

Regular Inverse & Pseudo Inverse Matrix Calculation using Singular Value Decomposition

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In order to calculate inverse kinematic solutions for robotic arm, we can use several methods. One of my favorite method is using the pseudo inverse of a jacobian matrix.

So what is pseudo inverse of a matrix and how it differs from the regular inverse of a matrix ?

Regular Inverse of a Square Matrix

Inversion a Matrix means a process to create a reciprocal version of a Matrix.

In math, the reciprocal of a number means inverse of a number, for example, we have a number called b , inverse of b number can be written as b^{-1} . b^{-1} comes from: $1/b$ (reciprocal of b).

Meanwhile for a matrix, e.g matrix A , inverse of A matrix can be written as A^{-1}

In order to be inverted, a matrix must meet 2 conditions :

1. the matrix has the same number of rows and columns (square matrix)
2. determinant of the matrix is not zero.

The Pseudo Inverse of a Matrix

The Pseudo inverse matrix is symbolized as A^\dagger .

However, sometimes there are some matrices that do not meet those 2 requirements, thus cannot be inverted. Where the condition is overdetermined, we can use a method called pseudo inverting to create a pseudo inverse matrix version of our original matrix.

We are going to use SVD (Singular Value Decomposition) Trick to decompose a non square matrix which doesn't have eigenvalue and eigenvector.

Inverting an Invertible Square Matrix

Inverting a square matrix is easy. For example, We have A matrix:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

In order to get inverse of A , we are going to use adjoint method :

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

Or

$$A^{-1} = \frac{1}{\det(A)} (\text{cofactor of matrix } A(A))^T$$

Inverse of A can be acquired from these steps:

- Step 1. calculating determinant of A
- Step 2. find cofactor of A
- Step 3. find adjoint of A
- Step 4. find the inverse

Based on <https://ringlayer.wordpress.com/2018/07/03/matrix-determinant-laplace-and-sarrus-method>, we found determinant of our matrix = -6.

So we can go straight ahead to step 2 : "find cofactor of A".

First of all, we need to get minor of A. Minor of A can be obtained by getting 2x2 possible matrix determinants of A :

Visually, we can do the trick a lot easier:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Next:

$$\begin{array}{ccc}
 \text{a} & \text{b} & \text{c} \\
 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} & \begin{pmatrix} 3 & 3 \\ 2 & 1 \end{pmatrix} \\
 \text{d} & \text{e} & \text{f} \\
 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\
 \text{g} & \text{h} & \text{i} \\
 \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}
 \end{array}$$

hence we got :

$$|a| = 2, |b| = 1, |c| = -3, |d| = 2, |e| = -1, |f| = -3, |g| = -1, |h| = -2, |i| = -3$$

Thus we got minor matrix:

$$\begin{pmatrix} 2 & 1 & -3 \\ 2 & -1 & -3 \\ -1 & -2 & -3 \end{pmatrix}$$

Cofactor matrix can be obtained easily:

$$\begin{pmatrix} +2 & -1 & +3 \\ -2 & +1 & -3 \\ +1 & -2 & +3 \end{pmatrix}$$

Hence we got cofactor matrix:

$$\begin{pmatrix} 2 & -1 & -3 \\ -2 & -1 & 3 \\ -1 & 2 & -3 \end{pmatrix}$$

Step 3, find adjoint of A :

Adjoint can be obtained by transforming cofactor matrix columns into rows, Adjoint :

$$\begin{pmatrix} 2 & -2 & -1 \\ -1 & -1 & 2 \\ -3 & 3 & -3 \end{pmatrix}$$

Finally based on our formula:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

$A^{-1} = 1/-6 * \text{adjoint matrix}$, Hence we got our inverse matrix :

$$\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Pseudo Inverse Matrix using SVD

Sometimes, we found a matrix that doesn't meet our previous requirements (doesn't have exact inverse), such matrix doesn't have eigenvector and eigenvalue. In order to find pseudo inverse matrix, we are going to use SVD (Singular Value Decomposition) method.

For Example, Pseudo inverse of matrix A is symbolized as A^{+}

When the matrix is a square matrix :

$$A^{+} = A^{-1}$$

when the matrix is overdetermined :

$$A^{+} = V \Sigma^{-1} U^t$$

V, Σ and U are matrices from SVD of A.

