Jensen-Tsallis Divergence for Supervised Classification under Data Imbalance

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Introduction

Due to its unique properties (such as boundedness, symmetry, and the ability to handle more than two distributions), the Jensen-Shannon Divergence (JSD) has been applied in various DL contexts: supervised classification, adversarial training, domain generalization, etc.

In this study, we analyze the use of JSD and Jensen-Tsallis Divergence (JTD) in supervised classification under data imbalance.

Main Contributions:

- Regularization interpretation of Jensen-Shannon Divergence.
- Extending this interpretation to Jensen-Tsallis Divergence, highlighting its added flexibility.
- Empirical evidence showing JTD enhances generalization in supervised classification.
- Investigation of JSD and JTD effects in imbalanced classification scenarios.

Related works

[Pereyra et al., 207] introduced a **confidence penalization** term at the output, penalizing low-entropy output distributions:

$$\mathcal{L}_p = D_{KL}(e^{(y)}||\tilde{p}) - \gamma\mathbb{H}[\tilde{p}], \tag{1}$$

where $e^{(y)}$ is a one-hot encoded vector, $\tilde{p}$ is the probability vector estimated by the network, γ is a hyperparameter that controls the strength of the confidence penalty, D_{KL} denotes the Kullback-Leibler Divergence, and $\mathbb{H}$ indicates Shannon's entropy.

According to [Mukhoti et al., 2020], Eq. (1) is related with the focal loss [Lin et al., 2017] (widely used to deal with calibration issues) as follows:

$$\mathcal{L}_f = -(1-\tilde{p}_y)^{\gamma}\log\tilde{p}_y \geq \mathcal{L}_p = D_{KL}(e^{(y)}||\tilde{p}) - \gamma\mathbb{H}[\tilde{p}]. \tag{2}$$

Theoretical Background

Assume a general functional class $\mathcal{F}$, where each $f \in \mathcal{F}$ maps an input $\boldsymbol{x} \in \mathbb{X}$ to the probability simplex Δ^{K-1}, i.e., to a categorical distribution over K classes $y \in \mathbb{Y} = \{1, 2, \ldots, K\}$. We seek $f^* \in \mathcal{F}$ that minimizes a risk, $R_{\mathcal{L}}(f) = \mathbb{E}_{\mathcal{D}}\left[\mathcal{L}\left(e^{(y)}, f(\boldsymbol{x})\right)\right]$, for some loss function $\mathcal{L}$ and joint distribution $\mathcal{D}$ over $\mathbb{X}\times\mathbb{Y}$, where $e^{(y)}$ is a K-dimensional vector with one at index y and zero elsewhere. In practice, $\mathcal{D}$ is unknown and, instead, we use $\mathcal{S} = \{(x_i, y_i)\}_{i=1}^N$, which are assumed to be identically and independently sampled from $\mathcal{D}$, to minimize an empirical risk $\frac{1}{N}\sum_{i=1}^N \mathcal{L}\left(e^{(y_i)}, f(x_i)\right)$.

Jensen-Tsallis Divergence

For two probability distributions, the JSD is defined as follows:

$$JSD^{\pi}(p,\rho) = \pi_1 D_{KL}(p||m) + \pi_2 D_{KL}(\rho||m) = \mathbb{H}[m] - \pi_1\mathbb{H}[p] - \pi_2\mathbb{H}[\rho], \tag{3}$$

where $p \in \Delta^{K-1}$ and $\rho \in \Delta^{K-1}$ are the two discrete probability distributions over K classes, $\pi \in \Delta$ is the weight distribution that assigns different importances to the two distributions, $m = \pi_1 p + \pi_2\rho$ is the weighted average distribution.

In the **typical supervised learning scenario**, where the labels follow a one-hot distribution $e^{(y)}$:

$$JSD^{\pi}(e^{(y)},\tilde{p}) = \mathbb{H}\left[\pi_1 e^{(y)} + \pi_2\tilde{p}\right] - \pi_2\mathbb{H}[\tilde{p}]. \tag{4}$$

The **Jensen-Tsallis Divergence** (JTD) is a generalization of the JSD that introduces the q-logarithm $log^{(q)}$:

$$JTD^{\pi}_q(e^{(y)},\tilde{p}) = \sum_{j=1}^{K}\left(\pi_1 e_j^{(y)} + \pi_2\tilde{p}_j^q\right)\log^{(q)}(m_j) - \pi_2\mathbb{H}_q[\tilde{p}], \tag{5}$$

