Chapter 4

Quantum states of light

4.1 Single mode

The quantized light field can be in different states. We start here with a single mode of the field. This may be a oversimplification, but single-mode fields have become part of the experimental reality with the advent of high-quality optical cavities. These devices give an electromagnetic field whose amplitude, in the region between two well-reflecting mirrors, is much higher at some resonant frequencies. The 'mode function' is in this case not a plane wave, of course, but a standing wave. In the transverse directions, one often has a gaussian profile. Around a cavity resonance, it is a frequent approximation to treat the full field as if it contained only a single mode. The coupling to other modes may be taken into account as a loss.

The electric field is given by

$$\mathbf{E}(\mathbf{x},t) = E_1 \boldsymbol{\varepsilon} \left(a(t) + a^\dagger(t) \right) \sin kz \tag{4.1}$$

where z is the coordinate along the cavity axis and $k = n_z \pi / L$. The factor E_1 can be called the 'electric field per photon'. From the multimode expansion, E_1 is given by the prefactor $E_k = (\hbar \omega_k / 2\varepsilon_0 V)^{1/2}$. The corresponding 'intensity' is

$$I_{1\mathrm{ph}} = \varepsilon_0 c E_k^2 = \frac{\hbar \omega_k c}{2V}. \tag{4.2}$$

In a cavity, we can take for V the volume 'filled' by the mode. For a transverse mode size of 1 micrometer and a cavity length of 1 cm, we get $I_{1\mathrm{ph}} \sim 10^3\,\mathrm{mW/cm^2}$ which is not really small. The total power, however, is quite small:

about $10^{-8}\,\mathrm{W}$. Note also that these numbers are based on very 'tight' (diffraction-limited) focussing — beams with larger cross-section have a smaller 'field per photon'.

In the Heisenberg picture, the field operator evolves as

$$\mathbf{E}(\mathbf{x},t) = E_1 \boldsymbol{\varepsilon} \left(a\,\mathrm{e}^{-\mathrm{i}\omega t} + a^\dagger\,\mathrm{e}^{-\mathrm{i}\omega t} \right) \sin kz \tag{4.3}$$

A combination of annihilation and creation operators like the one in parentheses is called a 'quadrature'. Quadratures always come in pairs. One can find a second quadrature variable by shifting the origin of time by one quarter period: $\propto -\mathrm{i}\,a\,\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{i}\,a^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}$. This corresponds to the magnetic field [compare eqs. (2.119) and (2.120)]. In analogy to the harmonic oscillator, one often uses the following quadrature variables

$$X = \frac{a + a^\dagger}{\sqrt{2}} \qquad P = \frac{a - a^\dagger}{\sqrt{2}\,\mathrm{i}} \tag{4.4}$$

or more generally

$$X_\theta = \frac{a\,\mathrm{e}^{-\mathrm{i}\theta} + a^\dagger\,\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{2}} \tag{4.5}$$

with $X_0 = X$ and $X_{\pi/2} = P$.

The ground state of the field mode is called the 'vacuum' (no photon, i.e., no excitation present). It is found by looking for the state that is annihilated by the annihilation operator: $a|\mathrm{vac}\rangle = 0$. Obviously, this is also an eigenstate of the photon number operator with zero photons: $|\mathrm{vac}\rangle = |0\rangle$. In the vacuum state, the electric field is also zero on average, of course.

But there are fluctuations around this average, called 'quantum noise'. In the vacuum state of the single mode (4.1), e.g., we get

$$\langle \mathbf{E}(\mathbf{x},t)^2 \rangle_0 = E_1^2 \sin^2 kz\, \langle 0| \left(a(t) + a^\dagger(t) \right) \left(a(t) + a^\dagger(t) \right) |0\rangle \tag{4.6}$$

and this combination of operators gives an average

$$\langle 0| \left(a(t) + a^\dagger(t) \right) \left(a(t) + a^\dagger(t) \right) |0\rangle = \langle 0|a(t)a^\dagger(t)|0\rangle = 1 \tag{4.7}$$

The 'vacuum noise' in our mode is thus given by the squared single photon field $E_1^2 \sin^2 kz$. Similarly, the other quadrature variable $a(t) - a^\dagger(t)$ shows a noise strength of unity. This is in accordance with Heisenberg's indeterminacy relation, since

$$\left[a(t) + a^\dagger(t),\, a(t) - a^\dagger(t) \right] = -2. \tag{4.8}$$

4.1.1 Number states

The simplest quantum states of the single mode field are given by the well-known stationary states of the harmonic oscillator. These quantum states are called 'Fock states' or 'number states' $|n\rangle$. They are eigenstates of the 'photon number operator'

$$\hat{n} = a^\dagger a = aa^\dagger - 1 \tag{4.9}$$

and are generated by applying the creation operator to the ground state of the oscillator:

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{4.10}$$

The expectation value of the annihilation operator is zero in a number state:

$$\langle a\rangle_n = \langle n|a|n\rangle = \sqrt{n}\langle n|n-1\rangle = 0 \tag{4.11}$$

The same is true for the creation operator. It follows that the electric field average vanishes not only in the vacuum state, but in any Fock state:

$$\langle \mathbf{E}(\mathbf{x},t)\rangle_n = 0 \tag{4.12}$$

In the exercises, you are asked to compute the variances of the quadrature operators X_θ in an arbitrary number state $|n\rangle$.

The quantum numbers n give an intuitive interpretation to the creation and annihilation operators: they connect states whose photon numbers differ by one. In this sense, the 'creation operator' $a^\dagger$ creates one photon since for example

$$\langle 1|a^\dagger|0\rangle = 1. \tag{4.13}$$

This matrix element plays an important role when one computed the probability amplitude that an excited atomic state emits a photon. For stimulated emission, one needs $\langle n+1|a^\dagger|n\rangle = \sqrt{n+1}$. Similarly, the 'annihilator' a destroys one photon:

$$\langle 0|a|1\rangle = 1. \tag{4.14}$$

This matrix element is needed to compute absorption, and in the general case, $\langle n-1|a|n\rangle = \sqrt{n}$.

4.1.2 Thermal states

This class of field states is more general than the 'pure' states described before. Strictly speaking, they are not "states", but density operators. The thermal state is the first example where one has to use both classical and quantum statistics, and this is achived with the concept of *density operator* that combines the two.

Density operators

A *density operator* is a hermitean operator $\hat{\rho}$ on the Hilbert space $\mathcal{H}$ of the quantum system under consideration, with the properties

- ρ is positive, i.e., $\langle\psi|\hat{\rho}|\psi\rangle \geq 0$ for all $\psi \in \mathcal{H}$
- ρ is a trace class operator, i.e., $\operatorname{tr}\hat{\rho} = \sum_n \langle n|\hat{\rho}|n\rangle = 1$ where the vectors $|n\rangle$ form a basis of $\mathcal{H}$.

It is easy to see the inequality $0 \leq \langle\psi|\hat{\rho}|\psi\rangle \leq 1$ for a normalized state vector. Physically, this means that this the real number can be interpreted as a probability: it is the probability to find the system in the state $|\psi\rangle$ when performing a measurement.

The expectation value of an operator A is now given by the rule

$$\langle A\rangle_{\hat{\rho}} = \operatorname{tr}(A\hat{\rho}) = \operatorname{tr}(\hat{\rho}A) \tag{4.15}$$

where the order under the trace can be changed because of cyclic permutations.

In a sense, thermal quantum states are a natural generalization of classical thermodynamics to the quantum world. One uses stationary states, hence the number states we found first, and imposes Boltzmann statistics to describe the field at thermal equilibrium.

For the single field mode we are discussing here, stationary states are the number states $|n\rangle$; they occur with a classical probability proportional to the Boltzmann factor $\mathrm{e}^{-n\hbar\omega/k_BT}$. The density operator is given by

$$\hat{\rho} = \frac{1}{Z}\sum_{n=0}^{\infty} \mathrm{e}^{-n\hbar\omega/k_BT}|n\rangle\langle n| \tag{4.16}$$

The normalization factor Z is found by requiring that the trace of this operator be unity:

$$Z = \operatorname{tr}\left(\sum_{n=0}^{\infty} \mathrm{e}^{-n\hbar\omega/k_BT}|n\rangle\langle n|\right) = \sum_{n=0}^{\infty} \mathrm{e}^{-n\hbar\omega/k_BT} = \frac{1}{1-\mathrm{e}^{-\hbar\omega/k_BT}}, \tag{4.17}$$

where a geometric series has been summed. You know this sum from classical thermodynamics as 'partition function' (*Zustandssumme*). The normalized probabilities

$$p_n(T) = (1-\mathrm{e}^{-\hbar\omega/k_BT})\mathrm{e}^{-n\hbar\omega/k_BT} \tag{4.18}$$

are simply the classical probability that the stationary state $|n\rangle$ is realized in the canonical ensemble.

We note that the terms $|n\rangle\langle n|$ in the sum (4.16) are also density operators: they are obviously positive and have trace unity. (In fact, the trace boils down to the norm squared of the state $|n\rangle$.) The thermal density operator is thus a probability-weighted, convex sum of density operators.[1] This convex summation is, in general, an allowed linear operation on the space of density operators.

The density operators $|n\rangle\langle n|$ are special because they are made up of a single state. These quantum states are called *pure*. A formal definition:

- A density operator $\hat{\rho}$ describes a *pure state* if $\hat{\rho}^2 = \hat{\rho}$.

In mathematics, operators with this property are called *projectors*. This is also what is suggested by the Dirac notation $|\psi\rangle\langle\psi|$: this operator acts on the Hilbert space by first projecting onto the state $|\psi\rangle$ and then gives back a vector proportional to $|\psi\rangle$, just what happens in geometry for the projection onto a vector.

