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Online portfolio optimization with transaction constraint

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«*Eagles do not fly with pigeons*»

Some guy in a movie theatre

Abstract

In 2021 Celli, Castiglioni and Kroer proposed their paper "*Online Learning with Knapsacks: the Best of Both Worlds*", dedicated to studying online learning problems. The focus was on maximizing expected rewards without violating a finite set of resource constraints. The research provided a novel framework, ensuring no-regret guarantees in both stochastic and adversarial scenarios. Today we are going to apply their algorithm in the realm of finance to introduce an innovative portfolio optimization method. This method aims to strike a balance between maximizing expected profits and minimizing transactions costs. This framework introduces two players: the decision maker and the dual player. The decision maker allocates resources to maximize rewards, while the dual player penalizes excessive transactions, striking a balance between strategy changes and cost management. We evaluate our online algorithm against established strategies, including the modern portfolio and buy-and-hold. Notably, our top-performing algorithm, which excludes the dual player, capitalizes on Tesla's exceptional performance, delivering remarkable profits but with increased risk. The PrimalDual Portfolio, incorporating both players, maintains a consistent transaction budget, harmoniously balancing risk and reward, and excelling in diversification.

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Introduction

How should an investor allocate funds among the possible investment choices? This so-called million-dollar question, or more precisely, 98 trillion dollar question [1], has been tried to be solved for over one century. In this context, portfolio optimization is a crucial concept in modern finance that aims to construct an optimal investment portfolio by reallocating its various assets, in order to balance risk and increase the expected return. However, an accurate estimation of optimal portfolio allocation is challenging due to the non-deterministic complexities of financial markets.

Generally, there have always been two opposing views about the cause of fluctuation of stock prices. Eugene Fama in his paper "Random Walks in Stock Market Prices" (1965) [2] presented empirical evidence suggesting that stock prices follow a random walk and that it is impossible to consistently predict short-term price movements. Fama's research challenged the notion of market inefficiency and provided support for the efficient market hypothesis, that states that stock prices incorporate all available information. In an efficient market, it is assumed that no investor can consistently outperform the market by predicting future price movements.

Alternatively, other economists concur that the market may be somewhat predicted, because prices follow patterns, and that analysis of past prices can be used to predict the direction of future prices. The presence of predictability in market prices challenges the assumptions of market efficiency and suggests that there may be room for investors to generate excess returns by exploiting these predictability patterns.

On the base of Fama's studies, academics suggested that fluctuation of stock prices is determined by endogenous and exogenous factors (Vladimir Filimonov and Didier Sornette [3]) The importance of external factors in influencing market behavior and outcomes has been underlined in numerous studies. For instance, Campbell and Shiller (1988)[4] have investigated the impact of macroeconomic variables on stock values, such as dividends, interest rates, and inflation. According to their research, these exogenous factors are essential in understanding changes in stock prices and predicted returns. Similar to this, Bekaert and Harvey (2000)[5] looked into how foreign investors affect emerging equity markets, highlighting the importance of external variables in determining market dynamics. However, numerous studies have disproved this claim, demonstrating that these exogenous factors have little effect on markets where behavior is primarily determined by endogenous factors (Bouchaud, 2011[6]) (David M. Cutler, James M. Poterba & Lawrence H. Summers, 1988[7])

Since the mid-twentieth century portfolio optimization has been the focus of many research areas, including finance, statistics, machine learning, and optimization. Its history can be traced back to the pioneering work of Harry Markowitz, who in 1952 introduced the Modern Portfolio Theory (MPT)[8], on which modern portfolio theory is still built upon. In general, the mean-variance framework of MPT formalizes the idea of the return-risk tradeoff, which states that investors should take both return and risk into account when deciding how much money to allocate to various investment options. It specifically advises to invest in the portfolio with the smallest variance out of the various portfolios that meet a specific return objective. All other portfolios are "inefficient" because they have a higher variance and, therefore, higher risk. Markowitz theory was quite revolutionary. By stating that portfolio's riskiness depends on the correlations of its constituents and not only on the average riskiness of its separate holdings, it introduced the concept of portfolio diversification.

MPT framework is the accepted common practice when creating portfolios in academics and the business world, however it is well known to be particularly prone to mistakes in the modeling of the random variables that predict asset returns. Since markets are known to exhibit non-stationary behavior, any static assumption about mean and variance, that underly the MPT framework, is unreliable[9].

As a result, in the next decades Markowitz's ideas have evolved, and researchers have made advancements in portfolio optimization techniques. For example, Fischer Black and Robert Litterman introduce in 1992 the Black-Litterman model[10] that incorporates the subjective view of investors to construct the optimal portfolio, and in 2000 Rockafellar, R. T., & Uryasev, S. presented the concept of Conditional Value-at-Risk (CVaR)[11] as a new risk measure to substitute the more traditional variance. With the advent of Online Learning, new techniques, called online portfolio strategies, came out. Online portfolio optimization refers to the continuous and dynamic process of updating and adjusting investment portfolios in real-time based on changing market conditions and new information. In this context, the Universal Portfolio introduced by Thomas Cover in 1991[12] is a significant contribution. The key idea behind the Universal Portfolio is to construct a portfolio that is competitive with any other fixed investment strategy over time, regardless of the specific market conditions or asset performance. It achieves this by adaptively reallocating portfolio weights based on the historical performance of assets.

More recently, thanks to the availability of massive amounts of data in the financial industry, the use of advanced data analysis tools to implement online portfolio strategies (OPSP) sparked, especially combining modern computer infrastructure with ML methods like Neural Networks or Deep Learning Algorithms. Algorithms for machine learning (ML) are purely based on empirical observations driven by the dynamic rise and fall of asset prices. A few algorithms (or strategies) that take advantage of these dynamic price fluctuations are Follow-the-Winner (Agarwal, Hazan, Kale, & Schapire, 2006 [13]; Li & Hoi, 2014[14]), Follow-the-Loser (Li & Hoi, 2012), and others. In short, an ML algorithm's approach to OPSP is to thoroughly examine the information on past asset prices that is accessible and, based on its unique strategy, make recommendations for how the portfolio should be allocated for the upcoming period to maximize the total cumulative wealth.

However, one aspect that is frequently overlooked is transaction costs. Many algorithms prioritize maximizing portfolio returns without considering the fact that, in the real world, it is often wiser to minimize the number of transactions, which can potentially result in substantial expenses.

In this paper we will propose an application of the algorithm proposed by Castiglioni, Celli and Croer, in their paper "Online Learning with Knapsacks: the Best of Both Worlds"[15]. In Game Theory fashion, they propose an innovative method to optimize the total stocks expected reward without violating a set of transaction budget constraints.

To tackle this problem, they present a framework with two players: the decision maker and the dual player. The decision maker decision maker has to satisfy supply or budget constraints. In particular, the decision maker is endowed with m $\geq$ 1 limited resources which are consumed over time. For each round t up to the time horizon T, the decision maker chooses a strategy ξ_t which defines a probability measure over the set of actions. Then, he observes some feedback about the reward and resource consumption incurred by playing ξ_t. The process stops at time horizon T, or when the total consumption of some resource exceeds its budget. The goal is to maximize the total reward. The dual player instead will penalize the first player for any excessive transaction with the final aim of finding the perfect balance between changing strategy and balance costs.

1. Online Learning

1.1 Foundations of Online Learning

Online Learning is a conceptual framework that addresses the challenges of sequential decision-making problems faced by agents in various environments. In this framework, an agent must make a series of consecutive actions, with each action resulting in a loss signal or reward from the environment, depending on the sign convention.

The purpose of this section is to present the general framework of Online Game Playing and to introduce the notation necessary for the development of the theory for Online Portfolio Optimization.

In the context, an agent (referred to as A) is tasked with predicting the outcome (y_t) based on a series of prior occurrences ($y_1, y_2, ..., y_{t1}$) that are a part of the outcome space (Y). At each time step, the agent makes a prediction (x_t) from the prediction space (D). The environment, in turn, determines the outcome (y_t) by selecting a loss function ($f(\cdot, y_t)$). The main objective of agent A is to identify the functions that effectively map the history of past outcomes to the new prediction. The prediction space can be finite allowing for a more exhaustive exploration, or infinite decision spaces that necessitate more efficient exploration strategies to balance the exploration of new actions and the exploitation of already known good actions.

