AJRR

Ejercicios T14

De aquí sale la integral del examen

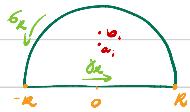
4.- $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = \frac{\pi(a+2b)}{2ab^3(a+b)^2}$ $\quad a \neq b$

$\Omega = \mathbb{C}$

- Sea $f(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)^2} \Rightarrow$ Polos: $x^2+a^2=0 \Leftrightarrow x = \pm ai$
$x^2+b^2=0 \Leftrightarrow x = \pm bi$

luego $A = \{\pm ai, \pm bi\} \Rightarrow A' \cap \Omega = \emptyset$ y $f \in \mathcal{H}(\Omega \setminus A)$

- Definimos ciclo: $\forall R > \max\{a,b\}$ $\Gamma_R = \gamma_R + \sigma_R$ con $\gamma_R : [-R,R] \to \mathbb{C}$ $\gamma_R(x) = x$
$\sigma_R : [0,\pi] \to \mathbb{C}$ $\sigma_R(x) = Re^{ix}$



Γ_R ciclo en $\Omega\setminus A$ y nul-homólogo con $\Omega \Rightarrow$ Tª de Residuos

- Aplicamos: $\displaystyle\int_{\Gamma_R} f(z)dz = 2\pi i\left[\cancelto{1}{Ind_{\Gamma_R}(ai)} Res(f(z),ai) + \cancelto{0}{Ind_{\Gamma_R}(-ai)} Res(f(z),-ai) + \cancelto{1}{Ind_{\Gamma_R}(bi)} Res(f(z),bi) + \cancelto{0}{Ind_{\Gamma_R}(-bi)} Res(f(z),-bi)\right]$

$$\int_{\Gamma_R} f(z)dz = 2\pi i\left[Res(f(z),ai) + Res(f(z),bi)\right]$$

Por otro lado, $\displaystyle\lim_{R\to+\infty}\int_{\Gamma_R} f(z)dz = \lim_{R\to+\infty}\int_{\gamma_R} f(z)dz + \lim_{R\to+\infty}\int_{\sigma_R} f(z)dz = \int_{-\infty}^{+\infty} f(z)dz + \cancelto{0}{\lim_{R\to+\infty}\int_0^\pi \frac{Re^{it}}{(R^2e^{2it}+a^2)(R^2e^{2it}+b^2)^2}} = \int_{-\infty}^{+\infty} f(z)dz$ $\quad \sigma_R'(t)$

Con lo cual $\displaystyle\int_{-\infty}^{+\infty} f(z)dz = \lim_{R\to+\infty}\int_{\Gamma_R} f(z)dz = 2\pi i\left[Res(f(z),ai) + Res(f(z),bi)\right]$

- Calculamos residuos: $\cdot\, ai$: polo de orden $1 \Rightarrow Res(f(z),ai) = \displaystyle\lim_{z\to ai}(z-ai)f(z) = \lim_{z\to ai}\frac{z-ai}{(z^2+a^2)(z^2+b^2)^2} = \lim_{z\to ai}\frac{1}{(z+ai)(z^2+b^2)^2}$

$$Res(f(z),ai) = \frac{1}{2ai(b^2-a^2)^2}$$

$\cdot\, bi$: polo de orden $2 \Rightarrow Res(f(z),bi) = \displaystyle\lim_{z\to bi}\frac{d}{dz}\left((z-bi)^2f(z)\right) = \lim_{z\to bi}\frac{d}{dz}\left[\frac{1}{(z^2+a^2)(z+bi)^2}\right] =$

$$= \lim_{z\to bi}\frac{-(2z(z+bi)^2 + 2(z+bi)(z^2+a^2))}{(z^2+a^2)^2(z+bi)^{\cancel{4}\,3}} = \lim_{z\to bi}\frac{-2z(z+bi)-2(z^2+a^2)}{(z^2+a^2)^2(z+bi)^3} =$$

$$= \frac{4b^2-2(a^2-b^2)}{(a^2-b^2)^2(2bi)^3} =$$

$$= \frac{a^2-3b^2}{4b^3i(a^2-b^2)^2}$$

Con lo que $\displaystyle\int_{-\infty}^{+\infty} f(x)dx = \cancel{2}\pi\cancel{i}\left[\frac{1}{\cancel{2}a\cancel{i}(b^2-a^2)^2} + \frac{a^2-3b^2}{\cancel{4}b^3\cancel{i}(a^2-b^2)^2}\right] = \pi\left[\frac{2b^3+a^3-3b^2a}{2ab^3(a^2-b^2)^2}\right]$

$$= \pi\left[\frac{\cancel{(a-b)^2}(a+2b)}{2ab^3(a+b)^2\cancel{(a-b)^2}}\right] \quad //$$

5.- $a \in \mathbb{R}^+$ $\qquad \displaystyle\int_{-\infty}^{+\infty} \frac{x^6\,dx}{(x^4+a^4)^2} = \frac{3\pi\sqrt{2}}{8a}$

Primero por partes

$$\int_{-\infty}^{+\infty} \frac{x^6}{(x^4+a^4)^2}\,dx = -1/4 \left.\frac{x^3}{x^4+a^4}\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} 1/4 \frac{3x^2}{x^4+a^4}\,dx$$

$$u = x^3 \Rightarrow du = 3x^2dx$$

$$dv = \frac{x^4dx}{(x^4+a^4)^2} \Rightarrow v = -1/4 \frac{1}{x^4+a^4}$$

Vemos que $\displaystyle\lim_{x\to-\infty} -1/4 \frac{x^3}{x^4+a^4} = \lim_{x\to+\infty} \frac{-1/4\, x^3}{x^4+a^4} = 0$

luego estudiamos $\displaystyle 3/4 \int_{-\infty}^{+\infty} \frac{x^2}{x^4+a^4}\,dx$

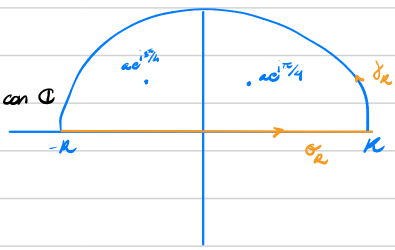
Sea $\Omega = \Omega^\circ = \mathbb{C}$ y $z^4+a^4=0 \Leftrightarrow z = \sqrt[4]{-a^4} = \sqrt[4]{a^4}\cdot\xi = a\cdot\xi$ con $\xi \in \{e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}\}$

Con $A = [\sqrt[4]{-a^4}] = \{ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, ae^{i7\pi/4}\} \Rightarrow A' = \emptyset \Rightarrow A' \cap \mathbb{C} = \emptyset$

Def $\displaystyle f(z) = \frac{z^2}{z^4+a^4}$ $\quad \forall z \in \Omega\setminus A \Rightarrow f \in \mathcal{H}(\Omega\setminus A)$

Usaremos semicirculo como cadena

con $R > a \Rightarrow \Gamma_R = \sigma_R + \gamma_R \Rightarrow \Gamma_R^* \subset \mathbb{C}\setminus A$ y nul-homólogo con $\mathbb{C}$



$\sigma_R(x) = x \quad \forall x \in [-R, R]$

$\gamma_R(x) = Re^{ix}$ $\quad \forall x \in [0,\pi]$

Aplicamos el T.ª de Residuos:

$$\int_{\Gamma_R} f(z)dz = 2\pi i \sum_{x\in A} \operatorname{Res}(f(z), x)\cdot \operatorname{Ind}_{\Gamma_R}(x) =$$

$$= 2\pi i\cdot[\operatorname{Res}(f(z), ae^{i\pi/4})\, \underbrace{\operatorname{Ind}_{\Gamma_R}(ae^{i\pi/4})}_{1} + \operatorname{Res}(f(z), ae^{i3\pi/4})\, \underbrace{\operatorname{Ind}_{\Gamma_R}(ae^{i3\pi/4})}_{1}]$$

las demás singularidades tendrían índice 0.

