forall$x$ ADELAIDE

forall*x*
Adelaide

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This book is a derivative work created by Antony Eagle, based upon Tim Button's 2016 Cambridge version of P.D. Magnus's forall$x$ (version 1.29). There are pervasive changes, big and small, substantive and cosmetic. You can find the most up to date version of this work at github.com/antonyeagle/forallx-adl.

The most up-to-date version of forall$x$ Cambridge is available at github.com/OpenLogicProject/forallx-cam – this has diverged noticeably from the 2016 version by now. Magnus' original is available at fecundity.com/logic.

Typesetting was carried out entirely in XeLaTeX using the `Memoir` class. The body text is set in Constantia; math is set in Cambria Math; sans serif in Calibri; and monospaced text in Consolas. The style for typesetting proofs is based on `fitch.sty` (v0.4) by Peter Selinger, University of Ottawa.

Contents

How to Use This Book

This book has been designed for use in conjunction with the University of Adelaide courses PHIL 1110 *Introduction to Logic* and PHIL 1111OL *Introduction to Logic Online*. But it is suitable for self-study as well. I have included a number of features to assist your learning.

› `forall`x is divided into seven chapters, each further divided into sections and subsections. The sections are continuously numbered.

  – Chapter 1 gives an overview of how I understand the project of formal logic;
  – Chapters **??**–**??** cover sentential or truth-functional logic;
  – Chapters **??**–**??** cover quantified or predicate logic;
  – and Chapters **??**–**??** cover the formal proof systems for our logical languages.

› The book contains many cross-references to other sections. So a reference to '§**??**' indicates that you should consult section 6, subsection 2 – you will find this on page **??**. Cross-references are hyperlinked, as are entries in the table of contents.

› Figure 1 shows how the sections depend on one another. For example, the arrows coming from §**??** in the diagram show that understanding that section requires familiarity with §**??** and §**??**, and also any sections on which they depend.

› Logical ideas and notation are pretty ubiquitous in philosophy, and there are a lot of different systems. We cannot cover all the alternatives, but some indication of other terminology and notation is contained in Appendix **??**.

› A quick reference to many of the aspects of the logical systems I introduce can be found in Appendix **??**.

› When I first use a new piece of technical terminology, it is introduced by writing it in small caps, LIKE THIS. You can find an index of defined terms in Appendix **??**.

› Each chapter in the book concludes with a box labelled 'Key Ideas in §n'. These are not a summary of the chapter, but contain some indication of what I regard as the main ideas that you should be taking away from your reading of the chapter.