In the aforementioned context, the idea of regret is crucial to understanding how to evaluate an agent's performance. By measuring the difference between the cumulative loss incurred by the agent's chosen actions and the cumulative loss that would have been experienced by an optimal decision-making strategy, the "expert", we can learn about the efficacy of the online algorithm. The theory of regret gives strong theoretical guarantees, since it assures that algorithms within this framework perform, in the long run, nearly as effectively as the top performer within the class of experts. The Experts framework works as follows: At each time step of the prediction game, each expert $e \in E$ predicts an element $x_{e,t} \in D$, and incurs in a loss $f(x_{e,t}, y_t)$, just as the agent A. The learner's objective is to minimize losses when compared to the top expert in the class E. This notion is precisely encapsulated in the concept of "regret." To provide a formal definition, we express the regret $R_{e,T}$ for the agent A concerning expert $e \in E$ as follows:

$$R_{e,T} = L_T - L_{e,T}$$

The regret observed by the agent A with respect to the entire class of experts E is defined as:

$$R_T = \sup R_{e,T} = L_T - \inf L_{e,T}$$

In our case, since it is clearly impossible to have a player behaving as an expert who can forecast the financial market, we are comparing our player with just the best action that can be taken in hindsight having knowledge of the stocks returns of the day.

Agent A's task is to determine a sequence x_t, which depends on the information gathered up to time t, with the aim of minimizing the regret y_T in comparison to any sequence $y_1, y_2, ...$ selected by the environment, aiming to achieve sublinear regret $R_T = o(T)$. Such a strategy is called Hannah-Consistent.[16] In simple terms, if an algorithm achieves sublinear regret, it means that as it continues to make decisions or predictions over time, its cumulative regret (the difference between its performance and the best possible performance) doesn't increase at a linear rate with the number of rounds. Instead, it increases at a slower rate. As proved by Cover [1966][17] strategies that achieve sub linear regret do not exist without implementing and additional strategy: The learner must add an element of randomness into its predictions. At each time step 't', the agent selects a probability distribution across the decision space and then decides 'x_t' based on this distribution. By introducing randomness into the decision-making process, if the original decision space was D with $|D| = N$, we effectively convert the decision space D into the $\Delta_{N-1} \in R_N$ probability simplex, making the domain D convex. This transformation is very significant in the context of online portfolio optimization as it enables the utilization of online convex optimization techniques. Convexity of the domain D and of the loss functions $f(\cdot, y)$ bound the problem geometry and let us derive simple and efficient learning procedures

In the next section we will present the problem framework and we will see how we can implement two algorithms from the family of online convex optimization, Online gradient descent (OGD) and Hedge, that guarantee regret minimization.

2. Model

2.1 Problem framework

In this section, we aim to introduce and outline the essential notations and structure of the portfolio optimization problem. This will serve as the foundation for our subsequent discussions and analyses.

With the primary goal of the analysis always at the forefront, our objective is twofold: to maximize the expected return of the portfolio while concurrently minimizing transaction costs. We address this challenge in an online learning setting, employing the algorithm developed by Castiglioni, Celli and Kroer, introducing the concept of a budget constraint. The framework assumes a simplified context where each transaction incurs a fixed transaction cost (or a percentage of its value), and there is an initial maximum transaction budget that remains constant regardless of portfolio performance. In the future, it may be desirable to have a budget that increases or decreases based on available cash and accumulated dividends.

We will now delve deeper into how transaction costs are impacted.

Types of Transaction Costs:

1. **Commissions**: Many brokerage firms charge a commission fee for each stock trade, which can be a fixed amount per trade or a percentage of the trade value. In this framework, the transaction cost is generalized as $2 per transaction, regardless of the quantity of stocks being purchased, or 0.2% of the transaction cost as a percentage.

2. **Spread**: The spread refers to the difference between the buying and selling prices of a stock. In this highly liquid market, the spread is considered to be zero, although in reality, it may be a small fraction of the transaction cost.

3. **Exchange Fees**: Since this is a simplified context, stock exchange fees are not considered.

4. **Regulatory Fees**: Some regulatory bodies impose fees on securities transactions, such as the Securities and Exchange Commission (SEC) fee for stock sales in the United States. In this framework, regulatory fees are considered to be part of the commissions fee.

To summarize, two potential ways to include transaction costs are \$2 per transaction or 0.2% of the transaction amount. Both transaction costs apply to both buying and selling stocks. The two transaction costs are equal for a transaction amount of \$1000. Therefore, if the initial budget is high, the algorithm will penalize less a high number of transactions with a fixed amount cost and more with a percentage cost, while the opposite will happen with a low initial budget.

2.1.1 Problem definition

We define the trading periods by $t_k = k\Delta t$, where k ranges from 0 to m. Here, Δt represents one day, as portfolio weights are updated daily, and m denotes the total number of participation cycles in transactions. Additionally, we denote the return vector of n assets from time t_{k-1} to t_k as R_k. The formula for calculating the return $R_{k,i}$ of the i-th asset is expressed as $R_{k,i} = \frac{P_{k,i}}{P_{k-1,i}}$, where $P_{k-1,i}$ and $P_{k,i}$ represent the prices of the i-th asset at times $t_{k-1}$and t_k, respectively.

Each trading day, the performance of the stocks can be described by a vector of returns denoted by $R = (r_1, r_2, \ldots, r_N)$, where r_i is the next day's return of the i-th stock. The value of an investment in stock i increases or decreases by the factor r_i.

A portfolio is defined by a weight vector $w = (w_1, w_2, \ldots, w_N)$ such that $w_i \geq 0$ and $\sum_{i=1}^{N} w_i = 1$. The i-th entry of a portfolio w represents the proportion of the total portfolio value invested in the i-th stock. Given a portfolio w and the returns r, investors using this portfolio increase or decrease their wealth from one morning to the next by a factor of $w \cdot r = \sum_{i=1}^{N} w_i r_i$.

The algorithm therefore aims to maximize $w \cdot r - c\left(\sum_i |W(t+1)_i - W(t)_i|\right)$, where

$W(t+1)_i$ denotes the weight of the i-th asset in the portfolio at time $t+1$

$W(t)_i$ represents the weight of the i-th asset in the portfolio at time t

c represents the transaction cost as a constant.

In this formula, the absolute value ensures that the differences are considered in a magnitude-agnostic manner, taking into account both increases and decreases in weight.

Adding the budget constraint B, the problem can also be defined as:

$$\max \sum_{t=1}^{T} w_t^\top r_t$$
$$s.t. \sum_{t=1}^{T} c_t(\|w_t - w_{t-1}\|_1) \leq B$$

Finally, it is important to emphasize that we will exclude cash as a viable asset within the portfolio, despite the potential inclusion in future works for the purpose of introducing a "hypothetical" stock with a stable price trajectory. Furthermore, for the sake of simplicity, we will abstain from incorporating dividend considerations, acknowledging that stock prices inherently reflect future cash flows derived from dividends. For the same reason, we will enforce non-negativity constraints on portfolio weights, thereby excluding short positions.

2.1.2 Other models

To verify the performance of our model, we will compare it against other well-established portfolio optimization methods that have been utilized over the past decades. These strategies include:

- **Minimum Variance Portfolio (MVP)**

  Mean-variance model is a strategy constructed in line with the Markowitz's theory. It captures the aforementioned risk-return trade-off.

$$W_k^{MV} = \arg\min_{W_k^1=1} W_k^T \sum_k W_k - R_k^T W_k \quad (1)$$

  where $R_k^T W_k$ is the expected return and $W_k^T \sum_k W_k$ is the variance of portfolio returns.