Por otro lado:

$$\int_{\Gamma_R} f(z)dz = \int_{\sigma_R} f(z)dz + \int_{\gamma_R} f(z)dz$$

$\gamma_R'(x) = Rie^{ix} \quad \forall x \in [0,\pi]$

- $\displaystyle\int_{\gamma_R} f(z)dz = \int_0^\pi f(\gamma_R(x))\gamma_R'(x)dx = \int_0^\pi \frac{R^2e^{2ix}}{R^4e^{4ix}+a^4} Rie^{ix}\,dx \Rightarrow \lim_{R\to+\infty} \int_{\gamma_R} f(z)dz = 0$

- $\displaystyle\int_{\sigma_R} f(z)dz = \int_{-R}^{R} f(x)dx = \int_{-R}^{R} \frac{x^2}{x^4+a^4}\,dx \Rightarrow \lim_{R\to+\infty} \int_{\sigma_R} f(z) = \int_{-\infty}^{+\infty} f(x)dx \Rightarrow$ la que buscamos

luego $\displaystyle\int_{-\infty}^{+\infty} f(x)dx = \lim_{R\to+\infty}\int_{\sigma_R} f(z)dz = \lim_{R\to+\infty}\int_{\Gamma_R} f(z)dz = \lim_{R\to+\infty} [2\pi i(\operatorname{Res}(f(z), ae^{i\pi/4}) + \operatorname{Res}(f(z), ae^{i3\pi/4}))] \Rightarrow$ No depende de R, luego el límite no afecta.

Calculamos residuos sabiendo que ambos son polos simples. Por la caracterización de los polos, como

- $\displaystyle\lim_{z\to ae^{i\pi/4}} (z-ae^{i\pi/4}) f(z) = \frac{0}{0} \overset{\text{L'Hôpital}}{=} \lim_{z\to ae^{i\pi/4}} \frac{\cancel{z^2}}{4z^{\cancel{3}}} = 1/4 \frac{1}{ae^{\pi/4 i}} = \frac{3\sqrt{2}}{4a}(1-i)^{-1}$

- $\displaystyle\lim_{z\to ae^{i3\pi/4}} (z-ae^{i3\pi/4}) f(z) = \frac{0}{0} \overset{\text{L'Hôpital}}{=} \lim_{z\to ae^{i3\pi/4}} \frac{\cancel{z^2}}{4z^{\cancel{3}}} = 1/4 \frac{1}{ae^{\pi 3/4 i}} = \frac{3\sqrt{2}}{4a}(-1-i)^{-1}$

$+ \left\} = \dfrac{\sqrt{2}}{4a}(-i)\right.$

Podemos calcular su residuo ya que para un polo de orden K: $\displaystyle\operatorname{Res}(f(z), w) = \frac{1}{(K-1)!}\lim_{z\to w} \frac{d^{K-1}}{dz^{K-1}}((z-w)^K f(z))$

luego para $K=1 \Rightarrow$ Coincide con los límites calculados

Por lo tanto

$$\int_{-\infty}^{+\infty} f(x)dx = 3/4\int_{-\infty}^{+\infty} \frac{x^2}{x^4+a^4}\,dx = 3/4\, \cancel{2}\pi \cancel{i}\, \frac{\sqrt{2}}{\cancel{8}\,a}(-\cancel{i}) = \frac{3\pi\sqrt{2}}{8a}$$

6.- $n \in \mathbb{N}$, $n > 2$ Frontera de $D(0,R) \cap \{z \in \mathbb{C}^* : 0 < \arg z < 2\pi/n\}$ $R \in \mathbb{R}^+$

$$\int_0^{+\infty} \frac{dx}{1+x^n} = \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n}$$

$\Omega = \mathbb{C}$ y $1+z^n = 0 \Longleftrightarrow z^n = -1 \Rightarrow z = [\sqrt[n]{-1}] = \{e^{i\pi/n + \frac{2k\pi i}{n}} ; k = 0 \dots n-1\} = A \Rightarrow A' = \emptyset \Rightarrow A' \cap \Omega = \emptyset$

Sea $f(z) = \dfrac{1}{1+z^n}$ $\forall z \in \Omega \setminus A \Rightarrow f \in H(\Omega \setminus A)$

Vemos que con

$$\gamma_{R_1}(x) = x \ \forall x \in [0,R] \quad \gamma_{R_2}(x) = Re^{ix} \ \forall x \in [0, 2\pi/n] \quad \gamma_{R_3}(x) = xe^{\frac{i2\pi}{n}} \ \forall x \in [0,R]$$

y con $\Gamma_R = \gamma_{R_1} + \gamma_{R_2} - \gamma_{R_3} \Rightarrow \Gamma_R^* \subset \Omega \setminus A$ $\forall R > 1$ y nul-homólogo a Ω.

Tª de Residuos

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \sum_{x \in A} \operatorname{Ind}_{\Gamma_R}(x) \operatorname{Res}(f(z), x) = 2\pi i [\overbrace{\operatorname{Ind}_{\Gamma_R}(e^{i\pi/n})}^{1} \operatorname{Res}(f(z), e^{i\pi/n})]$$ el resto $\operatorname{Ind}_{\Gamma_R} = 0$

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f(z), e^{i\pi/n})$$

Por otro lado,

$$\int_{\Gamma_R} f(z)\,dz = \sum_{i=1}^{3} \int_{\gamma_{R_i}} f(z)\,dz$$

•) $i = 1$: $\displaystyle\int_{\gamma_{R_1}} f(z)\,dz = \int_0^R f(x)\,dx = \int_0^R \frac{1}{1+x^n}\,dx \Rightarrow \lim_{R \to +\infty} \int_{\gamma_{R_1}} f(z)\,dz = \int_0^{+\infty} \frac{1}{1+x^n}\,dx = I$

$\left|\int_{\gamma_{R_2}} f(z)\,dz\right| \leq \frac{2\pi}{n} \frac{R}{R^n - 1} \xrightarrow{R \to +\infty} 0$ $x_1 - x_0$

•) $i = 2$: $\displaystyle\int_{\gamma_{R_2}} f(z)\,dz = \int_0^{2\pi/n} f(Re^{ix}) Rie^{ix}\,dx = \int_0^{2\pi/n} \frac{Re^{ix}}{1+R^n e^{inx}}\,dx \Rightarrow \lim_{R \to +\infty} \int_{\gamma_{R_2}} f(z)\,dz = 0$

•) $i = 3$: $\displaystyle\int_{\gamma_{R_3}} f(z)\,dz = \int_0^R f(xe^{i2\pi/n}) \cdot e^{i2\pi/n}\,dx = \int_0^R \frac{e^{i2\pi/n}}{1+x^n \cancel{e^{2\pi i}}^{\,1}} \Rightarrow \lim_{R \to +\infty} \int_{\gamma_{R_3}} f(z)\,dz = e^{i2\pi/n} I$

Buscamos I, sabiendo que

$$\lim_{R \to +\infty} \int_{\Gamma_R} f(z)\,dz = I - e^{i2\pi/n} I = I(1 - e^{i2\pi/n})$$

y como los residuos no dependen de R

$$I(1 - e^{i2\pi/n}) = \lim_{R \to +\infty} \int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f(z), e^{i\pi/n})$$

Vemos que claramente se trata de un polo simple. Calculamos residuo

$$\lim_{z \to e^{i\pi/n}} (z - e^{i\pi/n}) f(z) = \text{"}\frac{0}{0}\text{"} \overset{L'H}{=} \lim_{z \to e^{i\pi/n}} \frac{1}{nz^{n-1}} = \frac{1}{n e^{i\pi} e^{-i\pi/n}} = -\frac{e^{i\pi/n}}{n} = \operatorname{Res}(f(z), e^{i\pi/n})$$