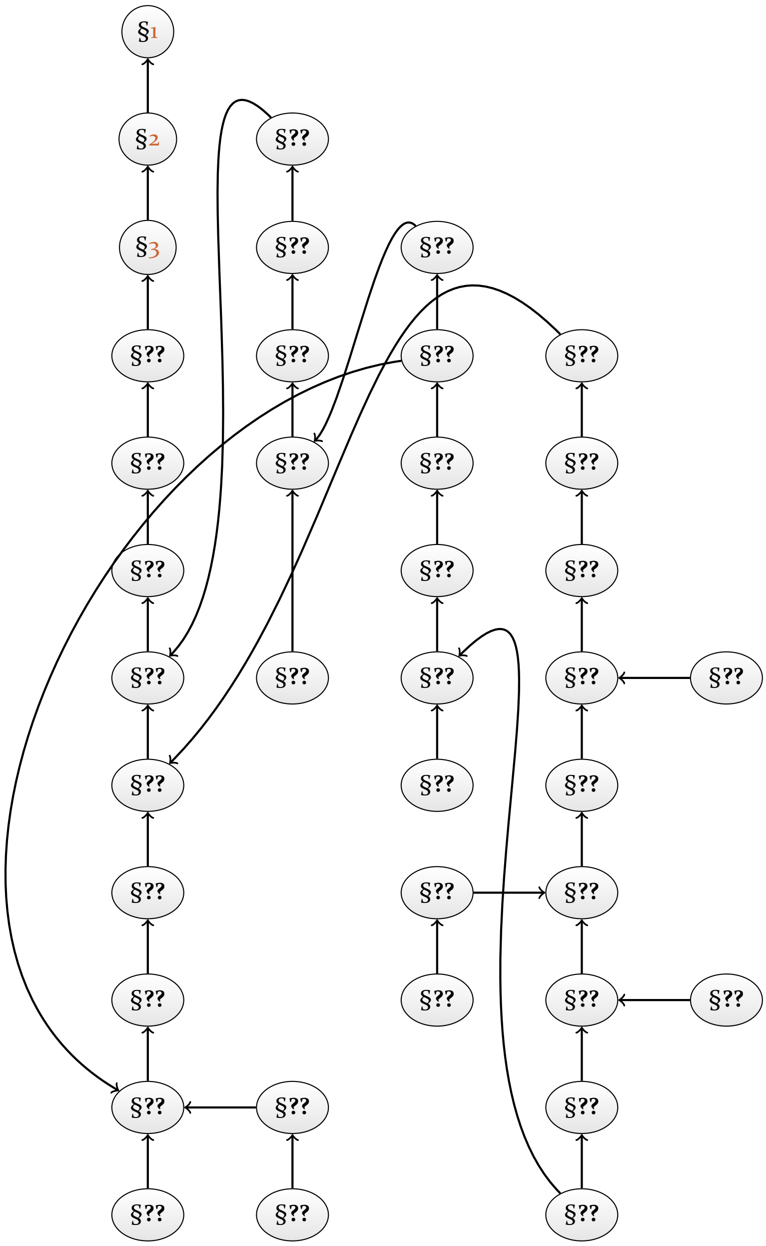

Figure 1: How the sections depend on one another.

› The book is the product of a number of authors. 'I' doesn't always mean me, but 'you' mostly means you, the reader, and 'we' mostly means you and me.

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Chapter 1

Key Notions

1

Arguments

Logic is the business of evaluating arguments – identifying some of the good ones and explaining why they are good. So what is an argument?

In everyday language, we sometimes use the word 'argument' to talk about belligerent shouting matches. Logic is not concerned with such teeth-gnashing and hair-pulling. They are not arguments, in our sense; they are disagreements.

Giving an argument, in the sense relevant to logic (and other disciplines, like law and philosophy), is something more like *making a case*. Giving an ARGUMENT, in our sense, involves presenting reasons that are intended to favour, or support, a specific claim. Consider this example of an argument that someone might give:

It is raining heavily.
If you do not take an umbrella, you will get soaked.
So: You should take an umbrella.

We here have a series of sentences. The word 'So' on the third line indicates that the final sentence expresses the CONCLUSION of the argument. The two sentences before that express PREMISES of the argument. If the argument is well-constructed, the premises provide reasons in favour of the conclusion. In this example, the premises do seem to support the conclusion. At least they do, given the tacit assumption that you do not wish to get soaked.

This is the sort of thing that logicians are interested in. We shall say that an argument is any collection of premises, together with a conclusion.[1]

In the example just given, we used individual sentences to express both of the argument's premises, and we used a third sentence to express the argument's conclusion. Many arguments are expressed in this way. But a single sentence can contain a complete argument. Consider:

I was wearing my sunglasses; so it must have been sunny.

[1] Because arguments are made of sentences, logicians are very concerned with the details of particular words and phrases appearing in sentences. Logic thus also has close connections with linguistics, particularly that subdisipline of linguistics known as SEMANTICS, the theory of meaning.

This argument has one premise followed by a conclusion.

Many arguments start with premises, and end with a conclusion. But not all of them. The argument with which this section began might equally have been presented with the conclusion at the beginning, like so:

> You should take an umbrella. After all, it is raining heavily. And if you do not take an umbrella, you will get soaked.

Equally, it might have been presented with the conclusion in the middle:

> It is raining heavily. Accordingly, you should take an umbrella, given that if you do not take an umbrella, you will get soaked.

When approaching an argument, we want to know whether or not the conclusion follows from the premises. So the first thing to do is to separate out the conclusion from the premises. As a guideline, the following words are often used to indicate an argument's conclusion:

so, therefore, hence, thus, accordingly, consequently

And these expressions often indicate that we are dealing with a premise, rather than a conclusion

since, because, given that

But in analysing an argument, there is no substitute for a good nose.

Key Ideas in §1

- An argument is a collection of sentences, divided into one or more premises and a single conclusion.
- The conclusion may be indicated by 'so', 'therefore' or other expressions; the premises indicated by 'since' or 'because'.
- The premises are supposed to support the conclusion, though whether they do so is another matter.

