Braket notation

State vector (Ket):

$$|a\rangle = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Conjugate transpose of state vector (Bra):

$$\langle a| = \begin{bmatrix} a_0^* & a_1^* & \dots & a_n^* \end{bmatrix}$$

Addition:

$$|a\rangle + |b\rangle = \begin{bmatrix} a_0 + b_0 \\ a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

Scalar Multiplication:

$$x\,|a\rangle = \begin{bmatrix} x \times a_0 \\ x \times a_1 \\ \vdots \\ x \times a_n \end{bmatrix}$$

Inner Product:

$$\langle a \,|\, b\rangle = a_0^* b_0 + a_1^* b_1 + \dots + a_n^* b_n$$

Outer Product:

$$|a\rangle \langle b| = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_0^* & b_1^* & \dots & b_n^* \end{bmatrix} = \begin{bmatrix} a_0 b_0^* & a_0 b_1^* & \dots & a_0 b_n^* \\ a_1 b_0^* & a_1 b_1^* & & \vdots \\ \vdots & & \ddots & \vdots \\ a_n b_0^* & a_n b_1^* & \dots & a_n b_n^* \end{bmatrix}$$

Tensor Product:

$$|ab\rangle = |a\rangle \otimes |b\rangle = \begin{bmatrix} a_0 & \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \\ a_1 & \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{bmatrix}$$

Qubit State Vector

$$|q\rangle = \alpha |0\rangle + \beta |1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$\alpha \in \mathbb{C}$ being the probability it collapse to 0 when measured
$\beta \in \mathbb{C}$ being the probability it collapse to 1 when measured

The state that will always measured to be 0:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The state that will always measured to be 1:

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The probability of measuring a state $|\psi\rangle$ in the state $|x\rangle$:

$$p(|x\rangle) = |\langle x|\psi\rangle|^2$$

Qubit represented in $\theta, \phi \in \mathbb{R}$:

$$|q\rangle = \cos\frac{\theta}{2} |0\rangle + e^{i\phi} \sin\frac{\theta}{2} |1\rangle$$

Single Quantum Gate

Pauli Gates

X-Gate: Rotate by π around the x-axis of Bloch sphere

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|$$

Equivalent to classical NOT when operated on classical states:

$$X|0\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle = |0\rangle\langle 1 \,|\, 0\rangle + |1\rangle\langle 0 \,|\, 0\rangle = |0\rangle \times 0 + |1\rangle \times 1 = |1\rangle$$

$$X|1\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle = |0\rangle\langle 1 \,|\, 1\rangle + |1\rangle\langle 0 \,|\, 1\rangle = |0\rangle \times 1 + |1\rangle \times 0 = |0\rangle$$

X-basis (eigenstates of X-gate):

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

syntax (qc is of object QuantumCircuit, qubit is index of qubit in the circuit to operate on):

```
qc.x(qubit)
```

Y-Gate: Rotate by π around the Y-axis of Bloch sphere

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$

Y-basis:

$$|\circlearrowleft\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$|\circlearrowright\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

syntax:

```
qc.y(qubit)
```

Z-Gate: Rotate by π around the Z-axis of Bloch sphere

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Z-basis:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

syntax:

```
qc.z(qubit)
```

Hadamard Gate (H-gate)

Rotation around the Bloch vector [1,0,1] (line between x and z-axis)

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = |+\rangle\langle 0| + |-\rangle\langle 1|$$

Transform Z-basis ($|0\rangle$ and $|1\rangle$) to X-basis ($|+\rangle$ and $|-\rangle$):

$H|0\rangle = (|+\rangle\langle 0| + |-\rangle\langle 1|)|0\rangle = |+\rangle\langle 0\,|\,0\rangle + |-\rangle\langle 1\,|\,0\rangle = |+\rangle \times 1 + |-\rangle \times 0 = |+\rangle$

$H|1\rangle = (|+\rangle\langle 0| + |-\rangle\langle 1|)|1\rangle = |+\rangle\langle 0\,|\,1\rangle + |-\rangle\langle 1\,|\,1\rangle = |+\rangle \times 0 + |-\rangle \times 1 = |-\rangle$

syntax:

```
qc.h(qubit)
```

R_ϕ-gate

Rotation of ϕ around z-axis

$$R_\phi = \begin{bmatrix}1 & 0\\0 & e^{i\phi}\end{bmatrix} = |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1|$$

syntax (phi is ϕ in radian):

```
qc.rz(phi, qubit)
```

Identity Gate (I-gate)

$$I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

syntax:

```
qc.i(qubit)
```