Computing Convex Hulls

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Overview

Hulls

Definition

Let $A \subset \mathbb{K}^n$, an affine combination of points in A is a linear combination $\sum_{i=1}^m \lambda_m a_m$ where $\lambda_m \in \mathbb{K}$ and $a \in A$ such that $\sum_{i=1}^m \lambda_m = 1$. The affine hull is the set of all such combinations.

Definition

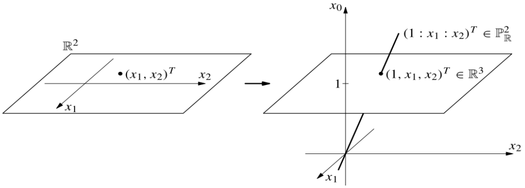
Let $A \subset \mathbb{R}^n$, a convex combination of points in A is an affine combination $\sum_{i=1}^m \lambda_m a_m$ where $\lambda_m \geq 0$. The convex hull is the set of all such combinations.

Definition

Let $A \subset \mathbb{R}^n$, a postive combination of points in A is a linear combination $\sum_{i=1}^m \lambda_m a_m$ where $\lambda_m \geq 0$. The positive hull is the set of all such combinations.

Definition

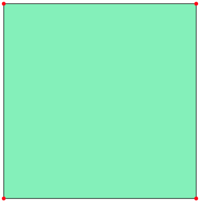
- Let $(x_0, \dots x_n) \in \mathbb{K}^{n+1} \setminus 0$, and let $x := \mathrm{lin}(x_0, \dots, x_n)$, for any element of $x \setminus 0$ we call $(x_0 : \dots : x_n)$ homegeneous coordinates for x, with $(y_0 : \dots y_n) \sim (x_0 : \dots x_n)$ if $\lambda(x_0, \dots, x_n) = (y_0, \dots, y_n)$ with $\lambda \neq 0$, we call equivalence classes with a the first coefficient 0 ideal points.
- Given a linear transformation $A \in \mathrm{GL}(\mathbb{K}, n+1)$ we call the induced transformation on homogeneous coordinates a projective transformation.
- We call a transformation affine if it sends ideal points to ideal points.

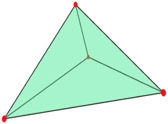


Polytopes

Definition

A set $P \subset \mathbb{R}^n$ is a polytope if it can be described as the convex hull of finitely many points. The dimension of P is defined to be the dimension of it's affine hull. A k-polytope is a k dimensional polytope. A k-simplex is the convex hull of $k+1$ affine independant points.





Faces

Definition

Given an n-polytope $P \subset \mathbb{R}^n$, the intersection $P \cap H$ with a supporting hyperplane H is called a proper face. A face of dimension k is called a k face, a 0-face is a vertex, 1-face an edge, $n-2$-face a ridge and an $n-1$-face a facet.

Remark

Proper faces are also polytopes with respect to their affine hull.

Theorem

The boundary of a full dimesional polytope is the union of all it's proper faces.

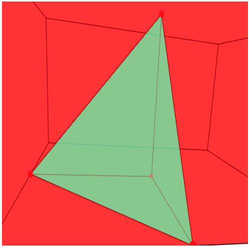
Half-spaces

Definition

Given an affine hyperplane $H \subset \mathbb{R}^n$ given in homogeneous coordinates as $[a_0 : \cdots : a_n]$, define the positive halfspace H^+ as $\{x \in \mathbb{R}^n \mid a_0 + a_1 x_1 + \cdots + a_n x_n \geq 0\}$.

Remark

For each facet f of a polytope P, there exists a positive halfspace H^+ such that $f = P \cap H$ and $P \subset H^+$



Polytope Descriptions

Theorem

Let H_i be the supporting hyperplanes for the facets of a polytope P. then $P = \cap_i^m H_i^+$

Theorem

Every polytope is the convex hull of it's vertices

Remark

We call $P = \text{conv}(v_1, \ldots, v_m)$ a V-description, and we call $P = \cap_{i=1}^k H_i^+$ an $\mathcal{H}$-description. An algorithm that finds a V-description from an $\mathcal{H}$-description is referred to as a convex hull computation. We call the pair $V, \mathcal{H}$ a double description of P.

Polarity and Duality

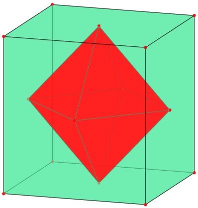
Definition

Given a set $X \subset \mathbb{R}^n$ define the polar set as $X^o = \{y \in \mathbb{R}^n \mid x_1 y_1 + \ldots x_n y_n \leq 1\}$

Theorem

If $P \subset \mathbb{R}^n$ is an n-polytope with $0 \in intP$ then P^o is also an n-polytope, and for V the vertex set of P we have

$$P^o = \bigcap_{v \in V} \{y \in \mathbb{R}^n \mid \langle v, y \rangle \leq 1\} = \bigcap_{v \in V} [1 : -v_0 : \cdots : -v_n]^+$$