Practice exercises

At the end of almost every sections, there are practice exercises that review and explore the material covered in the chapter. There is no substitute for actually working through some problems, because logic is more about cultivating a *way of thinking* than it is about memorising facts.

A. What is the difference between argument in the everyday sense, and in the logicians' sense? What is the point of logical arguments?

B. Highlight the phrase which expresses the conclusion of each of these arguments:

1. It is sunny. So I should take my sunglasses.
2. It must have been sunny. I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. You're the culprit!
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. And Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

2

Valid Arguments

In §1, we gave a very permissive account of what an argument is. To see just how permissive it is, consider the following:

	There is a bassoon-playing dragon in Rundle Mall.
So:	Salvador Dali was a poker player.

We have been given a premise and a conclusion. So we have an argument. Admittedly, it is a *terrible* argument. But it is still an argument.

2.1 Two ways that arguments can go wrong

It is worth pausing to ask what makes the argument so weak. In fact, there are two sources of weakness. First: the argument's (only) premise is obviously false. Rundle Mall has some interesting buskers, but not quite *that* interesting. Second: the conclusion does not follow from the premise of the argument. Even if there were a bassoon-playing dragon in Rundle Mall, we would not be able to draw any conclusion about Dali's predilection for poker.

What about the main argument discussed in §1? The premises of this argument might well be false. It might be sunny outside; or it might be that you can avoid getting soaked without taking an umbrella. But even if both premises were true, it does not necessarily show you that you should take an umbrella. Perhaps you enjoy walking in the rain, and you would like to get soaked. So, even if both premises were true, the conclusion might nonetheless be false. (If we were to add the formerly tacit assumption that you do not wish to get soaked as a further premise, then the premises taken together would provide support for the conclusion.)

The general point is as follows. For any argument, there are two ways that it might go wrong:

1. One or more of the premises might be false.
2. The conclusion might not follow from the premises – even if the premises were true, they would not support the conclusion.

To determine whether or not the premises of an argument are true is often a very important matter. But that is normally a task best left to experts in the field: as it might be, historians, scientists, or whomever. In our role as *logicians*, we are more concerned with arguments *in general*. So we are (usually) more concerned with the second way in which arguments can go wrong.

2.2 Conclusive arguments

As logicians, we want to be able to determine when the conclusion of an argument follows from the premises. One way to put this is as follows. We want to know whether, if all the premises were true, the conclusion would also have to be true. This motivates a definition:

> An argument is CONCLUSIVE if, and only if, the truth of the premises guarantees the truth of the conclusion.
> In other words: an argument is conclusive if, and only if: it is not possible for the premises of the argument to be true while the conclusion is false.

Consider another argument:

You are reading this book.
This is a logic book.
So: You are a logic student.

This is not a terrible argument. Both of the premises are true. And most people who read this book are logic students. Yet, it is possible for someone besides a logic student to read this book. If your housemate picked up the book and thumbed through it, they would not immediately become a logic student. So the premises of this argument, even though they are true, do not guarantee the truth of the conclusion. This is not a conclusive argument.

The crucial thing about a conclusive argument is that it is impossible, in a very strict sense, for the premises to be true whilst the conclusion is false. Consider this example:

Oranges are either fruits or musical instruments.
Oranges are not fruits.
So: Oranges are musical instruments.

The conclusion of this argument is ridiculous. Nevertheless, it follows from the premises. *If* both premises were true, *then* the conclusion would just have to be true. So the argument is conclusive.

Why is this argument conclusive? The most important factor for us in considering what makes an argument conclusive is to examine the argument's STRUCTURE – the grammatical forms of the premises and conclusion. An argument will be conclusive if its structure *guarantees* that its premises *support* the conclusion. In the present case, one premise says that oranges are in one of two categories; the other premise says that oranges are not in the first category. We conclude that they are in the second category. The premises and conclusion are about oranges. But it is plausible to think that *any* argument with this same sort of structure must be conclusive, whether we are talking about oranges, or cars – or anything really.

2.3 Reasons to believe

A conclusive argument, in the logician's sense, links the premises to the conclusion. It turns the reasons you have for accepting to the premises into reasons to accept its conclusion. But a conclusive argument need not provide you with a reason to believe the conclusion. One way this can happen is when you don't accept any of the premises in the first place. When the premises support the conclusion, that might just mean that they *would* be excellent reasons to accept the conclusion – *if only they were true*!