Con lo cual

$$I = \frac{-2\pi i e^{i\pi/n}}{n(1 - e^{2\pi i/n})} = \frac{\pi}{n} \cdot \frac{2ie^{i\pi/n}}{1 - e^{2\pi i/n}} = \frac{\pi}{n} \cdot 2i \cdot \frac{1}{e^{-i\pi/n} - e^{i\pi/n}} = \frac{\pi}{n} \cdot \frac{1}{\operatorname{sen}(\pi/n)} = \pi/n \operatorname{cosec}(\pi/n)$$

$\hookrightarrow \cos(\pi/n) - i\operatorname{sen}(\pi/n) - \cos(\pi/n) - i\operatorname{sen}\pi/n = -2i\operatorname{sen}\pi/n$

7.- $a, t \in \mathbb{R}^+$ $\qquad \displaystyle\int_{-\infty}^{+\infty} \frac{\cos(tx)}{(x^2+a^2)^2}\,dx = \pi/2a^3 (1+at)\, e^{-at}$

Sea $\Omega = \mathbb{C}$, $z^2 + a^2 = 0 \iff z = \pm ai \Rightarrow A = \{ai, -ai\} \Rightarrow A' = \emptyset \Rightarrow A' \cap \Omega = \emptyset$

Def. $f(z) = \dfrac{e^{itz}}{(z^2+a^2)^2}$ $\rightarrow$ luego nos quedamos con la parte real $\quad \forall z \in \Omega \setminus A \Rightarrow f \in \mathcal{H}(\Omega \setminus A)$

$\forall R > a \Rightarrow \Gamma_R = \gamma_R + \sigma_R$ con $\begin{array}{l} \gamma_R(x) = x \quad \forall x \in [-R, R] \\ \sigma_R(x) = Re^{ix} \quad \forall x \in [0, \pi] \end{array}$ $\Big\}$ Semicirculo

Tª Residuos: $\displaystyle\int_{\Gamma_R} f(z)dz = \int_{\gamma_R} f(z)dz + \int_{\sigma_R} f(z)dz = 2\pi i \,\mathrm{Res}(f(z), ai)$

$\displaystyle\lim_{R \to +\infty} \int_{\Gamma_R} f(z)dz = \int_{-\infty}^{+\infty} f(x)dx + 0 = 2\pi i\, \mathrm{Res}(f(z), ai)$

Como ai polo de orden 2: $\displaystyle \mathrm{Res}(f(z), ai) = \lim_{z \to ai} \frac{d}{dz}\left((z-ai)^2 f(z)\right) = \lim_{z \to ai} \frac{d}{dz} \frac{e^{itz}}{(z+ai)^2} = \lim_{z \to ai} \frac{ite^{itz}(z+ai)^2 - e^{itz}\,2(z+ai)}{(z+ai)^{\cancel{4}\,3}} =$

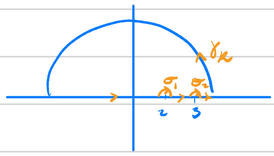
$\displaystyle = \lim_{z \to ai} \frac{e^{itz}(it(z+ai) - 2)}{(z+ai)^3} = \frac{e^{-at}(-2at-2)}{-8a^3 i} = \frac{e^{-at}(at+1)}{4a^3 i}$

luego $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos(tz)}{(z^2+a^2)^2}\,dz = \int_{-\infty}^{+\infty} \mathrm{Re} f(z)\,dz = \mathrm{Re}\left(\cancel{2}\pi \cancel{i}\, \frac{e^{-at}(at+1)}{\cancel{4}2 a^3 \cancel{i}}\right) = \pi/2a^3\,(1+at) \cdot e^{-at}$

Proposición: $\Omega = \Omega^\circ \subset \mathbb{C}$, $a \in \Omega$ $f \in \mathcal{H}(\Omega \setminus \{a\})$. Polo de orden 1 en a. $\forall \varepsilon > 0$

$\sigma_\varepsilon : [t_1, t_2] \to \mathbb{C}$ $\sigma_\varepsilon(t) = a + \varepsilon e^{it} \Rightarrow \lim_{\varepsilon \to 0} \int_{\sigma_\varepsilon} f(z)\,dz = i(t_2 - t_1)\operatorname{Res}(f(z), a)$

8 - $$\int_{-\infty}^{+\infty} \frac{x\,\operatorname{sen}(\pi x)}{x^2 - 5x + 6}\,dx = -5\pi$$



$x^2 - 5x + 6 = 0 \iff x = 2 \wedge x = 3 \Rightarrow$

Sea $\Gamma_{R,\varepsilon} = \sum_{i=1}^{6} \gamma_i$, $\Omega = \mathbb{C}$, $A = \{2, 3\}$ $f(z) = \dfrac{z e^{i\pi z}}{(z-3)(z-2)}$ $\forall z \in \Omega \setminus A$

T.ª de Residuos: $\int_{\Gamma_{R\varepsilon}} f(z)\,dz = 2\pi i \sum_{x \in A} \operatorname{Ind}_{\Gamma_{R\varepsilon}}(a) \operatorname{Res}(f(z), a) = 0$ Están fuera

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{R\varepsilon}} f(z)\,dz = \int_{[-R,R]} f(z)\,dz + \int_{\gamma_R} f(z)\,dz + \lim_{\varepsilon \to 0} \int_{\sigma_1} f(z)\,dz + \lim_{\varepsilon \to 0} \int_{\sigma_2} f(z)\,dz = 0$$

- $\lim_{\varepsilon \to 0} \int_{\sigma_1} f(z)\,dz = \lim_{\varepsilon \to 0} -\int_{-\sigma_1} f(z)\,dz = -i\pi \operatorname{Res}(f(z), 2) = i\pi 2$

$$\operatorname{Res}(f(z), 2) = \lim_{z \to 2} (z-2) f(z) = \lim_{z \to 2} \frac{2 e^{i\pi z}}{z - 3} = -2e^{i\pi 2} = -2$$

- $\lim_{\varepsilon \to 0} \int_{\sigma_2} f(z)\,dz = -i\pi \operatorname{Res}(f(z), 3) = i\pi 3$

$$\operatorname{Res}(f(z), 3) = \lim_{z \to 3} \frac{z e^{i\pi z}}{(z-2)} = -3$$

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{R,\varepsilon}} f(z)\,dz = \int_{[-R,R]} f(z)\,dz + \int_{\gamma_R} f(z)\,dz + 5\pi i = 0$$

- $0 \le \left| \int_{\gamma_R} f(z)\,dz \right| \le \int_{\gamma_R} \frac{|z e^{i\pi z}|}{|(z-2)(z-3)|}\,dz \le \int_{\gamma_R} \frac{R|e^{i\pi z}|}{(R-2)(R-3)}\,dz = \frac{R}{(R-2)(R-3)} \int_0^\pi |e^{i\pi R e^{it}} \cdot R i e^{it}|\,dt = \frac{R^2}{(R-2)(R-3)} \int_0^\pi e^{-\pi R \operatorname{sen} t}\,dt \xrightarrow{R \to +\infty} 0$

luego $\lim_{R \to +\infty} \int_{\gamma_R} f(z)\,dz = 0$

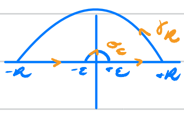
Con lo que

$$\lim_{R \to +\infty} \left[\lim_{\varepsilon \to 0} \int_{\Gamma_{R,\varepsilon}} f(z)\,dz \right] = \int_{-\infty}^{+\infty} f(x)\,dx + 5\pi i = 0 \Rightarrow \int_{-\infty}^{+\infty} f(x)\,dx = -5\pi i$$

y por lo tanto $\displaystyle\int_{-\infty}^{+\infty} \frac{x\,\operatorname{sen}(\pi x)}{x^2 - 5x + 6}\,dx = \int_{-\infty}^{+\infty} \operatorname{Im}(f(x))\,dx = \operatorname{Im}(-5\pi i) = -5\pi$ //

