CS 2300 : Discrete Computational Structures
Fall Semester, 2024
ASSIGNMENT #9
Due Date: Friday, November 22

Suggested Reading: Rosen Section 2.5; LLM Chapter 8.1

For the problems below, explain your answers and show your reasoning.

1. **[12 Pts]** Show that the following sets are countably infinite, by first giving an enumeration, and then defining a bijection between $\mathbb{N}$ and that set. You do not need to prove that your function is bijective.

   (a) **[6 Pts]** the set of integers divisible by 5

   (b) **[6 Pts]** $A \times \mathbb{Z}^+$ where $A = \{2, 3, 4\}$

2. **[10 Pts]** Argue that the set of all finite strings over the alphabet Σ, where

$$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, /\},$$

   is countable. Use this to argue that the set of positive rationals is countable.
   Hint: Represent any positive rational as a finite string.

3. **[7 Pts]** Prove that the set of functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable, by using a diagonalization argument.

4. **[9 Pts]** Give an example of two uncountable sets A and B (along with a justification) such that $A - B$ is (a) finite (b) countably infinite (c) uncountably infinite

5. **[12 Pts]** Determine whether the following sets are countable or uncountable. Prove your answer. To prove countable, describe your enumeration precisely, using dovetailing. There is no need to define a bijection.

   (a) **[6 Pts]** the set of real numbers with decimal representation consisting of all 5's (5.5 and 55.555... are such numbers).

   (b) **[6 Pts]** the set of real numbers with decimal representation consisting of 5's and 7's.

For more practice, you are encouraged to work on other problems in Rosen Sections 2.5 and in LLM Chapter 8.

1a.)
A set of integers that is divisible by 5 can be shown as:
$S = \{..., -1, -5, 0, 5, 10, ...\}$
Enumeration Step:
$s_1 = 0, s_2 = 5, s_3 = -5, s_4 = 10, s_5 = -10, s_6 = 15, s_7 = -15, ...$ Defining a bijection between N and the previous set we can express S as : $s_n = \{5 \cdot (n/2)$, if n is even $-5 \cdot (n + 1/2)$, if n is odd $\}$
CONCLUSION: I have shown that each integer is divisible by and since the function is both injective and surjective, S is countably infinite. 2a.)

Enumeration: Shown in pairs where $a \in A = \{2, 3, 4\}$ and $z \in Z^+ = \{1, 2, 3...\}$ so
A x $Z^+ = \{(2, 1), (2, 2), (2, 3), ...(3, 1), (3, 2), ..., (4, 1), (4, 2), ..\}$.
Defining the Bijection: We can define the bijection is $f : N =\rightarrow A$ x Z^+ which is $f(n) = (a, z)$,by computing $a = 2 + (n - 1)$ mod 3. Where a is going through the values of 2,3,4 and z is the row index of enumeration.

2.)
I can argue that for any fixed $n \geq 0$, there is a $|\sum|^n = 11^n$ is strings of length n and since $|\sum|^n$ is finite for each n then $\sum^n$ is also finite.
We can write the set of all finite strings as $S = \sum^0 \cup \sum^1 \cup \sum^2 \cup \sum^3$ ....,where $\sum^n$is the set of all strings of length n that is formed by using $\sum$.
Therefore the set of all finite string over $\sum$ is countable.

Showing that the set of positive rationals is countable:
We can say that for every positive rational number of $q \in Q^+$ can be written in the for $q = p/d$. where $p, d \in Z^+$ and dominator $(p, d) = 1$. Since every positive rational number corresponds to a string in $\sum$, the set of positive rational $Q^+$is a subset of the set of all finite strings over $\sum$
CONCLUSION: I have shown that both the strings over the alphabet $\sum$ is countable and that the positive rationals is countable.

3.)
We can use a contradiction to show that the set of functions from N x N is countable. Which means we can list the sequence f_1, f_2, f_3...
We can assume and create a new function g where $g : N \rightarrow N$ where $g(n) = f_n(n) + 1$, meaning that g will take the nth function in the list and apply it to n + 1. This creates a contradiction because we assumed that we have listed all the functions from N x N but we just constructed a function g that is not in the list. Which shows that our initial assumption from N x N is countable is false.
CONCLUSION: We have proven that the functions from N x N is uncountable by using diagonalization argument.]

4.)
Showing that $(A - B)$ is finite:
We consider set A where $A = [0, 1]$, the interval of real numbers between 0 and 1. This set is uncountable since the real numbers in any interval are uncountable.
We can let set $B = [0, 1]$ / $1/2$, which has the same interval as A but with a single number removed. By removing a finite number of elements from an uncountable set does not make the set countable, so B is uncountable.
CONCLUSION: The difference $(A - B = \{1/2\})$ is finite beucase it only contains one element and even though both A and B are both uncountable (A-B) is a finite set.

Showing that $(A - B)$ is countably infinite:
We can consider set A and let $(A = [0, 1])$, the interval of real numbers between 0 and 1 which is uncountable.
We can let set $(B = [0, 1]/(Q \cap [0, 1]))$. Where the interval (0,1) with all rational number between 0 and 1 removed. Since rational numbers are countable, removing them will have the set still be uncountable but the irrationals in (0,1) form an uncountable set.
CONCLUSION: The difference $(A - B = Q \cap [0, 1])$ which is the set of rational number

between 0 and 1. This set is countably infinite since the rationals are countable. So $(A - B)$ is countably infinite.

Showing that $(A - B)$ is uncountably infinite.
We can get $A =$ R(real numbers), and $B = [0, 1]$. $A = R$ is uncountable and $B = [0, 1]$ is uncountable because it is a real interval.
CONCLUSION: The difference $A - B = R/[0, 1]$ is uncountable because it includes the union of $(-\infty, 0)$ and $(1, \infty)$ both which are uncountable.

5a.)
We can show a digital representation of this by an example:
5.5
55.5
555.555
and so on
Mathematically this can be written as $S = \{\sum_{i=1}^{n} 5 \cdot 10^{n-i} | n \in N\} \cup \{\sum_{i=1}^{\infty} 5 \cdot 10^{-i}\}$
We show that each element of this set has a unique representation based on the number of 5's in the decimal expansion and for every $n \in N$ there is exactly one such number of the form 555...

5b.)
An example of a decimal representation of 5's and 7's are: 0.555
0.577
0.757
7.777
and so on
Each number can be represented as an infinite sequence:
$a_1.a_2a_3a_4....$, where $a_i \in \{5, 7\}$ for all $i \geq 1$
By proof by contradiction:
We assume that the set of number with digits 5 and 7 are countable. For each x_i we can consider different decimals representations:
$x_1 = 5.555.., x_2 = 5.757..,$, and so on
Every new number can be constructed as y such that the i-th digit of y is 5 if the i-th digit of x_i is 7
The i-th digit of y is 7 if the i-th digit of x_i is 5
And the new number y cannot appear in the list, as it differes from i-th number in the i-th digit.
CONCLUSION: This contradiction implies that the set is uncountable, so our set of real numbers with decimal representation of 5's is countable in (a) and the set of real numbers with decimal representation of 5's and 7's is uncountable (b).