- **Constant Weight Rebalance portfolio (CWR)**

  A CWR portfolio is a strategy in portfolio management where the weights assigned to different assets are periodically adjusted to maintain a fixed target allocation. In other words, it involves periodically rebalancing the portfolio back to its original target weights.

$$w_{i,k} = w_{i,t_0} \quad \text{for } k \text{ from } t+1 \text{ to } m \text{ and } \forall i \text{ from } 0 \text{ to } N \quad (2)$$

- **Equal Weight portfolio (EW)**

  EW simply ignores all data information and distributes the investment equally among all the assets:

$$w_k^{EW} = \frac{1}{n} \quad (2)$$

- **Value Weight portfolio (VW)**

  VW portfolio is an investment strategy that assigns weights to assets based on their market value. Assets with higher market values are assigned higher weights, while assets with lower market values have lower weights.

$$w_{i,k}^{VW} = \frac{m_{i,t}}{\sum_{i=0}^{N} m_{i,t}} \quad (3)$$

  where m_t is equal to the market capitalization value for stock i at time t.

- **Buy and Hold (BH)**

  BH strategy involves initializing all weights uniformly at time 0, with no rebalancing in the future

- **Online Moving Average Regression (OLMAR)**

  OLMAR, theorized first in 2012 by Li and Hoi[18], is characterized by its use of a moving average reversion approach. It maintains a moving average of the historical returns of assets in the portfolio and adjusts the portfolio weights based on the divergence between the current returns and the moving average.

$$w_{t+1} = w_t + \lambda \cdot \text{excess\_return} \quad (4)$$

where excess_return is equal to the difference between the moving average of each single stock and the mean of every stock and λ is a constraint that minimize excess transactions.

Additional portfolio strategies, that include Upper Confidence Bound 1 (UCB1), Maximum Probabilistic Sharpe ratio (MaxPSR)[19] and Probability Weighted UCB1 (PW-UCB1) can be added if necessary.

2.2 Model definition

Let $\gamma_t := (f_t, c_t)$ and $\gamma_T := (\gamma_t)_{t=1}^{T}$ be the sequence of inputs up to time T. At each step t, the decision maker can condition their decision on γ_{t-1} , and on the sequence of prior decisions $x1, ..., x_{t-1}$, but no information about future rewards or resource consumption is available. The repeated decision-making process stops at any round $\tau \leq T$ in which the total consumption of any resource i exceeds its budget B_i. At this point the weights will not be updated anymore. The core objective of the decision maker is to maximize its total reward, effectively managing budget expenditure by striking a balance between immediate wasteful spending and under utilization over the entire time horizon. The aim is to optimize the allocation of resources, ensuring that the budget is utilized efficiently and evenly distributed over time. Since the void action $\emptyset \in X$ of not updating the weights is possible, we are sure that there exists a feasible solution that does not violate the resources constraint.

A regret minimizer is utilized by the decision maker to update the new weights. At each time t the regret minimizer can perform two operations: (1)UPDATEWEIGHTS, that computes the new weights and the cost required to perform the transaction. If the cost does not exceed the remaining budget, the algorithm update the weights with the new ones. (2) UPDATELOSS, this procedure updates the internal state of the regret minimizer using the environment's feedback, in the form of a utility function $l_t : W \to R$. The utility function can depend adversarially on the sequence of outputs $w_1, ..., w_{t-1}$. The decision making process encoded by the regret minimizer is online: at each time t, the output of the regret minimizer can depend on the sequence $(w_{t'}, l_{t'})_{t'=1}^{t-1}$, but no information about future utilities is available. The objective of the regret minimizer is to output a sequence of points in W so that the cumulative regret

$$R^T := \sup_{w^* \in W} \sum_{t=1}^{T} (l_t(w^*) - l_t(w_t))$$

grows asymptotically sublinearly in the time T.

The algorithm is built to work in both bandit feedback and full feedback models. The first assumes that the player at each time t has information on the reward of only his choice and of the others in the decisions set (only$f_t(x_t)$, $c_t(x_t)$ is observed by the decision maker). In the Full feedback framework instead, f_t and c_t are both observed and the decision maker possesses knowledge of the potential reward that could have occurred had a different decision been made. At each time t we have full information on the returns of each weight composition, so we utilize the Full feedback framework.

We start by considering the set of probability measures on the Borel sets of X. We refer to this set as the set of strategy mixtures and denote it as Ξ. We assume that all possible functions f_t, c_t are measurable with respect to every probability measure $\xi \in \Xi$. Given a reward function $f : X \to [0, inf]$ and a cost function $c : X \to [0, \text{total budget}]^m$, we define the following linear program, which chooses the strategy ξ that maximizes the reward f , while keeping the expected consumption of every resource $i \in [m]$, given one of the selected cost functions mentioned in the previous section, below the transaction cost budget p:

$$\mathrm{OPT}_{f,c}^{\mathrm{LP}} := \begin{cases} \sup_{\xi \in \Xi} \mathbb{E}_{x \sim \xi}[f(x)] \\ \text{s.t. } \mathbb{E}_{x \sim \xi}[c(x)] \leq \rho \end{cases},$$

where $\mathbb{E}_{\boldsymbol{x} \sim \xi}\left[c_t(\boldsymbol{x})\right] = \left(\mathbb{E}_{\boldsymbol{x} \sim \xi}\left[c_t(\boldsymbol{x})[i]\right]\right)_{i=1}^{m} \in [0,1]^m$.

To solve this constraint optimization problem, we utilize the following Lagrangian function: $\Xi \times R_{\geq 0}^m \times I \to R$ is such that, for any $\xi \in \Xi, \lambda \in R_{\geq 0}^m, (f, c) \in I$ it holds

$$L(\xi, \lambda, f, c) := E_{x\sim\xi}[f(x)] + \langle \lambda, \rho - E_{x\sim\xi}[c(x)] \rangle$$

.

Originally, Celli and al. showed that we can restrict the set of admissible dual vectors to $D := \{\lambda \in R_{\geq 0} : \|\lambda\|_1 \leq 1/\rho\}$, while still maintaining strong duality. However, this assumption is only valid as long as the reward and cost functions codomains are bounded between 0 and 1. Since this is not the case, we cannot apply the same restriction it in a straightforward manner. In the next section, we will explore how the magnitude of these functions impacts the constraints imposed on the set of dual vectors.

Now that we have established the foundational basis, we can proceed to construct the actual algorithm. The Meta-Algorithm is based on the classic primal-dual approach (Balseiro et al., 2020b[20]; Immorlica et al., 2019[21]). Within this framework, there exist two distinct regret minimizer that have unique characteristics.

The first regret minimizer, denoted as RP, serves as the primal regret minimizer. It generates strategy mixtures within the set Ξ and receives feedback in the form of a linear utility function, denoted as $l_{Pt} : \Xi \to R$. This utility function evaluates each strategy mixture $\xi \in \Xi$ by computing the expected reward $E_{x\sim\xi}[ft(x)]$ minus the inner product of the dual variable λ_t and the expected cost $E_{x\sim\xi}[ct(x)]$, i.e., $l_{Pt}(\xi) := E_{x\sim\xi}[f_t(x)] - \langle \lambda_t, E_{x\sim\xi}[ct(x)] \rangle$. In our specific case, the loss at time t is equal to the vector composed by each stock returns at time $t(f_t(x))$, while the inner product is simply the product between lambda and the difference between the average daily budget and the transaction cost. On the other hand, the second regret minimizer, referred to as RD, operates as the dual regret minimizer. RD outputs points in the space of dual variables D and receives feedback through the linear utility function $l_{Dt} : D \to R$. This utility function, computed trought the use of Online gradient descent (OGD), assesses each dual variable $\lambda \in D$ by calculating the inner product of λ and the difference between the upper bound ρ and the expected cost $E_{x\sim\xi_t}[c_t(x)]$, i.e., $l_{Dt}(\lambda) := -\langle \lambda, \rho - E_{x\sim\xi_t}[ct(x)] \rangle$. Even in this in case, lambda at time t simply equals to the product between the lambda at time $t-1$ and the and the difference between the average daily budget and the transaction cost. Both minimizers are assumed to have full feedback.

