2D Isometries: An Introduction

Draft

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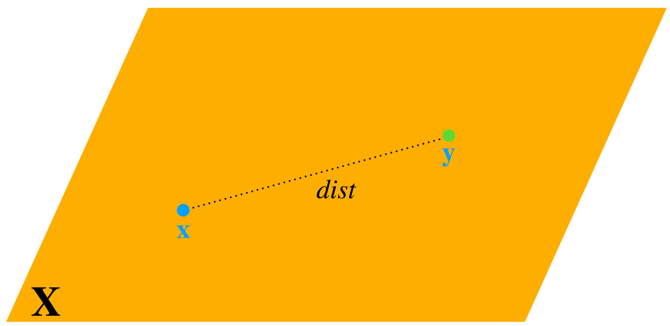
1 Isometry

1.1 Definition

We present an intuitive idea of what an isometry is. First, let us take a set X filled with generic elements. A very common property for a set of elements is the possibility to define a *distance* between two of its elements.



If we take a set X where we have defined a function $d(x_1, x_2)$ to calculate the distance between two points in X, we will call X a *metric space*[1].
The most familiar *metric space* is 3-dimensional Euclidean space, with the euclidean norm as *metric function*. In general, finding a space where it is not possible to define a metric is not very intuitive. But of course you can try.
Now, given a metric space, an isometry is a transformation $y = f(x)$ which maps elements to the same or another metric space such that the distance $d(f(x_1), f(x_2))$ between the image elements in the new metric space is equal to the distance between the elements in the original metric space $d(x_1, x_2)$.

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \tag{1}$$

The 2-dimensional Euclidean space together with the Euclidean norm is a metric space. In this context the isometries are all and only those transforms that include translations and rotations within them. By composing rotations and translations in the plane we can create generic isometries

1.2 Example: Express point coordinates in new reference frame

Many are the situations where isometries are used. Perhaps the most common is the change reference. Being able to compute the position of an object in a different reference system is fundamental: the position of an object and its movement are always relative to the point from where it is observed. This is

[1]If you want to know more: https://en.wikipedia.org/wiki/Metric_space

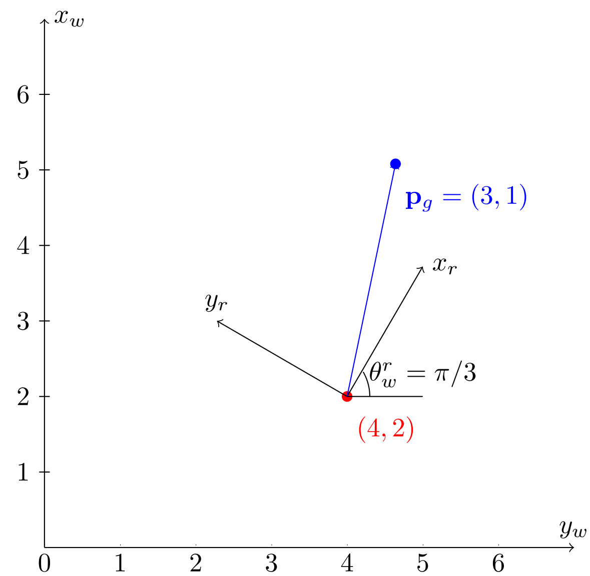


Figure 1: Problem set

relevant, since it is also one of the foundations of Einstein's theory of relativity. So, let's see a couple of practical applications of isometries.

1.3 Problem 1

Given the position of the point $\mathbf{p}_g$ in the reference frame (x_r, y_r) (r stands for robot) in figure 1, express the position of $\mathbf{p}_g$ in the world frame. Notice that:

- The robot frame w.r.t. the world frame is rotated and translated.
- Translation and angle of rotation are given.

1.4 Solution Problem 1

We need to express the position of the point p_r in the world frame. To this end it is useful to think about what is the simplest way to express this position, based on the data we have. This is quite simple: $\mathbf{p}_r$ (in world frame) is the vector resulting from the sum of the vector $\mathbf{t}_r^w$ and $\mathbf{t}_g^r$ (figure 2).
The only problem is that we know the vector $\mathbf{t}_g^r$ (or the position of $\mathbf{p}_g$ if you want) only in the robot reference frame. This means that we cannot simply add the two vectors in this forms. Doing so would result in $(7, 3)$ which is clearly wrong.
This happens because the vector $\mathbf{t}_g^r$ is in robot frame, which is rotated w.r.t.

the world frame. However if we are able to *rotate* $\mathbf{t}_g^r$ by an angle θ_r we have aligned the to reference frame and we can procede with the sum (see figure 3).