So: we are interested in whether or not a conclusion *follows from* some premises. Don't, though, say that the premises *infer* the conclusion. Entailment is a relation between premises and conclusions; inference is something we do. So if you want to mention inference when the conclusion follows from the premises, you could say that *one may infer* the conclusion from the premises.

But even this may be doubted. Often, when you believe the premises, a conclusive argument provides you with a reason to believe the conclusion. In that case, it might be appropriate for you to infer the conclusion from the premises.

But sometimes a conclusive argument shows that some premises support a conclusion you cannot accept. Suppose, for example, that you know the conclusion to be false. The fact that the argument is conclusive and has a false conclusion tells us that the premises cannot all be true. (Consider the argument from the previous section with the false conclusion 'Oranges are musical instruments': the second premise is as absurd as the conclusion.) In general, when an argument is conclusive

› the truth of all the premises guarantees the truth of the conclusion; and equally

› the falsity of the conclusion guarantees the falsity of at least one of the premises.

In this sort of situation, you might find that the argument gives you a better reason to abandon any belief in one of the premises than to accept the conclusion. A conclusive argument shows there is *some* reason to believe its conclusion, if you accept its premises; it doesn't mean there aren't *better* reasons to reject its premises, if you reject its conclusion.

The question, *what ought I to believe?* is one of the deepest in the area of philosophy known as EPISTEMOLOGY, the theory of knowledge. Logic is not able to answer that question all by itself. Even if logic tells you that there is a conclusive argument from premise $\mathcal{A}$ to conclusion $\mathcal{B}$, logic can't tell you whether you ought to believe both, or reject both.[1] However, logic will tell you something important, even if it is only a limited part of the answer to the question of rational belief. It will tell you that, when you know an argument to be conclusive, *you cannot both accept its premises while rejecting its conclusion*.

2.4 Conclusiveness for special reasons

An argument can be conclusive for reasons unrelated to its structure. Take this example:

Juanita is a vixen.
So: Juanita is a fox.

[1] This point is made forcefully by Gilbert Harman, *Change in View*, MIT Press, esp. ch. 2.

It is impossible for the premise to be true and the conclusion false. So the argument is conclusive. But this is not due to the structure of the argument. Here is an inconclusive argument with seemingly the same structure or form. The new argument is the result of replacing the word 'fox' in the first argument with the word 'cathedral', but keeping the overall grammatical structure the same:

Juanita is a vixen.
So: Juanita is a cathedral.

This might suggest that the conclusiveness of the first argument *is* keyed to the meaning of the words 'vixen' and 'fox'. But, whether or not that is right, it is not simply the FORM of the argument that makes it conclusive. Equally, consider the argument:

The sculpture is green all over.
So: The sculpture is not red all over.

Again, because nothing can be both green all over and red all over, the truth of the premise would guarantee the truth of the conclusion. So the argument is conclusive. But here is an inconclusive argument with the same form:

The sculpture is green all over.
So: The sculpture is not shiny all over.

The argument is inconclusive, since it is possible to be green all over and shiny all over. (I might paint my nails with an elegant shiny green varnish.) Plausibly, the conclusiveness of this argument is keyed to the way that colours (or colour-words) interact. But, whether or not that is right, it is not simply the form of the argument that makes it conclusive.

An argument can be conclusive due to its structure, and also be conclusive for other reasons. Arguably, this might be going on in the argument discussed at the end of §2.2, with the premise 'Oranges are not fruits'. Some people might think this premise has to be false, because of what oranges are. (Many will say that *being a fruit* is an essential part of what it is to be an orange.) But if the premise 'Oranges are not fruits' has to be false, it is not possible for the premises to be true. So it is not possible for premises to be true *while* the conclusion is false. Hence the argument is conclusive – both because it has a good structure, but also because it has a premsie that cannot be true.[2]

2.5 Validity

Logicians try to steer clear of controversial matters like whether there is a definition of an orange that requires it to be a fruit, or whether there is a 'connection in meaning' between being green and not being red. It is often difficult to figure such things out from the armchair (a logician's preferred habitat), and there may be widespread disagreement even among subject matter experts.

[2] When an argument has an impossible premise, any argument with that premise will be conclusive *no matter what* the conclusion is! So this is a weird kind of case of conclusiveness. But nothing much really turns on it, and it is simpler to simply count it as conclusive than to try and separate out such 'degenerate' cases of conclusive arguments. See also §3.3.