Purity

States that are not pure are called 'mixed'. This can be made more quantitative:

- The *purity* of a density operator $\hat{\rho}$ is defined by

$$\mathrm{Pu}(\hat{\rho}) = \mathrm{tr}\left(\hat{\rho}^2 - \hat{\rho}\right) + 1 = \mathrm{tr}\,\hat{\rho}^2 \tag{4.19}$$

  where the two definitions are equal if $\hat{\rho}$ is trace-normalized.

The purity is normalized such that for pure states, $\mathrm{Pu}(\hat{\rho}) = 1$. It is easy to see that the first term in (4.19), $\mathrm{tr}\,(\hat{\rho}^2 - \hat{\rho})$ is negative for mixed states. One can also show that all states with $\mathrm{Pu}(\hat{\rho}) = 1$ are necessarily pure.

To show these properties, evaluate the trace of $\hat{\rho}^2$ in the eigenbasis of $\hat{\rho}$. All eigenvalues are in the interval $[0, 1]$.

A thermal field mode

At optical frequencies and room temperature, the Boltzmann factor $\exp(-n\hbar\omega/k_BT)$ has a large negative argument for $n \geq 1$ so that the field is essentially at zero temperature. This is different for microwave radiation, e.g., or for star atmospheres.

[1] One talks about a *convex sum* if all coefficients are real numbers between zero and one.

Simple exercise: mean photon number. Let us apply the general rule (4.15):

$$\langle \hat{n} \rangle_T = \mathrm{tr}(\hat{n}\hat{\rho}_T) = \frac{1}{Z} \sum_{n=0}^{\infty} \langle n | \hat{n} \exp(-\hbar\omega \hat{n}/k_B T) | n \rangle \tag{4.20}$$

The number operators and the Boltzmann 'operator' act on their eigenvectors, hence

$$\langle \hat{n} \rangle_T = \frac{1}{Z} \sum_{n=0}^{\infty} n \exp(-\hbar\omega n/k_B T) = \frac{1}{\mathrm{e}^{\hbar\omega/k_B T} - 1}. \tag{4.21}$$

Exercise: photon number variance. Result:

$$(\Delta n)_T^2 = \frac{\mathrm{e}^{\hbar\omega/k_B T}}{\left(\mathrm{e}^{\hbar\omega/k_B T} - 1\right)^2} = \frac{1}{4 \sinh^2(\hbar\omega/2k_B T)}. \tag{4.22}$$

Exercise: discuss the purity $\mathrm{Pu}(T) = \mathrm{Pu}(\hat{\rho}_T)$. It is a function that goes to zero linearly when $\omega/T \to 0$ ('hot' or 'classical' limit) and reaches asymptotically unity for $\omega/T \to \infty$ ('cold' or 'quantum' limit).

Electric field fluctuations in a single mode at finite temperature:

$$\langle \mathbf{E}^2(\mathbf{x}, t) \rangle_T = E_1^2 \sin^2 kz \left\langle a(t)a^\dagger(t) + a^\dagger(t)a(t) \right\rangle_T = E_1^2 \sin^2 kz \, (2\langle \hat{n} \rangle_T + 1) \tag{4.23}$$

they are enhanced by a factor $2\langle \hat{n} \rangle_T + 1 = \coth(\hbar\omega/2k_B T)$ compared to zero temperature.

Three remarks on the advantages of the density operator formalism:

- the traces that are required for expectation values can be taken in any basis. One can choose a basis adapted to the operator whose average one its interested in.
- Second, the presence of the density operator $\hat{\rho}$ *under the trace* ensures that the trace exists even if the operator A has 'large matrix elements' (like the photon number operator). Well, this is in fact just a restriction on the observables and states that are mathematically allowed. Thermal states have the advantage that the expectation values exists for a broad class of observables because the matrix elements of the density operator become rapidly small for large n.
- The third advantage of using a density operator approach is that it gives a suitable description of a quantum system whose dynamics is not completely known and can only be specified by probabilities. In that case, one formulates an equation of motion for the density matrix from the solution of which the averages of all interesting quantities can be calculated.

Preparation of a thermal state with rate equations

As an example of the last remark, we sketch here a 'preparation scheme' for a thermal state. We are going to use 'rate equations': differential equations for the diagonal elements $p_n(t) = \langle n|\hat{\rho}(t)|n\rangle$:

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = -\kappa n p_n + \kappa' n p_{n-1} - \kappa'(n+1)p_n + \kappa(n+1)p_{n+1} \tag{4.24}$$

The constants κ and κ' can be interpreted as transition rates between states: the transition $|n\rangle \to |n-1\rangle$ happens with the rate κn (this rate appears as a negative term in $\dot{p}_n$ and as a positive term in $\dot{p}_{n-1}$). This process can be interpreted physically as the loss of one of the n photons. This photons goes into a 'thermostat' or 'environment' and is absorbed there. Similarly, the system described by $\hat{\rho}$ can absorb one photon from the thermostat – this happens with a 'Bose stimulation factor' because for the transition $|n-1\rangle \to |n\rangle$, the rate is $\kappa' n$. (To be read off from the second and third terms in Eq.(4.24).) Even the vacuum state can absorb a photon, hence not $n-1$, but n appears here.

If one waits long enough, the density matrix (more precisely, its diagonal elements) relax into a steady state given by the equations of 'detailed balance'

$$0 = -\kappa n p_n^{(\mathrm{ss})} + \kappa' n p_{n-1}^{(\mathrm{ss})} \tag{4.25}$$

This equation implies that $\dot{p}_n = 0$ in Eq.(4.24), but is slightly stronger. (One can probably show it by induction, starting from $n = 0$.) Eq.(4.25) gives a recurrence relation that links $p_n^{(\mathrm{ss})}$ to $p_{n-1}^{(\mathrm{ss})}$, whose solution is

$$p_n^{(\mathrm{ss})} \sim \left(\frac{\kappa'}{\kappa}\right)^n =: \mathrm{e}^{-n\hbar\omega/k_B T} \tag{4.26}$$

where we can identify the temperature T from the ratio of the rate constants κ'/κ. (One needs $\kappa' < \kappa$, otherwise, no stable equilibrium state is found.) Of course, this definition of temperature is linked to assigning an energy $n\hbar\omega$ to the state $|n\rangle$.

4.1.3 Coherent states (i)

The coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{4.27}$$

Since a is not an hermitean operator, α can be complex. *In a coherent state, the average electric field is nonzero:*

$$\langle \mathbf{E}(\mathbf{x},t)\rangle_\alpha = E_1 \sin kz \,\langle\alpha|\left(a(t) + a^\dagger(t)\right)|\alpha\rangle = E_1 \sin kz \,\left(\alpha\,\mathrm{e}^{-\mathrm{i}\omega t} + \alpha^*\,\mathrm{e}^{\mathrm{i}\omega t}\right). \tag{4.28}$$

We have assumed the field in a coherent state of the initial annihilator a. This expression is the same that we have used in chapter 1 for a classical, monochromatic field. The magnetic field quadrature also has on average its classical value in a coherent state. Coherent states are thus very useful to represent laser fields. We see that α measures the electric field strength in units of the 'single photon field' E_1. If we compute the average photon number in a coherent state, we get

$$\langle \hat{n} \rangle_\alpha = \langle a | a^\dagger a | \alpha \rangle = |\alpha|^2, \tag{4.29}$$

so that as an order of magnitude $\langle E \rangle \approx E_1 \langle \hat{n} \rangle^{1/2}$ (note the nonlinear dependence).

Coherent states are not stationary, but rotate in the complex α-plane: if $|\psi(0)\rangle = |\alpha\rangle$, then $|\psi(t)\rangle = |\alpha \, \mathrm{e}^{-\mathrm{i}\omega t}\rangle$. This can be shown using the expansion of a coherent state in terms of number states:

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{4.30}$$

Note that number states with arbitrarily high photon numbers are present in a coherent state. More specifically, we can introduce the probability $p_n(\alpha)$ of finding n photons in a coherent state:

$$p_n(\alpha) = |\langle n | \alpha \rangle|^2 = \mathrm{e}^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \tag{4.31}$$

which is a 'Poisson distribution' (the probability distribution of the sum of independent random bits). In the exercises, you are asked to compute the average photon number and its fluctuations (variance) in a coherent state.

The field quadratures also show quantum fluctuations around their classical average in a coherent state. This is inevitable because of the Heisenberg inequality. In the exercises, you are asked to show that these are equal to the quantum noise in the vacuum state (which is in fact a particular coherent state with $\alpha = 0$). This result can be displayed graphically in the complex α-plane by the sketch shown in fig. 4.1. We shall see that this plot gives the so-called Q-function (or Husimi function) of the state, see Eq.(4.44) below. This function provides a way to illustrate a quantum state by the analogy to the classical phase space. Note that since $a = (X + \mathrm{i}P)/\sqrt{2}$, we may identify the α-plane with the classical phase space of a harmonic oscillator. The gray area in this sketch indicates values for the position and momentum quadratures that are probable outcomes of measurements. This representation is of course schematic since X

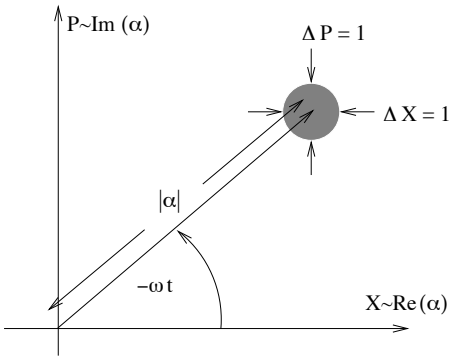

Figure 4.1: Representation of a coherent state in phase space.

and P cannot be measured simultaneously. We shall give it a precise meaning in section 4.2 where we show how coherent states can be used to expand any field state. (There are some subtleties related to the fact that they are not eigenstates of an hermitean operator.)