To address the primal regret minimizer RP, we can employ an algorithm specifically designed

Algorithm 1 Hedge algorithm

Input: parameters w = weights, α= learning rate, l = loss funct;
Initialization: $\vec{w}_1 \leftarrow \vec{w}$;
for $t = 1, 2, \ldots, T$ **do**
Update weights: pick $\vec{w}$ Calculate the exponentiated losses by exponentiating the negative product of the learning rate and the loss vector:

$$l_{exp} = e^{\alpha * l_t}$$

Update the weights by multiplying the current weights with the exponentiated losses and then normalizing the results by dividing them by the sum of the weights multiplied by the exponentiated losses:

$$\vec{w}_{t+1} = \vec{w}_t * l_{exp} / \sum w_t * l_{exp}$$

Get weights: update $\vec{w}$
$\vec{w} \leftarrow \vec{w}_{t+1}$

for regret minimization. There are several options available, including well-known algorithms such as Exp3 (Exponential-weight algorithm for Exploration and Exploitation), UCB (Upper Confidence Bound), and the Hedge Algorithm. For our purposes, we will select the Hedge Algorithm as our preferred choice. Hedge algorithm, first theorized by Freund and Schapire in their paper titled "A Decision-Theoretic Generalization of Online Learning and an Application to Boosting" in 1997[22], is a regret minimizer that assigns weights to a set of experts. It dynamically adjusts the weights based on the experts' performance, ensuring that the most accurate predictions receive higher weights, thereby minimizing regret. The hedge algorithm with a known fixed-horizon T with constant learning rate $\eta \, \alpha \, \rho log(M)/T$ suffers a regret upper bound of $\Theta(\sqrt{T}\sqrt{l}ogM)$ [23]. Our implementation of the algorithm adheres to logic of Algorithm 1.

Algorithm 2, instead, summarizes the structure of our meta-algorithm.

For each t, the meta-algorithm first computes a primal and dual decision through RP and RD, respectively (see the invocation of Get_weights() and Get_lambda()). The action played by the decision maker at t is going to be $x_t \sim \xi_t$. Then, (f_t, c_t) are observed, and the budget consumption is updated according to the realized cost vector $c_t(x_t)$. Finally, the internal state of the two regret minimizer is updated according to the feedback specified by l_t^P , l_t^D (see the invocation of Update_loss(). At each time t then the algorithms observes how much the new stocks allocation has gained compared to $t-1$ and updates the total wealth. When the temporal horizon is reached or the budget is finished, the algorithm ends.

After having set up the algorithm we proceed with the experiments with the real data.

Algorithm 2 Meta-Algorithm for strategy mixture Ξ

Input: parameters B, T, primal regret minimizer R^P, dual regret minimizer R^D
Initialization: $\forall i \in [m], B_{i,1} \leftarrow B, \rho \leftarrow 1 \times B/T,$ and Initalize R^D and R^P;
for $t = 1, 2, \ldots, T$ **do**
Primal decision: $\Xi \ni \xi \leftarrow R^P$.Get_weights(), State

$$x_t \leftarrow \begin{cases} x \sim \xi_t & \text{if} B_{i,t} \geq 1, \forall i \in [m] \\ \varnothing & \text{otherwise} \end{cases}$$

Dual decision: $D \ni \lambda_t \leftarrow R^D$.Get_lambda()

$$\lambda \leftarrow \begin{cases} 0 & \text{if } \lambda \leq 0, \\ \lambda & \text{if } 0 < \lambda < 1/\rho, \\ 1/\rho & \text{if } \lambda \geq 1/\rho \end{cases}$$

Observe request: observe (f_t, c_t) and update available resources: $B_{1,t+1} \leftarrow B_{1,t} - c_t(x_t)[i], \forall i \in [m]$.
Primal update:

- $l_t^p \leftarrow$ linear utility defined as

$$l_t^p : \Xi \ni \xi \mapsto \mathbb{E}_{x \sim \xi}[f_t(x)] - \langle \lambda_t, \mathbb{E}_{x \sim \xi}[c_t(x)] \rangle$$

- R^P.Update_loss(L_t^P)

Dual update:

- $l_t^d \leftarrow$ linear utility defined as

$$l_t^d : D \ni \lambda \mapsto -\langle \lambda, \rho - \mathbb{E}_{x \sim \xi}[c_t(x)] \rangle$$

- R^D.Update_Lambda(L_t^D)

3. Application

3.1 Basic framework

In this section, we will conduct experiments using real data from the 100 companies of the S&P100 index. Our simulation will cover a time horizon of 5 years. Throughout the experiments, we will explore various values for the learning rate, the magnitude of the returns vector (the loss vector for the hedge algorithm), and the transaction costs. The objective is to identify the optimal values that ensure a consistent expenditure of the budget over time. By systematically evaluating different combinations of these parameters, we aim to find the settings that result in a steady allocation of the budget throughout the simulation period. Moreover, the results will be compared with the same simulation of the other portfolio optimization methods mentioned in the setting section. It is important to mention that to ensure a fair comparison once the transaction budget is fully utilized, no further updates to the weights will be made for each method. By imposing this constraint, we can accurately assess the performance of each method in terms of budget expenditure. We begin our simulation by establishing an initial wealth equal to \$100,000, distributed equally across the 100 stocks, so that each initial weight is equal to 0.01. It is important to note that this initial weight distribution may not be applicable to models that require a specific initial weight allocation, such as the Value Weighted Portfolio, that will be initialized differently. Furthermore, as part of our simulation, we allocate an extra transaction budget of \$10,000 to each portfolio. This budget is specifically designated for covering transaction costs incurred during the trading process.

Before moving on to the actual simulations, it is essential to distinguish between two distinct types of online portfolios that are going to be discussed:

The first type employs a single scalar as the transaction cost, which is equivalent to the sum of the costs associated with each portfolio position. However, there is a critical limitation here. Since the loss is updated using the formula:

$$\text{loss} = \text{stock returns}_t + (\lambda * (\rho - \text{transaction costs}))$$

We cannot effectively exploit the Dual player functionality in this context. In fact, $\lambda * (\rho - \text{transaction costs}))$ is simply a scalar and the loss is reduced to a vector containing the stock returns shifted by this scalar. Consequently, when computing the new weights in the Hedge algorithm, we would lose any consideration for the change brought by this scalar shift, since the proportions between the weights would be the same as the case without the shift when multiplied by exponential value of the loss. Consequentially, when standardized such that their sums equal to 1, the weights would be the same in both cases. To better understand, let's write a simple example with just 3 positions in the portfolio. Setting the returns equal to $[0.01, -0.2, 0.7]$, the weights to $[0.1, 0.2, 0.7]$ and the λ equal to 1, we would have a loss equal to $[1.01, 0.8, 1.7]$. Running the hedge algorithm with the case in which λ equal to 1 or λ equal to 0 would give the same exact new weights, so that

$$\frac{\text{weights} * ([e^{1.01}, e^{0.8}, e^{1.7}])}{\sum(\text{weights} * [e^{1.01}, e^{0.8}, e^{1.7}])} = \frac{\text{weights} * ([e^{0.01}, e^{-0.2}, e^{0.7}])}{\sum(\text{weights} * [e^{0.01}, e^{-0.2}, e^{0.7}])}$$

On the other hand, the second type, which we will refer to as PrimalDual Portfolio, uses as the transaction cost, a vector, with each element representing the cost associated with each individual stock. In this way, the previous problem is avoided and the the dual player has a meaningful role in the algorithm. However, in this way we will penalize more the transactions of the stocks which price is higher, because they present a higher contribution to the loss function. For this reason we decide to build both types of online portfolios.

3.2 Simulations

For the first simulation we will start by considering the base case, in which we utilize the returns matrix where the values are expressed as percentage and the initial learning rate set to 0.05. Additionally, we only accept values for the dual vectors in the range from 0 to ρ, with ρ calculated as the ratio between \$10,000 and T; in other words we consider $D := \{\lambda \in R \geq 0 : ||\lambda||_1 \leq 1/\rho\}$.