So logicians do not study conclusive arguments in general, but rather concentrate on those conclusive arguments which have a good structure or form.[3] This is why the logic we are studying is sometimes called FORMAL LOGIC. We introduce a special term for the class of arguments logicians are especially interested in:

> An argument is VALID if, and only if, it is conclusive due to its structure; otherwise it is INVALID.

The notion of the structure of a sentence, or an argument, is an intuitive one. I make the notion more precise in §??. Relying on our intuitive grasp of the notion for now, however, we can see the argument about ogres on the right has the same form as the argument on the left about oranges (slightly tweaked from our earlier presentation in §2.2 to make its structure clearer). It is easy to see that both of these arguments are conclusive and valid:

	Either Oranges are fruits or oranges are musical instruments.		Either Ogres are fearsome or ogres are mythical.
	It is not the case that Oranges are fruits.		It is not the case that Ogres are fearsome.
So:	Oranges are musical instruments.	So:	Ogres are mythical.

The shared structure of these two arguments is something like this:

Either $\mathcal{A}$ or $\mathcal{B}$.
It is not the case that $\mathcal{A}$.
So: $\mathcal{B}$.

Any argument with this structure will be conclusive in virtue of structure, and hence valid. It does not matter, really, what sentences we put in place of '$\mathcal{A}$' and '$\mathcal{B}$'. (Within limits: you can't put a question or an exclamation and get a valid argument – see §3.1.)

This highlights that valid arguments do not need to have true premises or even true conclusions. We can put a true sentence in place of $\mathcal{A}$ and a false sentence in place of $\mathcal{B}$, and both premises and the conclusion will be false. The argument is still valid.

Conversely, having true premises and a true conclusion is not enough to make an argument valid. Consider this example:

London is in England.
Beijing is in China.
So: Paris is in France.

The premises and conclusion of this argument are, as a matter of fact, all true. But the argument is invalid. If Paris were to declare independence from the rest of France, then

[3] It can be very hard to tell whether an invalid argument is conclusive or inconclusive. Consider the argument 'The sea is full of water; so the sea is full of H_2O'. This is conclusive, since water just is the same stuff as H_2O. The sea cannot be full of that water stuff without being full of that exact same stuff, namely, H_2O stuff. But it took a lot of chemistry and ingenious experiments to figure out that water is H_2O. So it was not at all obvious that this argument was conclusive. On the other hand, it is generally very clear when an argument is conclusive due to its structure – you can just see the structure when the argument is presented to you.

the conclusion would be false, even though both of the premises would remain true. Thus, it is *possible* for the premises of this argument to be true and the conclusion false. The argument is therefore inconclusive, and hence invalid.

2.6 Soundness

The important thing to remember is that validity is not about the actual truth or falsity of the sentences in the argument. It is about whether the structure of the argument ensures that the premises support the conclusion. Nonetheless, we shall say that an argument is SOUND if, and only if, it is both valid and all of its premises are true. So every sound argument is valid and conclusive. But not every valid argument is sound, and not every conclusive argument is sound.

It is often possible to see that an argument is valid even when one has no idea whether it is sound. Consider this extreme example (after Lewis Carroll's *Jabberwocky*):

'Twas brillig, and the slithy toves did gyre and gimble in the wabe.
So: The slithy toves did gyre and gimble in the wabe.

This argument is valid, simply because of its structure (it has a premise conjoining two claims by 'and', and a conclusion which is one of those claims). But is it sound? That would depend on figuring out what all those nonsense words mean!

2.7 Inductive arguments

Many good arguments are inconclusive and invalid. Consider this one:

In January 1997, it rained in London.
In January 1998, it rained in London.
In January 1999, it rained in London.
In January 2000, it rained in London.
So: It rains every January in London.

This argument generalises from observations about several cases to a conclusion about all cases. Such arguments are called INDUCTIVE arguments. The argument could be made stronger by adding additional premises before drawing the conclusion: In January 2001, it rained in London; In January 2002.... But, however many premises of this form we add, the argument will remain inconclusive. Even if it has rained in London in every January thus far, it remains *possible* that London will stay dry next January.

The point of all this is that inductive arguments – even good inductive arguments – are not (deductively) valid. They are not *watertight*. Unlikely though it might be, it is *possible* for their conclusion to be false, even when all of their premises are true. In this book, we shall set aside (entirely) the question of what makes for a good inductive argument. Our interest is simply in sorting the (deductively) valid arguments from the invalid ones.