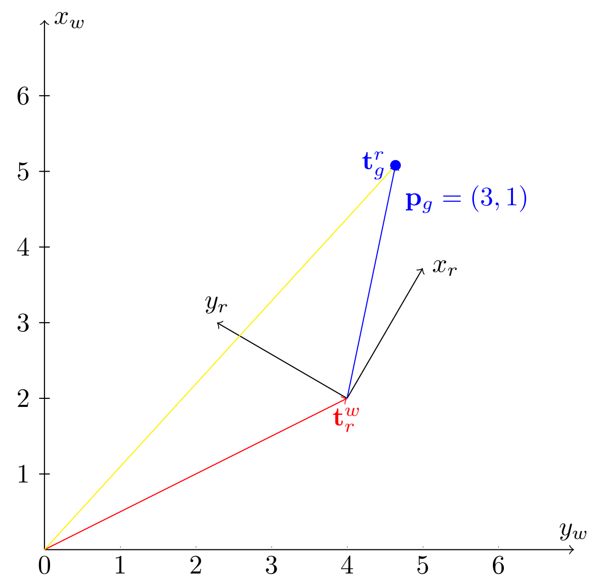


Figure 2: The vector in yellow is the solution we are looking for, and it is the sum of the two vector $\mathbf{t}_r^w$ and $\mathbf{t}_g^r$

In order to *rotate* $\mathbf{t}_g^r$ we need to use a rotation matrix. In linear algebra, a rotation matrix is a transformation matrix that is used to perform a rotation in Euclidean space (so exactly what we need). The column and row vectors of a rotation matrix express a set of unit vectors at 90 between each other. For example the matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

rotates points in the xy-plane counterclockwise by an angle θ with respect to the x axis about the origin of a two-dimensional Cartesian coordinate system. An important property of rotation matrix is that if you want to compute the rotation matrix of $-\theta$ (so to speek, the inverse rotation of the previous rotation), it is just given by the expression:

$$\mathbf{R}(-\theta) = \begin{bmatrix} cos(-\theta) & -sin(-\theta) \\ sin(-\theta) & cos(-\theta) \end{bmatrix} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$$

This means that the inverse rotation of a given rotation $\mathbf{R}$ it is just computed as the transpose of the rotation matrix $\mathbf{R}^T$. With the rotation matrices in place, let's come back to our problem and let us compute the rotation matrix that rotates $\mathbf{t}_g^r$ by $\frac{\pi}{3}$.

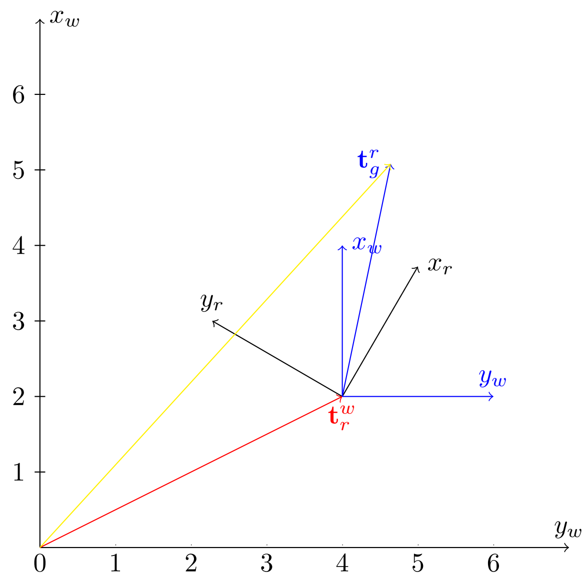


Figure 3: The right solution can be found with the sum of the vector $\mathbf{t}_r^w$ and the vector $\mathbf{t}_g^r$ express in the world reference frame

We know from the data given by the problem that the robot reference frame is rotated w.r.t the world reference frame by an angle of $\theta_w^r = \pi/3$.