The JTD can be expressed using the Tsallis divergence, $D_T^{(q)}$, as follows:

$$JTD^{\pi}_q(e^y,\tilde{p}) = \pi_1 D_T^{(q)}(e^y||m) + \pi_2 D_T^{(q)}(\tilde{p}||m)$$
$$= -\pi_1\log^{(q)}(\pi_1 + \pi_2\tilde{p}_y) - \pi_2\left(\sum_{j=1\wedge j\neq y}^{K}\tilde{p}_j\log^{(q)}(\pi_2) + \tilde{p}_y\log^{(q)}\left(\frac{\pi_1}{\tilde{p}_y} + \pi_2\right)\right).$$

Here, the last term plays a regularization role over the confidence output $\tilde{p}_y$ of the network.

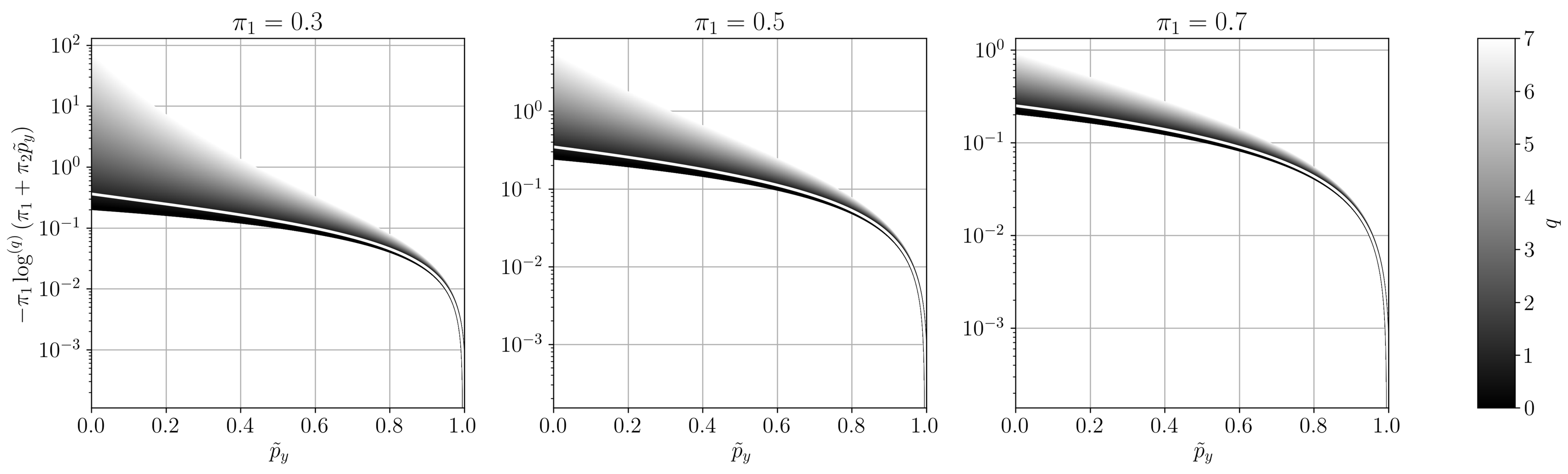


Figure: Numerical representation of the first term of the JTD $-\pi_1\log^{(q)}(\pi_1 + \pi_2\tilde{p}_y)$.