2.8 Making conclusive arguments valid

Some arguments which are conclusive but invalid can be turned into valid arguments. So consider again the argument 'The sculpture is green all over; therefore it is not red all over'. We can make a valid argument from this by adding a premise:

The sculpture is green all over.
If the sculpture is green all over, then it is not red all over.
So: The sculpture is not red all over.

This new argument has a premise which makes explicit a fact about green and red that was merely implicit in the original argument. Since the original argument was conclusive – since the fact about green and red is true just in virtue of the meaning of the words 'green' and 'red' (and 'not') – the new argument remains conclusive. (We can't undermine conclusiveness by adding further premises.) But the new argument is valid, because the additional premise we have added yields an argument with a structure that guarantees the truth of the conclusion, given the truth of the premises.

The original argument is sometimes thought to be merely an abbreviation of the expanded valid argument. An argument with an unstated premise, such that it can be seen to be valid when the premise is made explicit, is called an ENTHYMEME.[4] Many inconclusive arguments can be treated as enthymematic, if the unstated premise is obvious enough:

The Nasty party platform includes imprisoning people for chewing gum;
So: The Nasty party will not form the next government.

The unstated premise is something like 'If a party platform includes imprisoning people for chewing gum, then that party will win too few votes to form the next government'. The unstated premise may or may not be true. But if it is added, the argument is made valid.

Any conclusive argument you are likely to come across will either already be valid, or can be transformed into a valid argument by making some assumption on which it implicitly relies into an explicit premise.

Not every inconclusive argument should be treated as an enthymeme. In particular, many strong inductive arguments can be made weaker when they are treated as enythymematic. Consider:

In January 2017, it was hot in Adelaide.
In January 2018, it was hot in Adelaide.
So: In January 2019, it will be hot in Adelaide.

This argument is inconclusive. It can be made valid by adding the unstated premise 'Every January, it is hot in Adelaide'. But that unstated premise is extremely strong – we do not have sufficient evidence to conclude that it will be hot in Adelaide in January *for eternity*. So while the premises we have been given explicitly are good reason to think Adelaide will continue to have a hot January next year, we do not have good enough reason to think that every January will be hot. Treating the argument as an enthymeme makes it valid,

[4] The term is from ancient Greek; the concept was given its first philosophical treatment by Aristotle in his *Rhetoric*. He gives this example, among others: 'He is ill, since he has fever'.

but also makes it less persuasive, since the unstated premise on which it relies is not one many people will share.[5]

Key Ideas in §2

- An argument is conclusive if, and only if, the truth of the premises guarantees the truth of the conclusion.
- An argument is valid if, and only if, the form of the premises and conclusion alone ensures that it is conclusive. Not every conclusive argument is valid (though they can be made valid by addition of appropriate premises).
- An argument can be good and persuade us of its conclusion even if it is not conclusive; and we can fail to be persuaded of the conclusion of a conclusive argument, since one might come to reject its premises.

Practice exercises

A. What is a *conclusive* argument? What, in addition to being conclusive, is required for an argument to be *valid*? What, in addition to being valid, is required for an argument to be *sound*?

B. Which of the following arguments are valid? Which are invalid but conclusive? Which are inconclusive? Comment on any difficulties or points of interest.

1. Socrates is a man.
2. All men are carrots.

So: Therefore, Socrates is a carrot.

1. Abe Lincoln was either born in Illinois or he was once president.
2. Abe Lincoln was never president.

So: Abe Lincoln was born in Illinois.

1. Abe Lincoln was the president of the United States.

So: Abe Lincoln was a citizen of the United States.

1. If I pull the trigger, Abe Lincoln will die.
2. I do not pull the trigger.

So: Abe Lincoln will not die.

1. Abe Lincoln was either from France or from Luxembourg.
2. Abe Lincoln was not from Luxembourg.

So: Abe Lincoln was from France.

[5] If we had claim that the unstated premise was rather ‘If it has been hot in January for the previous two years, it will be hot the following year’, we would have had a valid argument and one that has a plausible unstated premise – though the unstated premise seems to be plausible only because the unamended inductive argument was fine to begin with!

1. If the world ends today, then I will not need to get up tomorrow morning.
2. I will need to get up tomorrow morning.

So: The world will not end today.