9.- $f(z) = \frac{1-e^{2iz}}{z^2}$ en frontera de la mitad superior de $A(0,\varepsilon,R)$ probar

$$\int_0^{+\infty} \frac{(\operatorname{sen} x)^2}{x^2}\,dx = \pi/2$$



$\Omega = \mathbb{C}$ $\quad A = \{0\}$ $\quad f \in H(\Omega \setminus A)$

Siendo $\Gamma_{R,\varepsilon} = [-R,-\varepsilon] + \sigma_\varepsilon + [\varepsilon, R] + \gamma_R$

Por el T^a de Residuos, como $0 \notin A(0;\varepsilon,R) \Rightarrow \overset{\forall \varepsilon, R>0}{\int_{\Gamma_{R,\varepsilon}} f(z)dz} = 2\pi i \sum_{x\in A} \operatorname{Ind}_{\Gamma_{R,\varepsilon}}(x) \operatorname{Res}(f(z),x) = 0$

Desarrollando, como este resultado no depende de ε o R

$$0 = \lim_{\varepsilon\to 0} \int_{\Gamma_{R,\varepsilon}} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\gamma_R} f(z)dz + \lim_{\varepsilon\to 0}\int_{\sigma_\varepsilon} f(z)dz$$

y comprobamos que es simple el polo en 0

$$\lim_{z\to 0} z\,\frac{1-e^{2iz}}{z^2} = \frac{0}{0} \overset{L'H\hat{o}p}{=} \lim_{z\to 0} \frac{-2ie^{2iz}}{1} = -2i \neq 0 \Rightarrow \text{Polo simple, con lo que}$$

$$\lim_{\varepsilon\to 0}\int_{\sigma_\varepsilon} f(z)dz = -\lim_{\varepsilon\to 0}\int_{-\sigma_\varepsilon} f(z)dz = -\pi i \operatorname{Res}(f(z),0) = -2\pi$$

luego $\quad 0 = -2\pi + \int_{[-R,R]} f(z)dz + \int_{\gamma_R} f(z)dz$

los resultados previos no dependen de R, luego

$$2\pi = \lim_{R\to+\infty}\left[\int_{[-R,R]} f(z)dz + \int_{\gamma_R} f(z)dz\right]$$

$\cdot \; \lim_{R\to+\infty} \int_{[-R,R]} f(z)dz = \int_{-\infty}^{+\infty} f(z)dz = \int_{-\infty}^{+\infty} \frac{1-e^{2iz}}{z^2}dz$

$\cdot \; \lim_{R\to+\infty}\int_{\gamma_R} f(z)dz = \lim_{R\to+\infty}\int_0^\pi \frac{1-e^{2iRe^{iz}}}{R^{\cancel{2}}e^{\cancel{2}iz}}\cancel{R}i\cancel{e^{iz}}\,dz = \lim_{R\to+\infty}\int_0^\pi \frac{1-e^{2iRe^{iz}}}{Re^{iz}}\,i\,dz = 0$

$$\left|\frac{1-e^{2iRe^{iz}}}{Re^{iz}}\,i\right| \le \frac{2}{R} \xrightarrow{R\to+\infty} 0 \Rightarrow \left|\int_0^\pi \frac{1-e^{2iRe^{iz}}}{Re^{iz}}\,i\,dz\right| \le \pi\cdot\frac{2}{R} \xrightarrow{R\to+\infty} 0$$

Con lo cual

$$\int_{-\infty}^{+\infty} \frac{1-e^{i2z}}{z^2}dz = 2\pi$$

$$e^{i2z} = (e^{iz})^2 = (\cos z + i\operatorname{sen} z)^2 = \cos^2 z + 2i\operatorname{sen} z\cos z - \operatorname{sen}^2 z$$

$$1 - e^{i2z} = 1 - \cos^2 z + 2i\operatorname{sen} z\cos z + \operatorname{sen}^2 z = 2\operatorname{sen}^2 z + 2i\operatorname{sen} z\cos z$$

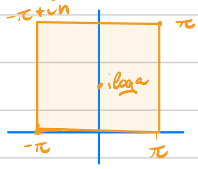
luego $\quad \operatorname{Re}(f(z)) = \frac{2\operatorname{sen}^2 z}{z^2} \Rightarrow \int_{-\infty}^{+\infty} \frac{\operatorname{sen}^2 x}{x^2}dx = \frac{1}{2}\operatorname{Re}\left(\int_{-\infty}^{+\infty}\frac{1-e^{i2z}}{z^2}dz\right) = \pi$

y como $x \longmapsto \frac{\operatorname{sen}^2 x}{x^2}$ par $\quad \int_{-\infty}^{0} \frac{\operatorname{sen}^2 x}{x^2}dx = \int_0^{+\infty}\frac{\operatorname{sen}^2 x}{x^2}dx = \frac{\pi}{2}$ //

10.- $a \in \mathbb{R}$, $a > 1$ $\quad f(z) = \dfrac{z}{a - e^{-iz}}$ sobre $[-\pi, \pi, \pi + in, -\pi + in, -\pi]$ con $n \in \mathbb{N}$ para

$$\int_{-\pi}^{\pi} \frac{x \operatorname{sen} x \, dx}{1 + a^2 - 2a\cos x} = \frac{2\pi}{a} \log\left(\frac{1+a}{a}\right)$$

$$a - e^{-iz} = 0 \iff e^{-iz} = a \iff -iz = \log a \iff z = -\frac{1}{i}\log a = i\log a$$



$\Omega = \mathbb{C}$ $\quad A = i\log a \Rightarrow f \in \mathcal{H}(\Omega \setminus A)$

$\Gamma_n = \underbrace{[-\pi, \pi]}_{\gamma_1} + \underbrace{[\pi, \pi+in]}_{\gamma_2} + \underbrace{[\pi+in, -\pi+in]}_{\gamma_3} + \underbrace{[-\pi+in, \pi]}_{\gamma_4}$

T.ª de Residuos $\displaystyle\int_\Gamma f(z)\,dz = \sum_{j=1}^{4} \int_{\gamma_j} f(z)\,dz = 2\pi i \operatorname{Res}(f(z), i\log a)\, \cancel{\operatorname{Ind}_\Gamma(i\log a)}^{\,1}$

Calculamos $\displaystyle\operatorname{Res}(f(z), i\log a) = \lim_{z \to i\log a} (z - i\log a) f(z) = \lim_{z \to i\log a} \frac{z^2 - i\log a\, z}{a - e^{-iz}} = \frac{0}{0} \overset{\text{L'Hôp}}{=}$

$$= \lim_{z \to i\log a} \frac{2z - i\log a}{ie^{iz}} = \frac{\cancel{i}\log a}{\cancel{i}a} = \frac{\log a}{a}$$

Por otro lado, vemos que $\displaystyle\int_{[-\pi,\pi]} f(z)dz = \int_{-\pi}^{\pi} \frac{z}{a - e^{-iz}}\,dz = \int_{-\pi}^{\pi} \frac{z(a - e^{iz})}{(a-e^{-iz})(a-e^{iz})}\,dz = \int_{-\pi}^{\pi} \frac{za - z\cos z - iz\operatorname{sen} z}{1 + a^2 - 2a\cos z}\,dz \Rightarrow$ la buscada $\displaystyle I = -\int_{[-\pi,\pi]} \operatorname{Im} f(z)\,dz$

$= a^2 - ae^{iz} - ae^{-iz} + 1 = 1 + a^2 - 2a\cos z$

Comprobamos $\gamma_2, \gamma_3, \gamma_4$

- γ_2 y γ_4: $\displaystyle\int_{\gamma_2} f(z)dz + \int_{\gamma_4} f(z)dz = \int_0^n f(\pi + ix)\,i\,dx - \int_0^n f(-\pi+ix)\,i\,dx = i\int_0^n \frac{\pi+ix}{a - e^{-i(\pi+ix)}} - \frac{-\pi+ix}{a - e^{-i(-\pi+ix)}}\,dx = i\int_0^n \frac{2\pi}{a - e^{\pi i + x}}\,dx = 2\pi i \int_0^n \frac{1}{a+e^x}\,dx$

$$\int_0^n \frac{1}{a+e^x}\,dx = \frac{1}{a}\int_0^n \frac{a + e^x - e^x}{a+e^x} = \frac{1}{a}\int_0^n 1 - \frac{e^x}{a+e^x}\,dx = \frac{1}{a}\left[x - \ln(a+e^x)\right]_0^n = \frac{\ln(a+1)}{a} + \frac{n - \ln(a+e^n)}{a} \xrightarrow{n\to+\infty} \frac{\ln(a+1)}{a}$$

$$= \frac{\ln e^n - \ln(a+e^n)}{a} = \frac{\ln\left(\frac{e^n}{a+e^n}\right)}{a} \xrightarrow{n\to+\infty} 0$$

