Solution to Jane Street Puzzle
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Problem

Their solution, along with a list of solvers, can be found at the following link: Jane Street Puzzle Archive. My name, Anton Sjöström, is included among the solvers.

Problem Statement

For a fixed p, independently label the nodes of an infinite complete binary tree 0 with probability p, and 1 otherwise. For what p is there exactly a $\frac{1}{2}$ probability that there exists an infinite path down the tree that sums to at most 1 (that is, all nodes visited, with the possible exception of one, will be labeled 0). Find this value of p accurate to 10 decimal places.

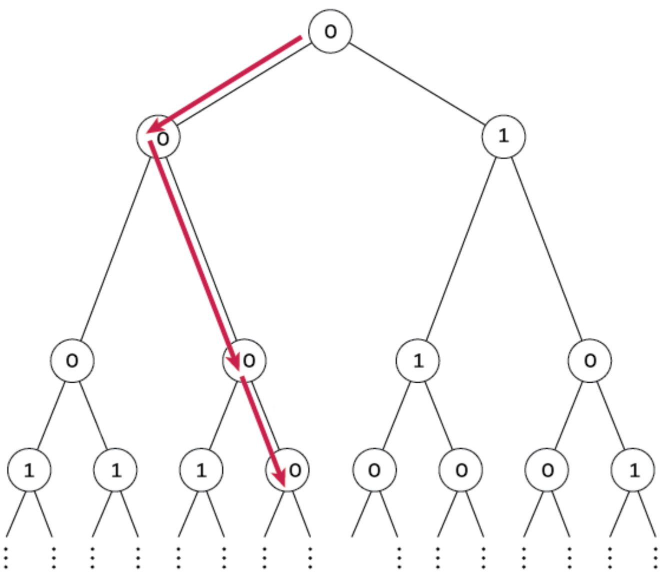

Figure 1: Illustration of the infinite binary tree.

Solution Outline

The solution proceeds in two main stages. First, we analyze a simplified scenario where the sum of the labels along the path must be exactly zero. This serves as a foundational sub-problem, providing insights and intermediate results that are useful in solving the full problem.

In the second stage, we generalize the analysis to the original problem, where the sum of the labels along the path must be at most one. This involves some more algebraic manipulation and probabilistic reasoning. After deriving the necessary equations, we apply numerical root-finding techniques to compute the value of p such that the probability of the existence of such a path is precisely $\frac{1}{2}$.

An interesting observation arises during the solution: for some critical non-zero value of p, and for all p below this threshold, the probability of finding a path with a sum of at most one is zero. This highlights a threshold phenomenon in the problem's structure.

Detailed Solution

In this section, we present the solution step by step, focusing on the key ideas and calculations while omitting overly tedious algebraic manipulations for clarity. The goal is to ensure the solution remains concise yet easy to follow.

Step 1: Initial Observations

The infinite nature of the tree might initially seem daunting. However, this very property allows the problem to have a recursive structure, which is key to deriving the solution. At any point in the tree, the situation is analogous to the starting point, enabling us to use recursion effectively.

Another key observation is that there are two distinct cases to consider: either the sum of the labels along the path is exactly zero, or it is at most one. To simplify the analysis, we begin with the case where the sum is zero, as it provides a more straightforward starting point.

Step 2: Case of Summing to Zero

Define P_0 as the probability of finding an infinite path in the tree such that the sum of the node labels along the path equals zero. Here, N_i represents the label of the i-th node along the path, starting with the root node at index zero.

To ensure the sum along an infinite path remains zero, every node on the path must be labeled 0. The recursive structure of the tree implies that for a given node, at least one of its child nodes must satisfy the same condition. Mathematically, this can be expressed as:

$$P_0 = 2pP_0 - p^2 P_0^2, \tag{1}$$

where:

- P_0 is the probability of finding an infinite path summing to zero,
- p is the probability that a node is labeled 0.

Rearranging, we obtain:

$$P_0(p^2 P_0 + (1 - 2p)) = 0. \tag{2}$$

This implies either $P_0 = 0$, or $p^2 P_0 + (1 - 2p) = 0$. Solving for P_0 in the non-trivial case, we find:

$$P_0 = \frac{2p - 1}{p^2}. \tag{3}$$

For P_0 to represent a valid probability, it must satisfy $0 \leq P_0 \leq 1$. This condition is only met when $p \geq \frac{1}{2}$. Thus, for $p < \frac{1}{2}$, the probability of finding an infinite path summing to zero is zero. This highlights a critical threshold phenomenon.

Step 3: Case of Summing to At Most One

Define P_1 as the probability of finding an infinite path such that the sum of the node labels along the path is at most one. At each step, two scenarios can occur: either we move to a node labeled 0 (probability p), or to a node labeled 1. In the latter case, the problem reduces to the sub-case where the sum along the path must be exactly zero. This recursive relationship can be expressed as:

$$P_1 = 2\big(pP_1 + (1 - p)P_0\big) - \big(pP_1 + (1 - p)P_0\big)^2, \tag{4}$$

where P_0 is as derived earlier. After some algebraic manipulation, we find:

$$P_1 = \frac{\sqrt{p}\sqrt{4p^3 - 12p^2 + 13p - 4} + 6p^2 - 7p + 2}{2p^3}. \tag{5}$$

Step 4: Total Probability and Numerical Solution

The total probability of success, P_s, is given by:

$$P_s = pP_1 + (1 - p)P_0, \tag{6}$$

since we need to account for the probability that the root is either labeled zero or one. Substituting the expressions for P_0 and P_1, we obtain:

$$P_s = 1 - \frac{1}{2p} + \sqrt{\frac{4p^3 - 12p^2 + 13p - 4}{4p^3}}. \tag{7}$$

To find the value of p such that $P_s = \frac{1}{2}$, we solve the equation numerically. The solution, accurate to 10 decimal places, is:

$$p = 0.5306035754. \tag{8}$$

Conclusion

This problem exhibits a fascinating threshold phenomenon: a non-zero value of p is required for P_s to become non-zero. By leveraging the recursive structure of the tree and breaking the problem into manageable sub-problems, we derived a solution.