1. Joe is right now 19 years old.
2. Joe (the same one) is also right now 87 years old.

So: Bob is now 20 years old.

C. Could there be:

1. A valid argument that has one false premise and one true premise?
2. A valid argument that has only false premises?
3. A valid argument with only false premises and a false conclusion?
4. A sound argument with a false conclusion?
5. An invalid argument that can be made valid by the addition of a new premise?
6. A valid argument that can be made invalid by the addition of a new premise?

In each case: if so, give an example; if not, explain why not.

3

Other Logical Notions

In §2, we introduced the idea of a valid argument. We will want to introduce some more ideas that are important in logic.

3.1 Truth values

As we said in §1, arguments consist of premises and a conclusion, where the premises are supposed to support the conclusion. But if premises are supposed to state reasons, and conclusions are supposed to state claims, then both of them have to be the sort of sentence which can be used to *say how things are*, truly or falsely.

So many kinds of English sentence cannot be used to express premises or conclusions of arguments. For example:

- **Questions**, e.g., 'are you feeling sleepy?'
- **Imperatives**, e.g., 'Wake up!'
- **Cohortatives**, e.g., 'Let's go to the beach!'
- **Exclamations**, e.g., 'Ouch!'

The common feature of these three kinds of sentence is that they cannot be used to make *assertions*: they cannot be true or false. It does not even make sense to ask whether a *question* is true (it only makes sense to ask whether the *answer* to a question is true).

The general point is that the premises and conclusion of an argument must be DECLARATIVE SENTENCES, capable of having a TRUTH VALUE. And the two truth values that concern us are just True and False. We need not know what the truth value of a sentence is for it to form part of an argument, but it must have the kind of grammatical structure that permits it to have a truth value.

3.2 Consistency

Consider these two sentences:

B1. Jane's only brother is shorter than her.

B2. Jane's only brother is taller than her.

Logic alone cannot tell us which, if either, of these sentences is true. Yet we can say that *if* the first sentence (B1) is true, *then* the second sentence (B2) must be false. And if B2 is true, then B1 must be false. It is impossible that both sentences are true together. These sentences are inconsistent with each other. And this motivates the following definition:

> Sentences are JOINTLY CONSISTENT if, and only if, it is possible for them all to be true together.

Conversely, B1 and B2 are JOINTLY INCONSISTENT.

We can ask about the consistency of any number of sentences. For example, consider the following four sentences:

G1. There are at least four giraffes at the wild animal park.

G2. There are exactly seven gorillas at the wild animal park.

G3. There are not more than two martians at the wild animal park.

G4. Every giraffe at the wild animal park is a martian.

G1 and G4 together entail that there are at least four martian giraffes at the park. This conflicts with G3, which implies that there are no more than two martian giraffes there. So the sentences G1–G4 are jointly inconsistent. They cannot all be true together. (Note that the sentences G1, G3 and G4 are jointly inconsistent. But if some sentences are already jointly inconsistent, adding an extra sentence to the mix will not make them consistent!)

There is an interesting connection between consistency and conclusive arguments. A conclusive argument is one where the premises guarantee the truth of the conclusion. So it is an argument where if the premises are true, the conclusion must be true. So the premises cannot be jointly consistent with the claim that the conclusion is false. Since the argument 'Dogs and cats are animals, so dogs are animals' is conclusive, that shows that the sentences 'Dogs and cats are animals' and 'Dogs are **not** animals' are jointly inconsistent. If an argument is conclusive, the premises of the argument taken together with the denial of the conclusion will be jointly inconsistent.

We just linked consistency to conclusive arguments. There is an analogous notion linked to valid arguments:

> Sentences are JOINTLY FORMALLY CONSISTENT if, and only if, considering only their structure, they can all be true together.

If some sentences are jointly consistent, they are also jointly formally consistent. But some formally consistent sentences are jointly inconsistent. Any conclusive but invalid argument will give us an example. Since 'The sculpture is green all over, so the sculpture is not red all over' is conclusive, the sentences 'The sculpture is green' and 'The sculpture is red' are jointly inconsistent. But those sentences are formally consistent – their inconsistency depends crucially on the meanings of the words 'red' and 'green', not on the overall structure of the sentences in which those words appear. (Again, more on the notion of structure invoked here in §**??**.)