Where SVD of A :

$$U \Sigma V^t$$

can be notated :

$$USV^t$$

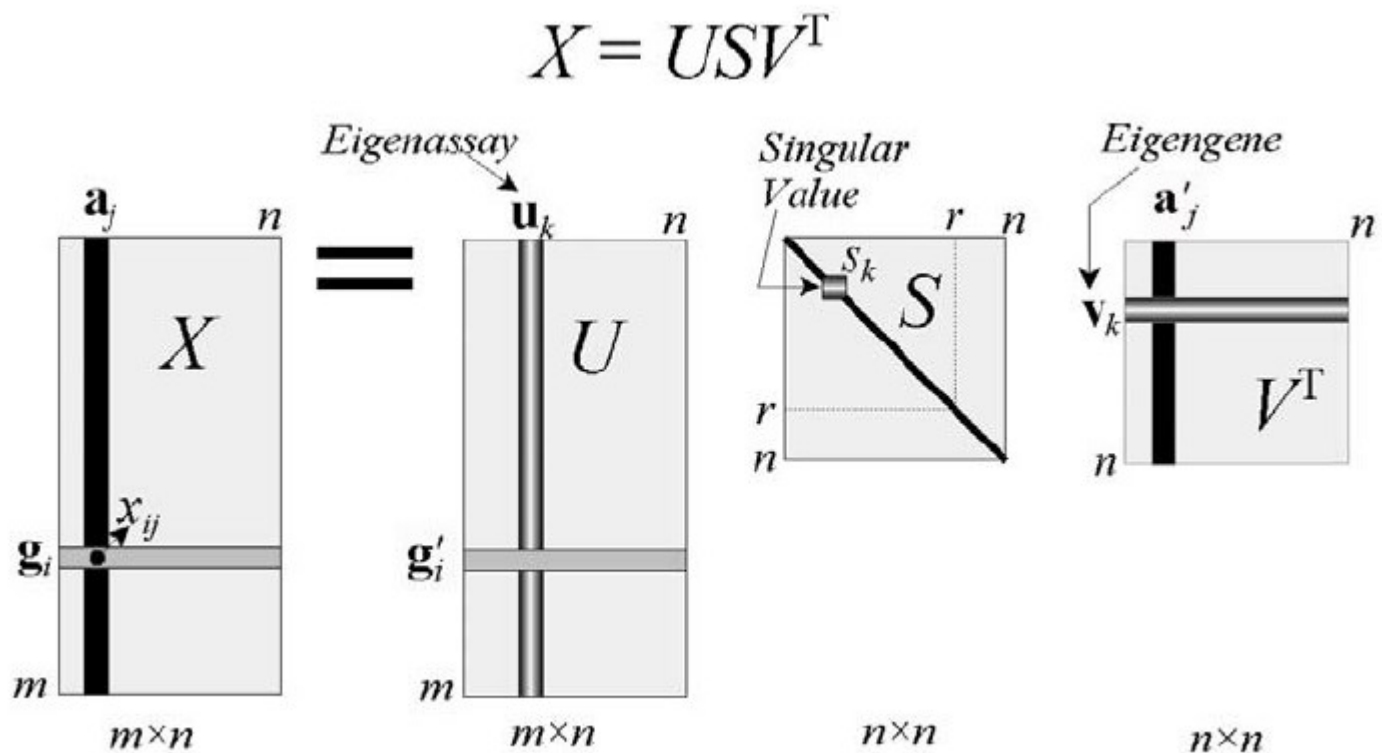
or

$$USV^t$$

or

$$UDV^t$$

Example of Pseudo Inverse Calculation using SVD Method



For example, We have a non squared matrix called "A" as follow :

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

U, s, VT of this matrix can be acquired manually or automatically using this simple python numpy script:

```
#!/usr/bin/env python3
import numpy as np
# example taken from Video Tutorials - All in One
# https://www.youtube.com/watch?v=P5mlg91as1c
a = np.array([[1, 2, 1],[2, 1, -1]])
np.set_printoptions(suppress=True)
np.set_printoptions(precision=3)
U, s, VT = np.linalg.svd(a, full_matrices=True)
print ("U:\n {}".format(U))
print ("s:\n {}".format(s))
print ("VT:\n {}".format(VT))
```

```
root@ringlayer-OptiPlex-780:/home/ringlayer/Desktop/BACAN/MATH/SVD# ./tes.py
A:
[[ 1  2  1]
 [ 2  1 -1]]
=====
U:
[[-0.707 -0.707]
 [-0.707  0.707]]
s:
[3.    1.732]
VT:
[[-0.707 -0.707  0.    ]
 [ 0.408 -0.408 -0.816]
 [ 0.577 -0.577  0.577]]
root@ringlayer-OptiPlex-780:/home/ringlayer/Desktop/BACAN/MATH/SVD#
```

or we can calculate SVD manually using this following steps:

