Computing Convex Hulls

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Overview

Hulls

Definition

Let $A \subset \mathbb{K}^n$, an affine combination of points in A is a linear combination $\sum_{i=1}^m \lambda_m a_m$ where $\lambda_m \in \mathbb{K}$ and $a \in A$ such that $\sum_{i=1}^m \lambda_m = 1$. The affine hull is the set of all such combinations.

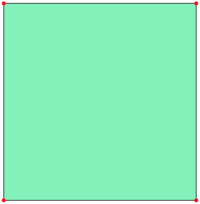
Definition

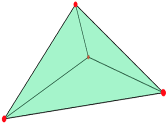
Let $A \subset \mathbb{R}^n$, a convex combination of points in A is an affine combination $\sum_{i=1}^m \lambda_m a_m$ where $\lambda_m \geq 0$. The convex hull is the set of all such combinations.

Polytopes

Definition

A set $P \subset \mathbb{R}^n$ is a polytope if it can be described as the convex hull of finitely many points. The dimension of P is defined to be the dimension of it's affine hull. A *k*-polytope is a k dimensional polytope. A *k*-simplex is the convex hull of $k + 1$ affine independant points.





Faces

Definition

Given an n-polytope $P \subset \mathbb{R}^n$, the intersection $P \cap H$ with a supporting hyperplane H is called a proper face. A face of dimension k is called a k face, a 0-face is a vertex, 1-face an edge, n-2 face a ridge and an n-1 face a facet.

Remark

Proper faces are also polytopes with respect to their affine hull.

Theorem

The boundary of a full dimesional polytope is the union of all it's proper faces.

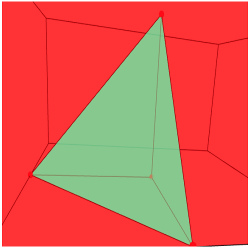
Half-spaces

Definition

Given an affine hyperplane $H \subset \mathbb{R}^n$ given in homegeneous coordinates as $[a_0, \ldots, a_n]$, define the positive halfspace H^+ as $\{x \in \mathbb{R}^n \mid a_0 + a_1x_1 + \cdots + a_nx_n \geq 0\}$.

Remark

For each facet f of a polytope P, there exists a positive halfspace H^+ such that $f = P \cap H$ and $P \subset H^+$



Polytope Descriptions

Theorem

Let H_i be the supporting hyperplanes for the facets of a polytope P. then $P = \cap_i^m H_i^+$

Theorem

Every polytope is the convex hull of it's vertices

Remark

We call $P = \text{conv}(v_1, \ldots, v_m)$ a $\mathcal{V}$-description, and we call $P = \cap_{i=1}^k H_i^+$ an $\mathcal{H}$-description. An algorithm that finds a $\mathcal{V}$-description from an $\mathcal{H}$-description is referred to as a convex hull computation. We call the pair $\mathcal{V}, \mathcal{H}$ a double description of P.

Polarity and Duality

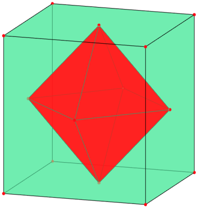
Definition

Given a set $X \subset \mathbb{R}^n$ define the polar set as $X^o = \{y \in \mathbb{R}^n \mid x_1 y_1 + \dots x_n y_n \leq 1\}$

Theorem

If $P \subset \mathbb{R}^n$ is an n-polytope with $0 \in intP$ then P^o is also an n-polytope, and for V the vertex set of P we have

$$P^o = \bigcap_{v \in V} \{y \in \mathbb{R}^n \mid \langle v, y \rangle \leq 1\} = \bigcap_{v \in V} [1 : -v_0 : \dots : -v_n]^+$$



Polarity and Duality

Theorem

Let $P \subset \mathbb{R}^n$ be an n polytope with $0 \in intP$ then

- $(P^o)^o = P$
- *For any point p on the boundary of P, $H = \{x \in \mathbb{R}^n \mid \langle p, x \rangle = 1\}$ is a supporting hyperplane of P^o*

Remark

If $0 \in \mathrm{int}P$, where $P = \bigcap_{i=1}^m H_i^+$, then we can write $H_i^+ = [1 : -h_1^{(i)} : \cdots : -h_n^{(i)}]$, then $P^o = \mathrm{conv}(h_1, \ldots, h_n)$. So, finding a half-space description of P^o will give us the vertices of P. So we can reduce the problem of finding a $\mathcal{V}$-description from an $\mathcal{H}$-description to a convex hull computation.

A basic Algorithm

Input: Finite point set $V \subset \mathbb{R}^n$ with dimension of $\operatorname{aff} V = n$
Output: Finite set of half-spaces H_i^+ such that $\operatorname{conv}(V) = \cap_{i=1}^{m} H_i^+$

$\mathcal{H} \leftarrow \emptyset$
for each n element subset $W \subset V$ with dimension $\operatorname{aff} W = n - 1$ **do**
    $H \leftarrow \operatorname{aff} W$
    if $V \subset H^+$ **then**
        $\mathcal{H} \leftarrow \mathcal{H} \cup H^+$
    else
        $\mathcal{H} \leftarrow \mathcal{H} \cup H^-$
    end if
end for
return $\mathcal{H}$

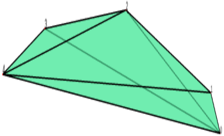
A Worst Case Example

Definition

The moment curve $\mu_n \to \mathbb{R}^n$ is defined as $\tau \to (\tau, \ldots, \tau^n)$. A polytope is called cyclic if it is the convex hull of points on the moment curve.

Remark

Notice that since any $n+1$ vertices lie in a distinct supporting hyperplane, that the number of facets is maximal given a fixed point of number vertices. So the worst case for any convex hull computation is $\Theta(m^{\lfloor n/2 \rfloor})$.



A Partitioning Lemma

Lemma

Let V_0, V_+, V_- be the partition of a point set V,defined by
$V_0 = V \cap H, V_+ = V \cap H^+ \setminus H, V_- = V \cap H^- \setminus H$ *where H is a hyperplane.*
Then we have $P \cap H = conv(V_0 \cup V_+ \cup \{[v, w] \cap H \mid v \in V_+, w \in V_-\})$

Input:
Output:

for $i = 1$ to N **do**
    for $j = 1$ to $JJJJ$ **do**
        $energy[i * JJJ + j] = interpolate(AAA[i * JJJ + j], ZZZ)$
    end for
end for

blah

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References

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