1.3 Quantum optics of the beamsplitter

recall scattering theory

transformation rules for mode operators, for quantum states

split a single photon (generate entanglement)

two-photon interference: Hong–Ou–Mandel experiment

homodyne measurement (local oscillator)

More details on multi-mode quantum fields can be found in Sec. 1.4.

1.3.1 Beamsplitter transformation

A beamsplitter is the most simple way to mix two modes, see Figure 1.5. From classical electrodynamics, one gets the following amplitudes for the

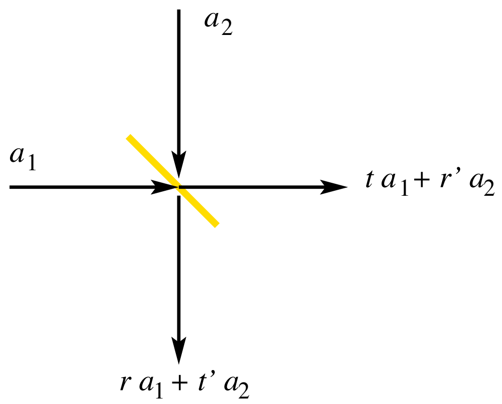


Figure 1.5: Mixing of two modes by a beam splitter.

outgoing modes:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{in}} \mapsto \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^{\text{out}} = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{in}}. \tag{1.87}$$

The recipe for quantization is now: ‘replace the classical amplitudes by annihilation operators’. If the outgoing modes are still to be useful for the quantum theory, they have to satisfy the commutation relations:

$$\left[A_i(\text{out}),\, A_j^\dagger(\text{out})\right] = \delta_{ij}. \tag{1.88}$$

These conditions give constraints on the reflection and transmission amplitudes, for example $|t'|^2 + |r'|^2 = 1$. Note that this is *not* identical to energy conservation for the incoming mode $a_1(\text{in})$ [that would read $|t|^2 + |r'|^2 = 1$]. But a sufficient condition is that the classical 'reciprocity relation' (*Umkehrung des Strahlengangs*) holds: $t = t'$.

We are now looking for a unitary operator S [the S-matrix] that implements this beamsplitter transformation in the following sense:

$$A_i = S^\dagger a_i S, \qquad i = 1, 2 \tag{1.89}$$

From this operator, we can also compute the transformation of the states: $|\text{out}\rangle = S|\text{in}\rangle$. Let us start from the general transformation (summation over double indices)

$$a_i \mapsto A_i = B_{ij} a_j \qquad \text{or} \quad \vec{a} \mapsto \vec{A} = \mathsf{B}\,\vec{a} \tag{1.90}$$

where we have introduced matrix and vector notation. For the unitary transformation, we make the *Ansatz*

$$S(\theta) = \exp\left(\mathrm{i}\theta J_{kl} a_k^\dagger a_l\right) \tag{1.91}$$

with J_{kl} a hermitean matrix (ensuring unitarity). The action of this unitary on the photon mode operators is now required to reduce to

$$a_i \mapsto A_i(\theta) \equiv S^\dagger(\theta) a_i S(\theta) \stackrel{!}{=} B_{ij} a_j. \tag{1.92}$$

We compute this 'operator conjugation' with the usual trick of a differential equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta} A_i(\theta) &= -\mathrm{i} J_{kl} S^\dagger(\theta) \left[a_k^\dagger a_l,\, a_i\right] S(\theta) \\ &= -\mathrm{i} J_{kl} S^\dagger(\theta) \left(-\delta_{ik} a_l\right) S(\theta) \\ &= \mathrm{i} J_{il} A_l(\theta). \end{aligned} \tag{1.93}$$