Step 1. Finding singular vector and singular value of U & V

in order to find singular vector and singular value of U & V, we have to find eigenvector and eigenvalue of kU & kV , where :

$$kU = A \cdot A^t$$

$$kV = A^t \cdot A$$

$$(A)^t \cdot (A) = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix}$$

Finding eigenvalue and eigenvector of kU :

$$| A - \lambda I | = 0$$

λ = scalar (eigenvalue corresponding to u)

I = identity matrix

$$\det \left(\begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

$$\det \begin{pmatrix} 6 - \lambda & 5 \\ 5 & 6 - \lambda \end{pmatrix} = 0$$

So :

$$(6 - \lambda)^2 - 25 = 0$$

we got 2 eigenvalue :

$$\lambda_1 = 11$$

and

$$\lambda_2 = 1$$

next, we are going to find eigenvector of kU.

eigenvector of kU = x

$$(A - \lambda.I) * x = 0$$

$$\begin{pmatrix} 6 - \lambda & 5 \\ 5 & 6 - \lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

for $\lambda = 11$:

$$\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$(-5 * x_1) + (5 * x_2) = 0$$

hence:

$$x_1 = x_2$$

$$x_1 = -1$$

$$x_2 = -1$$

for $\lambda = 1$:

$$\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$(5 * x_1) + (5 * x_2) = 0$$

hence:

$$x_1 - x_2 = 0$$

$$x_1 = -1$$

$$x_2 = 1$$

Finally we have collected enough informations for U:

$$\lambda_1 = 11 \rightarrow x_1 = -1 \text{ and } x_2 = -1$$

$$\lambda_2 = 1 \rightarrow x_1 = -1 \text{ and } x_2 = 1$$

$$Du = \begin{pmatrix} \sqrt{11} & 0 \\ 0 & \sqrt{1} \end{pmatrix}$$

$$u = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

in order to get U we need to normalize u (orthonormal) using this following steps:

We have our u matrix:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

to normalize u :

$$\frac{\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}}{\left((x_{11} \ x_{12}) \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} \right)^{1/2}}$$

hence:

$$\frac{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}{\left(\begin{pmatrix} -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right)^{\frac{1}{2}}}$$

we got:

$$\frac{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}{(2)^{\frac{1}{2}}}$$

2 power 1/2 = square root of 2.

$$\frac{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}{\sqrt{2}}$$

we got:

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

using the same method to normalize, another column, we got:

$$\frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

We got our normalize matrix:

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

so U =

$$\begin{pmatrix} -0.707 & -0.707 \\ -0.707 & 0.707 \end{pmatrix}$$

in order to find V, we can do the same steps just like U, we got :

$$dv = \begin{pmatrix} \sqrt{11} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.640 & 0.707 & 0 \\ -0.640 & -0.707 & 0 \\ 0.426 & 0 & 0 \end{pmatrix}$$

By transposing matrix V, we got:

$$V^t = \begin{pmatrix} -0.640 & -0.640 & 0.426 \\ 0.707 & -0.707 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2. Find D.

We can acquire D matrix by multiplying Du with Dv:

$$D = Du * Dv$$

$$= \begin{pmatrix} \sqrt{11} & 0 \\ 0 & \sqrt{1} \end{pmatrix} * \begin{pmatrix} \sqrt{11} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 3.317 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Finally we have collected enough informations to calculate pseudo inverse matrix using SVD method, we got this informations:

$$\mathbf{U} = \begin{pmatrix} -0.707 & -0.707 \\ -0.707 & 0.707 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} -0.640 & 0.707 & 0 \\ -0.640 & -0.707 & 0 \\ 0.426 & 0 & 0 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 3.317 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Based on our pseudo inverse matrix formula:

$$\mathbf{A}^+ = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^t$$

or

$$\mathbf{A}^+ = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^t$$

We can get our pseudo inverse matrix easily:

$$\mathbf{A}^+ = \begin{pmatrix} -0.640 & 0.707 & 0 \\ -0.640 & -0.707 & 0 \\ 0.426 & 0 & 0 \end{pmatrix} \cdot \left(\begin{pmatrix} 3.317 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right)^{-1} * \mathbf{U}^t$$

We got our Pseudo inverse matrix (A dagger):

$$\begin{pmatrix} -0.364 & 0.636 \\ 0.636 & -0.364 \\ -0.091 & -0.091 \end{pmatrix}$$