Polarity and Duality

Theorem

Let $P \subset \mathbb{R}^n$ be an n polytope with $0 \in intP$ then

- $(P^o)^o = P$
- *For any point p on the boundary of P, $H = \{x \in \mathbb{R}^n \mid \langle p, x \rangle = 1\}$ is a supporting hyperplane of P^o*

Remark

If $0 \in \text{int}P$, where $P = \bigcap_{i=1}^{m} H_i^+$, then we can write $H_i^+ = [1 : -h_1^{(i)} : \cdots : -h_n^{(i)}]$, then $P^o = \text{conv}(h_1, \ldots, h_n)$. So, finding a half-space description of P^o will give us the vertices of P. So we can reduce the problem of finding a V-description from an $\mathcal{H}$-description to a convex hull computation.

Polyhedra

Definition

$P \subset \mathbb{R}^n$ is called a polyhedron if it can be described by a finite intersection of closed affine half-spaces. A polyhedron that doesn't contain an affine line is called pointed.

Theorem

Every pointed polyhedron is projectively equivalent to a polytope.

$$T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ h_0^{(1)} & h_1^{(1)} & \dots & h_n^{(1)} \\ & \dots & \dots & \\ h_0^{(1)} & h_1^{(n)} & \dots & h_n^{(n)} \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$B : E_1^+ \cap \dots \cap E_n^+ \to E_1^+ \cap \dots \cap E_n^+ \cap [1 : -1 : \dots : -1]^+$$

Description of Polyhedra

Definition

Given two sets $X, Y \subset \mathbb{R}^n$, the Minkowski Sum is defined as
$X + Y = \{x + y \mid x \in X, y \in Y\}$

Theorem

Every polyhedron P can be expressed as the Minkowski sum

$$P = convV + posR$$

where V, R are finite.

A Trivial Algorithm

Input: Finite point set $V \subset \mathbb{R}^n$ with dimension of $\operatorname{aff} V = n$
Output: Finite set of half-spaces H_i^+ such that $\operatorname{conv}(V) = \cap_{i=1}^{m} H_i^+$

$\mathcal{H} \leftarrow \emptyset$
for each n element subset $W \subset V$ with dimension $\operatorname{aff} W = n - 1$ **do**
    $H \leftarrow \operatorname{aff} W$
    if $V \subset H^+$ **then**
        $\mathcal{H} \leftarrow \mathcal{H} \cup H^+$
    else
        if $V \subset H^-$ **then**
            $\mathcal{H} \leftarrow \mathcal{H} \cup H^-$
        end if
    end if
end for
return $\mathcal{H}$

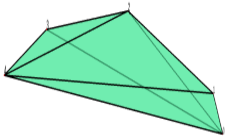
A Worst Case Example

Definition

The moment curve $\mu_n \to \mathbb{R}^n$ is defined as $\tau \to (\tau, \ldots, \tau^n)$. A polytope is called cyclic if it is the convex hull of points on the moment curve.

Remark

Notice that since any $n+1$ vertices lie in a distinct supporting hyperplane, each facet is an n-simplex. Hence we have many facets $\Theta(m^{\lfloor n/2 \rfloor})$, and cannot expect an algorithm that is polynomial in n and m.



A Partitioning Lemma

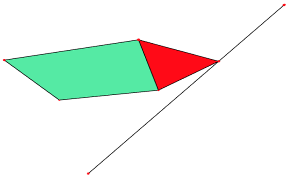
Lemma

Let $P = convV$, and let $P' = P \cap H^+$, where H is a hyperplane. Let V_0, V_+, V_- be the partition of a point set V, defined by

$$V_0 = V \cap H, V_+ = V \cap H^+ \setminus H, V_- = V \cap H^- \setminus H$$

where H is a hyperplane. Then we have

$$P' \cap H^+ = conv(V_0 \cup V_+ \cup \{[v, w] \cap H \mid v \in V_+, w \in V_-\})$$



A Basic Algorithm

Input: A set of affine half-spaces $\mathcal{H} = \{H_1^+, \ldots, H_m^+\}$ in $\mathbb{R}^n$ such that $P = \cap_{i=1}^m H_i^+$ is bounded and full dimensional and $P_{n+1} = \cap_{i=1}^{n+1} H_i^+$ is an n-simplex

Output: Point set V such that $\operatorname{conv} V = P$

1: $V_{n+1} \leftarrow$ set of vertices of P_{n+1}
2: **for** $k = n+2$ to m **do**
3: Construct V_k such that $\operatorname{conv} V_k = P_k = P_{k-1} \cap H_k^+$ as in the lemma
4: **end for**

- The basic algorithm is an improvement on the trivial one.
- The basic algorithm uses points that aren't vertices.
- At each iteration we may have that the points increase quadratically.
- Improvements can be made by noticing that vertices of P_k which are not vertices of P_{k-1} are generated by edges of P_{k-1} that intersect the hyperplane H_k