- γ_3: $\displaystyle\int_{\gamma_3} f(z)\,dz = \int_{-\pi}^{\pi} \frac{x+in}{a - e^{-i(x+in)}}\,dx = \int_{-\pi}^{\pi} \frac{x+in}{a - e^{-ix+n}}\,dx \xrightarrow{n\to+\infty} 0$

Con lo cual como los resultados del T.ª de Residuos no dependen de n

$$2\pi i \frac{\log a}{a} = \lim_{n\to+\infty} \int_{\Gamma_n} f(z)dz = 2\pi i \frac{\ln(a+1)}{a} + \int_{-\pi}^{\pi} f(z)dz$$

$$\int_{-\pi}^{\pi} f(z)dz = \frac{2\pi i}{a}(\ln(a) - \ln(a+1)) = \frac{2\pi i}{a}\ln\left(\frac{a}{a+1}\right)$$

$$I = \operatorname{Im}\left(\int_{-\pi}^{\pi} f(z)dz\right) = -2\pi/a \ln\left(\frac{a}{a+1}\right) = 2\pi/a \ln\left(\frac{a+1}{a}\right)$$

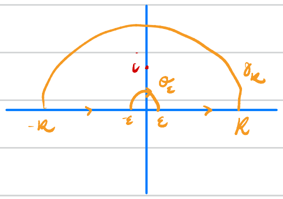
11.- $f(z)$ en frontera de mitad superior de $A(0;\varepsilon,R)$, $\forall \alpha \in]-1,3[$

$$\int_0^{+\infty} \frac{x^\alpha \, dx}{(1+x^2)^2} = \frac{\pi}{4}(1-\alpha)\sec\frac{\pi\alpha}{2}$$

$\to \log$ holomorfo en $\mathbb{C}^* \setminus \{z \in \mathbb{C} \; \arg(z) = -\pi/2\}$

$$z^\alpha = e^{\alpha \log z} \Rightarrow f(z) = \frac{e^{\alpha \log z}}{(1+z^2)^2}$$

$\Omega = \mathbb{C}^* \setminus \{z \in \mathbb{C} \; ; \arg(z) = -\pi/2\}$



$1+z^2 = 0 \Rightarrow z = \pm i \Rightarrow A = \{-i, i\} \Rightarrow A' = \emptyset \Rightarrow \Omega \cap A' = \emptyset$ y $f \in \mathcal{H}(\Omega \setminus A)$

Con $\Gamma_{\varepsilon,R} = [-R,-\varepsilon] + \sigma_\varepsilon + [\varepsilon, R] + \gamma_R \Rightarrow \Gamma^*_{\varepsilon,R} \subset \Omega \setminus A$ nul-homólogo a Ω

Por T^a de Residuos:

$$\int_{\Gamma_{\varepsilon,R}} f(z)dz = 2\pi i \sum_{x \in A} \operatorname{Ind}_{\Gamma_{\varepsilon,R}}(x) \operatorname{Res}(f(z), x) = 2\pi i \cdot \operatorname{Res}(f(z), i)$$

Polo de orden 2 $\Rightarrow \operatorname{Res}(f(z), i) = \lim_{z \to i} \frac{d}{dz}((z-i)^2 f(z)) =$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{e^{\alpha \log z}}{(z+i)^2}\right] = \lim_{z \to i} \frac{(z+i)^2 \alpha e^{(\alpha-1)\log z} - e^{\alpha \log z} 2(z+i)}{(z+i)^{\cancel{4}3}} =$$

$$= \frac{4i\alpha e^{(\alpha-1)i\pi/2} - e^{\alpha i \pi/2} \cdot 4}{-8i} =$$

$$= -1/4 \, i^{\alpha-1}(\alpha-1)$$

Con lo que $\int_{\Gamma_{\varepsilon,R}} f(z)dz = \frac{\pi}{2} i^\alpha (1-\alpha) \quad \forall \varepsilon, R > 0 \quad \varepsilon < 1 < R$

$$\frac{\pi}{2} i^\alpha (1-\alpha) = \frac{\pi}{2} e^{i\alpha\pi/2}(1-\alpha) = \frac{\pi}{2}(1-\alpha)\left[\cos\left(\frac{\alpha\pi}{2}\right) + i\operatorname{sen}\left(\frac{\alpha\pi}{2}\right)\right]$$

$$\lim_{\varepsilon\to 0} \int_{\Gamma_{\varepsilon,R}} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\gamma_R} f(z)dz + \lim_{\varepsilon \to 0}\int_{\sigma_\varepsilon} f(z)dz$$

$$\lim_{\varepsilon\to 0} \int_{\sigma_\varepsilon} f(z)dz = -\pi i \operatorname{Res}(f(z), 0)$$

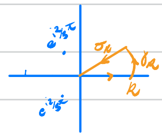
$$\operatorname{Res}(f(z), 0) = \lim_{z\to 0} z \frac{e^{\alpha \log z}}{(1+z^2)^2} =$$

12- $\alpha \in]0,2[$ $\quad \displaystyle\int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{1+e^{x}+e^{2x}} = \int_{0}^{+\infty} \frac{t^{\alpha-1}\,dt}{1+t+t^2} = \frac{2\pi}{\sqrt{3}} \, \frac{\operatorname{sen}(\pi(1-\alpha)/3)}{\operatorname{sen}(\pi\alpha)}$

$$\int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{1+e^x+e^{2x}}\,dx = \int_{0}^{+\infty} \frac{t^{\alpha-1}}{1+t+t^2}\,dt \qquad 1+t+t^2=0 \Rightarrow t = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{3}i}{2} = e^{\pm i 2/3 \pi}$$

$$e^x = t \Rightarrow e^x dx = dt \qquad f(t) = \frac{t^{\alpha-1}}{1+t+t^2} \quad \forall t \in \mathbb{C}\setminus\left\{\frac{-1\pm\sqrt{3}i}{2}\right\}$$

$$dx = \frac{dt}{t}$$



Sea $A = \left\{\frac{-1\pm\sqrt{3}i}{2}\right\} \Rightarrow A' = \emptyset \Rightarrow \Omega = \mathbb{C}$ $\quad \Omega\cap A' = \emptyset$ y $f \in \mathcal{H}(\Omega\setminus A)$

$\Sigma_R = [0,R] + \gamma_R + \sigma_R \Rightarrow \Sigma_R^* \subset \Omega\setminus A$ y nul-homólogo a Ω

Tª de Residuos, como $\operatorname{Ind}_{\Sigma_R}(x) = 0 \quad \forall x \in A \quad \forall R>0$

$$\int_{\Sigma_R} f(z)\,dz = 0$$

$$\int_{\Sigma_R} f(z)\,dz = \int_{[0,R]} f(z)\,dz + \int_{\gamma_R} f(z)\,dz + \int_{\sigma_R} f(z)\,dz$$

- $\displaystyle\lim_{R\to+\infty} \int_{[0,R]} f(z)\,dz = \int_0^{+\infty} f(z)\,dz = I \Rightarrow$ Int que buscamos

- $\displaystyle\int_{\gamma_R} f(z)\,dz = \int_0^{\pi/4} f(Re^{it})\,Rie^{it}\,dt = \int_0^{\pi/4} \frac{R^{\alpha} e^{it(\alpha-1)}}{1+Re^{it}+R^2e^{2it}}\,Rie^{it}\,dt$