$$\mathbf{R}(\pi/3) = \begin{bmatrix} cos(\pi/3) & -sin(\pi/3) \\ sin(\pi/3) & cos(\pi/3) \end{bmatrix}$$

Is the rotation matrix that rotates vector in the robot reference frame into the world reference frame. For this reason we are going to name this rotation matrix $\mathbf{R}_r^w$.
The $\mathbf{t}_g^r$ expressed i a frame oriented as the world frame is finally given by:

$$\mathbf{t}_g^w = \mathbf{R}_r^w \cdot \mathbf{t}_g^r \tag{2}$$

Once we have the vector $\mathbf{t}_g^w$ we have almost solved the problem: all we need to do is now translating $\mathbf{t}_g^w$ by $\mathbf{t}_w^r$. The position of the point $\mathbf{p}_g$ expressed in world coordinate is given by the expression:

$$\mathbf{p}_g^w = \mathbf{t}_r^w + \mathbf{t}_g^w = \mathbf{t}_r^w + \mathbf{R}_r^w \cdot \mathbf{t}_g^r \tag{3}$$

Substituting the numbers of the problem we got:

$$\begin{pmatrix} p_{g_x}^w \\ p_{g_y}^w \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{bmatrix} cos(\pi/3) & -sin(\pi/3) \\ sin(\pi/3) & cos(\pi/3) \end{bmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} p_{g_x}^w \\ p_{g_y}^w \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 0,63 \\ 3,09 \end{pmatrix} = \begin{pmatrix} 4,63 \\ 5,09 \end{pmatrix}$$

NB: It is straightforward to verify that the two vector (3,1) and (0.63,3.09) are actually the same, but rotated, vector, since the norm is preserved. Infact:

$$\sqrt{3^2 + 1^2} = \sqrt{0.63^2 + 3.09^2} \tag{5}$$

1.5 Problem 2

In the previous problem we have analyzed the situation with two different reference frames. But actually there is no limit on the number of reference frames that may be considered in a problem.
In this section we consider the situation where different reference frames (see the figure 4). Along with the reference frames we got two relative transformations that bring from one reference frame to another: in particular we have the transformation that brings from (x_w, y_w) to (x_r, y_r) and the one from (x_r, y_r) to (x_s, y_s). This means that we have both the translation vector that brings from one reference to the other ($\mathbf{t}_r^w$ and $\mathbf{t}_s^r$) and the angles that express the difference in orientation between the two reference frames (θ_r^w and θ_s^r).

NB: The transformations are relative to the starting reference frame. This means for example that the translation vector $\mathbf{t}_s^r$ is expressed in the reference frame (x_r, y_r).

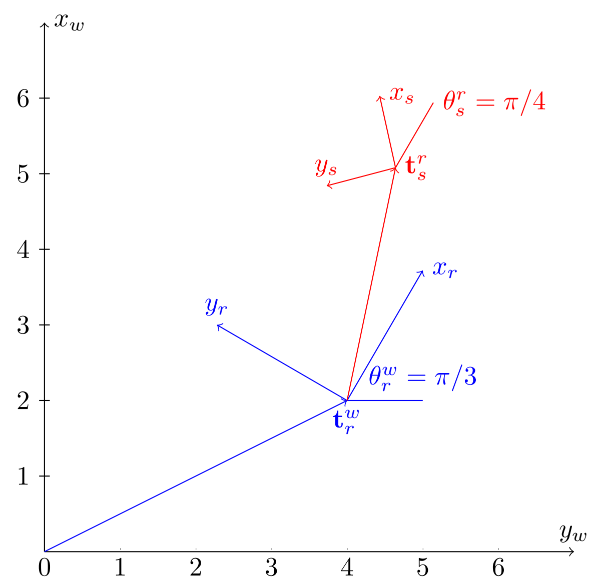


Figure 4: We ha three reference frames (x_w, y_w), (x_r, y_r), (x_s, y_s), and the two relative transformation that bring from (x_w, y_w) to (x_r, y_r) and from (x_r, y_r) to (x_s, y_s)