Edge Detecting Lemma

Definition

Let $W \subset V$ be a point set and define $\mathcal{H}(W) = \{H \mid H = \partial H^+ \text{ for } H \in \mathcal{H}^+ \text{ and } W \subset H\}$. For simplicity we denote $H(\{v, w\})$ as $H(v, w)$

Lemma

Let $(V, \mathcal{H})$ be a double description of an n-polytope P. Given two distinct points $v, w \in V$ the set $aff\{v, w\} \cap P$ is an edge of P if and only if

$$\cap\mathcal{H}(v, w) = aff\{v, w\}.$$

When v, w are vertices then

$$conv\{v, w\} = P \cap (\cap\mathcal{H}(v, w))$$

Finding a Data Structure

- We would like to find the right data structure that allows us to take advantage of lemma
- We would like to change finding dimension to finding the rank of a matrix
- Using homogeneous coordinates allows us to change from affine space to a linear space
- We will need to extend convex hull problem to pointed polyhedron

Homogenizing

We now let P be an n-dimensional point polyhedron and homogenize by considering

$$Q = \{(\lambda, \lambda x) \mid x \in P\}$$

We know $P = \operatorname{conv} V + \operatorname{pos} R$, and so Q can be described as

$$Q = \operatorname{pos}(\{(1, v) \mid v \in V\} \cup \{(0, r) \mid r \in R\})$$

Definition

- We define W to be the set of vectors that generate Q, and we store W as an $(n+1) \times m$ matrix, where the columns are the vectors $w^{(i)}$.
- We define the $k \times (n+1)$ matrix $\mathcal{H}$ to be the matrix whose rows are the linear half-spaces $h^{(i)}$.

Incidence Matrix

Definition

Let $(W, \mathcal{H})$ be a double description of a pointed cone $Q \subset \mathbb{R}^{n+1}$ with $W \in \mathbb{R}^{(n+1)\times m}$ and $\mathcal{H} \in \mathbb{R}^{k\times(n+1)}$. The incidence matrix $I(W, \mathcal{H})$ is defined as

$$I_{ij} = \begin{cases} 1 & \text{if } w^{(j)} \in H_i = \partial H_i^+, i.e., h^{(j)}(w^{(i)}) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Remark

We use $I(W, \mathcal{H})$ to quickly determine the set $\mathcal{H}(w^{(s)}, w^{(t)})$. Then we can calculate the dimension of the intersection of the half-spaces as $n+1$ minus the rank of the submatrix $\mathcal{H}(w^{(s)}, w^{(t)})$

Double Description Algorithm

Input: Matrix $\mathcal{H} \in \mathbb{R}^{k\times(n+1)}$ with row vectors $h^{(1)}, \ldots, h^{(k)}$ such that
$Q = \{x \in \mathbb{R}^{n+1} : \mathcal{H}x \geq 0\}$ is a full-dimensional pointed cone and
$Q_{n+1} := \{x \in \mathbb{R}^{n+1} : h^{(1)}x \geq 0, \ldots, h^{(n+1)}x \geq 0\}$ is a simplicial cone.

Output: Set W of vectors with pos $W = Q$

Let $W_{n+1} \in \mathbb{R}^{(n+1)\times(n+1)}$ be a matrix whose columns positively generate Q_{n+1}.
for $i \leftarrow n+2, \ldots, k$ **do**
  Create W^+_{i-1} from those columns of W_{i-1} that lie on the positive side of $h^{(i)}$ and create W^-_{i-1} from the columns on the negative side.
  if $W^-_{i-1} = \emptyset$ **then**
    $W_i \leftarrow W_{i-1}$
  else
    $X \leftarrow \emptyset$
    foreach Pair (w, w') of columns of W^+_{i-1} and W^-_{i-1} **do**
      if rank $\mathcal{H}_{i-1}(w, w') = n-1$ **then**
        Choose x as generator of the kernel of the matrix $\mathcal{H}'_{i-1}(w, w')$ that consists of the rows of $\mathcal{H}_{i-1}(w, w')$ and $h^{(i)}$.
        $X \leftarrow X \cup \{x\}$
    Let W_i be the matrix consisting of the columns of W_{i-1} without the columns of W^-_{i-1} and enhanced by the column vectors from X.
return W_k

Bad News

- It's hard.
- No globally optimal algorithm is known.
- Difficult to say which algorithm works best on what input.
- It is not known if there exists a polynomial total time algorithm (polynomial in input and output)
- No incremental algorithm can run in polynomial total time

Good News

State of the art algorithms are available in `polymake`, hence OSCAR. Here is a short list on noteable algorithms

- Beneath and beyond, "beneath_beyond" (incremental).
- Double description, "cdd", "ppl"(incremental).
- Pyramid Decomposition, "libnormaliz" (incremental).
- Reverse Search "lrs".

Rules of Thumb

- If you don't know anything try double description.
- If you expect the output to be extremely large and if partial information is useful.
- Use double description when looking for facets of 0/1 polytopes.
- Beneath and beyond often behaves well on random input.

Thank you!

References

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