$\gamma_R(t) = Re^{it} \quad \forall t\in[0,\pi/4]$

$\gamma_R'(t) = Rie^{it}$

$$\left|\frac{R^{\alpha} i e^{it\alpha}}{1+Re^{it}+R^2e^{2it}}\right| \le \frac{R^{\alpha}}{R^2-R-1} \xrightarrow[\alpha<2]{R\to+\infty} 0 \Rightarrow \left|\int_0^{\pi/4} \frac{R^{\alpha} i e^{it\alpha}}{1+Re^{it}+R^2e^{2it}}\,dt\right| \le \frac{\pi}{4}\,\frac{R^{\alpha}}{R^2-R-1} \xrightarrow{R\to+\infty} 0$$

Luego $\displaystyle\lim_{R\to+\infty}\int_{\gamma_R} f(z)\,dz = 0$

- $\displaystyle\int_{\sigma_R} f(z)\,dz = \int_0^R \frac{t^{\alpha-1}e^{i\pi/4(\alpha-1)}}{1+te^{i\pi/4}+t^2e^{i\pi/2}}\,e^{i\pi/4}\,dt = \int_0^R \frac{t^{\alpha-1}e^{i\pi/4\,\alpha}}{1+te^{i\pi/4}+it^2}\,dt$???

$\sigma_R(t) = te^{i\pi/4} \quad \forall t\in[0,R]$

$\sigma_R'(t) = e^{i\pi/4}$

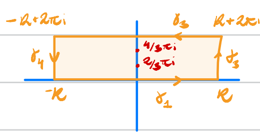
Intento con la exponencial

$$f(z) = \frac{e^{\alpha z}}{1+e^z+e^{2z}} \Rightarrow 1+e^z+e^{2z} = 0$$

$$1+w+w^2 = 0 \Rightarrow w = \frac{-1\pm\sqrt{3}i}{2} \Rightarrow z \in \operatorname{Log}\left(\frac{-1+\sqrt{3}i}{2}\right) = i\frac{2\pi}{3} + 2\pi i\mathbb{Z}$$

$$\Rightarrow z \in \operatorname{Log}\left(\frac{-1-\sqrt{3}i}{2}\right) = -i\tfrac{2}{3}\pi + 2\pi i\mathbb{Z} = i\tfrac{4}{3}\pi + 2\pi i\mathbb{Z}$$

Con $A = i\frac{2}{3}\pi + 2\pi i\mathbb{Z} \cup i\frac{4}{3}\pi + 2\pi i\mathbb{Z} \Rightarrow A' = \emptyset \Rightarrow \Omega = \mathbb{C}$ $\quad \Omega\cap A' = \emptyset$ y $f\in\mathcal{H}(\Omega\setminus A)$



$\Sigma_R = \sum_{i=1}^{4} \gamma_i \Rightarrow \Sigma_R^* \subset \Omega\setminus A$ nul-homólogo

Tª Res $\displaystyle\int_{\Sigma_R} f(z)\,dz = 2\pi i\left(\operatorname{Ind}_{\Sigma_R}(\tfrac{2}{3}\pi i)^{\,1}\operatorname{Res}(f(z),\tfrac{2}{3}\pi i) + \operatorname{Ind}_{\Sigma_R}(\tfrac{4}{3}\pi i)^{\,1}\operatorname{Res}(f(z),\tfrac{4}{3}\pi i)\right)$

- $\displaystyle\operatorname{Res}(f(z),\tfrac{2}{3}\pi i) = \lim_{z\to 2/3\pi i}(z-\tfrac{2}{3}\pi i)f(z) = \lim_{z\to 2/3\pi i}\frac{(z-\frac{2}{3}\pi i)e^{\alpha z}}{1+e^z+e^{2z}} = \text{``}\frac{0}{0}\text{''} \overset{\text{L'Hôpital}}{=} \lim_{z\to 2/3\pi i}\frac{e^{\alpha z}+(z-\frac{2}{3}\pi i)\alpha e^{\alpha z}}{e^z+2e^{2z}} = \frac{e^{2\pi/3\, i\alpha}}{e^{4\pi/3\, i}-1}$

- $\displaystyle\operatorname{Res}(f(z),\tfrac{4}{3}\pi i) = \lim_{z\to 4/3\pi i}(z-\tfrac{4}{3}\pi i)f(z) = \lim_{z\to 4/3\pi i}\frac{(z-\frac{4}{3}\pi i)e^{\alpha z}}{1+e^z+e^{2z}} = \frac{0}{0} \overset{\text{L'Hôpital}}{=} \lim_{z\to 4/3\pi i}\frac{e^{\alpha z}+(z-\frac{4}{3}\pi i)\alpha e^{\alpha z}}{e^z+2e^{2z}} = \frac{e^{4\pi/3\, i\alpha}}{e^{8\pi/3\, i}-1}$

Estudiamos ahora la integral de cada arco

$$\int_{\Sigma_R} f(z)\,dz = \sum_{i=1}^{4}\int_{\gamma_i} f(z)\,dz$$

- γ_1: $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} f(z)\,dz = \int_{-R}^{R}\frac{e^{\alpha z}}{1+e^z+e^{2z}}\,dz = I_R$

- γ_3: $\displaystyle\int_{\gamma_3} f(z)\,dz = \int_{R}^{-R}\frac{e^{\alpha(z+2\pi i)}}{1+e^{z+2\pi i}+e^{2(z+2\pi i)}}\,dz = \int_{R}^{-R}\frac{e^{\alpha z}e^{\alpha 2\pi i}}{1+e^z+e^{2z}} = -e^{\alpha 2\pi i}I_R$

$$\gamma_2 : \left| \int_{\gamma_2} f(z)dz \right| = \left| \int_0^{2\pi} \frac{e^{\alpha(R+ti)}}{1+e^{R+ti}+e^{2R+2ti}}\, i\, dt \right| \leq \int_0^{2\pi} \left| \frac{e^{\alpha(R+ti)}}{1+e^{R+ti}+e^{2R+2ti}} \right| \leq 2\pi \frac{e^{\alpha R}}{e^{2R}-e^{R}-1} \xrightarrow[\alpha<2]{R\to+\infty} 0$$

$\gamma_2(t) = R+ti \quad \forall t\in[0,2\pi]$

$\gamma_2'(t) = i$

$$\gamma_4 : \left| \int_{\gamma_4} f(z)dz \right| = \left| -\int_{-\gamma_4} f(z)dz \right| = \left| -\int_0^{2\pi} \frac{e^{-\alpha R+\alpha ti}}{1+e^{-R+ti}+e^{-2R+2ti}}\, i \right| dt \leq 2\pi \frac{e^{-\alpha R}}{1-e^{-R}-e^{-2R}} \xrightarrow{R\to+\infty} 0$$

Luego $\lim\limits_{R\to+\infty} \int_{\gamma_R} f(z)dz = (1-e^{\alpha 2\pi i})\int_{-\infty}^{\infty} f(z)dz = 2\pi i \left[\frac{e^{\frac{2\pi i}{3}\alpha}}{e^{\frac{4\pi i}{3}}-1} + \frac{e^{4/3\pi i\alpha}}{e^{2/3\pi i}-1} \right] = 2\pi i\, e^{2\pi/3 i\alpha} \left[\frac{1}{e^{4\pi/3 i}-1} + \frac{e^{2/3\pi i\alpha}}{e^{2/3\pi i}-1} \right] =$

$= 2\pi i e^{2\pi/3 i\alpha} \left(-1/2 + \frac{\sqrt{3}}{6} i + \left(-1/2 - \frac{\sqrt{3}}{6} i\right) e^{2/3\pi i\alpha}\right)$

...