As mentioned before, we still do not have any theorical guarantee that this is the correct interval. To verify its validity, the budget expenditure graph will be plotted. If the graph shows a consistent line without significant jumps and tends to reach \$10,000 at the end (at T), we can conclude that the algorithm is effectively balancing the transactions throughout the entire period. This visual representation will provide insights into the behavior of the budget expenditure over time and help assess the algorithm's performance. While tuning the best parameters our primary focus is on evaluating and optimizing the budget expenditure progression of the algorithm, without taking into consideration its overall performance. This approach helps to avoid introducing bias and ensures that the analysis remains objective. Nevertheless, it is reasonable to expect to have the best results with the right parameters. Finally, only after having found these parameters, we will compare our portfolio with the results of other models.

Figure 1 displays the progression of total portfolio wealth plotted over time, ranging from 0 to T. Figure 2 illustrates the budget expenditure, showcasing how the allocated budget is utilized over time. This graph provides insights into the pattern of expenditure, allowing us to analyze the consistency and efficiency of budget allocation strategies employed by the algorithms. Lastly, in Figure 3, we present the graph displaying the portfolio profit over time. This metric is calculated as the difference between the portfolio's wealth and the transaction costs incurred.

From Figure 2, it is evident that not all transaction budget is consumed throughout the temporal horizon, but only approximately three tenth of it is utilized. This observation indicates that the base parameters used may not be optimal for achieving the desired outcome. It suggests that modifications to the parameters are necessary to achieve a more satisfactory result. Since we want to incentive more budget expenditure, we can opt for three ways:

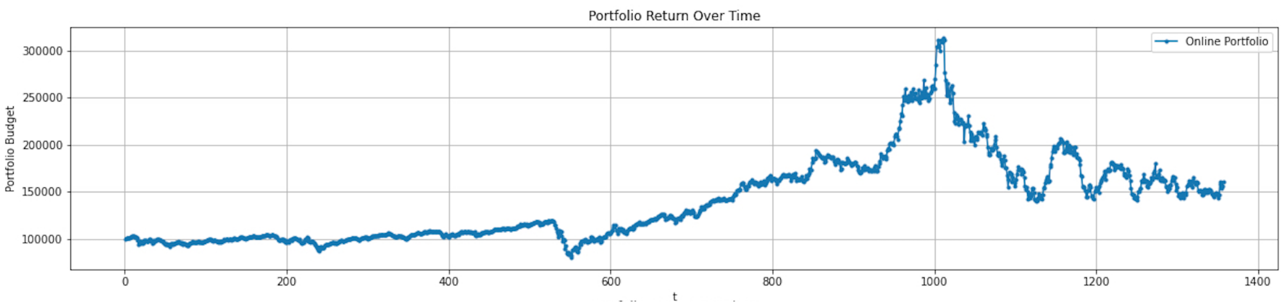

Figure 1: Online Portfolio Return Over Time

1. **Increase the learning rate of hedge algorithm:** A higher learning rate would lead to a larger adjustment of the weights in each iteration, potentially resulting in more frequent

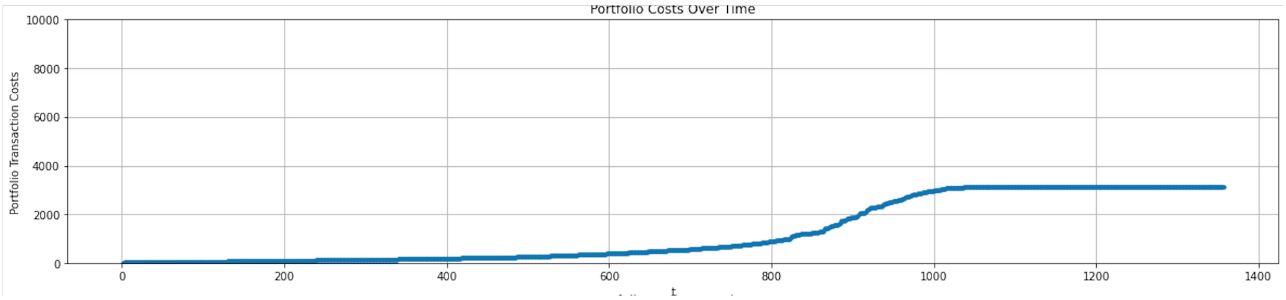

Figure 2: Online Portfolio Costs Over Time

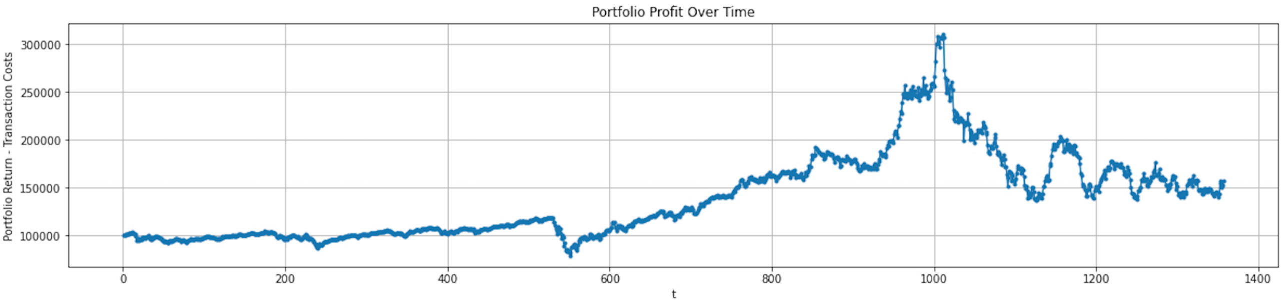

Figure 3: Online Portfolio Profit Over Time

trading and transaction costs, impacting the allocation and utilization of the budget. This would be achieved in the following way: the exponentiated losses in Algorithm 1 (l_{exp}) will decrease more rapidly as the learning rate increases, putting greater emphasis on recent losses in the weight adjustment process. Subsequently, the division by the sum of weights multiplied by exponentiated losses ($\sum(w_t * l_{exp})$) will be influenced by the larger learning rate, so that the weights will be adjusted more drastically in response to the losses.

However, this method presents some downsides too. Firstly, there is a greater risk of overfitting the historical data. The algorithm might become overly sensitive to noise or short-term patterns, resulting in suboptimal performance when faced with new or unseen market conditions. Moreover, it can also introduce more volatility in the algorithm's weight updates. This can cause larger swings in portfolio allocations, potentially leading to greater fluctuations in portfolio performance.

2. **Increase the magnitude of the loss vector:** a higher magnitude of the loss vector would make the algorithm more responsive to change in stock prices. This means that even small losses or raises would have a more significant impact on the weight updates. Consequently, the algorithm would adjust its weights more aggressively, potentially leading to a more aggressive investment strategy and more transactions. This effect is straightforward even in this case. Directly studying its effect on the hedge algorithm, it

is easy to see that the larger loss vector will be more influential in the weight adjustment process as it will contribute to a more significant decrease in the exponentiated losses. As the previous case, this will lead to more pronounced adjustments in the weights based on the amplified losses. The division by the sum of weights multiplied by exponentiated losses ($\sum(w_t * l_{exp})$) will indeed be affected by the increased magnitude of the loss vector.

3. **Increase the upper bound of the dual vectors set:** Only working in the case in which we use the transaction cost in vector form, enabling higher values for the dual vectors would result in a greater magnitude of the stock allocation update. When the lambda values are increased, the size of the loss vector values would in fact be amplified in the subsequent iterations. This enhancement is verifiable from the update in the algorithm, loss $=$ stock returns$_t$ $+ (\lambda * (\rho -$ transaction costs$))$. In the loss a higher lambda value would result in a larger contribution from this penalty term, $(\lambda * (\rho -$ transaction costs$))$, to the overall loss. In other words, the loss function would become more sensitive to deviations between the average daily budget and the effective transaction cost. Even small differences between ρ and transaction costs would contribute more to the overall loss. This increased sensitivity could potentially lead to more precise adjustments in the optimization process. As proved before, amplifying the loss vector would result in an increase number of transactions.