Finally, coherent states are not orthogonal. This is again a consequence of being the eigenstate of a non-hermitean operator. Let us calculate the overlap

$$\begin{aligned}\langle\alpha|\beta\rangle &= \sum_n \mathrm{e}^{-|\alpha|^2/2-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^{*n}\beta^n}{n!}\\ &= \exp\left[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2-\alpha^*\beta-\alpha\beta^*-\alpha^*\beta+\alpha\beta^*\right)\right]\\ &= \exp\left[-\frac{1}{2}|\alpha-\beta|^2+\mathrm{i}\,\mathrm{Im}\,\alpha^*\beta\right] \qquad (4.32)\end{aligned}$$

Here, we have split the complex overlap into its magnitude: a Gaussian with maximum at $\beta=\alpha$ and a phase factor. If we consider the Gaussian as a function of α, we get a peaked function in the phase space plane, with a typical width (the same in all directions) of the order of $\frac{1}{2}$ or 1.

Coherent states can be prepared by feeding the field mode with a "classical source". This could be a classical oscillating dipole, as it happens in a so-called "free electron laser". Or the field of a intense laser which is often approximated by a classical field. More details including the calculation of the time evolution operator for a classical source are given in Sec.4.2.2.

4.1.4 Squeezed states

You should have got the feeling up to now that the quantized field essentially differs from a classical field by its ('quantum') fluctuations. So people have thought whether it is possible to reduce the quantum noise in a field quadrature to get something even 'more classical' – or having less noise. This can be achieved in part, to 50%, say. Of course, one cannot beat the Heisenberg inequality, and the reduced fluctuations in one quadrature have to be paid by enhanced noise in the other one.

In the exercises, you are asked to show that the following operator

$$S(\xi) = \exp\left(\xi a^{\dagger 2} - \xi^* a^2\right) \tag{4.33}$$

'squeezes' the fluctuations of the vacuum state such that one quadrature component (fixed by the phase of the complex number ξ) has less quantum noise. A general 'quadrature' variable may be defined by

$$X_\theta = \frac{a\,\mathrm{e}^{-\mathrm{i}\theta} + a^\dagger\,\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{2}} \tag{4.34}$$

Two 'orthogonal' quadratures are then given by X_θ and $X_{\theta+\pi/2}$. The usual position and momentum quadratures correspond to $\theta = 0, \pi/2$. In the state $S(\xi)|0\rangle$ with $\xi = |\xi|\,\mathrm{e}^{-2\mathrm{i}\theta}$, the uncertainties of the quadratures are

$$\Delta X_\theta^2 = \frac{\mathrm{e}^{2|\xi|}}{2}, \qquad \Delta X_{\theta+\pi/2}^2 = \frac{\mathrm{e}^{-2|\xi|}}{2}, \tag{4.35}$$

their product being unchanged.

Note also that the 'squeezed vacuum' $S(\xi)|0\rangle$, when expanded in the Fock basis, contains only even photon number states:

$$S(\xi)|0\rangle = \sum_{m=0}^{\infty} c_{2m}|2m\rangle. \tag{4.36}$$

This is because the squeezing operator (4.33) only contains evens powers of the creation operator.

Exercise: compute the coefficients c_{2m}. Answer (details may be wrong):

$$c_{2m} = N\left(\frac{(2m-1)!!}{(2m)!!}\right)^{1/2} \mathrm{e}^{-2\mathrm{i}m\phi}\,\tanh^m r$$

where N is a normalization, $\xi = (r/2)\mathrm{e}^{2\mathrm{i}\phi}$, and $n!!$ is the product $n(n-2)\cdots$ of all numbers with the same parity up to n.

Preparation of a squeezed state

How can one prepare a squeezed state? The "cheating way of it" is just a re-scaling of the position and momentum quadratures:

$$X' = \eta X, \quad P' = \eta^{-1} P \tag{4.37}$$

This generates operators X' and P' that obey the same commutation relations. However, the energy of the field mode will not be proportional to $a'^{\dagger}a' \sim X'^2 + P'^2$, but involve terms of the form $(a')^2$ and $(a'^{\dagger})^2$. So the "ground state" $|\psi\rangle$ defined by $a'|\psi\rangle = 0$ will not be a stationary state of this Hamiltonian. This example illustrates, however, that (i) squeezed states evolve in time and are not stationary and (ii) that the quadratic terms $(a')^2$ and $(a'^{\dagger})^2$ play a key role.

The second way is to find a way to add these terms to the Hamiltonian. This can be done with a nonlinear medium. The 'squeezing' operator (4.33) can be realized with the interaction Hamiltonian

$$H_{\text{int}} = \mathrm{i}\hbar \left(g\, \mathrm{e}^{-2\mathrm{i}\omega t}\, a^{\dagger 2} - g^*\, \mathrm{e}^{2\mathrm{i}\omega t}\, a^2 \right) \tag{4.38}$$

with the squeezing parameter given by $\xi = \int \mathrm{d}t\, g(t)$. This interaction occurs in nonlinear optics. To get a qualitative understanding, imagine a medium with a field-dependent dielectric constant ('$\chi^{(2)}$ nonlinearity'). This is usually forbidden for symmetry reasons, but it happens in some special cases. In the electromagnetic energy density, one has

$$u = \frac{\varepsilon(|\mathbf{E}|)}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \tag{4.39}$$

where the linearization

$$\varepsilon(|\mathbf{E}|) = \varepsilon_0 \left(1 + n_2 |\mathbf{E}|\right)^2 \approx \varepsilon_0 \left(1 + 2 n_2 |\mathbf{E}|\right)$$

is often appropriate. In the quantum picture, this gives a contribution to the Hamiltonian with a term of third order in the field:

$$H_3 = \varepsilon_0 n_2 \int_V \mathrm{d}^3 x\, |\mathbf{E}(\mathbf{x}, t)|^3 \tag{4.40}$$

Let us now pick out two spatial modes of the field and put one of it into a coherent state $|\alpha\, \mathrm{e}^{-\mathrm{i}\omega_p t}\rangle$ with a 'large' amplitude $|\alpha| \gg 1$. The index 'p' is for 'pump field'. Let us call the other mode (the 'quantum' one) the 'signal'. The electric field is then

$$\mathbf{E}(\mathbf{x}, t) = E_p a_p \boldsymbol{\varepsilon}_p\, \mathrm{e}^{-\mathrm{i}(\omega_p t - \mathbf{k}_p \cdot \mathbf{x})} + E_1 \boldsymbol{\varepsilon} a(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + \text{h.c.} \tag{4.41}$$

The interaction Hamiltonian thus generates cross terms of the form[2]

$$H_{\rm int} = \ldots + \hbar\left(g\,{\rm e}^{-{\rm i}\omega_p t}\,a_p a^{\dagger 2} + g^*\,{\rm e}^{{\rm i}\omega_p t}\,a_p^\dagger a^2\right) \tag{4.42}$$

$$\hbar g = 3\varepsilon_0 n_2 E_p E_1 \boldsymbol{\varepsilon}_p\cdot\boldsymbol{\varepsilon}^*\int_V {\rm d}^3x\,{\rm e}^{{\rm i}(\mathbf{k}_p-2\mathbf{k})\cdot\mathbf{x}} \tag{4.43}$$

One often ignores the quantum fluctuations of the pump mode and replaces its annihilation operator a_p by the coherent state amplitude α. The interaction (4.42) then looks quite like our model Hamiltonian (4.38).

The nonlinear squeezing parameter $g\alpha$ is nonzero when the pump and signal modes are 'phase matched', i.e., $\mathbf{k}_p = 2\mathbf{k}$. For collinear modes, this is achieved by taking $\omega_p = 2\omega$. The spatial integral actually runs only over the region where the nonlinear index n_2 is different from zero. We also see from (4.42) that one 'pump photon' with energy $\hbar\omega_p = 2\hbar\omega$ can 'decay' into a pair of signal photons. We already anticipated this behaviour in the number state expansion (4.36).

We finally get a time-independent Hamiltonian by assuming that the pump mode is in a coherent state, $a_p \mapsto \alpha_p$ and by going into a rotating frame at half the pump frequency, $a(t) = {\rm e}^{-{\rm i}\omega_p t/2}\tilde{a}(t)$. If one works in addition at exact resonance, the time evolution operator is $U(t) = S(\xi)$ with $\xi = g\alpha_p t$. In practice, one does not get infinite squeezing as $t\to\infty$ because of damping.

4.2 Phase space distribution functions

The quantum states of the radiation field can be characterized by their behaviour in phase space. Fig.4.1 is one example for a coherent state. Can a similar picture be also constructed for the vacuum state? Yes: the vacuum is a special coherent state, $|{\rm vac}\rangle = |0\rangle$. What about number states or thermal states?

4.2.1 Overview

There are several possibilities to construct distribution functions on the 'phase space' spanned by the quadratures X and P. This is rooted in the fact that these are non-commuting operators.

[2]We are actually cheating with the polarization vector $\boldsymbol{\varepsilon}$. An accurate desription replaces n_2 by a third-rank tensor that produces a scalar out of three vectors.

The Q-function

The overlap we calculated (4.32) motivates the following function to characterize a quantum state the *Husimi or Q-function*

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \tag{4.44}$$

where the meaning of the prefactor $1/\pi$ will become clear soon. Each density operator $\hat{\rho}$ defines a Q-function and more generally, the Q-function just provides an alternative characterization of the quantum state.

The Q-function has the following nice property: it is positive $Q(\alpha) \geq 0$ for all α and any density operator $\hat{\rho}$. This directly follows from $\hat{\rho}$ being a density operator and the coherent state $|\alpha\rangle$ being a normalizable Hilbert space vector.

For a pure coherent state, $\hat{\rho} = |\beta\rangle\langle\beta|$, the Q-function is a Gaussian centered at $\alpha = \beta$ and a spread of order unity, see (4.32).