$$\text{Si } \alpha = 1 \Rightarrow \int_{-\infty}^{+\infty} \frac{e^x}{1+e^x+e^{2x}}dx = \int_0^{+\infty} \frac{dt}{1+t+t^2} = \int_0^{+\infty} \frac{dt}{\frac{3}{4}+(t+1/2)^2} = \int_0^{+\infty} \frac{4/3}{1+(\frac{2}{\sqrt{3}}(t+1/2))^2} dt = \frac{2}{\sqrt{3}} \left[\operatorname{arctg}\left(\frac{2}{\sqrt{3}}(t+1/2)\right)\right]_0^{+\infty} = \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3}\frac{2}{\sqrt{3}}$$

$(t+1/2)^2 = t^2+t+1/4$

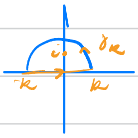
13.- $f(z) = \dfrac{\log(z+i)}{1+z^2}$ por frontera de $\{z\in\mathbb{C} : |z|<R, \operatorname{Im} z>0\}$ y $R>1$

$$\int_{-\infty}^{+\infty} \frac{\log(1+x^2)}{1+x^2}\,dx$$

Sea $\Omega = \mathbb{C}\setminus\{z\in\mathbb{C} : z+i \in \mathbb{R}^- \cup \{0\}\}$

$1+z^2=0 \iff z=\pm i \Rightarrow A=\{\pm i\}$ $\quad A'=\emptyset \Rightarrow \Omega\cap A'=\emptyset$ y $f\in\mathcal{H}(\Omega\setminus A)$

Sea $\Gamma_R = [-R,R]+\gamma_R \Rightarrow \Gamma_R^* \in \Omega\setminus A$ y nul-homólogo a Ω



T.ª de Residuos $\displaystyle\int_{\Gamma_R} f(z)\,dz = 2\pi i\, \overset{1}{\cancel{\operatorname{Ind}_{\Gamma_R}(i)}}\operatorname{Res}(f(z),i) = \pi\ln 2 + \frac{\pi^2}{2} i \quad \forall R>1$

$$\operatorname{Res}(f(z),i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{\log(z+i)}{(z+i)} = \frac{\log(2i)}{2i} = \frac{\ln 2 + \frac{\pi}{2}i}{2i} = -\frac{i\ln 2}{2}+\frac{\pi}{4}$$

Estudiamos $[-R,R]$ y γ_R

· $\displaystyle\lim_{R\to+\infty}\int_{[-R,R]} f(z)\,dz = \int_{-\infty}^{+\infty}\frac{\log(i+t)}{1+t^2}dt = \int_{-\infty}^{+\infty}\frac{\ln|i+t|+i\arg(i+t)}{1+t^2}dt = \int_{-\infty}^{+\infty}\frac{\frac12\ln(t^2+1)}{1+t^2}dt + i\int_{-\infty}^{+\infty}\frac{\arg(i+t)}{1+t^2}dt$

· $\displaystyle\left|\int_{\gamma_R} f(z)\,dz\right| = \left|\int_0^\pi \frac{\log(i+Re^{it})}{1+R^2e^{2it}}Rie^{it}\,dt\right| \le \pi\frac{\ln(1+R)R}{1+R^2} \xrightarrow{R\to+\infty} 0 \Rightarrow \lim_{R\to+\infty}\int_{\gamma_R}f(z)\,dz=0$

$\gamma_R(t)=Re^{it}$ $\forall t\in[0,\pi]$

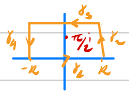
$\gamma_R'(t) = Rie^{it}$ $\forall t\in[0,\pi]$

Con lo cual $\displaystyle\int_{-\infty}^{+\infty}\frac{\ln(t^2+1)}{t^2+1}dt = 2\cdot\lim_{R\to+\infty}\operatorname{Re}\left(\int_{\Gamma_R}f(z)\,dz\right) = 2\pi\ln 2 = \pi\ln 4$

14.- $[-R, R, R+\pi i, -R+\pi i, -R]$. Calcular $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos x \, dx}{e^x + e^{-x}}$

$e^z + e^{-z} = 0 \iff e^{2z} = -1 \Rightarrow 2z = \pi i + 2\pi i \mathbb{Z}$, $z = \frac{\pi}{2} i + \pi i \mathbb{Z}$

$A = \{\frac{\pi}{2} i + \pi i \mathbb{Z}\} \Rightarrow A' = \emptyset$ y con $\Omega = \mathbb{C} \Rightarrow \Omega \cap A' = \emptyset$, y con $f(z) = \dfrac{e^{iz}}{e^z + e^{-z}}$ $\forall z \in \Omega \setminus A$ $f \in \mathcal{H}(\Omega \setminus A)$

$\Gamma_R = [-R, R, R+i\pi, -R+i\pi, -R] = \sum_{j=1}^{4} \gamma_j$  $\forall R > 0$

por ser homólogamente conexo

Vemos que $\Gamma_R^* \subset \Omega \setminus A$ y nul-homólogo con $\Omega \Rightarrow$ T^a de Residuos.

$\displaystyle\int_{\Gamma_R} f(z) dz = 2\pi i \, \cancel{\operatorname{Ind}_{\Gamma}(\tfrac{\pi}{2} i)}^{1} \operatorname{Res}(f(z), \tfrac{\pi i}{2}) = \pi e^{-\pi/2} \Rightarrow$ Esto $\forall R > 0$

- $\operatorname{Res}(f(z), \frac{\pi}{2} i) = \lim_{z \to \frac{\pi}{2} i} (z - \frac{\pi}{2} i) f(z) = \lim_{z \to \frac{\pi}{2} i} \dfrac{z e^{iz} - \frac{\pi}{2} i e^{iz}}{e^z + e^{-z}} = \text{"}\dfrac{0}{0}\text{"} \overset{\text{L'Hôpital}}{=} \lim_{z \to \frac{\pi}{2} i} \dfrac{e^{iz} + z i e^{iz} + \frac{\pi}{2} e^{iz}}{e^z - e^{-z}} = \dfrac{e^{-\pi/2} - \cancel{\frac{\pi}{2} e^{-\pi/2}} + \cancel{\frac{\pi}{2} e^{-\pi/2}}}{e^{\frac{\pi}{2} i} - e^{-\frac{\pi}{2} i}} = \dfrac{e^{-\pi/2}}{i + i} = -\frac{1}{2} i e^{-\pi/2}$

Calculamos ahora integrales:

$\displaystyle\int_{\Gamma_R} f(z) dz = \sum_{j=1}^{4} \int_{\gamma_j} f(z) dz$

- $\displaystyle\lim_{R \to +\infty} \int_{\gamma_1} f(z) dz = \int_{-\infty}^{+\infty} f(t) dt = \int_{-\infty}^{+\infty} \frac{e^{it}}{e^t + e^{-t}} dt = \int_{-\infty}^{+\infty} \frac{\cos t}{e^t + e^{-t}} dt + i \int_{-\infty}^{+\infty} \frac{\operatorname{sen} t}{e^t + e^{-t}} dt = I$

  $\gamma_1(t) = t \; \forall t \in [-R, R]$

- $\displaystyle\left|\int_{\gamma_2} f(z) dz\right| = \left|\int_0^{\pi} \frac{e^{iR-t}}{e^{R+it} + e^{-R-it}} dt\right| \leq \frac{\pi}{e^R - e^{-R}} \xrightarrow{R \to +\infty} 0 \Rightarrow \lim_{R \to +\infty} \int_{\gamma_2} f(z) dz = 0$

  $\gamma_2(t) = R + it \; \forall t \in [0, \pi]$

- $\displaystyle\int_{\gamma_3} f(z) dz = -\int_{\tilde{\gamma}_3} f(z) dz = -\int_{-R}^{R} \frac{e^{i(t+i\pi)}}{e^{t+i\pi} + e^{-t-i\pi}} dt = e^{-\pi} \int_{-R}^{R} \frac{e^{it}}{e^t + e^{-t}} dt \Rightarrow \lim_{R \to +\infty} \int_{\gamma_3} f(z) dz = e^{-\pi} I$