This is a system of linear differential equations with constant coefficients, so that we get as solution

$$\vec{A}(\theta) = \exp\left(\mathrm{i}\theta \mathsf{J}\right) \vec{A}(0) = \exp\left(\mathrm{i}\theta \mathsf{J}\right) \vec{a}\,, \tag{1.94}$$

where $\exp\left(\mathrm{i}\theta \mathsf{J}\right)$ is a matrix exponential. We thus conclude that the so-called generator J of the beam splitter matrix is fixed by

$$\mathsf{B} = \exp\left(\mathrm{i}\theta \mathsf{J}\right). \tag{1.95}$$

If the transformation B is part of a continuous group and depends on θ as a parameter, we can expand it around unity. Doing the same for the matrix exponential, we get

$$\mathsf{B} \approx \mathbb{1} + \mathrm{i}\theta\mathsf{J} + \ldots$$

This equation explains the name *generator* for the matrix J: it actually generates a subgroup of matrices $\mathsf{B} = \mathsf{B}(\theta)$ parametrized by the angle θ. The unitary transformation we are looking for is thus determined via the same generator J.

For the two-mode beam splitter, an admissible transformation is given by

$$\mathsf{B} = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} = \begin{pmatrix} \cos\theta & \mathrm{i}\sin\theta \\ \mathrm{i}\sin\theta & \cos\theta \end{pmatrix}. \tag{1.96}$$

The factor i is just put for convenience so that the reflection amplitudes are the same for both sides, $r = r'$, as expected by symmetry.[2] Expanding for small θ, the generator is

$$\mathsf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \tag{1.97}$$

so that the unitary operator for this beamsplitter reads

$$S(\theta) = \exp\left[\mathrm{i}\theta(a_1^\dagger a_2 + a_2^\dagger a_1)\right]. \tag{1.98}$$

Note that indeed, one has the identity

$$\exp(\mathrm{i}\theta\sigma_1) = \cos\theta + \mathrm{i}\sigma_1\sin\theta = \begin{pmatrix} \cos\theta & \mathrm{i}\sin\theta \\ \mathrm{i}\sin\theta & \cos\theta \end{pmatrix} \tag{1.99}$$

Funnily enough, one has to put $\theta = \pi/4$ into the exponent of S [Eq.(1.98)] instead of $1/\sqrt{2}$ to get a 50:50 beam splitter.

With the beamsplitter transformation $\exp\left[\mathrm{i}\theta(a_1^\dagger a_2 + a_2^\dagger a_1)\right]$ [Eq.(1.98)], we have added another element to the group of transformations on two modes. The two-mode squeezing operation $\exp(\xi a_1^\dagger a_2^\dagger - \xi^* a_1 a_2)$ [Eq.(1.85)] was our first example. The whole

[2] The relation $1 + r = t$, familiar from the continuous matching of fields, cannot be true in general for a beamsplitter because its thickness is nonzero. For the parameters chosen above, one just needs a global phase $r = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\sin\theta$ and $t = \mathrm{e}^{\mathrm{i}\theta}\cos\theta$.

group of transformations that leaves the commutation relations between the a_i and $a_j^\dagger$ invariant, is called the canonical (or symplectic) group on four phase-space coordinates, Sp(4). We are dealing here with its 'metaplectic' representation, unitary and of infinite dimension. In total, there are ten generators in Sp(4) if the count is made for hermitean generators (real parameters): two directions of squeezing per mode, one rotation per mode, that makes six. There are two (real) parameters for two-mode squeezing: eight, and actually there are two independent beam-splitter transformations (rotation in the $a_1 a_2$ plane) that correspond to the Pauli matrices σ_1 (found above) and σ_2: ten. The beam splitter generated by σ_2 must be included in the group because it is generated from σ_1 by local phase shifts (independent rotations in the complex a_1-plane and a_2-plane).

1.3.2 Examples

Splitting a single photon state

What is the state of the two-mode system if one photon is incident in mode 1 on the beam splitter? Initial state $|\mathrm{in}\rangle = |1,0\rangle = a_1^\dagger|0,0\rangle$. The final state is then, using Eq.(1.98) for small θ

$$\begin{aligned} |\mathrm{out}\rangle &= S|1,0\rangle \approx |1,0\rangle + \mathrm{i}\theta(a_1^\dagger a_2 + a_2^\dagger a_1)|1,0\rangle \\ &= |1,0\rangle + \mathrm{i}\theta|0,1\rangle. \end{aligned} \tag{1.100}$$

For finite θ, the higher powers also contribute. The calculation gets easy with the beam splitter transformation of the creation operators.

