CS 2300 : Discrete Computational Structures

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Homework Assignment #3

Due Date: Friday, Sept 20

Suggested Reading: Lecture Notes Proofs 1 - 3; Rosen Sections 1.7 - 1.8; Lehman et al. Chapter 1

For the problems below, explain your answers and show your reasoning. You may not get full credit unless you show all your work. Use paragraph-style proofs as shown in class.

1. [**8 Pts**] Prove that for all integers p, p is even if and only if $p^3 + 7$ is odd. *Note: You need to do two proofs, one in each direction. You may not use the theorems shown in class regarding odd and even products.*

2. [**7 Pts**] Prove that if $x/5$ is irrational then $2x$ is irrational. What proof technique did you use and why?

3. [**7 Pts**] Let m and n be positive integers. Prove that if $mn > 35$, then $m \geq 8$ or $n \geq 6$. What proof technique did you use and why?

4. [**7 Pts**] Let x and y be non-zero rational numbers and let z be an irrational number. Prove that xy/z is irrational. Can you use a direct proof? Why or why not?

5. [**7 Pts**] Suppose your student organization has 40 scheduled meetings for the calendar year. Prove that at least four of the meetings fall on the same month. What proof technique do you use?

6. [**7 Pts**] Prove that if $p \geq 3$ or $p \leq -7$ then $(p+2)^2 \geq 25$. What proof technique did you use and why?

7. [**7 Pts**] Prove that there exist irrational numbers x and y whose sum is rational. Is your proof constructive or non-constructive? Explain. *Note: If you state that a number is irrational that was not proven to be in lecture or recitation, you must prove it is irrational.*

1.)

a.) Proof 1 (Forward Direction)

We suppose that p is even iff $p^3 + 7$ is odd
$\exists k$ such that $p = 2k$
We then get the expression $(2k)^3 + 7 = 8k^3 + 7$
Since we suppose p is even we can factor out a 2
The expression then becomes $8k^3 + 7 = 2(4k^3 + 3) + 1$ $8k^3$ is even, and adding 7 will result in an odd number.
Therefore $p^3 + 7$ is odd

b.) Proof 2(Backward Direction)

We suppose that $p^3 + 7$ is odd iff p is even
We can first assume that p is odd so that p=$2k + 1$ for some integer k
Then when we compute $p^3 + 7$ we get that $p = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1$
Each term in the expression is divisible by 2 which makes it even.
p^3 = (even number) + 1 is equal to an odd number and $p^3 + 7$= (odd number) + 7 is equal to and even number.
The expression contradicts the assumption that $p^3 + 7$ is odd, therefore p cannot be odd, and it must be EVEN.
CONCLUSION: for all integers p, p is even if and only if p3 + 7 is odd is TRUE

2.)

Proving "if $x/5$ is irrational then $2x$ is irrational ($p \to q$)"
Assume the opposite, "if $x/5$ is irrational then $2x$ is rational", so we have that $2x = a/b$ for some integer a and b, where $b \neq$0
Solving for x, x=$a/2b$
We can use $x/5$ and substitute it into x, which gets us $x/5$=$(1/5) * (a/2b) = a/10b$
$a/10b$ is a rational number since a and b are both integers.
CONCLUSION: Using proof by contradiction, we assume from the start that $x/5$ is irrational, but proved that $x/5 = a/10b$ is rational. Using proof by contradiction we can show that any assumption contrary to the given conditions will violate the logical structure of the problem.

3.)

Proving that "if $mn > 35$ then $m \geq 8$ or $n \geq 6$ $(p \to q)$"
Using proof by contrapositive we get "if $m < 8$ and $n < 6$, then $mn \leq 35$
1. We assume that $m < 8$ and $n < 6$, which implies that $m \leq 7$ and $n \leq 5$ since m and n are positive integers.
2. Computing for the max of m and the max of n we have $m * n = 7 * 5 = 35$
Since $mm \leq 35$, assuming that $m < 8$ and $n < 6$, we have proven the contrapositive is true, meaning the original statement is true.
CONCLUSION: This technique simplifies the problem by working with more manageable inequalities.

4.)
Since x and y are irrational numbers, we can write them into expressions as $x = a/b$ and $y = c/d$ where a, b, c, d are integers and b and $d \neq 0$
1. We multiply x and y, $xy = (a/b) * (c/d) = ac/bd$; Since a, b, c, d are integers, $xy = ac/bd$ is a rational number because the product of 2 rational number is rational 2. If we divide by the irrational number, z, we would need to prove that xy/z is irrational.
To prove that it is irrational we substitute in xy=ac/bd into xy/z, which will get us $xy/z = (ac/bd)/z = ac/bdz$.
Since z is irrational, it makes bdz irrational because of multiplication if a non-zero rational number with a irrational number results in a irrational number CONCLUSION: xy/z is irrational by using direct proofs because we can manipulate the expression and apply known properties of rational and irrational numbers.

5.)
Letting m be months and there are 12 months in a year we can say that $m \geq 4$, meaning that each month has at least 4 meetings a month. So in a year there are 48 meetings, $12 * 4 =$ 48. 48 is more than what is scheduled in the calendar year. So we can assume the opposite of what we would like to prove, which is: Assume that no month has 4 or more meetings. The assumption of not having 4 or more meetings it would be that each month would have 3 meetings a month, 12 * 3 = 36 meetings.
CONCLUSION: Since 40 is greater than 36 it is not possible to have all the months to have 3 meetings, which means that there is at least one month with 4 meetings.
It leads to a contradiction. Showing that eh original assumption is false and at least one month must have 4 or more meetings.

6.)
Using my proof by cases we can use each statement split into cases making it easier to work on.

Case 1:
if $p \geq 3$ then $(p+2)^2 \geq 25$
Using the contrapositive: if$(p+2)^2 < 25$, then $p < 3$
$(p+2) < \sqrt{25}$ $(p+2) < 5$
CONCLUSION: $p < 3$
Case 2:
if $p \leq -7$ then $(p+2)^2 \geq 25$
Using the contrapositive: if $(p+2)^2 < 25$ then $p > -7$
$(p+2) < \sqrt{25}$
$-(p+2) < 5$
$(p+2) > -5$

CONCLUSION: $p > -7$

7.)
We let $x = \sqrt{2}$ **and** $y = -\sqrt{2}$
$\sqrt{2}$ **is an irrational number**
The sum of $\sqrt{2} + (-\sqrt{2}) = 0$**, but zero is a rational number.**
CONCLUSION: Since 0 is a rational number this problem serves as a constructive proof for the existence of irrational numbers whose sum is rational with the specified values of x and y.

For more practice, work on the problems from Rosen Sections 1.7 - 1.8; Lehman et al. Chapter 1.