CS 230 : Discrete Computational Structures
Fall Semester, 2024
ASSIGNMENT #7
Due Date: Wednesday, October 30

Suggested Reading: Rosen Sections 5.2 - 5.3; Lehman et al. Chapters 5, 6.1 - 6.3

For the problems below, explain your answers and show your reasoning.

1. **[8 Pts]** Give an inductive definition for the following sequences $a_0, a_1, \ldots$.
   (a) $3, 9, 15, 21, \ldots$
   (b) $4, 11, 25, 53, \ldots$

2. **[8 Pts]** Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ for all positive integers n, where f_i are the Fibonacci numbers.

3. **[8 Pts]** Consider the following state machine with the states $\{0, 1, 2, 3, 4, 5, 6\}$. The start state is 0. The transitions are $0 \to 1$, $0 \to 4$, $1 \to 2$, $2 \to 3$, $3 \to 0$, $4 \to 5$, $5 \to 6$, and $6 \to 0$.

   Prove that if we take n steps in the state machine we will end up in state 0 if and only if n is divisible by 4. Argue that to prove the statement above by induction, we first have to *strengthen the induction hypothesis.* State the strengthened hypothesis and prove it.

4. **[9 Pts]** Consider the following game. Suppose you have a stack of n bricks. In a sequence of moves, you will split the stack of n bricks into n stacks of 1 brick each. You will get a score for each move and you want to maximize your total score.

   In each move, you take a stack and split it into two non-empty stacks. For any $a, b \geq 1$, if you split a stack of $a + b$ bricks into one stack of a bricks and one stack of b bricks, you get ab points.

   Prove by strong induction that the total score will be $n(n-1)/2$ regardless of the order in which the bricks are split.

5. **[9 Pts]** A robot wanders around a 2-dimensional grid. He starts out at (1,1). At any state it is in, it can take the following steps: (+1,-4), (-2,+2), (+4,-1) and (0,+3). Define a state machine for this problem. Then, define a Preserved Invariant and prove that the robot can never get to (0,0).

6. **[8 Pts]** Show that if a predicate $P(n)$ can be proven true for all positive integers n by strong induction, then it can be proven true also by regular induction, once you strengthen the inductive hypothesis. In other words, Strong Induction isn't really stronger than Regular Induction. *Hint: Given any statement $P(n)$, define a new (stronger) statement $Q(n)$ so that proving $P(n)$ by strong induction is similar to proving $Q(n)$ by regular induction.*

For more practice, you are encouraged to work on other problems in Rosen Sections 5.2 and 5.3 and in LLM Chapter 5, 6.1 - 6.3.

1.)
a.) Each term increases by 6 so we can say
$a_0 = 3$ which is the base case.
The inductive step would be for $n \geq 1$ for the next term a_n is:
CONCLUSION: $a_n = a_{n-1} + 6$

b.)
We can notice a pattern, that when we take take the difference while it is growing, we get that $11 - 4 = 7, 25 - 11 = 14, 53 - 25 = 28....$ The terms are added by an increasing power of 2.
We can take the base: $a_0 = 4$ and for the inductive definition:
CONCLUSION: $a_n = a_{n-1} + 2^{n+1}$ **for** $n \geq 1$

2.)
Base Case:
$n = 1$ so that $f_1 = f_1 = 1$ and that $f_{2*1} = f_2 = 1$ Inductive Step:
We assume that k holds for some positive integer k so that
$f_1 + f_3 + f_5 + ... + f_{2k-1} = f_{2k}$ and show that it implies $f_1 + f_3 + f_5 + ... + f_{2k-1} + f_{2k+1} = f_{2k+2}$
By IH we know that $f_1 + f_3 + f_5 + ... + f_{2k-1} = f_{2k}$ so expanding the left side, we can add f_{2k+1} to both sides to get $f_1 + f_3 + f_5 + ... + f_{2k-1} + f_{2k+1} = f_{2k} + f_{2k+1}$
Simplify:
By the definition of the Fib sequence we have that $f_{2k} + f_{2k+1} = f_{2k+1}$
CONCLUSION: Inductive Step: $f_1 + f_3 + f_5 + ... + f_{2k-1} + f_{2k+1} = f_{2k+2}$ **which means that** $f_1 + f_3 + f_5 + ... + f_{2n-1} = f_{2n}$ **for all positive integers n**