$$\begin{aligned} |\mathrm{out}\rangle &= S a_1^\dagger|0,0\rangle \\ &\overset{(1)}{=} S a_1^\dagger S^\dagger|0,0\rangle \\ &\overset{(2)}{=} (a_1^\dagger\cos\theta + \mathrm{i}a_2^\dagger\sin\theta)|0,0\rangle \\ &= \cos\theta|1,0\rangle + \mathrm{i}\sin\theta|0,1\rangle \end{aligned} \tag{1.101}$$

In step (1), we have used that the unitary operator S leaves the vacuum state unchanged. (This is because we have written the exponent in normal order.) In step (2), we have used that S implements the transformation inverse to $S^\dagger$ (unitarity). Re-introducing the transmission amplitudes, we find

$$|1,0\rangle \mapsto t|1,0\rangle + r'|0,1\rangle \tag{1.102}$$

so that the probability amplitudes to find the photon in either output mode correspond exactly, for this incident one-photon state, to the classical transmission and reflection amplitudes. In other words: *for single-photon states, a beamsplitter turns the amplitudes of classical electrodynamics into the probability amplitudes of quantum mechanics. A single photon is transmitted 'and' reflected with 'and' having the meaning of quantum-mechanical superposition*.

It is quite complicated to show in the same way as in Eq.(1.100) the following property of a 'bi-coherent state'

$$S|\alpha,\beta\rangle = |\alpha',\beta'\rangle, \qquad \begin{pmatrix}\alpha'\\ \beta'\end{pmatrix} = \mathsf{B}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \tag{1.103}$$

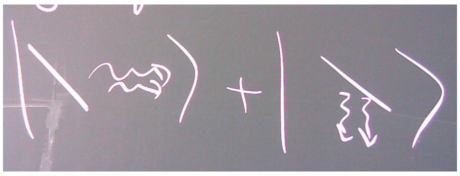
that remains bi-coherent after the beam splitter. But the proof is quite simple with the unitary transformation of the mode operators.

Splitting a two-photon state (Hong–Ou–Mandel interference)

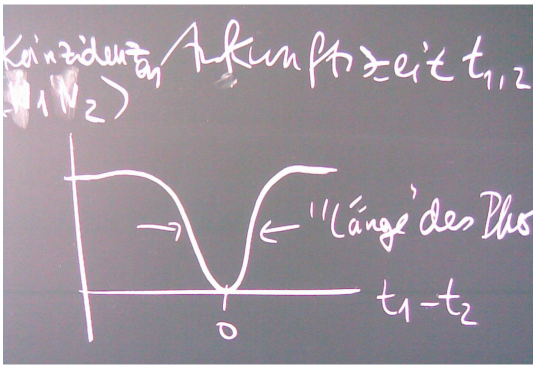
Two-photon states do not behave as 'intuitively'. Let us consider two single-photon states incident on the same beam splitter as before, $|\mathrm{in}\rangle = |1,1\rangle$. Then, by the same trick,

$$\begin{aligned} |\mathrm{out}\rangle &= S|\mathrm{in}\rangle = Sa_1^\dagger S^\dagger S a_2^\dagger S^\dagger S|0,0\rangle \\ &= (a_1^\dagger\cos\theta + \mathrm{i}a_2^\dagger\sin\theta)(a_2^\dagger\cos\theta + \mathrm{i}a_1^\dagger\sin\theta)|0,0\rangle \\ &= (|2,0\rangle - |0,2\rangle)\,\frac{\sin 2\theta}{\sqrt{2}} + |1,1\rangle\cos 2\theta \end{aligned} \tag{1.104}$$

Hence, for a 50/50 beam splitter ($\cos\theta = \sin\theta$ or $\theta = 45°$), the last term cancels and the photons are transmitted in 'bunches': they come out together at either output port. There are zero 'coincidences' of one photon in port A_1 and the other in A_2.