Exercise: for a thermal state, $Q_T(\alpha)$ is a Gaussian centered at $\alpha = 0$ with a width of order $[\langle\hat{n}\rangle_T + 1]^{1/2}$.

How would the Q-function look for a number state? A first guess is a 'ring', since the photon number (or energy) is fixed and shows no fluctuations. This is not far from the precise answer that we have already calculated:

$$Q_n(\alpha) = |\langle\alpha|n\rangle|^2 = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{4.45}$$

where now the Poisson distribution has to be read as a function of α. It is manifestly isotropic, increases like a power law $|\alpha|^{2n}$ near the origin and decays in a gaussian manner for large α. The maximum indeed occurs for $|\alpha|^2 \approx n$. The rim of this 'volcano distribution' becomes narrower and narrower as n increases.

The P-function

This function, also called Glauber-Sudarshan distribution provides an expansion of the density operator in the basis of coherent states. There are two variants: the ('simple') P-function (Sudarshan, 1963)

$$\hat{\rho} = \int \mathrm{d}^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha| \tag{4.46}$$

(the integration measure is $\mathrm{d}^2\alpha = \mathrm{d}(\mathrm{Re}\,\alpha)\,\mathrm{d}(\mathrm{Im}\,\alpha)$) and the 'positive P-function' (Glauber, 1963)

$$\hat{\rho} = \int \mathrm{d}^2\alpha\,\mathrm{d}^2\beta\, P(\alpha,\beta^*)|\alpha\rangle\langle\beta| \tag{4.47}$$

It is actually surprising that any density operator (well, there are some restrictions) can be represented as a sum of projectors $|\alpha\rangle\langle\alpha|$ on coherent states. This is related to the coherent states being not orthogonal. The price to pay is also that the P-function can be a quite singular distribution, containing δ-functions and derivatives of δ-functions.

Example: for a coherent state,

$$\hat{\rho} = |\beta\rangle\langle\beta| : \qquad P_{\hat{\rho}}(\alpha) = \delta(\alpha - \beta) \tag{4.48}$$

where the δ-function is defined with respect to the integration measure: $\delta(\alpha) = \delta(\mathrm{Re}\,\alpha)\,\delta(\mathrm{Im}\,\alpha)$.

It is easy to see, by taking the expectation value in a coherent state, that *the Q-function is a Gaussian convolution* (Faltung) *of the P-function*:

$$Q(\alpha) = \int \mathrm{d}^2\beta\, P(\beta) \exp(-\tfrac{1}{2}|\alpha - \beta|^2) \tag{4.49}$$

This explains why the Q-function behaves always 'less singularly' than the P-function.

4.2.2 Coherent states (ii) Displacement in phase space

How is it possible to generate a coherent state physically? One possible answer is 'never' because to this end, one must be able to control the phase of the complex number α, or equivalently, the origin of time (recall the discussion before Eq.(4.30)). In practice, however, it is at least useful, if not necessary, to think 'as if' the phase of a light field were controlled, for example in a laser field. For an instructive discussion, see two papers by Klaus Mølmer 1997 where he talks about a 'convenient fiction'. A physical example where it is plausible that the phase of a light field can be controlled is the 'free electron laser' where a beam of electrons (in an accelerator ring) is modulated ('wiggler') in a controlled way and made to emit photons into a laser cavity.

This example comes close to the folllowing single-mode Hamiltonian

$$H = \hbar\omega \left(a^\dagger a + \tfrac{1}{2}\right) + \mathrm{i}\hbar \left(\mathrm{e}^{-\mathrm{i}\omega_s t}\, g a^\dagger - \mathrm{e}^{\mathrm{i}\omega_s t}\, g^* a\right) \tag{4.50}$$

where the first term is the energy of our mode and the second term describes, e.g., the coupling of a classical dipole oscillator at frequency ω_s with the field

mode (in the electric-dipole interaction, replace the dipole operators $\sigma_\pm$ by c-numbers), or of a classical current density $\mathbf{j}(x,t)$ with the vector potential of the mode (via the minimal coupling interaction). We are going to see that *classical sources generate coherent states*.

In the interaction representation, the first term is transformed away and the exponentials $\mathrm{e}^{\pm\mathrm{i}(\omega-\omega_s)t}$ appear.[3] If we choose resonant conditions, $\omega_s = \omega$, we thus get a Schrödinger equation with a time-independent Hamiltonian. The solution is thus given by (in the interaction picture)

$$|\tilde{\psi}(t)\rangle = \exp[t(ga^\dagger - g^* a)]|\tilde{\psi}(0)\rangle = \hat{D}(gt)|\tilde{\psi}(0)\rangle \tag{4.51}$$

where $\hat{D}(\alpha)$ is the so-called displacement operator

$$\hat{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{4.52}$$

Let us assume that the mode starts in the vacuum state, we thus find using the Baker-Campbell-Hausdorff identity[4]

$$|\psi(t)\rangle = \exp[t(ga^\dagger - g^* a)]|0\rangle = \mathrm{e}^{-|g|^2 t^2/2}\,\mathrm{e}^{gt\,a^\dagger}\,\mathrm{e}^{-g^* t\,a}|0\rangle \tag{4.53}$$

Now the annihilation operator gives 0 when acting on the vacuum state, so that its exponential reduces to unity here. Expanding the exponential with the creation operator in a power series, we find

$$|\psi(t)\rangle = \mathrm{e}^{-g^2 t^2/2} \sum_{n=0}^{\infty} \frac{(gta^\dagger)^n}{n!}|0\rangle = |gt\rangle \tag{4.54}$$

This interaction thus generates a coherent state with amplitude $\alpha = gt$ that grows linearly in time. To obtain a stationary result, either the 'oscillator amplitude' g can be made time-dependent, or loss processes have to be added.

We have just shown that coherent states can be obtained by applying a 'displacement operator' to the vacuum state:

$$|\alpha\rangle = D(\alpha)|0\rangle \qquad D(\alpha) = \exp\left\{\alpha a^\dagger - \alpha^* a\right\} \tag{4.55}$$

This unitary operator also displaces the creation and annihilation operators as follows (to prove by deriving a differential equation in the 'Heisenberg picture',

[3] The others, involving $\mathrm{e}^{\pm\mathrm{i}(\omega+\omega_s)t}$ are neglected in the Hamiltonian by making the resonance or 'rotating wave' approximation.

[4] If the commutator $[A,B]$ commutes with A and B: $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]}$.

setting $\alpha = gt$)

$$D^\dagger(\alpha)\, a\, D(\alpha) = a + \alpha \tag{4.56}$$
$$D^\dagger(\alpha)\, a^\dagger\, D(\alpha) = a^\dagger + \alpha^*. \tag{4.57}$$

This identity is useful to show that the field quadrature fluctuations in a coherent state are those of the vacuum state.

The displacement operators provide a mapping from the complex numbers into unitary operators on the single-mode Hilbert space. Complex numbers can be added, and operators be applied sequentially. So how do the two operations compare? The answer lies in the equation

$$D(\alpha)D(\beta) = \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\alpha\beta^*)} D(\alpha+\beta) \tag{4.58}$$

that can be easily proven with the Baker-Campbell-Hausdorff formula (footnote 4). If the phase factor were not there, this equation would make the mapping $\alpha \mapsto \hat{D}(\alpha)$ a *representation* (*Darstellung*) of the additive group in $\mathbb{C}$ in the space of unitary operators $\mathcal{U}(\mathcal{H})$ over the (infinite-dimensional) Hilbert space $\mathcal{H}$ of the single mode: either one applies the displacement operators one after the other (left-hand side) or one adds the complex numbers and applies a single displacement (right-hand side), one gets the same result.

Now, there is a phase factor, involving $\mathrm{Im}(\alpha\beta^*)$. The mapping $\alpha \mapsto \hat{D}(\alpha)$ is then not a ('proper') representation, but only a *projective representation*. This must be so because the additive group in $\mathbb{C}$ is finite-dimensional and commutative, while the unitary operators $D(\alpha)$ form a non-commutative and infinite-dimensional group. And more precisely, the *generators* of the two groups do not have the same algebra (a Lie algebra formed by their commutators). For the additive group and its action on $\mathbb{C}$ itself, the generators can be taken as unit vectors parallel to the x and p axes. The addition of these vectors is, of course, commutative. For the 'image' formed by the $D(\alpha)$, acting on the Hilbert space of state vectors, the corresponding generators are (expand for small $\alpha = x + \mathrm{i}p$ with real parameters x and p)

$$D(\alpha) \approx \mathbb{1} + x(a^\dagger - a) + \mathrm{i}p(a^\dagger + a) \tag{4.59}$$

so we identify the generators $(a^\dagger - a)/\mathrm{i}$ and $(a^\dagger + a)$ whose commutator is twice $\mathrm{i}\mathbb{1}$. (One likes to choose hermitean generators, this explains the factors i. The commutator is hermitean after multiplication with i as well.) This means that the group structure is fundamentally different: the algebra spanned by the generators does not close, and a proper representation is not possible. In fact, the

additional phase factor that appears in the formula for the projective representation can be understood by enlarging the Lie algebra (and the group) to include also the unit operator.

To conclude, the phase factor appearing in Eq.(4.58) could be argued to have no physical significance: after all, changing a state vector by a ('global') phase does not change the quantum-mechanical predictions. But if a superposition can be constructed where the phase appears only in one term, then the phase becomes observable. A typical example is the 'geometric Berry phase'. We are not aware whether there is a link between this concept and the projective phase for the displacement operators.

4.3 Quantum theory of the beam splitter

More details on multi-mode quantum fields can be found in Sec.4.4.

