Spring 2021 ECON200C: Discussion 1

April 2, 2021

Bayes' Rule

- Bayes' Rule: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A) \cdot P(A)}{P(B)}$

- π_s: **prior** probability of state s
- q_m: unconditional probability of receiving message m

$$q_m = \sum_{s \in S} q_{m,s} \pi_s$$
$$= q_{m,s_1} \pi_{s1} + q_{m,s_2} \pi_{s2} + q_{m,s_3} \pi_{s3} + \ldots$$

- $j_{s,m}$: joint probability of state s and message m, e.g. $j_{s,m} = \pi_s q_{m,s}$
- $q_{m,s}$: conditional probability of message m, given state s
- $\pi_{s,m}$: **posterior** probability of state s, given message m

$$\pi_{s,m} = \frac{j_{s,m}}{q_m} = \pi_s \frac{q_{m,s}}{q_m}$$

Lecture Example

Consider lecture example of oil drill. We have three **states** of the oil field, $S = \{s_1, s_2, s_3\}$. Each state corresponds to some probability of having oil underground, but we need some further information to have more precise estimate. Thus, a drill will serve as a **message**, so we have two potential messages, $M = \{wet, dry\}$. All the information we have is the **prior** belief over the states, π_s, and the probability of having oil underground conditional on each state, which is essentially $q_{m,s}$

State s_i	$q_{wet,s}$	$q_{dry,s}$	π_s
s_1	90%	10%	10%
s_2	30%	70%	50%
s_3	0%	100%	40%

Lecture Example

- Q: Calculate out the posterior matrix $\Pi = [\pi_{s,m}]$?
- Note: It is important to identify what is your state and message. In the exam questions, the question might not explicitly tell what is the message m, and neither the information about $q_{m,s}$, you need to learn how to interpret the text and translate into the information you need.

Lecture Example

- First calculate $q_m = \sum_{s \in S} q_{m,s}\pi_s$

$$\begin{aligned} q_{wet} &= q_{wet,s_1}\pi_{s_1} + q_{wet,s_2}\pi_{s_2} + q_{wet,s_3}\pi_{s_3} \\ &= 0.9 * 0.1 + 0.3 * 0.5 + 0 * 0.4 = 0.24 \\ q_{dry} &= q_{dry,s_1}\pi_{s_1} + q_{dry,s_2}\pi_{s_2} + q_{dry,s_3}\pi_{s_3} \\ &= 0.1 * 0.1 + 0.7 * 0.5 + 1 * 0.4 = 0.76 \end{aligned}$$

Lecture Example

- Then calculate posteriors $\pi_{s,m} = \frac{j_{s,m}}{q_m} = \pi_s \frac{q_{m,s}}{q_m}$

$$\pi_{s_1,wet} = \frac{\pi_{s_1} q_{wet,s_1}}{q_{wet}} = \frac{0.9 * 0.1}{0.24} = 0.375$$

$$\pi_{s_2,wet} = \frac{\pi_{s_2} q_{wet,s_2}}{q_{wet}} = \frac{0.3 * 0.5}{0.24} = 0.625$$

$$\pi_{s_3,wet} = \frac{\pi_{s_3} q_{wet,s_3}}{q_{wet}} = \frac{0 * 0.4}{0.24} = 0$$

$$\pi_{s_1,dry} = \frac{\pi_{s_1} q_{dry,s_1}}{q_{dry}} = \frac{0.1 * 0.1}{0.76} = \frac{1}{76}$$

$$\pi_{s_2,dry} = \frac{\pi_{s_2} q_{dry,s_2}}{q_{dry}} = \frac{0.7 * 0.5}{0.76} = \frac{35}{76}$$

$$\pi_{s_3,dry} = \frac{\pi_{s_3} q_{dry,s_3}}{q_{dry}} = \frac{1 * 0.4}{0.76} = \frac{40}{76}$$