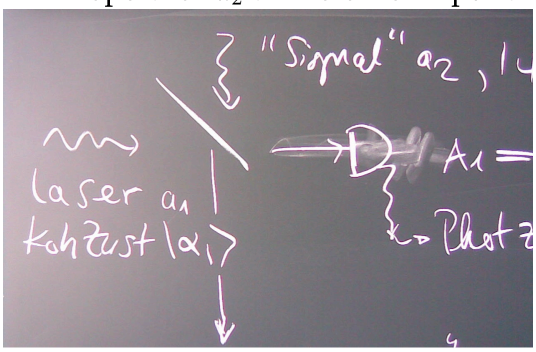
This is due to a destructive interference between two indistinguishible histories for the two photons from source to detector – this is called the 'Hong-Ou-Mandel dip'.



The dip in the coincidence signal can be observed by tuning a parameter (like a delay time) that makes the two photons (in)distinguishable. This is way to measure the 'length' of a single photon (relative to another one, so it is actually a joint length). How far the dip goes to zero, below the signal level at large delay, is a measure of how well the initial two-photon state has been prepared.

1.3.3 Homodyne detection

Introduce coherent state $|\alpha_1\rangle$, simplest model for an intense laser beam. And "signal beam" with operator a_2 at the other input arm.



Discuss output operators $t\alpha_1 + r'a_2$ after a beam splitter: "mixing" of signal with "local oscillator" (= laser beam). The quadratures X_θ appear in the "beating" (interference) when a signal mode a is mixed on a beam splitter with a large-amplitude coherent state $|\alpha_1\rangle$ ("local oscillator", "reference beam"). Indeed, in one output of a 50:50 beam splitter, we have approximately a square $(\alpha_1 + a_2)^\dagger(\alpha_1 + a_2)$ containing a mixed term. The quadrature phase can be chosen from the phase of α_1, in other words, the quadratures of a_2 are measured relative to the phase of the local oscillator.

("Only relative phases are measurable.")

Picture of average quadratures vs. phase angle for different states of signal mode a_2: vacuum state, number state, coherent state.

Detect the squeezing of the signal mode: need to look at variances of the photon number $N = A^\dagger A$, that involve the variances $\langle \Delta X_\theta^2 \rangle$. If these variances depend on the phase θ of the local oscillator, then we see squeezing.

1.4 Two modes, many modes

Material not covered in SS 2013. Kept here for information.

1.4.1 Multi-mode Hilbert space and observables

The state space of a two-mode field is the tensor product of the Fock spaces of two harmonic oscillators. In terms of number states, the basis vectors of this space can be written

$$|n_1; n_2\rangle = |n_1\rangle_{\text{mode}\,1} \otimes |n_2\rangle_{\text{mode}\,2}$$

where the first mode contains n_1 and the second mode n_2 photons. These states are called 'product states'. That have expectation values of products of operators pertaining to mode 1 and 2, that factorize, e.g.,

$$\langle \hat{n}_1 \hat{n}_2 \rangle = \langle \hat{n}_1 \rangle \langle \hat{n}_2 \rangle.$$

But due to the possibility of forming superpositions, there is much more 'space' in the multi-mode Hilbert space. For example, it is possible that two modes 'share' a single photon:

$$\frac{1}{\sqrt{2}} \left(|0; 1\rangle + |1; 0\rangle \right) \tag{1.105}$$

This state is called 'entangled' if no change of basis for the mode expansion exists such that the state is mapped onto a product state (this may be very difficult to check in practice).[3] The state is by no means unphysical, however, since it is

[3] It is simple to see, however, that the expectation value of $\hat{n}_1 \hat{n}_2$ does not factorize. Indeed, $\langle \hat{n}_1 \rangle = \frac{1}{2} = \langle \hat{n}_2 \rangle$ while $\langle \hat{n}_1 \hat{n}_2 \rangle = 0$ since in each component of the state (1.105), at least one mode has zero photons.

generated by

$$\frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger)|0;0\rangle \tag{1.106}$$

where $|0;0\rangle$ is the two-mode vacuum. Such sums of creation operators occur always in the mode expansion of the quantized field. We have seen an example when a single photon is sent onto a beamsplitter, Eq.(1.102). The decay of an excited atomic state generates even a continuous superposition of one-photon states where an infinite number of modes share a single photon.