4.3.1 Homodyne detection

Quadratures X_θ appear in the "beating" (interference) when a signal mode $a \mapsto a+\beta$ is mixed on a beam splitter with a large-amplitude coherent state $|\beta\rangle$ ("local oscillator", "reference beam"). The quadrature phase can be chosen from the phase of β.

4.3.2 State transformation

A beamsplitter is the most simple way to mix two modes, see Figure 4.2. From classical electrodynamics, one gets the following amplitudes for the outgoing modes:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{in}} \mapsto \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{out}} = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{in}}. \tag{4.60}$$

The recipe for quantization is now: 'replace the classical amplitudes by annihilation operators'. If the outgoing modes are still to be useful for the quantum theory, they have to satisfy the commutation relations:

$$\left[a_i(\text{out}),\, a_j^\dagger(\text{out})\right] = \delta_{ij}. \tag{4.61}$$

These conditions give constraints on the reflection and transmission amplitudes, for example $|t^2| + |r^2| = 1$. This is *not* identical to energy conservation for the

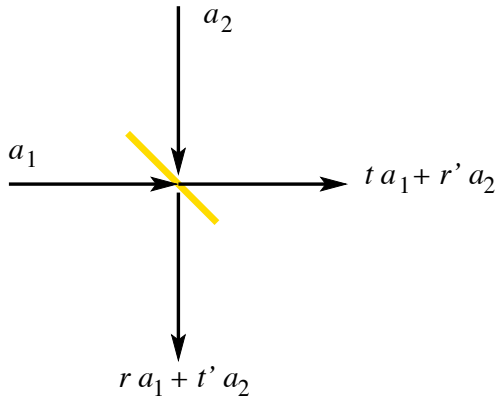

Figure 4.2: Mixing of two modes by a beam splitter.

incoming mode $a_1(\mathrm{in})$. But a sufficient condition is that the classical 'reciprocity relation' (*Umkehrung des Strahlengangs*) holds: $t = t'$.

We are now looking for a unitary operator U that implements this beamsplitter transformation in the following sense:

$$a_i' = U^\dagger a_i U, \qquad i = 1, 2 \tag{4.62}$$

From this operator, we can also compute the transformation of the states: $|\mathrm{out}\rangle = U|\mathrm{in}\rangle$. Let us start from the general transformation

$$a_i \mapsto A_i = S_{ij} a_j \qquad \text{or} \quad \vec{a} \mapsto \vec{A} = \mathsf{S}\,\vec{a} \tag{4.63}$$

where we have introduced matrix and vector notation. For the unitary transformation, we make the *Ansatz*

$$U(\tau) = \exp\left(\mathrm{i}\theta B_{jk} a_j^\dagger a_k\right) \tag{4.64}$$

with B_{jk} a hermitean matrix (ensuring unitarity). The action of this unitary on the photon mode operators is now required to reduce to

$$a_i \mapsto A_i(\theta) \equiv U^\dagger(\theta) a_i U(\theta) \stackrel{!}{=} S_{ij} a_j. \tag{4.65}$$

We compute this 'conjugated operator' with the usual trick via a differential equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta} A_i(\theta) &= -\mathrm{i} B_{jk} U^\dagger(\theta) \left[a_j^\dagger a_k,\, a_i\right] U(\theta) && (4.66)\\ &= -\mathrm{i} B_{jk} U^\dagger(\theta) \left(-\delta_{ij} a_k\right) U(\theta) && (4.67)\\ &= \mathrm{i} B_{ik} A_k(\theta). && (4.68) \end{aligned}$$

This is a system of linear differential equations with constant coefficients, so that we get as solution

$$\vec{A}(\theta) = \exp\left(\mathrm{i}\theta \mathsf{B}\right) \vec{A}(0) = \exp\left(\mathrm{i}\theta \mathsf{B}\right) \vec{a}. \tag{4.69}$$

We thus conclude that the matrix B is fixed by

$$\mathsf{S} = \exp\left(\mathrm{i}\theta\mathsf{B}\right). \tag{4.70}$$

If the transformation S is part of a continuous group and depends on a parameters ϵ, we can expand it around unity. Doing the same for the matrix exponential, we get

$$\begin{aligned} \mathsf{S} &\approx \mathbb{1} + \mathrm{i}\epsilon\mathsf{T} + \ldots \\ \exp\left(\mathrm{i}\theta\mathsf{B}\right) &\approx \mathbb{1} + \mathrm{i}\theta\mathsf{B} + \ldots \\ \theta\mathsf{B} &= \epsilon\mathsf{T}. \end{aligned} \tag{4.71}$$

The unitary transformation is thus determined via the *generator* of the mode transformation. More precisely, the unitary $U(\epsilon) = \exp(\mathrm{i}\epsilon B_{jk} a_j^\dagger a_k)$ implements the 'one-parameter subgroup' of mode transformations $\mathsf{S}(\epsilon)$ that is the solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathsf{S}(\epsilon) = \mathrm{i}\mathsf{T}\mathsf{S}(\epsilon).$$

A formal solution can be written as $\mathsf{S}(\epsilon) = \exp(\mathrm{i}\epsilon\mathsf{T})\mathsf{S}(0)$ so that with Eq.(4.70), we find indeed $\theta\mathsf{B} = \epsilon\mathsf{T}$.

For the two-mode beam splitter, an admissible transformation is given by

$$\mathsf{S} = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{4.72}$$

Its generator is given by, for small θ,

$$\mathsf{T} = \theta \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \theta\sigma_2 \tag{4.73}$$

so that the corresponding unitary operator reads

$$U(\theta) = \exp\left[\mathrm{i}\theta(-\mathrm{i}a_1^\dagger a_2 + \mathrm{i}a_2^\dagger a_1)\right] = \exp\left[\theta(a_1^\dagger a_2 - a_2^\dagger a_1)\right]. \tag{4.74}$$

The value of the parameter θ can be fixed via the identity

$$\exp\left(\mathrm{i}\theta\sigma_2\right) = \cos\theta + \mathrm{i}\sigma_2\sin\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{4.75}$$

so that θ is indeed the rotation angle for a general (not infinitesimal) transformation.

Example: splitting a single photon state

Finally, compute the state of the two-mode system if one photon is incident in mode 1 on the beam splitter: initial state $|\mathrm{in}\rangle = |1,0\rangle = a_1^\dagger|0,0\rangle$. The final state is then, using Eq.(4.74) for small θ

$$\begin{aligned} |\mathrm{out}\rangle &= U(\theta)|1,0\rangle \approx |1,0\rangle + \theta(a_1^\dagger a_2 - a_2^\dagger a_1)|1,0\rangle && (4.76)\\ &= |1,0\rangle - \theta|0,1\rangle. && (4.77)\end{aligned}$$

For finite θ, the higher powers also contribute. The calculation gets easy with the beam splitter transformation of the creation operators.

$$\begin{aligned} |\mathrm{out}\rangle &= U(\theta)a_1^\dagger|0,0\rangle && (4.78)\\ &\overset{(1)}{=} U(\theta)a_1^\dagger U^\dagger(\theta)|0,0\rangle && (4.79)\\ &\overset{(2)}{=} \left(a_1^\dagger\cos\theta - a_2^\dagger\sin\theta\right)|0,0\rangle && (4.80)\\ &= \cos\theta|1,0\rangle - \sin\theta|0,1\rangle && (4.81)\end{aligned}$$

In step (1), we have used that the unitary operator leaves the vacuum state unchanged. (This is because we have written the exponent in normal order.) In step (2), we have used that $U^\dagger(\theta)$ implements the transformation inverse to $U(\theta)$. Re-introducing the transmission amplitudes, we find

$$|1,0\rangle \mapsto t|1,0\rangle + r|0,1\rangle \tag{4.82}$$

so that the probability amplitudes to find the photon in either output mode correspond exactly, for this incident one-photon state, to the classical transmission and reflection amplitudes.

It is quite complicated to show in the same way the following property of a 'bi-coherent state'

$$U|\alpha,\beta\rangle = |\alpha',\beta'\rangle, \qquad \begin{pmatrix}\alpha'\\ \beta'\end{pmatrix} = \mathsf{S}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \tag{4.83}$$

that remains bi-coherent after the beam splitter. But the proof is quite simple with the unitary transformation of the mode operators.

Example: splitting a two-photon state

Two-photon states do not behave as 'intuitively'. Let us consider two single-photon states incident on the same beam splitter as before, $|\mathrm{in}\rangle = |1,1\rangle$. Then, by the same trick,

$$\begin{aligned} |\mathrm{out}\rangle &= U|\mathrm{in}\rangle = Ua_1^\dagger U^\dagger U a_2^\dagger U^\dagger|0,0\rangle\\ &= (a_1^\dagger\cos\theta - a_2^\dagger\sin\theta)(a_2^\dagger\cos\theta + a_1^\dagger\sin\theta)|0,0\rangle\\ &= (|2,0\rangle - |0,2\rangle)\,\frac{\sin 2\theta}{2} + |1,1\rangle\cos 2\theta && (4.84)\end{aligned}$$

Hence, for a 50/50 beam splitter ($\cos\theta = \sin\theta$ or $\theta = 45°$), the last term cancels and the photons are transmitted in 'bunches': they come out together at either output port.