Opting for approach 2, we advance by augmenting the magnitude of the loss vector. Following multiple attempts, we decide to use the stock returns matrix as the loss vector, with its values multiplied by a factor of 2.5. As before, figure 4 shows the trend of portfolio value over the given time period, whereas figure 5 represents the transaction costs. In the case of utilizing the new loss vector, we can observe that the algorithm effectively distributes the transaction budget evenly across the entire time horizon. This demonstrates the correctness of the algorithm's functionality in maintaining a balanced allocation of resources. Given these promising results, we can consider adopting this model as our benchmark for future comparisons with other models. Furthermore, as hypothesized, this model performs better than the base one. It starts with a total wealth of \$100,000 and finishes with \$491,037, an increase of almost 500% in just 5 years. Surprisingly, the portfolio initially starts with an equal distribution among all 100 stocks in the indices but it eventually converges to a situation where only two stocks, Ford and Tesla, remain with weights of approximately 0.109223 and 0.890777 respectively. This contradicts the principle of diversification put forth by Markowitz, who emphasized that spreading investments across assets with varying risk and return characteristics can lower overall portfolio risk without compromising potential returns. The algorithm's approach of concentrating on just two stocks deviates from this theory, which could pose significant risks if one of the two stocks experiences a sudden decline.

Moreover, we build two additional models using different losses. For the "Weekly Online Portfolio" we use the weekly stocks moving average return and for the "4 days Online Portfolio" the four days moving average, as the name suggests.

Finally, aiming to fully leverage our model, we consider an online portfolio strategy that employs a cost function represented by a vector encompassing the individual stock costs at each time interval 't'(As explained before, this is the only way for which we can use the dual player to penalize the excess of transactions). However, it is important to note that this approach introduces a penalty for the transactions of the stocks with higher price, because they contribute with a heavier weight on the transaction expenses. Our model will be benchmarked against the results of the previously detailed methods in the first section. This comparative analysis will encompass a range of metrics aimed at assessing both profitability and risk tolerance.

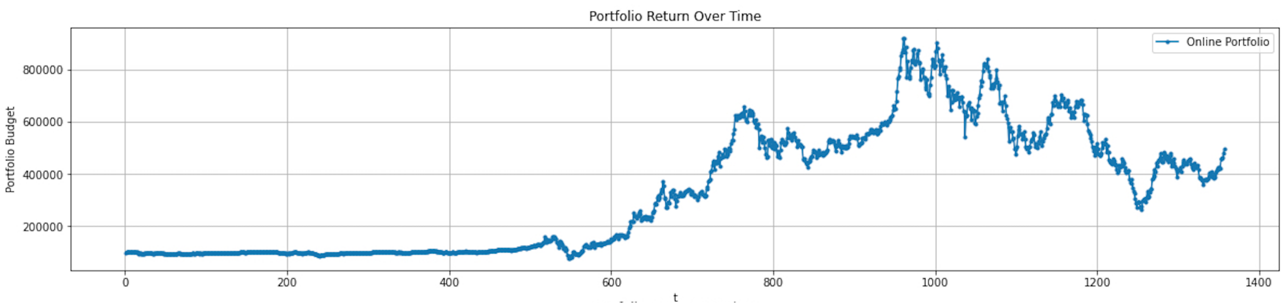

Figure 4: Portfolio Return Over Time

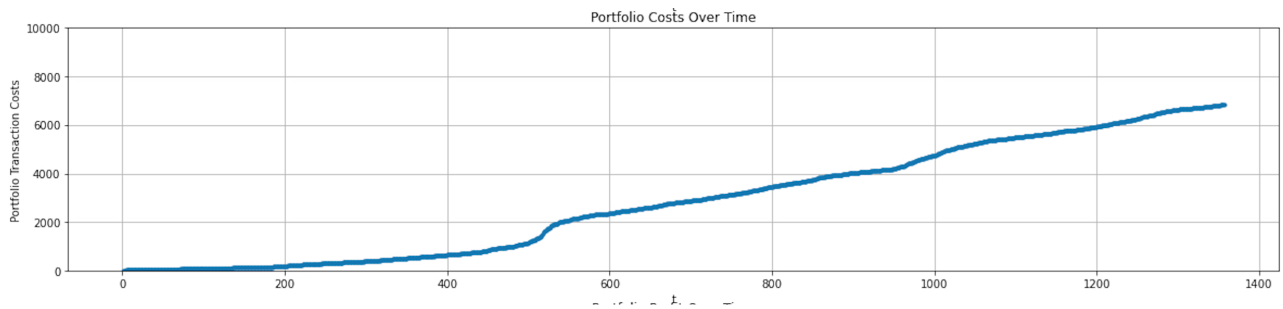

Figure 5: Portfolio Costs Over Time

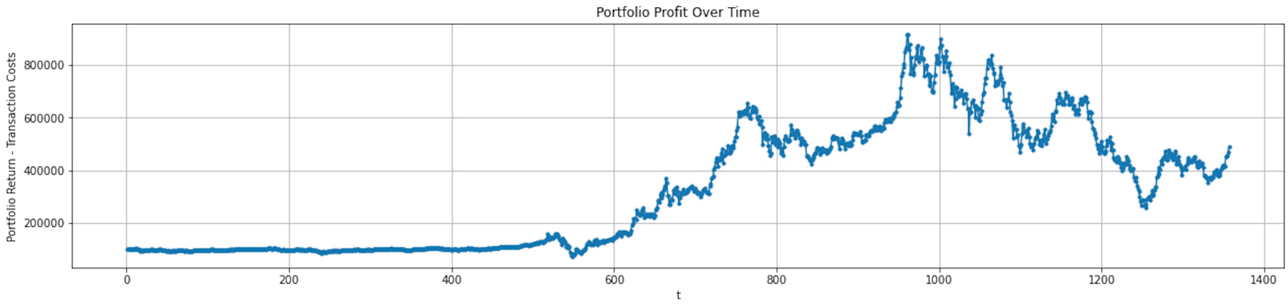

Figure 6: Portfolio Profit Over Time

The various metrics include:

1. **Profitability Metrics**:

   a. Profit Percentage (%): This metric measures the total return on investment, indicating how much profit was generated relative to the initial investment.

   b. Sharpe Ratio: The Sharpe Ratio quantifies the risk-adjusted return, helping us understand how efficiently returns are generated concerning the level of risk taken.

   c. Sortino Ratio: Similar to the Sharpe Ratio, the Sortino Ratio assesses risk-adjusted returns but focuses solely on downside risk, providing insights into how well a portfolio performs in protecting against losses.

   d. Information Ratio: The Information Ratio evaluates the excess returns generated by a portfolio in comparison to a benchmark index, in this case the Equally weighted Portfolio, indicating whether a strategy adds value beyond the market.

2. **Risk Aversion Metrics**:

   e. Max Drawdown: This metric represents the most significant peak-to-trough decline in portfolio value expressed in percentage, offering insights into the maximum loss an investor could have experienced.

   f. Ulcer Ratio: The Ulcer Ratio measures the depth and duration of drawdowns, giving a more nuanced view of downside risk and portfolio stress.

   g. Winning Percentage: This metric calculates the proportion of profitable periods or trades relative to the total number, offering insights into the consistency of returns over time.

Table 1 shows the results of other portfolio optimization methods.

Image 7 illustrates the portfolios' returns and transaction costs over the given time horizon. It is evident that the other optimization methods do not account for the consistent expenditure of the budget, which is necessary to maintain balance throughout the time horizon.

