Universally Free Numerical Semigroups

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- $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^+$, $\mathcal{S} = \{u_1 a_1 + \cdots + u_n a_n \mid u_i \in \mathbb{N}\}$.
- $\deg_A(\mathbf{u}) = u_1 a_1 + \cdots + u_n a_n$.
- A-homogeneous prime ideals of height $n-1$,
  $I_A = \langle x^u - x^v \mid deg_A(x^u) = deg_A(x^v)\rangle$.
- $A' = \{a_1/\gcd(A), \ldots, a_1/\gcd(A)\}$, $I'_A = I_A$.
- $(S) = \langle A' \rangle$ is a numerical semigroup.
- Looks at universally free numerical semigroup and study it's implications.

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- A is the gluing of partitions A_1, A_2 if $\operatorname{lcm}(\gcd(A_1), \gcd(A_2)) \in \langle A_1 \rangle \cup \langle A_2 \rangle$
- $\mathcal{S}$ is free for the arrangement $\{a_1, \ldots, a_n\}$ if for each $\{i, \ldots, n\}$ the set $\{a_i, \ldots a_n\}$ is the gluing of $\{a_i\}$ with $\{a_{i+1}, \ldots a_n\}$
- $\mathcal{C}_A$ set of circuits of I_A

Proposition

Let S be a universally free numerical semigroup then I_A is generated by $n-1$ circuits. And some other more technical properties, involving Betti degree and critical binomials.

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Proposition

If $\mathcal{S}$ is a universally free numerical semigroup then $\mathcal{C}_A \subset \mathcal{M}_A$

Conjecture

$\mathcal{S}$ is a universally free numerical semigroup if and only if $\mathcal{C}_A \subset \mathcal{M}_A$. Large 15 day computation suggesting true.

Theorem

Let $\mathcal{S}$ be a numerical semigroup, $\mathcal{S}$ is universally free if and only if every reduced Gröbner basis of I_A has $n-1$ elements.

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- A degrees of any Markov basis are invariant and called the Betti degrees of I_A.
- If A is a finite set of positive integers, $\mathcal{C}_A \subset \mathcal{U}_A \subset Gr_A$ and $Cr_A \subset \mathcal{M}_A \subset Gr_A$.
- A is Betti divisible if its Betti degrees are ordered by divisibility.
- If $\mathcal{S}$ is a Betti divisible numerical semigroup minimally generated by A then $\mathcal{C}_A = \mathcal{U}_A \subset Cr_A = \mathcal{M}_A = Gr_A$.
- Gives characterization of 3 generated universally free numerical semigroups.