ASRR

Resumen T2 2ª Parte: Integración

función peso nodos

$$\alpha(f) = \int_a^b w(x) f(x)\,dx = \sum_{i=0}^{n} \alpha_i f(x_i) + R(f) \rightarrow \text{error}$$

Formas de obtener coefs:

- Integrando polinomios de Lagrange

$$\alpha_i = \int_a^b w(x)\,\ell_i(x)\,dx \qquad \ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$$

n = num nodos − 1 n+1 nodos $(x_0, x_1, \dots, x_n)$

- Obligando exactitud en $\{1, x, \dots, x^n\}$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & & & \\ x_0^n & x_1^n & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \int_a^b w(x)\,dx \\ \int_a^b x\,w(x)\,dx \\ \vdots \\ \int_a^b x^n w(x)\,dx \end{pmatrix}$$

- Integrando al interpolante si p interpola a $f \Rightarrow p(x_i) = f(x_i)\ \forall i = 0 \dots n$

$$\int_a^b p(x)\,dx = \alpha_0 p(x_0) + \dots + \alpha_n p(x_n)$$

Calcular el error

$$R(f) = \int_a^b w(x) f[x_0, \dots, x_n, x]\,\pi(x)\,dx \quad \text{con } \pi(x) = (x-x_0)\dots(x-x_n)$$

$\Downarrow$

Integral generalizado

- Teorema de Valor Medio: si f cont. en $[a,b]$ y g integrable y no cambia de signo en $[a,b]$, entonces $\exists \mu \in [a,b]$ tq

$$\int_a^b f(x)g(x)\,dx = f(\mu)\int_a^b g(x)\,dx$$

$\Downarrow$

Si $\pi(x)$ no cambia de signo en $[a,b]$

$$R(f) = \int_a^b w(x) f[x_0, \dots, x_n, x]\,\pi(x)\,dx = \frac{f^{(n+1)}(\xi)}{(n+1)!}\int_a^b w(x)\pi(x)\,dx$$

$\pi(x)$ cambiaría en x_1

Si $\pi(x)$ cambia de signo

- Sea $\{x_0 = a, x_1, x_2 = b\}$ $f[a, x_1, x_1, b, x] = \dfrac{f[a, x_1, b, x] - f[a, x_1, x_1, b]}{x - x_1}$

$$f[a, x_1, b, x] = f[a, x_1, x_1, b] + f[a, x_1, x_1, b, x](x - x_1)$$

$\dfrac{f^{iv}(\xi)}{4!}$

$f[a, x_1, x_1, b, \mu]\int \dots$

$$R(f) = \int_a^b f[a, x_1, b, x]\,w(x)\pi(x)\,dx = f[a, x_1, x_1, b]\int_a^b w(x)\pi(x)\,dx + \int w(x) f[a, x_1, x_1, b, x](x - x_1)\pi(x)\,dx$$

Fórmulas simples usuales

		$R(f)$	Exact.
• Rectang. Izq.	$\simeq (b-a)f(a)$	$\frac{(b-a)^2}{2}f'(\xi)$	$\mathbb{P}_0$
• Rectang. Der.	$\simeq (b-a)f(b)$	$-\frac{(b-a)^2}{2}f'(\xi)$	$\mathbb{P}_0$
• Punto medio	$\simeq (b-a)f(\frac{a+b}{2})$	$\frac{(b-a)^3}{24}f''(\xi)$	$\mathbb{P}_1$
• Trapecio	$\simeq \frac{b-a}{2}(f(a)+f(b))$	$-\frac{(b-a)^3}{12}f''(\xi)$	$\mathbb{P}_1$
• Simpson	$\simeq \frac{b-a}{6}(f(a)+4f(\frac{a+b}{2})+f(b))$	$-\frac{(b-a)^5}{2880}f^{iv}(\xi)$	$\mathbb{P}_3$

Fórmulas de Newton-Cotes

Se basan en distribución equiespaciada de nodos en $[a,b]$

Para $h = \frac{b-a}{n}$

a, b nodos

• Cerradas: $\int_a^b f(x)dx = \sum_{i=0}^{n} \alpha_i f(a+ih) + R(f)$

a, b no nodos

• Abiertas: $\int_a^b f(x)dx = \sum_{i=1}^{n-1} \alpha_i f(a+ih) + R(f)$

Integración Romberg

Sea T_n fórmula del trapecio para n intervalos

$$\int_a^b f(x)dx = T_n + \underbrace{a_1h^2 + a_2h^4 + \dots + a_mh^{2m}}_{R(f)} \quad \tilde{1}$$