  $\tilde{\gamma}_3(t) = -\gamma_3(t) = t + i\pi \; \forall t \in [-R, R]$

- $\displaystyle\left|\int_{\gamma_4} f(z) dz\right| = \left|-\int_{\tilde{\gamma}_4} f(z) dz\right| = \left|\int_0^{\pi} \frac{e^{i(-R+it)}}{e^{-R+it} + e^{R-it}} dt\right| = \left|\int_0^{\pi} \frac{e^{-iR-t}}{e^{-R+it} + e^{R-it}} dt\right| \leq \frac{\pi}{e^R - e^{-R}} \xrightarrow{R \to +\infty} 0 \Rightarrow \lim_{R \to +\infty} \int_{\gamma_4} f(z) dz = 0$

  $\tilde{\gamma}_4(t) = -\gamma_4(t) = -R + it \; \forall t \in [0, \pi]$

Por lo tanto

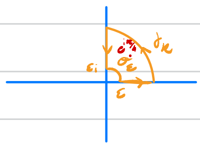
$\displaystyle\pi e^{-\pi/2} = \int_{\Gamma_R} f(z) dz = \lim_{R \to +\infty} \int_{\Gamma_R} f(z) dz = (1 + e^{-\pi}) I \Rightarrow I = \frac{\pi e^{-\pi/2}}{1 + e^{-\pi}} \Rightarrow \int_{-\infty}^{+\infty} \frac{\cos x}{e^x + e^{-x}} dx = \operatorname{Re}(I) = \frac{\pi e^{-\pi/2}}{1 + e^{-\pi}}$ //

15.- Frontera de $\{z\in\mathbb{C};\ \varepsilon<|z|<R,\ 0<\arg z<\pi/2\}$ $\quad 0<\varepsilon<1<R$

$$\int_0^{+\infty}\frac{\log x}{1+x^4}\,dx$$

$1+x^4=0 \Rightarrow x\in[\sqrt[4]{-1}]=\{e^{i\pi/4},\ e^{i3\pi/4},\ e^{i5\pi/4},\ e^{i7\pi/4}\}=A$

$A'=\emptyset \Rightarrow \Omega=\mathbb{C}\setminus\mathbb{R}^-\cup\{0\}$ $\quad \Omega\cap A=\emptyset$ $\quad$ y $\quad f(z)=\dfrac{\log z}{1+z^4}\ \forall z\in\Omega\setminus A$ $\quad$ y $\quad f\in\mathcal{H}(\Omega\setminus A)$



Sea $\Gamma_{\varepsilon,R}=[\varepsilon,R]+\gamma_R+[Ri,\varepsilon i]-\sigma_\varepsilon \Rightarrow \Gamma_{R,\varepsilon}^*\subset\Omega\setminus A$ y $\Gamma_{R,\varepsilon}$ nul-homólogo con Ω, homól. conexo

T^a de Residuos $\Rightarrow \displaystyle\int_{\Gamma_{\varepsilon,R}} f(z)\,dz = 2\pi i\,\overset{1}{\cancel{\operatorname{Ind}_{\Gamma_{\varepsilon,R}}}}(e^{i\pi/4})\operatorname{Res}(f(z),\ e^{i\pi/4}) = 2\pi i\cdot\pi(1-i)\sqrt{2}\cdot\frac{1}{32\cdot 16}=\frac{\pi^2}{16}(1+i)\sqrt{2}$

$\cdot\ \operatorname{Res}(f(z),\ e^{i\pi/4}) = \lim\limits_{z\to e^{\pi/4 i}}(z-e^{\pi/4 i})f(z) = \text{“}\frac{0}{0}\text{”} \overset{\text{L'Hôp}}{=} \lim\limits_{z\to e^{\pi/4 i}}\dfrac{\log z+(z-e^{i\pi/4})\frac{1}{z}}{4z^3} =$

$$=\frac{\pi/4\, i}{4e^{3\pi/4 i}}=\frac{\pi/4\, i}{4(-\sqrt{2}/2+\sqrt{2}/2\, i)}=\frac{\pi(1-i)\sqrt{2}}{32}$$

Obtenemos los límites $\varepsilon\to0$, $R\to+\infty$:

$$\lim_{\varepsilon\to0}\int_{\Gamma_{\varepsilon,R}} f(z)\,dz=\int_{[0,R]}f(z)\,dz+\int_{\gamma_R}f(z)\,dz+\int_{[Ri,0]}f(z)\,dz-\lim_{\varepsilon\to0}\int_{\sigma_\varepsilon}f(z)\,dz$$

$\cdot\ \displaystyle\lim_{R\to+\infty}\int_{[0,R]}f(z)\,dz=\int_0^{+\infty}\frac{\log z}{1+x^4}\,dz=\mathcal{I}$

$\cdot\ \displaystyle\left|\int_{\gamma_R}f(z)\,dz\right|=\left|\int_0^{\pi/2}\frac{\log Re^{it}}{1+R^4e^{4it}}Rie^{it}\,dt\right|\le\pi/2\cdot\frac{R\ln R}{1+R^4}\xrightarrow{R\to+\infty}0\Rightarrow\lim_{R\to+\infty}\int_{\gamma_R}f(z)\,dz=0$

$\gamma_R(t)=Re^{it}\quad\forall t\in[0,\pi/2]$

$\cdot\ \displaystyle\int_{[Ri,0]}f(z)\,dz=-\int_0^R\frac{\log it}{1+t^4}\,i\,dt=-\int_0^R\frac{\log i+\log t}{1+t^4}\,i\,dt=-\int_0^R\frac{\frac{\pi}{2}i+\log t}{1+t^4}\,i\,dt=-i\int_0^R\frac{\log t}{1+t^4}\,dt+\frac{\pi}{2}\int_0^R\frac{1}{1+t^4}\,dt$

$[Ri,0]=-\tilde{\gamma}\Rightarrow\tilde{\gamma}(t)=it\quad\forall t\in[0,R]$

Por el ej 6: $\displaystyle\int_0^{+\infty}\frac{1}{1+t^4}\,dt=\frac{\pi}{4}\operatorname{cosec}\frac{\pi}{4}=\frac{\pi}{4}\sqrt{2}$

Luego

$$\lim_{R\to+\infty}\int_{[Ri,0]}f(z)\,dz=-i\mathcal{I}+\pi/2\ \pi/4\sqrt{2}=-i\mathcal{I}+\frac{\pi^2}{8}\sqrt{2}$$

$\cdot\ \displaystyle\left|\int_{\sigma_\varepsilon}f(z)\,dz\right|=\left|\int_0^{\pi/2}\frac{\log\varepsilon e^{it}}{1+\varepsilon^4e^{i4t}}\varepsilon ie^{it}\,dt\right|\le\frac{\pi}{2}\frac{\log\varepsilon+\pi/2}{1-\varepsilon^4}\varepsilon\xrightarrow{\varepsilon\to0}0\Rightarrow\lim_{\varepsilon\to0}\int_{\sigma_\varepsilon}f(z)\,dz=0$

$\sigma_\varepsilon(t)=\varepsilon e^{it}\quad\forall t\in[0,\pi/2]$

Por lo tanto:

$$(1+i)\frac{\pi^2}{16}\sqrt{2}=\lim_{R\to+\infty}\lim_{\varepsilon\to0}\int_{\Gamma_{\varepsilon,R}}f(z)\,dz=(1-i)\mathcal{I}+\frac{\pi^2}{8}\sqrt{2}$$

$$(-1+i)\frac{\pi^2}{16}\sqrt{2}=(1-i)\mathcal{I}\Rightarrow\mathcal{I}=-\frac{\pi^2}{16}\sqrt{2}\ //$$

16.- $\int_{-\infty}^{+\infty} \frac{e^{x/2}}{e^x+1}\,dx = \int_0^{+\infty} \frac{2\cancel{t}}{(t^2+1)\cancel{t}}\,dt = 2\,\mathrm{arctg}(t)\Big]_0^{+\infty} = 2\frac{\pi}{2} = \pi$

$t = e^{x/2} \Rightarrow dt = 1/2\, e^{x/2}\,dx$

$\frac{2}{t}\,dt = dx$