What we want to find is the transformation that brings from the world frame to (x_s, y_s). So given the three reference frames and their relative transformation:

$$W \longrightarrow R \longrightarrow S$$

We want to find:

$$\mathbf{R}_s^w, \mathbf{t}_s^w$$

1.6 Solution 2

1.6.1 Composed Translation

Let's start finding the composed translation vector $\mathbf{t}_s^w$. This is the sum of the two vectors $\mathbf{t}_r^w$ and $\mathbf{t}_s^r$. But again, as in the previuos problem, the vector $\mathbf{t}_s^r$ is expressed in a different reference frame w.r.t the world. So we need to transform it; this means rotating it:

$$\mathbf{t}_s^w = \mathbf{R}_r^w \cdot \mathbf{t}_s^r$$

where the notation $\mathbf{t}_s^w$ stands for the vector $\mathbf{t}_s$ oriented as world frame. The rotation matrix is the same as before so:

$$\mathbf{t}_s^w = \begin{bmatrix} cos(\pi/3) & -sin(\pi/3) \\ sin(\pi/3) & cos(\pi/3) \end{bmatrix} \cdot \mathbf{t}_s^r$$

The composed translation vector so is given by the expression:

$$\mathbf{t}_{tot}^w = \mathbf{R}_r^w \cdot \mathbf{t}_s^r + \mathbf{t}_r^w = \mathbf{t}_s^w + \mathbf{t}_r^w \tag{6}$$

1.6.2 Composed Rotation

Now let's calculate the composed rotation. This is simply the composition of the two rotation $\mathbf{R}_w^r$ (rotation from W to R) and $\mathbf{R}_r^s$ (rotation from R to S). This actually quite intuitive: the rotation matrix that brings us from $W \longrightarrow S$ can be seen as the rotation matrix the at first brings us from (x_w, y_w) to (x_r, y_r) and then (x_r, y_r) to (x_s, y_s). Thus the solution is given by:

$$\mathbf{R}_w^s = \mathbf{R}_r^s \cdot \mathbf{R}_w^r = \begin{bmatrix} cos(\theta_s^r) & sin(\theta_s^r) \\ -sin(\theta_s^r) & cos(\theta_s^r) \end{bmatrix} \cdot \begin{bmatrix} cos(\theta_r^w) & sin(\theta_r^w) \\ -sin(\theta_r^w) & cos(\theta_r^w) \end{bmatrix}$$

And remember that the rotation matrix that gives us the rotation from the reference frame S to the world is the inverse of the rotation matrix just computed. Notice that, due to properties of orthonormal matrix, the inverse of $\mathbf{R}$ is just $\mathbf{R}^T$

1.7 General Solution

Suppose now that you have n reference frames, and for each of them you have the relative transformation that brings from the reference $i-1$ to i where $i = 0, ..., n$. We want to find the composed transformation (translation and rotation) that brings from the first reference frame to the last one:

$$< \mathbf{t}_n^0; \mathbf{R}_n^0 >$$

1.7.1 Rotation

For what concern the total rotation the solution is straightforward. As in the case with $n = 2$ the solution is simply the composition of the n rotations.

$$\mathbf{R}_n^0 = \prod_{i=1}^{n} \mathbf{R}_i^{i-1}$$

1.7.2 Translation

The translation part is a little bit more complicated. Let's take back the example with $n = 2$ and the equation 6 and substitute letters with numbers, in order to be extended easier:

$$\mathbf{t}_2^0 = \mathbf{R}_1^0 \cdot \mathbf{t}_2^1 + \mathbf{t}_1^0 \tag{7}$$

Now suppose that $n = 3$. The equation become:

$$\mathbf{t}_3^0 = \mathbf{R}_1^0 \cdot \mathbf{R}_2^1 \cdot \mathbf{t}_3^2 + \mathbf{R}_1^0 \cdot \mathbf{t}_2^1 + \mathbf{t}_1^0 \tag{8}$$

This because the vector $\mathbf{t}_3^2$ belongs to a reference frame that is subject to two rotation w.r.t the reference frame zero.
Now we are ready to write down the general solution:

$$\mathbf{t}_n^0 = \sum_{i=1}^{n} \left(\prod_{k=0}^{i-1} \mathbf{R}_k^{k-1} \right) \mathbf{t}_i^{i-1} \tag{9}$$

where $\mathbf{R}_0^{-1} = \mathbf{I}$.
We have now the final solution and we can write:

$$< \mathbf{t}_n^0; \mathbf{R}_n^0 > = < \sum_{i=1}^{n} \left(\prod_{k=0}^{i-1} \mathbf{R}_k^{k-1} \right) \mathbf{t}_i^{i-1};\ \prod_{i=1}^{n} \mathbf{R}_i^{i-1} > \tag{10}$$