4.4 Two modes, many modes

4.4.1 Multi-mode Hilbert space and observables

The state space of a two-mode field is the tensor product of the Fock spaces of two harmonic oscillators. In terms of number states, the basis vectors of this space can be written

$$|n_1; n_2\rangle = |n_1\rangle_{\text{mode 1}} \otimes |n_2\rangle_{\text{mode 2}}$$

where the first mode contains n_1 and the second mode n_2 photons. These states are called 'product states'. That have expectation values of products of operators pertaining to mode 1 and 2, that factorize, e.g.,

$$\langle \hat{n}_1 \hat{n}_2 \rangle = \langle \hat{n}_1 \rangle \langle \hat{n}_2 \rangle.$$

But due to the possibility of forming superpositions, there is much more 'space' in the multi-mode Hilbert space. For example, it is possible that two modes 'share' a single photon:

$$\frac{1}{\sqrt{2}} \left(|0; 1\rangle + |1; 0\rangle \right) \tag{4.85}$$

This state is called 'entangled' if no change of basis for the mode expansion exists such that the state is mapped onto a product state (this may be very difficult to check in practice).[5] The state is by no means unphysical, however, since it is generated by

$$\frac{1}{\sqrt{2}} (a_1^\dagger + a_2^\dagger) |0; 0\rangle \tag{4.86}$$

where $|0; 0\rangle$ is the two-mode vacuum. Such sums of creation operators occur always in the mode expansion of the quantized field. The decay of an excited atomic state, for example, generates a continuous superposition of one-photon states where an infinite number of modes share a single photon.

Many-mode single-photon states are also generated when an atom is illuminated by a single photon: the scattering of this photon by the atom generates, as in the classical electromagnetic theory, a continuous angular distribution of modes with a nonzero amplitude for one-photon excitations.

[5] It is simple to see, however, that the expectation value of $\hat{n}_1 \hat{n}_2$ does not factorize. Indeed, $\langle \hat{n}_1 \rangle = \frac{1}{2} = \langle \hat{n}_2 \rangle$ while $\langle \hat{n}_1 \hat{n}_2 \rangle = 0$ since in each component of the state (4.85), at least one mode has zero photons.

Finally, what about the density matrix for a multi-mode field? Let us start with the simple case of two modes of the same frequency in thermal equilibrium. According to the general rule, the density matrix is a sum of projectors onto the stationary states $|n_1; n_2\rangle$ of the two-mode system, each weighted with a probability proportional to $\mathrm{e}^{-\beta(n_1+n_2)}$. (Use $\beta = \hbar\omega/k_B T$.) Since the energy is made additively from single-mode energies, we can factorize this density operator:

$$\begin{aligned} \hat{\rho} &= Z^{-1} \sum_{n_1,n_2} \mathrm{e}^{-\beta(n_1+n_2)} \, |n_1; n_2\rangle\langle n_1; n_2| \\ &= Z \sum_{n_1} \mathrm{e}^{-\beta n_1} \, |n_1\rangle\langle n_1| \otimes \sum_{n_2} \mathrm{e}^{-\beta n_2} \, |n_2\rangle\langle n_2| \\ &= Z^{-1} \, \tilde{\rho}_1 \otimes \tilde{\rho}_2 \end{aligned} \tag{4.87}$$

where the $\tilde{\rho}_{1,2}$ are un-normalized density matrices. The tensor product of the projectors is defined by coming back to the tensor product of states

$$|n_1\rangle\langle n_1| \otimes |n_2\rangle\langle n_2| = (|n_1\rangle \otimes |n_2\rangle) \, (\langle n_1| \otimes \langle n_2|) \, .$$

The trace of the two-mode density matrix (4.87) also factorizes because the matrix elements of a tensor product operator are, by definition, the products of the individual matrix elements

$$\begin{aligned} \mathrm{tr}\,(\hat{\rho}) &= Z^{-1} \sum_{n_1,n_2} \langle n_1; n_2|\tilde{\rho}_1 \otimes \tilde{\rho}_2|n_1; n_2\rangle \\ &= Z^{-1} \sum_{n_1,n_2} \langle n_1|\tilde{\rho}_1|n_1\rangle\langle n_2|\tilde{\rho}_2|n_2\rangle \\ &= Z^{-1} \, (\mathrm{tr}\,\tilde{\rho}_1) \, (\mathrm{tr}\,\tilde{\rho}_2) \end{aligned} \tag{4.88}$$

and therefore $Z = Z_1 Z_2 = (1 - \mathrm{e}^{-\beta})^{-2}$.

Since the density matrix of this thermal two-mode state factorizes, this state is not entangled (averages of products of single-mode operators factorize). This is no longer true, however, if we allow for an interaction between the modes. Then the energy is no longer a sum of single-mode energies, and the previous factorization does no longer work. This is by the way a general rule: interactions between quantum systems lead to entangled states. For this reason, entangled states are much more frequent in Nature than are factorized states. It is a nontrivial task, however, to decide whether a given density matrix describes an entangled state or not.

Digression (*Einschub*): tensor product states and operators

It is somewhat tricky to guess the right formulas for multimode field states and operators. The general rule is the following:

Field operator	$\leftrightarrow$	sum of modes
Field state	$\leftrightarrow$	product of modes

For example, the electric field operator for a two-mode field is given by

$$\mathbf{E}(\mathbf{x},t) = E_1\boldsymbol{\varepsilon}_1 a_1(t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + E_2\boldsymbol{\varepsilon}_2 a_2(t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{h.c.}$$

while a typical state is for example the product state $|n_1;n_2\rangle = |n_1\rangle\otimes|n_2\rangle$. The general rule gets complicated (1) when we allow for superpositions (sums) of product states and (2) when we consider measurements that involve products of different mode operators.

In calculations, one often needs products of operators, like $\mathbf{E}^2(\mathbf{x},t)$. These are computed in the usual way, one has just to take care that operators sometimes do not commute. But this is only relevant for operators acting on the same mode, $[a_1, a_1^\dagger] = 1$, while for different modes

$$[a_1, a_2^\dagger] = 0$$

because they correspond to independent degrees of freedom.

Operator averages in product states. Let us consider the average electric field for the two-mode case written above. Using the mode expansion, we find terms like $\langle a_i(t)\rangle$ ($i = 1, 2$) and their adjoints. Now the operator $a_1|\psi\rangle$ is evaluated by letting a_1 act on the first factor of a product state:

$$a_1|n_1;n_2\rangle = (a_1|n_1\rangle)\otimes|n_2\rangle$$

If $|\psi\rangle$ is a sum of product states (entangled state), then this procedure is done for every term in this sum. Sometimes this is formalized by writing the operator as $a_1\otimes\mathbb{1}$, thus indicating that for the second mode nothing happens. The action of such operator tensor products is apparently defined as

$$A_1\otimes B_2|n_1;n_2\rangle = A_1|n_1\rangle\otimes B_2|n_2\rangle \tag{4.89}$$

by letting each operator factor act on the respective state factor. This notation allows to avoid the subscripts 1 and 2 as the relevant mode is indicated by the position in the operator product.

Similarly, the scalar product of tensor products of states is defined by

$$\langle n_1;n_2|m_1;m_2\rangle = \langle n_1|\otimes\langle n_2|m_1\rangle\otimes|m_2\rangle = \langle n_1|m_1\rangle\langle n_2|m_2\rangle$$

by taking the scalar product of the corresponding factors.

The average of the electric field for a product of number states is thus zero, as for a single-mode field, because $\langle n|an\rangle = 0$, and this is true for both modes. What about

a product state of two coherent states, $|\psi\rangle = |\alpha;\beta\rangle$? It is simple to see that we get the classical result (we assume that both modes have the same frequency ω)

$$\langle \mathbf{E}(\mathbf{x},t)\rangle = E_1\boldsymbol{\varepsilon}_1\alpha\,\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + E_2\boldsymbol{\varepsilon}_2\beta\,\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{c.c.} \tag{4.90}$$

(Note that 'c.c.' and not 'h.c.' occurs.) As a general rule, classical fields can be described by tensor products of coherent states.

Last example where we go quantum: a superposition of coherent product states,

$$|\psi\rangle = c|\alpha;\beta\rangle + d|\beta;\alpha\rangle$$

with some complex amplitudes c, d. Then we find

$$\langle a_1\rangle = |c|^2\alpha + |d|^2\beta$$

if $\langle\alpha|\beta\rangle = 0$. (This is actually never exactly the case, but can be achieved to a very good precision if $|\alpha-\beta| \gg 1$.) This result is an average over the two possible coherent amplitude, weighted with the corresponding probabilities. The average field thus becomes:

$$\langle \mathbf{E}(\mathbf{x},t)\rangle = E_1\boldsymbol{\varepsilon}_1\left(|c|^2\alpha + |d|^2\beta\right)\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + E_2\boldsymbol{\varepsilon}_2\left(|c|^2\beta + |d|^2\alpha\right)\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{c.c.}$$

Question: this result does not allow to distinguish this state from an 'incoherent mixture' of coherent product states like in (4.90), each state occurring with a probability $|c|^2, |d|^2$. This mixture would be described by the density operator

$$\hat{\rho}_{\mathrm{mix}} = |c|^2|\alpha;\beta\rangle\langle\alpha;\beta| + |d|^2|\beta;\alpha\rangle\langle\beta;\alpha|$$

and gives the same average electric field (exercise). If the coherent amplitudes α, β are closer together, then due to the nonzero overlap $\langle\alpha|\beta\rangle$, one can distinguish superposition and mixture (exercise). Are there observables that can make the difference in the case $\langle\alpha|\beta\rangle = 0$?

Average of single-mode operator. Let us calculate as another example the average photon number in mode 1 for a two-mode field in the entangled state (4.85). The relevant photon number operator is given by $a_1^\dagger a_1$ or, to be more precise, $a_1^\dagger a_1 \otimes \mathbb{1}$. Its action on the entangled state is worked out using linearity and the operator product rule (4.89)

$$\begin{aligned}
&\frac{1}{\sqrt{2}} a_1^\dagger a_1 \otimes \mathbb{1}\left(|0;1\rangle + |1;0\rangle\right)\\
&= \frac{1}{\sqrt{2}}\left(a_1^\dagger a_1|0\rangle\otimes|1\rangle + a_1^\dagger a_1|1\rangle\otimes|0\rangle\right)\\
&= \frac{1}{\sqrt{2}}|1\rangle\otimes|0\rangle = \frac{1}{\sqrt{2}}|1;0\rangle
\end{aligned}$$

Taking the scalar product with the original state, we find

$$\langle \hat{n}_1 \rangle = \frac{1}{2} \left(\langle 0;1| + \langle 1;0| \right) |1;0\rangle = \frac{1}{2}.$$

Once you have done this calculation, you can use the shorter rule: all we need are the probabilities of having $n_1 = 0, 1, \ldots$ photons in mode 1. For this, collect all product states in the state with the same number of photons n_1 and compute the squared norm of these states. From the probabilities for n_1 photons, you get the average photon number.