Many-mode single-photon states are also generated when an atom is illuminated by a single photon: the scattering of this photon by the atom generates, as in the classical electromagnetic theory, a continuous angular distribution of modes with a nonzero amplitude for one-photon excitations.

Finally, what about the density matrix for a multi-mode field? Let us start with the simple case of two modes of the same frequency in thermal equilibrium. According to the general rule, the density matrix is a sum of projectors onto the stationary states $|n_1;n_2\rangle$ of the two-mode system, each weighted with a probability proportional to $\mathrm{e}^{-\beta_1 n_1 - \beta_2 n_2}$. (Use $\beta_i = \hbar\omega_i/k_B T$.) Since the energy is made additively from single-mode energies, we can factorize this density operator:

$$\begin{aligned}
\hat{\rho} &= Z^{-1} \sum_{n_1,n_2} \mathrm{e}^{-\beta_1 n_1 - \beta_2 n_2} \, |n_1;n_2\rangle\langle n_1;n_2| \\
&= Z \sum_{n_1} \mathrm{e}^{-\beta_1 n_1} \, |n_1\rangle\langle n_1| \otimes \sum_{n_2} \mathrm{e}^{-\beta_2 n_2} \, |n_2\rangle\langle n_2| \\
&= Z^{-1} \, \tilde{\rho}_1 \otimes \tilde{\rho}_2
\end{aligned} \tag{1.107}$$

where the $\tilde{\rho}_{1,2}$ are un-normalized density matrices. (This would also hold if the two modes had different frequencies.) The tensor product of the projectors is defined by coming back to the tensor product of states

$$|n_1\rangle\langle n_1| \otimes |n_2\rangle\langle n_2| = (|n_1\rangle \otimes |n_2\rangle)\,(\langle n_1| \otimes \langle n_2|)\,.$$

The trace of the two-mode density matrix (1.107) also factorizes because the matrix elements of a tensor product operator are, by definition, the products of the individual matrix elements

$$\begin{aligned}
\mathrm{tr}\,(\hat{\rho}) &= Z^{-1} \sum_{n_1,n_2} \langle n_1;n_2|\tilde{\rho}_1 \otimes \tilde{\rho}_2|n_1;n_2\rangle \\
&= Z^{-1} \sum_{n_1,n_2} \langle n_1|\tilde{\rho}_1|n_1\rangle\langle n_2|\tilde{\rho}_2|n_2\rangle \\
&= Z^{-1}\,(\mathrm{tr}\,\tilde{\rho}_1)\,(\mathrm{tr}\,\tilde{\rho}_2)
\end{aligned} \tag{1.108}$$

and therefore $Z = Z_1 Z_2 = (1 - \mathrm{e}^{-\beta_1})^{-1}(1 - \mathrm{e}^{-\beta_2})^{-1}$.

Since the density matrix of this thermal two-mode state factorizes, this state is not entangled (averages of products of single-mode operators factorize). This is no longer true, however, if we allow for an interaction between the modes. Then the energy is no longer a sum of single-mode energies, and the previous factorization does no longer work. This is by the way a general rule: interactions between quantum systems lead to entangled states. For this reason, entangled states are much more frequent in Nature than are factorized states. It is a nontrivial task, however, to decide whether a given density matrix describes an entangled state or not.

Digression (*Einschub*): tensor product states and operators

It is somewhat tricky to guess the right formulas for multimode field states and operators. The general rule is the following:

Field operator	$\leftrightarrow$	sum of modes
Field state	$\leftrightarrow$	product of modes

For example, the electric field operator for a two-mode field is given by

$$\mathbf{E}(\mathbf{x},t) = E_1 \boldsymbol{\varepsilon}_1 a_1(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + E_2 \boldsymbol{\varepsilon}_2 a_2(t)\, \mathrm{e}^{\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{h.c.}$$

while a typical state is for example the product state $|n_1; n_2\rangle = |n_1\rangle \otimes |n_2\rangle$. The general rule gets complicated (1) when we allow for superpositions (sums) of product states and (2) when we consider measurements that involve products of different mode operators.