Table 1: Performance Metrics

	Profit (%)	Sharpe Ratio	Max Drawdown (%)	Sortino Ratio	Information Ratio	Ulcer Ratio	Winning %
Equally Weighted Portfolio	51.57	0.0152	31.12	0.0231	0	1.0135	0.5475
Value Weighted Portfolio	69.50	0.0217	**28.42**	0.0328	0.0221	1.0588	0.5512
Buy and Hold Portfolio	61.43	0.0184	32.08	0.0278	0.0160	0.8709	**0.5593**
Weekly Online Portfolio	**517.26**	**0.0496**	72.29	**0.0843**	**0.0508**	1.8864	0.5232
4 Days Online Portfolio	481.42	0.0485	72.51	0.0823	0.0495	1.8283	0.5225
OLMAR Portfolio	65.22	0.0195	31.79	0.0298	0.0127	0.8729	0.5512
Markowitz Portfolio	77.31	0.0218	48.23	0.0337	0.0177	**0.7057**	0.5387
PrimalDual Portfolio	76.96	0.0237	36.07	0.0367	0.0302	1.1110	0.5379
Online Portfolio	391.04	0.0458	70.93	0.0769	0.0472	1.6970	0.5335

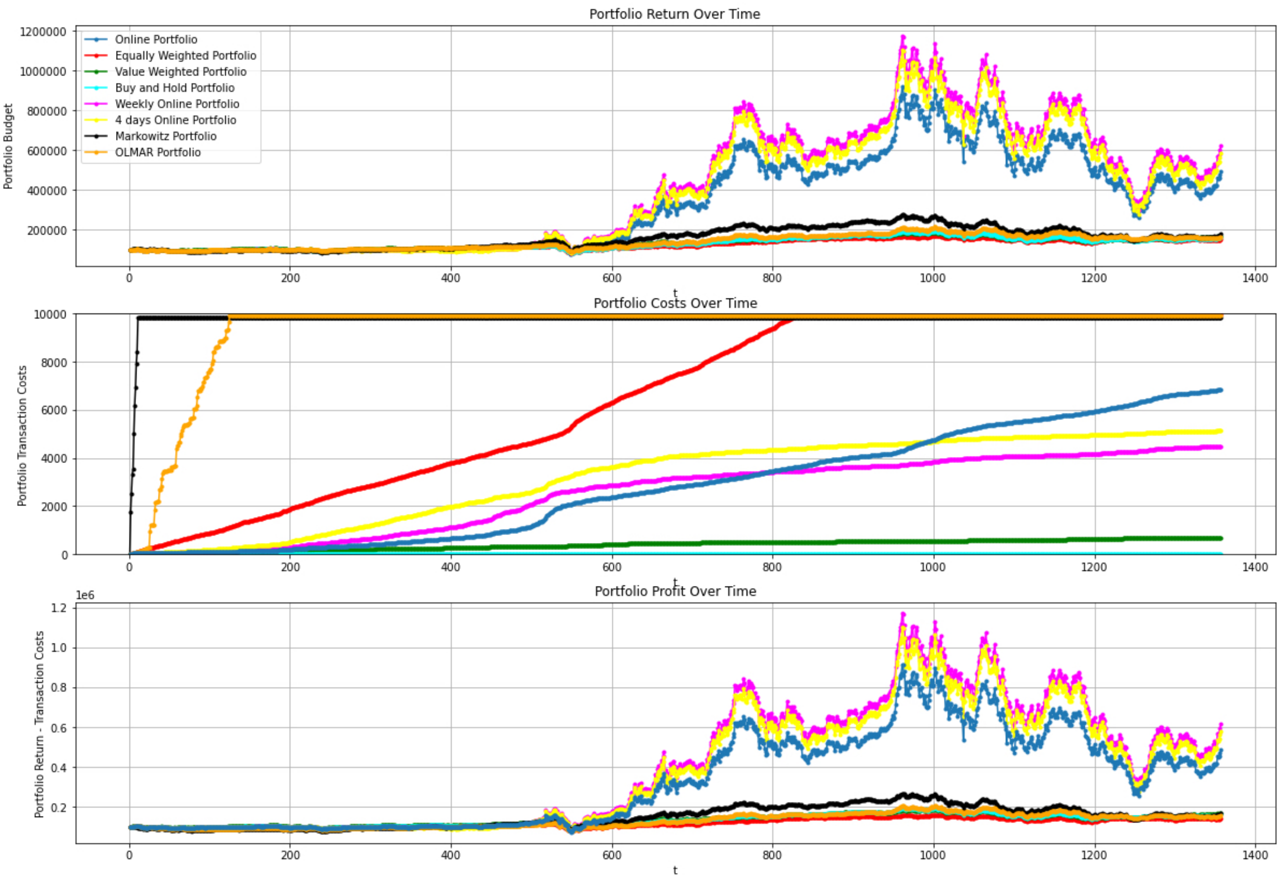

Figure 7: Benchmark infographics

We decide to designate the equally weighted portfolio as the basic benchmark. It provides moderate profitability, but the risk-adjusted return (Sharpe ratio) is relatively low. The portfolio has a moderate Ulcer ratio, indicating modest downside volatility. It has a winning percentage of 54.75%, suggesting that it is profitable more than half the time. Overall, this method is the worst performing in terms of profitability.

The value-weighted portfolio offers better profitability and risk-adjusted performance compared to the equally weighted portfolio. It also has a slightly higher Ulcer ratio and it stands out with the lowest maximum drawdown among the presented strategies, indicating relatively lower downside risk. Similarly, the buy and hold portfolio has a moderate profitability with a relatively low Ulcer ratio, suggesting again lower downside risk. The Sharpe ratio is also moderate, indicating a balanced risk-return profile.

The weekly online portfolio and it exhibits exceptional profitability, 517%. However, it comes with significantly higher volatility and drawdown, as indicated by the high Ulcer ratio and Max Drawdown. In terms of success, it is the best performing model and it follows the same logic as our basic Online Portfolio. It gradually distributes all the weights in Tesla, ending up with 98% of Tesla and 2% of Ford.

The four days online portfolio performs really similar to the weekly one, with a slightly worse final value.

The Markowitz portfolio utilizes modern portfolio theory to find the optimal asset allocation. It offers good profitability and, as expected, has the lowest Ulcer ratio.

Finally, the online portfolio that uses a vector as transaction costs instead of a scalar, (Primal-Dual Portfolio), performs better than the benchmark, 77% profit against 52%, and it is less risky compared to the other online portfolios, standing out for its diversification. In fact, rather than ending up with just 2 stocks, it is composed by 23 different ones. Table 2 gives an overview of the final composition of this diversified portfolio.

Table 2: Weights Distribution

Stock	**Value**	**Stock**	**Value**
AAPL	0.002631	BAC	0.051033
BK	0.014541	CL	0.002157
CMCSA	0.041290	CSCO	0.024997
EXC	0.025057	F	0.077458
GM	0.028392	GOOGL	0.025031
INTC	0.028061	KO	0.078274
MDLZ	0.083782	MET	0.083990
MS	0.047531	NEE	0.079249
NKE	0.010021	ORCL	0.064011
PFE	0.011531	SO	0.096107
T	0.074572	USB	0.001003
VZ	0.049281		

Conclusions

Throughout the experiment, we have tested different versions of our Online algorithm against strategies that are already established, such as the modern portfolio and the buy and hold portfolio. We have noticed different results that can be attractive to different types of investors. Our top performing algorithm, that significantly exceed the other portfolios in terms of profitability, is the online portfolio that does not incorporate the dual player in its strategy. It simply updates the weights inside the portfolio based on the stocks performances of the previous day or their moving average. Doing this it is able to almost completely "bet" on Tesla stock, that had a great run during the last 5 years, making the portfolio value skyrocket. This type of portfolio is not able to strike a balance between maximizing profitability and minimizing the risk, making it less appealing to more risk averse investors. Nevertheless, its incredible profit cannot be overseen and further tests using different time periods or stock indices could give us more insight on its performance. Moreover we will a regularization constraint to the weights update to allow for more diversification: following the same logic of its implemenation for the OLMAR portfolio we could utilize the deviation from the mean of the daily returns instead of just the daily returns as the loss to get a less drastic update.

On the other hand, the portfolio PrimalDual Portfolio that use both the primal and dual players has a more modest run, although it still outperforms all other benchmarks. It is able to spend the transaction budget constantly over time and effectively maintains an harmonious balance in risk management. Furthermore, it excels in diversifying the portfolio by strategically shedding underperforming stocks while retaining the best-performing ones. As previously mentioned, our next steps involve applying this algorithm to a variety of stocks, allowing us to gain a more comprehensive understanding of its genuine performance.

