ledoit, O. and Wolf, M. (2008), 'Robust performance hypothesis testing with the sharpe ratio', *Journal of Empirical Finance* **15**(5), 850–859.

Maillard, S., Roncalli, T. and Teiletche, J. (2010), 'The properties of equally weighted risk contribution portfolios', *The Journal of Portfolio Management* **36**(4), 60–70.

Merton, R. C. (1980), 'On estimating the expected return on the market: An exploratory investigation', *Journal of Financial Economics* **8**(4), 323–361.

Michaud, R. O. (1989), 'The markowitz optimization enigma: Is 'optimized'optimal?', *Financial Analysts Journal* **45**(1), 31–42.

Roncalli, T. (2016), *Introduction to risk parity and budgeting*, Chapman and Hall/CRC.

Rosenberg, B., Reid, K. and Lanstein, R. (1985), 'Persuasive Evidence of Market Inefficiency', *Journal of Portfolio Management* **11**, 9–11.

Appendix A. A risk budget portfolio with long and short positions

Consider a set of n assets. For each asset i, s_i equals 1 (-1) if the corresponding positions should be long (short). We are interested in the following risk budget problem:

$$\begin{cases} \mathcal{RC}_i = b_i \\ x_i > 0 \quad \text{if} \quad s_i > 0 \\ x_i < 0 \quad \text{if} \quad s_i < 0 \\ \sum_i |x_i| = 1 \end{cases} . \tag{A.1}$$

Define the following optimization problem:

$$\begin{aligned} \min_x \quad & \sigma_P(x) \\ \text{subject to} \quad & \begin{cases} \sum_{i=1}^n b_i \ln |x_i| \geq c \\ x_i s_i > 0 \end{cases} , \end{aligned} \tag{A.2}$$

The Lagrangian of this problem is

$$\mathcal{L}(x;\lambda) = \sigma_P(x) - \sum_{i=1}^n \lambda_i x_i s_i - \lambda_c \left(\sum_{i=1}^n b_i \ln |x_i| - c \right). \tag{A.3}$$

The first-order condition is

$$\begin{aligned} \frac{\partial \mathcal{L}(x;\lambda)}{\partial x_i} &= \frac{\partial \sigma_P(x)}{\partial x_i} - \lambda_i s_i - \lambda_c \frac{b_i}{|x_i|} \frac{x_i}{|x_i|} = 0 \\ &= \frac{\partial \sigma_P(x)}{\partial x_i} - \lambda_i s_i - \lambda_c \frac{b_i}{x_i} = 0. \end{aligned} \tag{A.4}$$

The Kuhn-Tucker conditions imply that

$$\begin{cases} \lambda_i x_i s_i = 0 \\ \lambda_c \left(\sum_{i=1}^n b_i \ln |x_i| - c \right) = 0. \end{cases} \tag{A.5}$$

Since $x_i \neq 0$, if follows that $\lambda_i = 0$ for all i. Also, since $x = 0$ cannot be a solution, the constraint $\sum_{i=1}^n b_i \ln |x_i| = c$

must be reached, implying $\lambda_c > 0$. Therefore, (A.6) becomes

$$\begin{aligned} \frac{\partial \sigma_P(x)}{\partial x_i} &= \lambda_c \frac{b_i}{x_i} && \text{(A.6)} \\ \Rightarrow x_i \frac{\partial \sigma_P(x)}{\partial x_i} &= \lambda_c b_i, && \text{(A.7)} \end{aligned}$$

which implies marginal risk contributions are proportional to risk budgets. The vector x can be rescaled so that $\sum_j |x_i| = 1$.