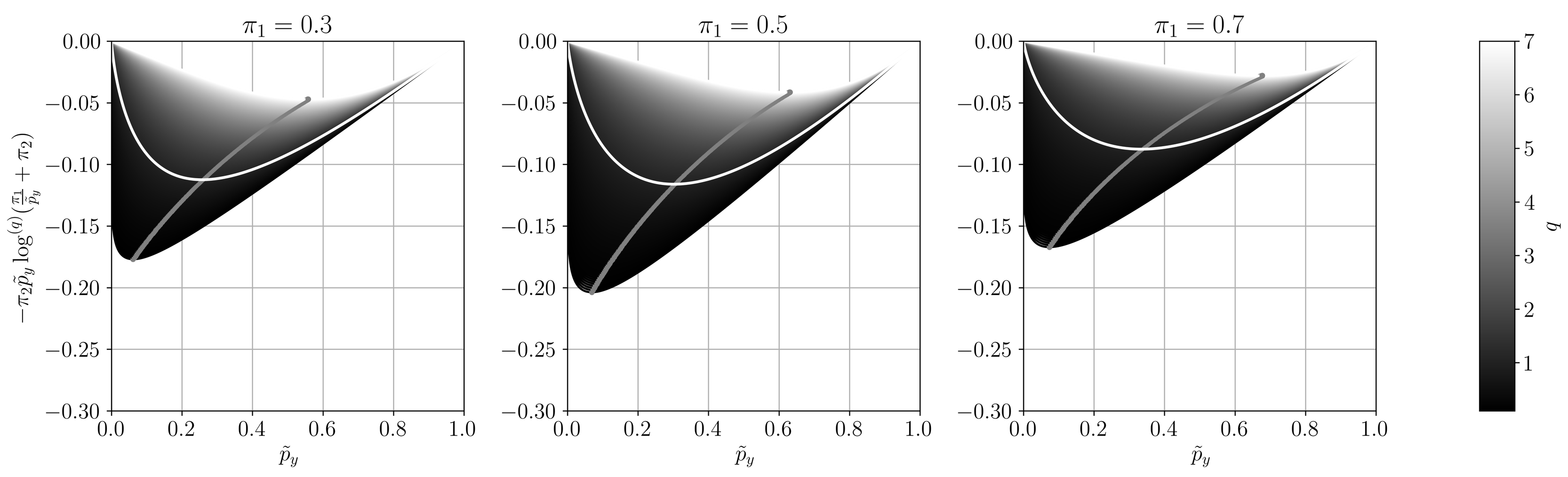


Figure: Numerical representation of regularization component $-\pi_2\tilde{p}_y\log^{(q)}\left(\frac{\pi_1}{\tilde{p}_y} + \pi_2\right)$.

Numerical Simulations

Illustrative Experiment – CIFAR-10: The network trained using the CE achieves 100% accuracy on the training set, whereas the ones trained using the JSD and the JTD functions do not. However, networks trained using the JTD with larger values of q outperform the others on the test set.

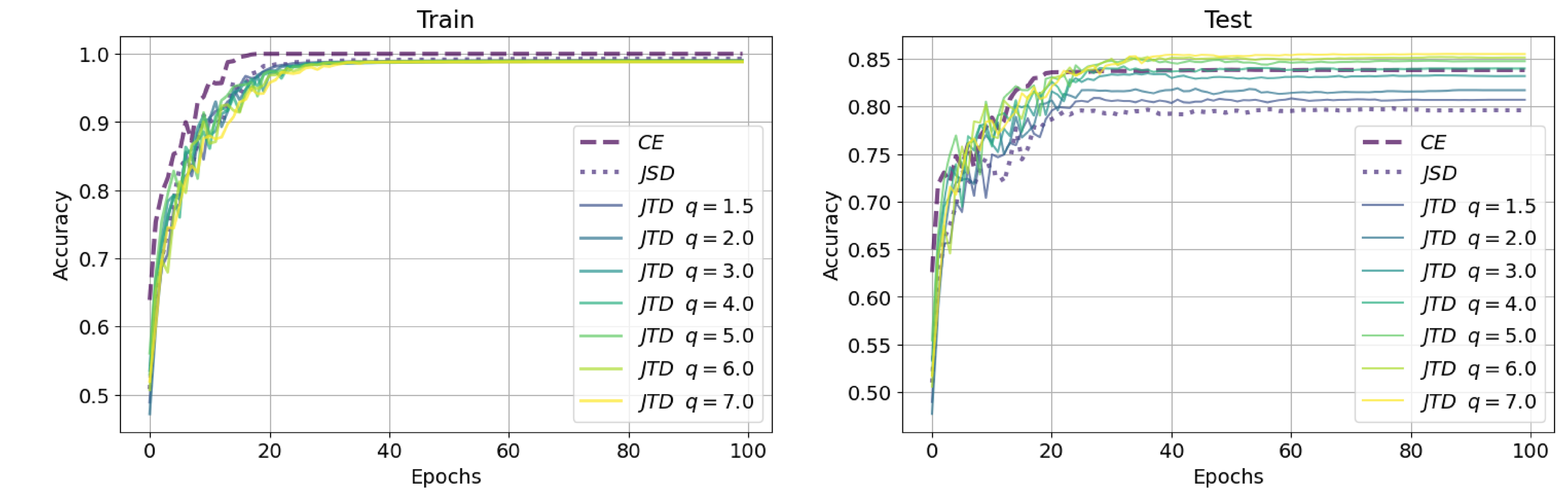
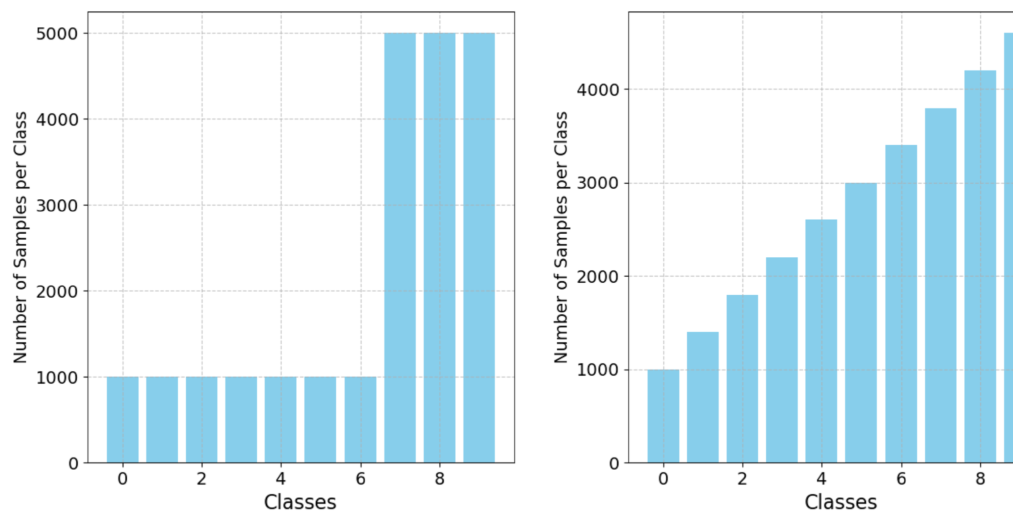