In calculations, one often needs products of operators, like $\mathbf{E}^2(\mathbf{x},t)$. These are computed in the usual way, one has just to take care that operators sometimes do not commute. But this is only relevant for operators acting on the same mode, $[a_1, a_1^\dagger] = 1$, while for different modes

$$[a_1, a_2^\dagger] = 0$$

because they correspond to independent degrees of freedom.

Operator averages in product states. Let us consider the average electric field for the two-mode case written above. Using the mode expansion, we find terms like $\langle a_i(t)\rangle$ ($i = 1,\, 2$) and their adjoints. Now the operator $a_1|\psi\rangle$ is evaluated by letting a_1 act on the first factor of a product state:

$$a_1|n_1; n_2\rangle = (a_1|n_1\rangle) \otimes |n_2\rangle$$

If $|\psi\rangle$ is a sum of product states (entangled state), then this procedure is done for every term in this sum. Sometimes this is formalized by writing the operator as $a_1 \otimes \mathbb{1}$, thus indicating that for the second mode nothing happens. The action of such operator tensor products is apparently defined as

$$A_1 \otimes B_2 |n_1; n_2\rangle = A_1|n_1\rangle \otimes B_2|n_2\rangle \tag{1.109}$$

by letting each operator factor act on the respective state factor. This notation allows to avoid the subscripts 1 and 2 as the relevant mode is indicated by the position in the operator product.

Similarly, the scalar product of tensor products of states is defined by

$$\langle n_1; n_2 | m_1; m_2\rangle = \langle n_1| \otimes \langle n_2 | m_1\rangle \otimes |m_2\rangle = \langle n_1|m_1\rangle\langle n_2|m_2\rangle$$

by taking the scalar product of the corresponding factors.

The average of the electric field for a product of number states is thus zero, as for a single-mode field, because $\langle n|an\rangle = 0$, and this is true for both modes. What about a product state of two coherent states, $|\psi\rangle = |\alpha; \beta\rangle$? It is simple to see that we get the classical result (we assume that both modes have the same frequency ω)

$$\langle \mathbf{E}(\mathbf{x}, t)\rangle = E_1 \boldsymbol{\varepsilon}_1 \alpha \, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + E_2 \boldsymbol{\varepsilon}_2 \beta \, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{c.c.} \tag{1.110}$$

(Note that 'c.c.' and not 'h.c.' occurs.) As a general rule, classical fields can be described by tensor products of coherent states.

Last example where we go quantum: a superposition of coherent product states,

$$|\psi\rangle = c|\alpha; \beta\rangle + d|\beta; \alpha\rangle$$

with some complex amplitudes c, d. Then we find

$$\langle a_1\rangle = |c|^2\alpha + |d|^2\beta$$

if $\langle\alpha|\beta\rangle = 0$. (This is actually never exactly the case, but can be achieved to a very good precision if $|\alpha - \beta| \gg 1$.) This result is an average over the two possible coherent amplitude, weighted with the corresponding probabilities. The average field thus becomes:

$$\langle \mathbf{E}(\mathbf{x}, t)\rangle = E_1 \boldsymbol{\varepsilon}_1 \left(|c|^2\alpha + |d|^2\beta\right) \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}_1\cdot\mathbf{x}} + E_2 \boldsymbol{\varepsilon}_2 \left(|c|^2\beta + |d|^2\alpha\right) \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}_2\cdot\mathbf{x}} + \text{c.c.}$$

Question: this result does not allow to distinguish this state from an 'incoherent mixture' of coherent product states like in (1.110), each state occurring with a probability $|c|^2, |d|^2$. This mixture would be described by the density operator

$$\hat{\rho}_{\text{mix}} = |c|^2|\alpha; \beta\rangle\langle\alpha; \beta| + |d|^2|\beta; \alpha\rangle\langle\beta; \alpha|$$

and gives the same average electric field (exercise). If the coherent amplitudes α, β are closer together, then due to the nonzero overlap $\langle\alpha|\beta\rangle$, one can distinguish superposition and mixture (exercise). Are there observables that can make the difference in the case $\langle\alpha|\beta\rangle = 0$?