2 Homogeneous Transformation

Let's come back on the first problem (see figure 1 and 2).
In this scenario we have two reference frames (the world-frame and the robot-frame) and it is given to us a point whose coordinates lie in the robot frame. The problem asked to find the coordinates of the point in the world coordinate frame and we have seen that the solution is given by:

$$\mathbf{p}_g^w = \mathbf{t}_r^w + \mathbf{R}_r^w \cdot \mathbf{t}_g^r$$

If we analyze this solution we notice that it is composed of two parts, which are the rotation and the translation (as already said many times).
The question we ask ourselves now is this: would it be possible to merge rotation and translation in a single operation? The answer is yes of course, and in order to construct this new solution we use *Homogeneous transformation*. Let's now un-pack and simplify the notation of the previous equation. This will helps to think about these concepts in a more general way:

$$\begin{pmatrix} p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} t_x \\ t_y \end{pmatrix} + \mathbf{R} \cdot \begin{pmatrix} p_x \\ p_y \end{pmatrix} \tag{11}$$

So we have the point (p_x', p_y') that is the result of sum of a translation vector t and a point (p_x, p_y) to which we applied a generic rotation matrix $\mathbf{R}$ whose elements can be named as:

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

So equation 11 can be written as:

$$\begin{pmatrix} p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} t_x \\ t_y \end{pmatrix} + \begin{pmatrix} r_{11} \cdot p_x + r_{12} \cdot p_y \\ r_{21} \cdot p_x + r_{22} \cdot p_y \end{pmatrix} = \begin{pmatrix} r_{11} \cdot p_x + r_{12} \cdot p_y + t_x \\ r_{21} \cdot p_x + r_{22} \cdot p_y + t_y \end{pmatrix} \tag{12}$$

Homogeneous transformation takes the right part of equation 12 and express it as a single matrix transformation. To be able to do this merge operation we need to augment the dimension of our point $\mathbf{p}$ and $\mathbf{p}'$. This new dimension is completely fictitious, has no physical meaning, its value is always equal to one and it simply serves to make this merging operation possible. So if we have a point with coordinates $\mathbf{p} = (p_x, p_y)$. Its homogeneous coordinates is:

$$\mathbf{p}_{hom} = \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} \tag{13}$$

With the same reasoning the homogeneous coordinates of $\mathbf{p}'$ is given by:

$$\mathbf{p}_{hom}' = \begin{pmatrix} p_x' \\ p_y' \\ 1 \end{pmatrix}$$

Even if can sound as a strange operation to do, this little trick allows to merge translation and rotation in to a single operation. In fact, we can define the matrix:

$$\mathbf{T} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{R}} & \mathbf{t} \\ \hline 0 & 0 & 1 \end{array}\right] = \begin{pmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

And if we apply the *homogeneous transformation* $\mathbf{T}$ to the vector $\mathbf{p}_{hom}$ we obtain:

$$\mathbf{p}'_{hom} = \mathbf{T} \cdot \mathbf{p}_{hom} = \begin{pmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} \cdot p_x + r_{12} \cdot p_y + t_x \\ r_{21} \cdot p_x + r_{22} \cdot p_y + t_y \\ 1 \end{pmatrix} \tag{15}$$

At this point one might wonder why one would want to use homogeneous transformations. The reason lies in the problem we addressed in section 1.7:
We have n reference frame, and for each of them you ave the relative transformation that brings from the reference $i-1$ to i where $i = 0, ..., n$. We want to find the composed transformation (translation and rotation) that brings from the first reference frame to the last one.
We have seen that the solution to this problem is given by the equation 10. But this is solution is actually quite hard to handle.
Instead, if we transform the coordinates of our problem in homogeneous coordinates, the solution to the problem is just given by the multiplication of each homogeneous transformation!
So if we have n transformations, the composed transformation is given by:

$$\mathbf{T}_0^n = \mathbf{T}_0^1 \cdot \mathbf{T}_1^2 \cdot ... \cdot \mathbf{T}_{n-1}^n \tag{16}$$