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Acknowledgments

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Sed commodo posuere pede. Mauris ut est. Ut quis purus. Sed ac odio. Sed vehicula hendrerit sem. Duis non odio. Morbi ut dui. Sed accumsan risus eget odio. In hac habitasse platea dictumst. Pellentesque non elit. Fusce sed justo eu urna porta tincidunt. Mauris felis odio, sollicitudin sed, volutpat a, ornare ac, erat. Morbi quis dolor. Donec pellentesque, erat ac sagittis semper, nunc dui lobortis purus, quis congue purus metus ultricies tellus. Proin et quam. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Praesent sapien turpis, fermentum vel, eleifend faucibus, vehicula eu, lacus.

Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Donec odio elit, dictum in, hendrerit sit amet, egestas sed, leo. Praesent feugiat sapien aliquet odio. Integer vitae justo. Aliquam vestibulum fringilla lorem. Sed neque lectus, consectetuer at, consectetuer sed, eleifend ac, lectus. Nulla facilisi. Pellentesque eget lectus. Proin eu metus. Sed porttitor. In hac habitasse platea dictumst. Suspendisse eu lectus. Ut mi mi, lacinia sit amet, placerat et, mollis vitae, dui. Sed ante tellus, tristique ut, iaculis eu, malesuada ac, dui. Mauris nibh leo, facilisis non, adipiscing quis, ultrices a, dui.

Morbi luctus, wisi viverra faucibus pretium, nibh est placerat odio, nec commodo wisi enim eget quam. Quisque libero justo, consectetuer a, feugiat vitae, porttitor eu, libero. Suspendisse

sed mauris vitae elit sollicitudin malesuada. Maecenas ultricies eros sit amet ante. Ut venenatis velit. Maecenas sed mi eget dui varius euismod. Phasellus aliquet volutpat odio. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque sit amet pede ac sem eleifend consectetuer. Nullam elementum, urna vel imperdiet sodales, elit ipsum pharetra ligula, ac pretium ante justo a nulla. Curabitur tristique arcu eu metus. Vestibulum lectus. Proin mauris. Proin eu nunc eu urna hendrerit faucibus. Aliquam auctor, pede consequat laoreet varius, eros tellus scelerisque quam, pellentesque hendrerit ipsum dolor sed augue. Nulla nec lacus.

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

Sed feugiat. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Ut pellentesque augue sed urna. Vestibulum diam eros, fringilla et, consectetuer eu, nonummy id, sapien. Nullam at lectus. In sagittis ultrices mauris. Curabitur malesuada erat sit amet massa. Fusce blandit. Aliquam erat volutpat. Aliquam euismod. Aenean vel lectus. Nunc imperdiet justo nec dolor.

Etiam euismod. Fusce facilisis lacinia dui. Suspendisse potenti. In mi erat, cursus id, nonummy sed, ullamcorper eget, sapien. Praesent pretium, magna in eleifend egestas, pede pede pretium lorem, quis consectetuer tortor sapien facilisis magna. Mauris quis magna varius nulla scelerisque imperdiet. Aliquam non quam. Aliquam porttitor quam a lacus. Praesent vel arcu ut tortor cursus volutpat. In vitae pede quis diam bibendum placerat. Fusce elementum convallis neque. Sed dolor orci, scelerisque ac, dapibus nec, ultricies ut, mi. Duis nec dui quis leo sagittis commodo.

Aliquam lectus. Vivamus leo. Quisque ornare tellus ullamcorper nulla. Mauris porttitor pharetra tortor. Sed fringilla justo sed mauris. Mauris tellus. Sed non leo. Nullam elementum, magna in cursus sodales, augue est scelerisque sapien, venenatis congue nulla arcu et pede. Ut suscipit enim vel sapien. Donec congue. Maecenas urna mi, suscipit in, placerat ut, vestibulum ut, massa. Fusce ultrices nulla et nisl.

Etiam ac leo a risus tristique nonummy. Donec dignissim tincidunt nulla. Vestibulum rhoncus molestie odio. Sed lobortis, justo et pretium lobortis, mauris turpis condimentum augue, nec ultricies nibh arcu pretium enim. Nunc purus neque, placerat id, imperdiet sed, pellentesque nec, nisl. Vestibulum imperdiet neque non sem accumsan laoreet. In hac habitasse platea dictumst. Etiam condimentum facilisis libero. Suspendisse in elit quis nisl aliquam dapibus. Pellentesque auctor sapien. Sed egestas sapien nec lectus. Pellentesque vel dui vel neque bibendum viverra. Aliquam porttitor nisl nec pede. Proin mattis libero vel turpis. Donec rutrum mauris et libero. Proin euismod porta felis. Nam lobortis, metus quis elementum commodo, nunc lectus elementum mauris, eget vulputate ligula tellus eu neque. Vivamus eu dolor.

Nulla in ipsum. Praesent eros nulla, congue vitae, euismod ut, commodo a, wisi. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Aenean nonummy magna non leo. Sed felis erat, ullamcorper in, dictum non, ultricies ut, lectus. Proin vel arcu a odio lobortis euismod. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Proin ut est. Aliquam odio. Pellentesque massa turpis, cursus eu, euismod nec, tempor congue, nulla. Duis viverra gravida mauris. Cras tincidunt. Curabitur eros ligula, varius ut, pulvinar in, cursus faucibus, augue.

Nulla mattis luctus nulla. Duis commodo velit at leo. Aliquam vulputate magna et leo. Nam vestibulum ullamcorper leo. Vestibulum condimentum rutrum mauris. Donec id mauris. Morbi molestie justo et pede. Vivamus eget turpis sed nisl cursus tempor. Curabitur mollis sapien condimentum nunc. In wisi nisl, malesuada at, dignissim sit amet, lobortis in, odio. Aenean consequat arcu a ante. Pellentesque porta elit sit amet orci. Etiam at turpis nec elit ultricies imperdiet. Nulla facilisi. In hac habitasse platea dictumst. Suspendisse viverra aliquam risus. Nullam pede justo, molestie nonummy, scelerisque eu, facilisis vel, arcu.

Curabitur tellus magna, porttitor a, commodo a, commodo in, tortor. Donec interdum. Praesent scelerisque. Maecenas posuere sodales odio. Vivamus metus lacus, varius quis, imperdiet quis, rhoncus a, turpis. Etiam ligula arcu, elementum a, venenatis quis, sollicitudin sed, metus. Donec nunc pede, tincidunt in, venenatis vitae, faucibus vel, nibh. Pellentesque wisi. Nullam malesuada. Morbi ut tellus ut pede tincidunt porta. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Etiam congue neque id dolor.

Donec et nisl at wisi luctus bibendum. Nam interdum tellus ac libero. Sed sem justo, laoreet vitae, fringilla at, adipiscing ut, nibh. Maecenas non sem quis tortor eleifend fermentum. Etiam id tortor ac mauris porta vulputate. Integer porta neque vitae massa. Maecenas tempus libero a libero posuere dictum. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Aenean quis mauris sed elit commodo placerat. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos hymenaeos. Vivamus rhoncus tincidunt libero. Etiam elementum pretium justo. Vivamus est. Morbi a tellus eget pede tristique commodo. Nulla nisl. Vestibulum sed nisl eu sapien cursus rutrum.

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Nulla ac nisl. Nullam urna nulla, ullamcorper in, interdum sit amet, gravida ut, risus. Aenean ac enim. In luctus. Phasellus eu quam vitae turpis viverra pellentesque. Duis feugiat felis ut enim. Phasellus pharetra, sem id porttitor sodales, magna nunc aliquet nibh, nec blandit nisl mauris at pede. Suspendisse risus risus, lobortis eget, semper at, imperdiet sit amet, quam. Quisque scelerisque dapibus nibh. Nam enim. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Nunc ut metus. Ut metus justo, auctor at, ultrices eu, sagittis ut, purus. Aliquam aliquam.