Average of single-mode operator. Let us calculate as another example the average photon number in mode 1 for a two-mode field in the entangled state (1.105). The relevant photon number operator is given by $a_1^\dagger a_1$ or, to be more precise, $a_1^\dagger a_1 \otimes \mathbb{1}$. Its action on the entangled state is worked out using linearity and the operator product rule (1.109)

$$\begin{aligned}
&\frac{1}{\sqrt{2}} a_1^\dagger a_1 \otimes \mathbb{1} \left(|0;1\rangle + |1;0\rangle\right) \\
&= \frac{1}{\sqrt{2}} \Big(a_1^\dagger a_1 |0\rangle \otimes |1\rangle + a_1^\dagger a_1 |1\rangle \otimes |0\rangle \Big) \\
&= \frac{1}{\sqrt{2}} |1\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}} |1;0\rangle
\end{aligned}$$

Taking the scalar product with the original state, we find

$$\langle \hat{n}_1 \rangle = \frac{1}{2} \left(\langle 0;1| + \langle 1;0| \right) |1;0\rangle = \frac{1}{2}.$$

Once you have done this calculation, you can use the shorter rule: all we need are the probabilities of having $n_1 = 0, 1, \ldots$ photons in mode 1. For this, collect all product states in the state with the same number of photons n_1 and compute the squared norm of these states. From the probabilities for n_1 photons, you get the average photon number.

Product operators. As a second example, let us compute the average value of the product $a_i^\dagger a_j$ (i, $j = 1$, 2) in a thermal two-mode state. This object occurs when you measure the two-mode field with a photodetector (see paragraph **??** below). The tensor product notation is more cumbersome here and gives

$$a_1^\dagger a_1 \otimes \mathbb{1} \quad \text{or} \quad \mathbb{1} \otimes a_2^\dagger a_2 \quad \text{or} \quad a_1^\dagger \otimes a_2 \quad \text{or} \quad a_1 \otimes a_2^\dagger.$$

The density matrix is a tensor product of thermal single-mode density matrices. We shall see that the result is:

$$\langle a_i^\dagger a_j \rangle_T = \delta_{ij} \bar{n}(T) \tag{1.111}$$

where $\bar{n}(T)$ is the average photon number in a single mode. How does this come about?

When $i = j$, we are left with the calculation of the average photon number for a single mode:

$$\langle a_i^\dagger a_i \rangle = \sum_{n_1,n_2} \langle n_1; n_2 | a_i^\dagger a_i \hat{\rho}_1 \otimes \hat{\rho}_2 | n_1; n_2 \rangle$$

The action of the product density operators factorizes:

$$\hat{\rho}_1 \otimes \hat{\rho}_2 | n_1; n_2 \rangle = \hat{\rho}_1 | n_1 \rangle \otimes \hat{\rho}_2 | n_2 \rangle$$

Each single-mode density operator, acting on a number state, gives the corresponding occupation probability:

$$\hat{\rho}_1 | n_1 \rangle = \sum_{m_1} p_{m_1}(T) | m_1 \rangle \langle m_1 | n_1 \rangle = p_{n_1}(T) | n_1 \rangle,$$

so that we have, using the result for the photon number of one mode

$$\begin{aligned}
\langle a_i^\dagger a_i \rangle &= \sum_{n_1,n_2} p_{n_1}(T) p_{n_2}(T) \langle n_1; n_2 | a_i^\dagger a_i | n_1; n_2 \rangle \\
&= \sum_{n_1,n_2} p_{n_1}(T) p_{n_2}(T) n_i \\
&= \sum_{n_i} p_{n_i}(T) n_i \sum_{n_j} p_{n_j}(T)
\end{aligned}$$

In the last step, we have noted that the double sum can be factorized ($j \neq i$ is the other index). The second sum gives unity because the probabilities are normalized, the first sum gives the average photon number $\bar{n}(T)$ at temperature T and does no longer depend on the mode label (this is because we assumed equal frequencies for both modes). This completes the proof in the case $i = j$.

A similar calculation shows that the average of $a_1^\dagger a_2$ vanishes: indeed, we have

$$\langle n_1; n_2 | a_1^\dagger a_2 | n_1; n_2 \rangle = \langle n_1 | a_1^\dagger | n_1 \rangle \langle n_2 | a_2 | n_2 \rangle = 0.$$

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