Optimal Decision

- The value of an information signal is the marginal benefit you could have by choosing the optimal decision x after observing the signal.
- In other words, you need to compare:

$$\mathbb{E}(u(x_0)) \quad vs \quad \mathbb{E}(\mathbb{E}_m u(x_m))$$

where $x_0 = \arg\max_x \mathbb{E}(u(x,s)) = \arg\max_x \sum_s \pi_s u(x,s)$
and $x_m = \arg\max_x \mathbb{E}_m(u(x,s)) = \arg\max_x \sum_s \pi_{s,m} u(x,s)$

- There is a difference between *ex ante* and *ex post* cases
  - In *ex ante*, one could only decide to buy a message service
  - In *ex post*, one already received the message and the optimal action is based on the specific message

Lecture Example

Consider the drill oil example again, but this time, there are only two states $S = \{wet, dry\}$, and corresponding probability $\pi_{wet} = 0.24, \pi_{dry} = 0.76$.
There are two possible actions, $X = \{drill, no\ drill\}$. If you drill and it is wet, you gain \$1,000,000, if it is dry, you lose \$400,000. If you don't drill, you have a lost of \$50,000 regardless of the state.
There is also a message service, which provides a not accurate estimate of state, giving the likelihood matrix below:

$q_{m,s}$		Message	
		Wet	Dry
State	Wet	0.6	0.4
	Dry	0.2	0.8

Lecture Example

What is x_0, x_m?

- $x_0 = \arg\max_x \sum_s \pi_s u(x, s)$

$$
\begin{aligned}
E(u(drill)) &= \pi_{wet} u(drill, wet) + \pi_{dry} u(drill, dry) \\
&= 0.24 * 1,000,000 + 0.76 * (-400,000) = -64,000 \\
E(u(no\ drill)) &= -50,000
\end{aligned}
$$

- So choose $x_0 =$ no drill

Lecture Example

What is x_0, x_m?

- $x_m = \arg\max_x \mathbb{E}_m(u(x,s))$
- $m =$ dry:

$$\begin{aligned} E(u(drill)|m = dry) &= \pi_{dry,dry} u(drill, dry) + \pi_{wet,dry} u(drill, wet) \\ &= \frac{\pi_{dry} q_{dry,dry}}{q_{dry}} u(drill, dry) + \frac{\pi_{wet} q_{dry,wet}}{q_{dry}} u(drill, wet) \\ &= 0.86 * (-400,000) + 0.14 * 1,000,000- \\ &= -204,000 \\ E(u(no\ drill)) &= -50000 \end{aligned}$$

- So choose $x_{dry} =$ no drill

Lecture Example

What is x_0, x_m?

- $m =$ wet:

$$
\begin{aligned}
E(u(drill)|m = wet) &= \pi_{dry,wet} u(drill, dry) + \pi_{wet,wet} u(drill, wet) \\
&= \frac{\pi_{dry} q_{wet,dry}}{q_{wet}} u(drill, dry) + \frac{\pi_{wet} q_{wet,wet}}{q_{wet}} u(drill, wet) \\
&= 0.51 * (-400,000) + 0.49 * 1,000,000 = 286,000
\end{aligned}
$$

- So choose $x_{wet} =$ drill

$$
\begin{aligned}
\mathbb{E}(\mathbb{E}_m u(x_m)) &= q_{wet} \mathbb{E}_{wet} u(x_{wet}) + q_{dry} \mathbb{E}_{dry} u(x_{dry}) \\
&= [q_{wet,dry} \pi_{dry} + q_{wet,wet} \pi_{wet}] * 286,000 \\
&+ [q_{dry,dry} \pi_{dry} + q_{dry,wet} \pi_{wet}] * (-50,000) \\
&= 0.296 * 286,000 - 0.704 * 50,000 = 49,456
\end{aligned}
$$

- Value of Info = 49,456 - (-50,000) = 99,456