$\Downarrow$ $2n \Rightarrow h \to h/2$

$$\int_a^b f(x)dx = T_{2n} + a_1\frac{h^2}{4} + a_2\frac{h^4}{16} + \dots + a_m\frac{h^{2m}}{2^m} \quad \tilde{2}$$

Con $(\tilde{1})\cdot(-1/3) + 4/3\cdot(\tilde{2})$ =

$$\int_a^b f(x)dx = \frac{4T_{2n} - T_n}{3} + \underbrace{b_2h^4 + \dots + b_mh^{2m}}_{R(f)} \Rightarrow \text{de orden 2 a 4}$$

Integración de Romberg: construir la tabla

$$\begin{array}{llll}
R(0,0) & & & \\
R(1,0) & R(1,1) & & \\
\vdots & \vdots & \ddots & \\
R(N,0) & R(N,1) & \cdots & R(N,N)
\end{array}$$

$R(j,0) = T_{2^j}$

$R(j,k) = \frac{4^k R(j,k-1) - R(j-1,k-1)}{4^k - 1}$

$\Downarrow$

Teorema: si $f \in C[a,b] \Rightarrow \lim_{N\to\infty} R(N,k) = \int_a^b f(x)dx \quad \forall k$

$$T_{2n} = \frac{1}{2}(T_n + h\sum_{i=0}^{n-1} f_{i+1/2})$$

Integración adaptativa

Sea $S(a,b) = \frac{h}{3}(f(a) + 4f(m) + f(b))$ $\qquad h = \frac{b-a}{2}$ $\quad m = \frac{a+b}{2}$

Simpson: $\int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} f^{iv}(\mu)$

$\Downarrow$

Si $|S(a,b) - S(a,m) - S(m,b)| < 15\varepsilon$ (ó 10ε) $\Rightarrow \int_a^b f(x)dx \simeq S(a,m) + S(m,b)$ y $|R(f)| < \varepsilon$

Cuadratura Gaussiana

Teorema: No existe fórmula $\int_a^b w(x)f(x)dx = \alpha_0 f(x_0) + \dots + \alpha_n f(x_n)$ con exactitud $> 2n+1$

max $= 2n+1$ ($n = $ nº nodos -1)

$\hookrightarrow$ Una fórmula tiene exact. $= n+q \iff \mathcal{L}(x^j \pi(x)) = 0 \quad \forall j = 0 \dots q-1 \quad \mathcal{L}(x^q \pi(x)) \neq 0$

$= 2 \cdot$ núm nodos -1

Teorema: $\exists! \, x_0, x_1, \dots x_n \in [a,b]$ tq la fórmula tiene exactitud $2n+1$

Proceso:

1. Obtener nodos $\quad \pi(x) = (x-x_0)(x-x_1)\dots = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \dots + \lambda_{n-1}x^{n-1} + x^n$

   $\mathcal{L}(\pi(x)) = \mathcal{L}(x\pi(x)) = \dots = \mathcal{L}(x^{n-1}\pi(x)) = 0 \Rightarrow$ Encuentras $\lambda_i \; i = 0..n$

   $\pi(x) = \dots = (x-x_0)(x-x_1)\dots$

2. Obtener coefs: Los métodos del principio

3. Error: $R(f) = \frac{1}{(2n+2)!} f^{(2n+2)}(\xi)\mathcal{L}(\pi^2)$

Fórmulas gaussianas

$$\langle f, g \rangle = \mathcal{L}(fg) = \int_a^b w(x)f(x)g(x)dx$$

Se puede obtener $\pi(x)$:

$$\alpha_n P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)$$

$$\alpha_n = \frac{\langle xP_n, P_{n+1}\rangle}{\langle P_{n+1}, P_{n+1}\rangle} \qquad \beta_n = \frac{\langle xP_n, P_n\rangle}{\langle P_n, P_n\rangle} \qquad \gamma_n = \frac{\langle xP_n, P_{n-1}\rangle}{\langle P_{n-1}, P_{n-1}\rangle}$$

Fórmulas gaussianas clásicas

	$[a,b]$	$w(x)$	Recurrencia
Gauss-Legendre	$[-1,1]$	1	$P_0 = 1 \quad P_1 = x \quad P_{n+1} = \frac{2n+1}{n+1} x P_n - \frac{n}{n+1} P_{n-1}$
Gauss-Chebyshev	$[-1,1]$	$\frac{1}{\sqrt{1-x^2}}$	$T_0 = 1 \quad T_1 = x \quad T_{n+1} = 2xT_n - T_{n-1}$