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Robot Programming
C++ Autodiff Example

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Derivatives

In many cases one has to compute derivatives of complicated multivariate functions

Computing them by hand is

- tedious
- error prone
- requires a lot of time

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}$$

Computers like doing boring stuff

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

Numerical Differentiation

Use the definition of derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Given a set of perturbation vectors

$$\epsilon_1 = \begin{pmatrix} \epsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \epsilon_2 = \begin{pmatrix} 0 \\ \epsilon \\ 0 \\ \vdots \end{pmatrix}, \quad \ldots, \quad \epsilon_m = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \epsilon \end{pmatrix}$$

Symmetry around linearization point leads to lower numerical errors

we compute the i^th column by the following

$$\frac{\partial \mathbf{f}}{\partial x_i} \simeq \frac{\mathbf{f}(\mathbf{x}+\epsilon_i) - \mathbf{f}(\mathbf{x}-\epsilon_i)}{2\epsilon}$$

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}$$

Each column requires evaluating ***f*** 2 times

Numerical Differentiation

Choosing epsilon might be non trivial

- too small leads to machine precision errors
- too large poor derivative
- computation might be lowered

However

- easy to implement
- most of the times it works well
- can be used to check your hand-computed derivatives

Automatic Differentiation

Can we get the computer giving us the exact value of the derivative at a point, **without** computing the derivatives analytically?

Derivatives are mechanic!

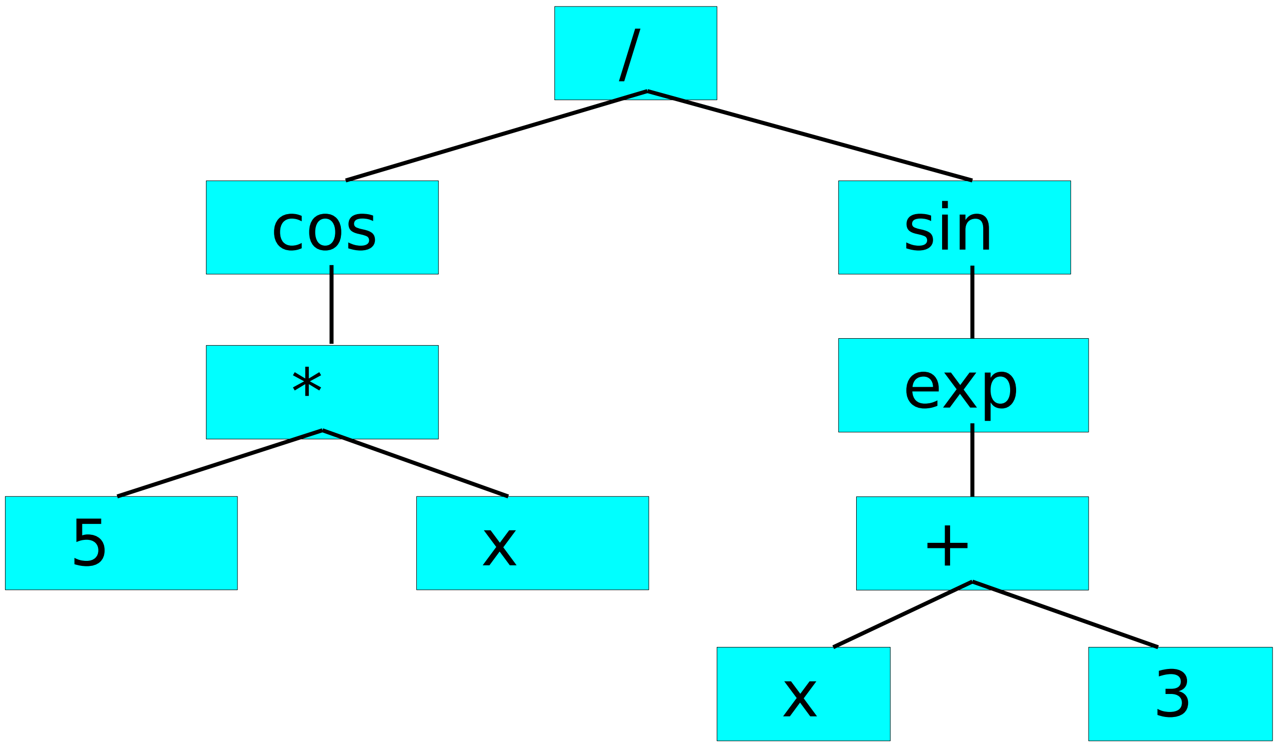
Can't we get the computer **evaluating** them for us?

Parsing Tree

Consider the following expression

```
cos(5*x)/(sin(exp(x+3)))
```

Its parsing tree looks like that



Chain Rule

To **evaluate** the derivative of a nested function

$$f(g(x))$$

in a point $\breve{x}$ we need to know

- The **formula** of the derivative of *f*

$$\frac{\partial f(y)}{\partial y} = f'(y)$$

- The **value** of the argument of *f*

$$y = g(\breve{x})$$

- The **value** of the derivative of the argument

$$y' = g'(\breve{x})$$

Chain rule tells us that

$$\frac{\partial f(g(x))}{\partial x}\Big|_{x=\breve{x}} = f'(y)y'$$

Parsing Tree for Derivatives

For a generic function, we can use the parsing tree to compute

- the value of a function
- the value of the derivative of the function

We need to replace the basic type (float/double) with a pair (u, u'), and redefine the operators consistently

Atoms and Unary Functions

Atoms

- The variable used for differentiation becomes a pair [x, 1]
- All constants become a pair [c_i, 0]

Transcendental functions

- sin([u,u'])=[sin(u),cos(u)*u']
- cos([u,u'])=[cos(u),-sin(u)*u']
- exp([u,u'])=[exp(u),exp(u)*u']
- ....

Binary Functions (operators)

Sum, Subtraction, Multiplications and Division are implemented by applying the derivative rules on their arguments

- [u,u']+[v,v']=[u+v,u'+v']
- [u,u']-[v,v']=[u-v,u'-v']
- [u,u']*[v,v']=[uv,u'v+v'u]
- [u,u']/[v,v']=[u/v,(u'v-v'u)/v^2]

Autodiff in C++

Redefining the operators over a new type allow us to exploit the parsing tree constructed by to compute a function AND its derivative

All we need to do is

- to define a new type DualValue, which will contain both x, and x'.
- to redefine all operators and "standard" math functions

Autodiff in C++

We introduced new elements

operator T(): defines what happens when a cast to T is enforced;

we had to make it explicit which of the default functions should be called for the basic types (the compiler got confused)

Exercise

Write a program that computes all **values** and **derivatives** of the function in the range x=[-1:1], y=[-1,1] with step size 0.1, with respect to x and y (2 evaluations)

$$\mathbf{f}(x, y) = \frac{\sin x \cdot \cos y}{\ln(2 + \sin x)}$$

Hint: when deriving w.r.t. x, y is a constant (x.derivative =0). when deriving w.r.t y, x is a constant (y.derivative=0).