Figure: Learning curves over CIFAR-10. ResNet34, trained for 100 epochs, batch size of 256, starting learning rate of 0.01, and a cosine decay learning rate applied. Stochastic Gradient Descent (SGD) with Nesterov momentum 0.9 and weight decay $1e-4$ was employed. No data augmentation was applied to showcase the regularization effect of the cost function.

Imbalanced Data Experiment: Varying degrees of artificial imbalance to different open datasets based on the framework of [Buda et al., 2018]

$$\rho = \frac{\max_j\{|C_j|\}}{\min_j\{|C_j|\}}$$
$$\mu = \frac{|\{j \in \{1,\ldots,K\}: C_j \text{ is minority}\}|}{K}$$



((a)) Step $\rho = 5$ $\mu = 0.7$ ((b)) Linear $\rho = 5$

Figure: Number of samples in each class after applying the imbalance strategy on CIFAR-10. C_j is the set of samples contained in the dataset belonging to class j, and K is the total number of classes.

Table: Average accuracy and relative standard deviation over 5 runs - CIFAR10

CIFAR10		Imbalance Type					
		Linear (ρ)			Step (ρ - μ)		
Loss		2.0	10.0	50.0	2.0 - 0.5	10.0 - 0.5	50.0 - 0.5
BL	92.43±0.81	90.77±0.27	85.62±0.13	80.48±0.47	89.62±0.44	75.83±0.22	55.01±0.23
CE	**92.90±0.71**	**91.44±0.80**	**86.82±0.29**	**81.66±0.27**	**90.68±0.25**	**77.50±0.20**	55.63±0.12
FL	92.45±0.45	90.74±0.36	85.87±0.16	80.06±0.47	89.94±0.40	75.74±0.22	54.33±0.19
JSD	91.24±0.44	89.21±0.31	84.07±0.23	78.52±0.46	88.54±0.30	74.00±0.19	54.43±0.21
JTD	92.63±0.55	90.93±0.55	**86.71±0.23**	**82.07±0.35**	**90.40±0.27**	**77.81±0.13**	**59.00±0.16**
MAE	84.12±1.98	78.15±7.59	77.14±4.60	72.21±5.90	74.53±6.72	68.86±5.16	55.89±4.55

Table: Average accuracy and relative standard deviation over 10 runs for MNIST - Step Imbalance