Product operators. As a second example, let us compute the average value of the product $a_i^\dagger a_j$ ($i, j = 1, 2$) in a thermal two-mode state. This object occurs when you measure the two-mode field with a photodetector (see paragraph **??** below). The tensor product notation is more cumbersome here and gives

$$a_1^\dagger a_1 \otimes \mathbb{1} \quad \text{or} \quad \mathbb{1} \otimes a_2^\dagger a_2 \quad \text{or} \quad a_1^\dagger \otimes a_2 \quad \text{or} \quad a_1 \otimes a_2^\dagger.$$

The density matrix is a tensor product of thermal single-mode density matrices. We shall see that the result is:

$$\langle a_i^\dagger a_j \rangle_T = \delta_{ij} \bar{n}(T) \tag{4.91}$$

where $\bar{n}(T)$ is the average photon number in a single mode. How does this come about?

When $i = j$, we are left with the calculation of the average photon number for a single mode:

$$\langle a_i^\dagger a_i \rangle = \sum_{n_1, n_2} \langle n_1; n_2 | a_i^\dagger a_i \hat{\rho}_1 \otimes \hat{\rho}_2 | n_1; n_2 \rangle$$

The action of the product density operators factorizes:

$$\hat{\rho}_1 \otimes \hat{\rho}_2 | n_1; n_2 \rangle = \hat{\rho}_1 | n_1 \rangle \otimes \hat{\rho}_2 | n_2 \rangle$$

Each single-mode density operator, acting on a number state, gives the corresponding occupation probability:

$$\hat{\rho}_1 | n_1 \rangle = \sum_{m_1} p_{m_1}(T) | m_1 \rangle \langle m_1 | n_1 \rangle = p_{n_1}(T) | n_1 \rangle,$$

so that we have, using the result for the photon number of one mode

$$\begin{aligned} \langle a_i^\dagger a_i \rangle &= \sum_{n_1, n_2} p_{n_1}(T) p_{n_2}(T) \langle n_1; n_2 | a_i^\dagger a_i | n_1; n_2 \rangle \\ &= \sum_{n_1, n_2} p_{n_1}(T) p_{n_2}(T) n_i \\ &= \sum_{n_i} p_{n_i}(T) n_i \sum_{n_j} p_{n_j}(T) \end{aligned}$$

In the last step, we have noted that the double sum can be factorized ($j \neq i$ is the other index). The second sum gives unity because the probabilities are normalized, the first sum gives the average photon number $\bar{n}(T)$ at temperature T and does no longer depend on the mode label (this is because we assumed equal frequencies for both modes). This completes the proof in the case $i = j$.

A similar calculation shows that the average of $a_1^\dagger a_2$ vanishes: indeed, we have

$$\langle n_1; n_2 | a_1^\dagger a_2 | n_1; n_2 \rangle = \langle n_1 | a_1^\dagger | n_1 \rangle \langle n_2 | a_2 | n_2 \rangle = 0.$$

4.5 Nonclassical states, correlations, and entanglement

4.5.1 Nonclassical states

Long tradition in this area: *Arbeitsgruppe Nichtklassische Strahlung* led by Harry Paul at HU Berlin.

Definition of Sudarshan (1963): any state whose P-function is regular (not more singular than a δ-function) has a classical interpretation.

Related to representation of normally ordered averages as integrals over the P-function, for example:

$$\langle a^\dagger a \rangle = \ldots = \int \mathrm{d}^2\alpha \, \alpha^* \alpha P(\alpha) \tag{4.92}$$

If $P(\alpha) \geq 0$ and not more singular than a δ-function, we can read this integral as a classical average over "existing values" α and α^* for the operators a and $a^\dagger$. This works only if the operators are in normal order, however. Any observable can be brought into normal order, however, using the commutation relations.

An alternative criterion with is "easier" to verify experimentally: a state is non-classical if for some point α in the phase space plane, the Wigner function is negative, $W(\alpha) < 0$. The Wigner function is related to the P-function by a gaussian convolution

$$W(\alpha) = \int \frac{\mathrm{d}^2\beta}{\pi} \, \mathrm{e}^{-\frac{1}{2}|\alpha-\beta|^2} P(\alpha) \tag{4.93}$$

(Convolution: smoothen a photo and make it less "sharp".) The singular features of a non-classical P-function are smoothened by this convolution because the Wigner function $W(\alpha)$ is a continuous function. But the non-classical singularities do not appear completely, because $W(\alpha)$ can be negative.

Wigner function: provides a statistical interpretation of symmetrically ordered averages, for example

$$\frac{1}{2}\langle aa^\dagger + a^\dagger a\rangle = \int \mathrm{d}^2\alpha \, \frac{\alpha\alpha^* + \alpha^*\alpha}{2} W(\alpha) \tag{4.94}$$

For the vacuum state, this is not zero because the Wigner function has a finite width around $\alpha = 0$ (a disk of "vacuum fluctuations"). In Eq.(4.92) for the P-function, the result is zero because $P(\alpha)$ is a δ-peak centered at zero for the vacuum state ("no fluctuations").

Q-function: anti-normally ordered average.

4.5.2 Quasi-probabilities

Characteristic functions (moment generating functions). For the P-function, normally ordered form of displacement operator

$$\tilde{P}(\alpha) = \langle : \hat{D}(\alpha) : \rangle = \langle \mathrm{e}^{\alpha a^\dagger} \mathrm{e}^{-\alpha^* a} \rangle \tag{4.95}$$

where $: \ldots :$ means: write all operator products in $\ldots$ in normally ordered form, ignoring commutators.[6] Expansion in powers of α and α^* gives normally ordered operator products. Inverse Fourier transform from $\tilde{P}(\alpha)$ gives the P-function, as with the characteristic function in classical statistics.

Similarly for the Wigner function: generating function for symmetrically ordered products is the displacement operator itself

$$\tilde{W}(\alpha) = \langle \hat{D}(\alpha) \rangle \tag{4.96}$$

The Campbell-Baker-Hausdorff formula allows to re-order the operators in $\hat{D}(\alpha)$ and to bring it in normal order, see Eq.(4.53). This gives the relation

$$\tilde{Q}(\alpha) = \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \tilde{W}(\alpha) = \mathrm{e}^{-|\alpha|^2} \tilde{P}(\alpha) \tag{4.97}$$

The characteristic functions of the Wigner and Q-functions are therefore smaller for large values of α. This implies for their Fourier transforms (the Wigner and Q-functions) that they are "smoother" (a picture with few high k-vectors is blurred, out of focus, *unscharf*). From a mathematical viewpoint, this means that the Fourier transforms exist as ordinary functions for Wigner and Q, but that the P-function does not necessarily exist (as an ordinary function). One must then take recourse to the singular functions of distribution theory.

[6] This is a subtle issue. Example: $: a^\dagger a :=: aa^\dagger := a^\dagger a$.

4.5.3 Entanglement and correlations

Entanglement is a property of two observables A and B or of two subsystems described by A and B. The two subsystems are called *entangled* when A and B show "non-classical correlations", i.e., correlations that cannot be explained by classical statistics.

This formulation is similar to the negative (or singular) values of certain quasi-probabilities. When these cannot be interpreted as classical probabilities, we encounter non-classical states. The examples discussed below illustrate for example that squeezing in one mode can be used to entangle two modes – the "non-classicality" of one input state is a "resource" that provides "entangling power".

EPR correlations

Output $a_{1,2} = (a \pm b)/\sqrt{2}$ of a balanced beamsplitter with squeezed vacuum state in mode a. This gives for suitable position and momentum uncertainties

$$\Delta(X_1 - X_2)\Delta(P_1 + P_2) < 1 \tag{4.98}$$

because the variance of the difference, $\Delta(X_1 - X_2)$, is just related to the squeezed variance $\Delta X < 1/\sqrt{2}$ of the input mode a. The other variable $P_1 + P_2$ has a variance related to the state of input mode b, it can be brought to a minimum uncertainty of order 1 with a coherent state. The inequality (4.98) is not inconsistent with the Heisenberg relations because the sum $P_1 + P_2$ and the difference $X_1 - X_2$ are commuting operators.

In other words, Eq.(4.98) tells us that the combination "squeezed vacuum + coherent state" sent onto a beam splitter provides two beams whose X-quadratures are correlated better than what is allowed by the standard vacuum fluctuations (or the fluctuations around a coherent = quasi-classical state). This is the criterion for a non-classical correlation.

Einstein, Podolski, and Rosen (1935) or "EPR" have discussed this arrangement in a slightly different form and came to the conclusion that quantum mechanics must be an incomplete theory. They mixed up, however, that the correlations we have here do not require some "instantaneous action at a distance" between the systems A and B (the two output beams after the beam splitter). Nonlocal correlations of this kind already appear in classical physics: hide a red and a blue ball in two boxes, move one box to the moon and open it. You know

immediately the color of the other box, whereever it is. This correlation cannot be used to transmit information, however.