3.3 Necessity and contingency

In assessing whether an argument is conclusive, we care about what would be true *if* the premises were true. But some sentences just *must* be true. Consider these sentences:

1. It is raining.
2. If it is raining, water is precipitating from the sky.
3. Either it is raining here, or it is not.
4. It is both raining here and not raining here.

In order to know if sentence 1 is true, you would need to look outside or check the weather channel. It might be true; it might be false.

Sentence 2 is different. You do not need to look outside to know that it says something true. Regardless of what the weather is like, if it is raining, water is precipitating – that is just what rain *is*, metereologically speaking. That is a NECESSARY TRUTH. Here, a necessary connection in meaning between 'rain' and 'precipitation' makes what the sentence says true in every circumstance.

Sentence 3 is also a necessary truth. Unlike sentence 2, however, it is the structure of the sentence which makes it necessary. No matter what 'raining here' means, 'Either it is raining here or it is not raining here' will be true. The structure 'Either it is … or it is not …', where both gaps ('…') are filled by the same phrase, must be a true sentence.

Equally, you do not need to check the weather, or even the meaning of words, to determine whether or not sentence 4 is true. It must be false, simply as a matter of structure. It might be raining here and not raining across town; it might be raining now but stop raining even as you finish this sentence; but it is impossible for it to be both raining and not raining in the same place and at the same time. So, whatever the world is like, it is not both raining here and not raining here. It is a NECESSARY FALSEHOOD.

A sentence which is capable of being true or false, but which says something which is neither necessarily true nor necessarily false, is CONTINGENT.

If a sentence says something which is sometimes true and sometimes false, it will definitely be contingent. But something might *always* be true and still be contingent. For instance, it seems plausible that whenever there have been people, some of them habitually arrive late. 'Some people are habitually late' is always true. But it is contingent, it seems: human nature could have been more punctual. If so, the sentence would have been false. But if something is really necessary, it will always be true.[1]

[1] Here's an interesting example to consider. It seems that, whenever anyone says the sentence 'I am here now', they say something true. That sentence is, whenever it is uttered, *truly uttered*. But does it say something necessary or contingent?

If some sentences contain amongst themselves a necessary falsehood, those sentences are jointly inconsistent. At least one of them cannot be true, so they cannot all be true together. Accordingly, if an argument has a premise that is a necessary falsehood, or its conclusion is a necessary truth, or both, then the argument is conclusive – its premises and the denial of its conclusion will be jointly inconsistent.

Key Ideas in §3

- Arguments are made up of declarative sentences, all of which are either true or false.
- Some declarative sentences are formally consistent if, and only if, their structures don't rule out the possibility that they are all true together.
- Some declarative sentences can only have one truth value – they are either necessary or impossible. Others are contingent, having one truth value in some circumstances and the other truth value in other circumstances.

Practice exercises

A. Which of the following sentences are capable of being true or false?

1. Earth is the third planet from the Sun.
2. Pluto is the ninth planet from the Sun.
3. Have you been feeding the lions?
4. Socrates said, 'Be as you wish to seem'.
5. 'Have you been feeding the lions?' is a sentence.
6. Always forgive your enemies; nothing annoys them so much.

B. For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon.
2. Someone once crossed the Rubicon.
3. No one has ever crossed the Rubicon.
4. If Caesar crossed the Rubicon, then someone has.
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon.
6. If anyone has ever crossed the Rubicon, it was Caesar.

C. Look back at the sentences G1–G4 in this section (about giraffes, gorillas and martians in the wild animal park), and consider each of the following:

1. G2, G3, and G4
2. G1, G3, and G4
3. G1, G2, and G4
4. G1, G2, and G3

Which are jointly consistent? Which are jointly inconsistent?

D. Could there be:

1. A conclusive argument, the conclusion of which is necessarily false?
2. An inconclusive argument, the conclusion of which is necessarily true?
3. Jointly consistent sentences, one of which is necessarily false?
4. Jointly inconsistent sentences, one of which is necessarily true?

In each case: if so, give an example; if not, explain why not.

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In the Introduction to his *Symbolic Logic*, Charles Lutwidge Dodson advised:

> When you come to any passage you don't understand, *read it again*: if you *still* don't understand it, *read it again*: if you fail, even after *three* readings, very likely your brain is getting a little tired. In that case, put the book away, and take to other occupations, and next day, when you come to it fresh, you will very likely find that it is *quite* easy.

The same might be said for this volume, although readers are forgiven if they take a break for snacks after *two* readings.