MNIST	Imbalance Type								
	Step (ρ - μ)								
Loss	10.0 - 0.5	25.0 - 0.5	50.0 - 0.5	100.0 - 0.5	250.0 - 0.5	500.0 - 0-5	1000.0 - 0.5	2500.0 - 0.5	5000.0 - 0.5
BL	97.44±0.19	95.96±0.44	94.34±0.46	91.79±0.6	85.8±1.21	79.49±1.33	72.7±1.89	60.4±3.56	**54.58±1.84**
CE	97.83±0.13	96.53±0.27	94.88±0.42	92.79±0.57	87.1±1.27	81.18±1.5	73.73±1.76	61.5±2.79	**54.72±1.83**
FL	97.64±0.22	96.17±0.31	94.37±0.46	91.99±0.74	86.28±1.43	79.96±1.47	71.38±2.07	60.06±3.26	**53.69±1.73**
JSD	96.5±0.4	94.71±0.61	92.88±0.61	90.69±0.77	85.46±1.66	81.69±1.6	**74.72±1.8**	**63.63±3.74**	**55.5±2.13**
JTD	**98.34±0.14**	**97.33±0.16**	**96.17±0.35**	**94.05±1.33**	**89.06±1.24**	**83.96±1.18**	74.54±2.42	61.22±3.73	**55.28±2.1**
MAE	86.75±8.13	82.61±7.56	82.55±7.16	77.57±9.51	68.92±11.3	64.66±8.24	56.31±8.46	45.55±6.98	37.52±9.29

Table: Average accuracy and relative standard deviation over 10 runs for MNIST - Linear Imbalance

MNIST	Imbalance Type								
	Linear (ρ)								
Loss	10.0	25.0	50.0	100.0	250.0	500.0	1000.0	2500.0	5000.0
BL	98.29±0.09	98.02±0.11	97.88±0.18	97.5±0.29	96.61±0.66	95.67±0.58	94.29±0.85	91.63±1.54	90.45±1.2
CE	98.52±0.11	98.26±0.12	98.05±0.22	97.74±0.31	96.81±0.37	95.81±0.57	94.57±1.13	92.36±1.48	90.98±1.35
FL	98.46±0.11	98.12±0.13	97.91±0.19	97.54±0.3	96.52±0.48	95.53±0.65	94.21±1.28	91.93±1.4	90.3±1.28
JSD	97.3±0.27	96.99±0.2	96.8±0.24	96.51±0.29	95.88±0.55	95.09±0.62	93.67±1.09	91.79±1.46	89.93±1.56
JTD	**98.66±0.1**	**98.42±0.13**	**98.29±0.13**	**98.18±0.34**	**97.45±0.47**	**96.26±0.68**	**95.31±1.3**	**93.22±1.42**	**91.56±1.32**
MAE	88.59±7.71	89.45±6.88	88.25±7.67	84.49±8.93	86.13±7.99	83.28±9.63	83.42±8.39	78.94±10.86	80.16±8.12

Table: Sensitivity analysis of JTD generalization with respect π and q over CIFAR10 with ResNet34

π_1	q							
	0.5	1.0	2.0	3.0	4.0	5.0	6.0	7.0
0.1	87.36	88.43	90.99	91.63	87.35	57.76	10	10
0.2	89.85	90.44	91.38	91.79	91.97	88.24	10	10
0.3	90.17	90.68	91.69	92.19	92.52	82.50	89.72	83.74
0.4	90.39	91.10	92.15	92.64	92.64	92.85	92.24	92.79
0.5	90.85	91.18	91.90	92.27	92.47	92.69	92.47	92.69
0.6	90.55	91.53	91.83	91.99	91.80	90.79	84.80	58.25
0.7	90.57	91.08	91.75	90.93	87.89	43.77	24.35	19.02
0.8	90.12	90.53	91.34	87.01	41.80	29.09	10	18.25
0.9	88.63	90.32	90.61	59.28	29.52	18.24	10	10

Conclusions

- JSD and JTD can be interpreted as loss functions with intrinsic confidence regularization in supervised learning applications, and also as a priors over the output confidence of the network.
- JTD can outperform the JSD, CE, and other loss functions by tuning the parameter q.
- Analysis of imbalanced data classification scenarios conducted using different datasets, highlighting that the JTD emerges as one of the best loss functions for generalization in this case.

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