Bell correlations

The reasoning of EPR has been made more precise by John Bell (1987) who invented a systematic way of deriving inequalities (upper limits) to correlations between observables A and B. The "classical" assumption is that these take definite values (those that appear as outcomes of single measurements), but determined by some other "hidden variables" that obey classical statistics. If these "hidden variable theories" are formulated in a non-local way, any quantum correlation can be reproduced. But this would require assumptions that are not natural from the "local" viewpoint that has become familiar to us from relativity. An example of a local hidden variable theory provides an upper limit to spin correlations measured on two two-level systems with spin operators $\boldsymbol{\sigma} \otimes \mathbb{1}$ (system A) and $\mathbb{1} \otimes \boldsymbol{\sigma}$ (system B). More precisely, let us take four unit vectors: $\mathbf{n}$, $\mathbf{n}'$ (for system A) and $\mathbf{m}$, $\mathbf{m}'$ (for system b). Then the following inequality holds (Clauser, Horne, Shimony, and Holt 1969)

$$|\langle \mathbf{n}\cdot\boldsymbol{\sigma} \otimes \mathbf{m}\cdot\boldsymbol{\sigma}\rangle + \langle \mathbf{n}'\cdot\boldsymbol{\sigma} \otimes \mathbf{m}\cdot\boldsymbol{\sigma}\rangle + \langle \mathbf{n}'\cdot\boldsymbol{\sigma} \otimes \mathbf{m}'\cdot\boldsymbol{\sigma}\rangle - \langle \mathbf{n}\cdot\boldsymbol{\sigma} \otimes \mathbf{m}'\cdot\boldsymbol{\sigma}\rangle| \leq 2 \tag{4.99}$$

where the pattern of signs is to be noted. One central idea in the proof is that one can "locally choose" between $\mathbf{n}$ and $\mathbf{n}'$ (i.e. two different components of the Bloch vector for system A), and that the outcome for system B is not affected by this choice (this is a "local theory of hidden variables").

A classical, perfect correlation can always be achieved between detectors in a fixed direction, $\mathbf{n} = \mathbf{m}$, say. This is within the scope of the CHSH inequality (4.99), however. Take for example $\mathbf{n}\cdot\boldsymbol{\sigma} = \sigma_3$ with eigenstates $|e\rangle$, $|g\rangle$ and consider the statistical mixture

$$\rho = \frac{1}{2}\left(|e,g\rangle\langle e,g| + |g,e\rangle\langle g,e|\right) \tag{4.100}$$

Then perfect anti-correlation holds $\langle \sigma_3 \otimes \sigma_3\rangle = \mathrm{tr}\left[(\sigma_3 \otimes \sigma_3)\,\rho\right] = -1$. This does not produce any correlations for the Bloch components $\sigma_{1,2}$, however. Check that one gets for the CHSH correlation

$$\begin{aligned} \mathrm{CHSH} &= -n_3 m_3 - n_3' m_3 - n_3' m_3' + n_3 m_3' \\ &= -(n_3 + n_3')m_3 - (n_3' - n_3)m_3' \end{aligned} \tag{4.101}$$

Let us look for the maximum value of this expression. The components of the unit vectors are in the range $-1 \ldots +1$. For $-1 \leq n_3 < n_3' < 0$, both parentheses are negative, and we get a maximum by choosing $m_3 = m_3' = 1$. But then, $\mathrm{CHSH} = -2n_3' \leq 2$. Along similar lines, one can prove the inequality (4.99).

The power of this reasoning is that the inequality applies to any choice of state, i.e., of choice of "hidden variables" or classical correlations between the outcomes.

Quantum mechanics gives a different answer, however, sometimes. Take the "maximally entangled state"

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|e, g\rangle - |g, e\rangle \right) \tag{4.102}$$

whose density operator $|\psi\rangle\langle\psi|$ differs from Eq.(4.100) because of the off-diagonal terms $|e, g\langle\rangle g, e|$. In this state, one has the perfect anti-correlation

$$\langle\psi|\mathbf{n} \cdot \boldsymbol{\sigma} \otimes \mathbf{n} \cdot \boldsymbol{\sigma}|\psi\rangle = -1 \tag{4.103}$$

in any choice of basis $\mathbf{n}$. (This is related to the "singlet" or zero total spin character of the state $|\psi\rangle$.)[7] For two different orientations at A and B, one gets the result

$$\langle\psi|\mathbf{n} \cdot \boldsymbol{\sigma} \otimes \mathbf{m} \cdot \boldsymbol{\sigma}|\psi\rangle = -\mathbf{n} \cdot \mathbf{m} \tag{4.104}$$

where $\mathbf{n} \cdot \mathbf{m}$ is the standard scalar product. The CHSH correlation then becomes

$$\mathrm{CHSH} = -(\mathbf{n} + \mathbf{n}') \cdot \mathbf{m} - (\mathbf{n}' - \mathbf{n}) \cdot \mathbf{m}' \tag{4.105}$$

which can be maximized by choosing a suitable "tetrad" of unit vectors[8] up to a value $2\sqrt{2}$. The range of CHSH correlations

$$2 < |\mathrm{CHSH}| \leq 2\sqrt{2} \qquad \text{Bell inequality violated} \tag{4.106}$$

is therefore called the "non-classical" domain which cannot be interpreted in terms of a classical theory (more precisely: a local hidden variable model). The number $|\mathrm{CHSH}| - 2$ can be taken as a quantitative measure of entanglement between system A and B: it quantifies the degree of "non-classicality" of the correlations between A and B.

[7] The form of the state $|\psi\rangle$ in a different cartesian basis for the spin vectors $\boldsymbol{\sigma}$ is generated by the three operators $\sigma_i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_i$) $(i, = 1, 2, 3)$. However, their action on $|\psi\rangle$ gives zero: hence $|\psi\rangle$ is invariant under rotation (a "singlet state"). Hence the perfect anticorrelation for the components $\sigma_3 \otimes \sigma_3$ carries over onto any direction.

[8] Exercise: choose $\mathbf{m}$ and $\mathbf{m}'$ opposite to the directions of $\mathbf{n} \pm \mathbf{n}'$, respectively. Then $\mathrm{CHSH} = \sqrt{2 + 2\cos\theta} + \sqrt{2 - 2\cos\theta}$ with $\cos\theta = \mathbf{n} \cdot \mathbf{n}'$. This quantity varies between 2 and $2\sqrt{2}$. The maxima are obtained for $\cos\theta = 0$, hence orthogonal directions $\mathbf{n}$ and $\mathbf{n}'$. The directions $\mathbf{m}$ and $\mathbf{m}'$ are then orthogonal as well, and one bisects that angle between $\mathbf{n}$ and $\mathbf{n}'$.

Further reading

On entanglement between quadratures or position and momentum variables in general (so-called “continuous variables”): Eisert & Plenio (2003) and Plenio & Virmani (2007). An introduction to the EPR paradox and applications: Reid & al. (2009).

Bibliography

J. S. Bell (1987). On the Einstein–Podolsky–Rosen paradox. in *Speakable and unspeakable in quantum mechanics: collected papers in quantum mechanics*, pages 14–21. Cambrige University Press, Cambridge. [*Physics* **1** (1964) 195–200].

J. F. Clauser, M. A. Horne, A. Shimony & R. A. Holt (1969). Proposed experiment to test local hidden-variable theories, *Phys. Rev. Lett.* **23**, 880–84. Erratum: **24** (1970) 549.

C. Cohen-Tannoudji, J. Dupont-Roc & G. Grynberg (1987). *Photons et atomes — Introduction à l'électrodynamique quantique*. InterEditions, Paris. English translation: *Photons and Atoms — Introduction to Quantum Electrodynamics* (Wiley, New York 1989).

D. P. Craig & T. Thirunamachandran (1984). *Molecular Quantum Electrodynamics*. Academic Press, London.

J. Dalibard, Y. Castin & K. Mølmer (1992). Wave-Function Approach to Dissipative Processes in Quantum Optics, *Phys. Rev. Lett.* **68**, 580–583.

P. A. M. Dirac (1927). The Quantum Theory of the Emission and Absorption of Radiation, *Proc. Roy. Soc. (London) A* **114**, 243–65.

A. Einstein, B. Podolsky & N. Rosen (1935). Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?, *Phys. Rev.* **47**, 777–780.

J. Eisert & M. B. Plenio (2003). Introduction to the basics of entanglement theory in continuous-variable systems, *Int. J. Quant. Inf.* **1** (4), 479–506.

R. J. Glauber (1963). Coherent and Incoherent States of the Radiation Field, *Phys. Rev.* **131**, 2766–2788.

H. Haken & H. C. Wolf (2000). *The Physics of Atoms and Quanta*. Advanced Texts in Physics. Springer, Heidelberg Berlin, 6th edition.

C. Henkel (2007). Laser theory in manifest Lindblad form, *J. Phys. B: Atom. Mol. Opt. Phys.* **40**, 2359–71.

C. Itzykson & J. B. Zuber (2006). *Quantum field theory*. Dover, New York.

K. Mølmer (1997a). Optical coherence, a convenient fiction, *Phys. Rev. A* **55**, 3195–3203.

K. Mølmer (1997b). Quantum entanglement and classical behaviour, *J. Mod. Opt.* **44**, 1937–1956.

M. A. Nielsen & I. L. Chuang (2000). *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge.

M. B. Plenio & S. Virmani (2007). An introduction to entanglement measures, *Quant. Inf. Comp.* **7** (1–2), 1–51.

M. D. Reid, P. D. Drummond, W. P. Bowen, E. G. Cavalcanti, P. K. Lam, H. A. Bachor, U. L. Andersen & G. Leuchs (2009). Colloquium: The Einstein-Podolsky-Rosen paradox: From concepts to applications, *Rev. Mod. Phys.* **81**, 1727—51.

E. C. G. Sudarshan (1963). Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, *Phys. Rev. Lett.* **10** (7), 277–279.

W. H. Zurek (1991). Decoherence and the transition from quantum to classical, *Physics Today* **44** (10), 36–44. Update: quant-ph/0306072.