3.)
By strengthening the IH we need to track the values of n module 4, along withe the values when n is divisible by 4. By doing this we can prove that the behavior for each case of n is increasing by 1 step.
Base Case: $n = 0$ which is state 0, since $0 \equiv 0$
Inductive Step: We assume that for some $k \geq 0$ Showing the strengthened hypothesis with mod 4:
If $n \equiv 0$, the state is 0
If $n \equiv 1$, the state is 1
If $n \equiv 2$, the state is 2
If $n \equiv 3$, the state is 3
Continuing the inductive step we need to show that the IH holds for k+1
So going one-by-one for each state: 1. If $k \equiv 0$ then by the IH the state k steps is 0 then the next state would be 1, concluding that after k+1 step we are in state 1, where $(k + 1) \equiv 1$
2. If $k \equiv 1$ then by the IH the state k steps is 1 then the next state would be 2, concluding that after k+1 step we are in state 2, where $(k + 1) \equiv 2$
3. If $k \equiv 2$ then by the IH the state k steps is 2 then the next state would be 3, concluding that after k+1 step we are in state 3, where $(k + 1) \equiv 3$
4. If $k \equiv 3$ then by the IH the state k steps is 3 then the next state would be 0, concluding that after k+1 step we are in state 0, where $(k + 1) \equiv 0$
After completing the inductive step the hypothesis holds for $k + 1$
CONCLUSION: The strengthened hypothesis si proven for all $n \geq 0$**. Therefore we return to state 0 if and only if n is divisible by 4**

4.)
Base Case: $n = 1$, meaning that there is only one stack of 1 brick, so in the game no moves can be made. Making a formula we can get the base would be:
$S(1) = 0$, and expanding it we get that $S(1) = 1 * (1 - 1)/2 = 0$
Inductive Hypothesis:
We can assume that for all k where $1 \leq k \leq n$ the score of $S(k)$ when splitting the stack of k bricks into k stacks of 1 is given by the equation:
$S(k) = k(k - 1)/2$
Inductive Step:
To show that $S(n+1) = ((n+1)n)/2$ we can start with a stack of $n+1$ bricks and after first move it will split the stack into two stacks of a and b bricks, where $a+b = n+1$ and $a, b \geq 1$ would give the score of ab
After the split we will have 2 separate stacks of a and b, the stack will be labeled $S(a)$ and $S(b)$, giving the total score of $S(n + 1) = ab + S(a) + S(b)$
Using IH we know that $S(a) =$ and $S(b) = b(b - 1)/2$
Simplifying it we get $S(n + 1) = ab + (a(a - 1)/2) + (b(b - 1)/2)$
We can simplify even more by combing terms and factoring, which gets us:$S(n + 1) = ab + (a^2 + b^2 - a - b)/2$
Continuing we get that we can combine and simplify: $a^2 + b^2 = (n + 1)^2 - 2ab$
CONCLUSION: $S(n + 1) = ab + ((n + 1)^2 - 2ab - (n + 1))/2$

5.)
State Machine:
With 2 variables, $2^2 = 4$, that gives the robot 4 possible moves:
$(x, y) \rightarrow (x + 1, y - 4)$
$(x, y) \rightarrow (x - 2, y + 2)$
$(x, y) \rightarrow (x + 4, y - 1)$
$(x, y) \rightarrow (x, y + 3)$
Preserved Invariant:
Proving that the bot can never reach $(0, 0)$, we find a condition that remains true after each move the bot makes.
Initially we can assume $I(x, y) \equiv 2 \pmod 3$
Iterating through the moves the bot makes with the assumption and checking if the invariant is preserved after each move:
1. MOVE(1,-4):
$I(x + 1, y4) = (x + 1) + (y4) = x + y3 \equiv x + y$, keeps (x, y) unchanged
2. MOVE (-2,2)
$I(x2, y + 2) = (x2) + (y + 2) = x + y \equiv x + y$, keeps (x, y) unchanged
3. MOVE (4,-1)
$I(x + 4, y1) = (x + 4) + (y1) = x + y + 3 \equiv x + y$, keeps (x, y) unchanged
4. MOVE (0, 3)
$I(x, y + 3) = x + (y + 3) = x + y + 3 \equiv x + y$
We can also prove that$I(0, 0) = 0+0 = 0$, and it does not satisfy the invariant $I(x, y) = x+y(\text{mod } 3)$
CONCLUSION: The invariant remains constant at 2(mod 3) for all positions the robot can reach, starting from (1,1). Since (0,0) does not satisfy this invariant, the robot can never reach (0,0).

6.)
We can use a new stronger statement $Q(n)$ that implies $P(n)$, where we can use it to prove $Q(n)$ using regular induction will imply $P(n)$ for all of n.
We define that $Q(n) : P(1), P(2), ..., P(n)$ are all true.

Proving Q(n) by regular induction:
Base Case: For $n = 1, Q(1)$ states that $P(1)$ is true. Since strong induction allows us to assume that $P(1)$ is true as part of its base case, we have $P(1)$ is true, and thus $Q(1)$ holds.
Inductive Step:
We assume that $Q(k)$ is true from $k \geq 1$ and assume $P(1), P(2), ..., P(k)$ are all true, and we want to show $Q(k+1)$ is true and also show $P(1), P(2), ..., P(k), P(k+1)$ are all true.
Using the string induction hypothesis that $P(1), P(2), ..., P(k)$ are true to prove that $P(k+1)$. Therefore by using string induction, $P(k+1)$ is true if $P(1), P(2), ..., P(k)$ are true
CONCLUSION: Firstly by showing regular induction, we have shown that $Q(n)$ is true for all $n \geq 1$. Since $Q(n)$ implies P(n), it would conclude that $P(n)$ is true for all n