2.1 Properties of Homogeneous Transformation

Homogeneous transformation forms a group [2]. Therefore the following properties are satisfied:

- Given the hom. transformations $\mathbf{T}_0$ and $\mathbf{T}_1$, then $\mathbf{T}_0 \cdot \mathbf{T}_1$ is an hom. transformation.
- The identity matrix is identity-homogeneous transformation
- $\mathbf{I} = \mathbf{T} * \mathbf{T}^{-1}$
- Given $\mathbf{T}_a^b$ and $\mathbf{T}_b^c$, then $\mathbf{T}_a^c = \mathbf{T}_a^b \cdot \mathbf{T}_b^c$, and $\mathbf{T}_c^a = (\mathbf{T}_a^c)^{-1} = (\mathbf{T}_c^b)^{-1} \cdot (\mathbf{T}_a^b)^{-1}$

[2] What is a group: https://en.wikipedia.org/wiki/Group_(mathematics)

2.2 General Case and Approach

We will now analyze a more general case, together with the de-facto approach for tackling easely this type of problems.
For simplicity in the figure 2.2, we have used the following notation:

- With the blue spot we indicate a reference frame. We have five different reference frame.
- With the arrows we indicate the homogeneous transformation between two reference frame. The direction of the arrow indicates the direction of the transformation.

The general approach that we propose to find a transformation between two generic reference frames A_i, A_j is the following:

1. Find a path that leads from A_i to A_j and chain the transformation from left to right.
2. Take the direct transformation $\mathbf{T}$ if the arrow goes with the flow, otherwise take the inverse $\mathbf{T}^{-1}$.

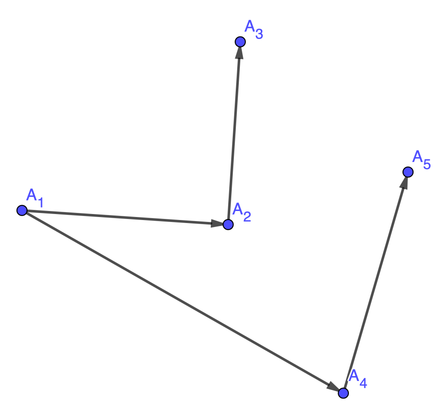


Figure 5: A_i with $i = 1, ..., 5$ represent the reference frames, and the arrows represent the homogeneous transformations between two reference frames.

With this approach in mind now we want to find to new composed transformations: the $A_4 \longrightarrow A_3$ and the $A_2 \longrightarrow A_5$. These two transformations are represented in the figure 2.2.1.

2.2.1 Solution

The solution for the two transformations, given the previous approach is quite simple. For the $A_2 \longrightarrow A_5$ transformation the solution is given by:

$$\mathbf{T}_2^5 = (\mathbf{T}_1^2)^{-1} \cdot \mathbf{T}_1^4 \cdot \mathbf{T}_4^5 \tag{17}$$

For the $A_4 \longrightarrow A_3$ transformation instead, the solution is:

$$\mathbf{T}_4^3 = (\mathbf{T}_4^1)^{-1} \cdot \mathbf{T}_1^2 \cdot \mathbf{T}_2^3 \tag{18}$$

Observe that, given the solution 17, the transformation $\mathbf{T}_4^3$ can be expressed also as:

$$\mathbf{T}_4^3 = (\mathbf{T}_4^5) \cdot (\mathbf{T}_2^5)^{-1} \cdot \mathbf{T}_2^3 \tag{19}$$

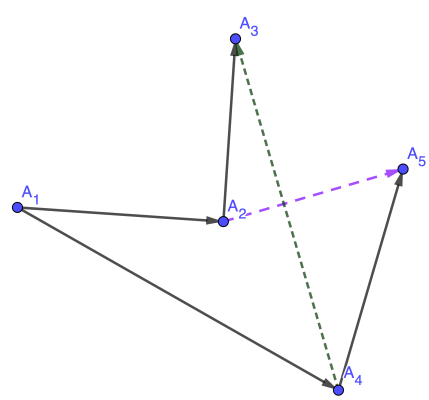


Figure 6: Find the transformation $A_4 \longrightarrow A_3$ and $A_2 \longrightarrow A_5$