CS 2300 : Discrete Computational Structures

Fall Semester, 2024

HOMEWORK ASSIGNMENT #5

Due Date: Wednesday, Oct 16

Suggested Reading: Lecture Notes Functions 2, Relations 1-3; Rosen Sections 9.1 and 9.5; Lehman et al. Chapter 10.5, 10.6 and 10.10 For the problems below, explain your answers and show your reasoning.

1. [**10 Pts**] Let g be a total function from A to B and f be a total function from B to C.

   (a) If $f \circ g$ is one-to-one, then is f one-to-one? Prove or give a counter-example.

   (b) If $f \circ g$ is onto, then is f onto? Prove or give a counter-example.

2. [**10 Pts**] For each of these relations decide whether it is reflexive, anti-reflexive, symmetric, anti-symmetric and transitive. Justify your answers. R_1 and R_2 are over the set of real numbers.

   (a) $(x, y) \in R_1$ if and only if $xy = 2$

   (b) $(x, y) \in R_2$ if and only if $y = 5x$

3. [**7 Pts**] Consider relation R_3 defined below on the set of real numbers. Prove that R_3 is an equivalence relation. What is the equivalence class of 2? of π? Describe all the equivalence classes using set-builder notation.

$$(x, y) \in R_3 \text{ if and only if } x = y - 4n \text{ for some integer } n.$$

4. [**5 Pts**] Consider relation R_4 defined below on the set of positive real numbers. Prove that R_4 is a partial order, *i.e.*, it is reflexive, anti-symmetric and transitive.

$$(x, y) \in R_4 \text{ if and only if } y/x \in \mathbb{Z}$$

5. [**8 Pts**] Let R_5 be the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ where $((a, b), (c, d)) \in R_5$ if and only if $c/a = d/b$.

   (a) Prove that R_5 is an equivalence relation.

(b) Define a function f such that $f(a,b) = f(c,d)$ if and only if $((a,b),(c,d)) \in R_5$.

(c) Define the equivalence class containing $(1,1)$.

(d) Describe the equivalence classes using set-builder notation.

6. **[10 Pts]** Prove that these relations on the set of all functions from $mathbbZ$ to $\mathbb{Z}$ are equivalence relations. Describe the equivalence classes.

   (a) $R_6 = \{(f,g) \mid f(0) = g(0) \text{ and } f(1) = g(1)\}$

   (b) $R_7 = \{(f,g) \mid \exists C \in \mathbb{Q}^+, \forall x \in \mathbb{Z}, f(x) = Cg(x)\}$

For more practice, work on the problems from Sections 9.1 and 9.5; Lehman et al. Chapter 10.5, 10.6 and 10.10

1a.) Using a contrapositive, we assume that f is not one-to-one so then $f \circ g$ is not one-to-one. This would mean there exist elements $b_1, b_2 \in B$ such that $f(b_1) = f(b_2)$ but $b_1 \neq b_2$

We can use g as a function from A to B using $a_1, a_2 \in A$ such that $g(a_1) = b_1$ and $g(a_2) = b_2$

After computing $f \circ g$ for a_1, a_2:

$(f \circ g)(a_1) = f(g(a_1)) = f(b_1)$

$(f \circ g)(a_2) = f(g(a_2)) = f(b_2)$

But since $f(b_1) = f(b_2)$, we get that $(f \circ g)(a_1) = (f \circ g)(a_2)$

CONCLUSION:

$a_1 \neq a_2$ **because** $g(a_1) = b_1$ **and** $g(a_2) = b_2$**. This would show that** $f \circ g$ **is not one-to-one contradicting the assumption initially. So therefore** f **must be one-to-one for** $f \circ g$ **to be one-to-one.**

1b.) Using a counter example, we can let: $A = \{1,2\}, B = \{1,2,3\}$, and $C = \{1,2\}$. Defining the following functions $g : A \to B$ and $f : B \to C$

$g(1) = 1, g(2) = 2$ which would show that g is onto

$f(1) = 1, f(2) = 2$, and $f(3) = 2$ shows that f is not onto

When we consider $f \circ g$: $f \circ g(1) = f(g(1)) = f(1) = 1$

$f \circ g(2) = f(g(2)) = f(2) = 2$

CONCLUSION:

$f \circ g$ maps to A onto C, which would mean that $f \circ g$ is onto, but f itself is not onto because $1 \in C$. Therefore $f \circ g$ being onto does not imply that f is onto

2a.) R_1 is reflexive if $(x, x) \in R_1$ for all $x \in R$, which would mean that x * x = 2 for all x.
But since for any real number x, x * x = 2 is not true, for example if x = 1, 1 *1 = 1 $\neq$ 2.
CONCLUSION: R_1 is not reflexive.

R_1 is anti-reflexive if $(x, x) \notin R_1$ for all $x \in R$ since no real number satisfies x * x = 2, but $+-\sqrt{2}$ but that is specific value
CONCLUSION: Therefore R_1 is anti-reflexive

R_1 is symmetric if $(x, y) \in R_1$ and $(y, x) \in R_1$ which means that x * y = 2, then y * x = 2
Since the use of multiplication is commutative if $(x, y) \in R_1$ then $(y, x) \in R_1$
CONCLUSION: There for R_1 is symmetric.

R_1 is anti-symmetric if $(x, y) \in R_1$ and $(y, x) \in R_1$ imply that $x = y$
We can say that $x = 1$ and $y = 2$, then $1 * 2 = 2$ so $(1, 2) \in R_1$ and since $2 * 1 = 2$, $(2, 1) \in R_1$
But $1 \neq 2$
CONCLUSION: Therefore R_1 is not anti-symmetric

R_1 is transitive if $(x, y) \in R_1$ and $(y, z) \in R_1$ imply $(x, z) \in R_1$
We can consider that $(x, y) \in R_1$, so $x * y = 2$ and $(y, z) \in R_1$, so $y * z = 2$.
But if we let $x = 1$, $y = 2$, and $z = 1/2$. Then $x * y = 2$ and $y = 2$ but $x * z = 1 \neq 2$
CONCLUSION: Therefore R_1 is not transitive

2b.) R_2 is reflexive if $(x, x) \in R_2$ for all $x \in R$, which would mean that x = 5x for all x.
CONCLUSION: Easily conclude that it is only true when $x = 0$ so R_2 is not reflexive.

R_2 is anti-reflexive if $(x, x) \notin R_1$ for all $x \in R$

Since $x, x) \in R_2$ only when $x = 0$, **CONCLUSION: R_2 is not anti-reflexive because** $(0, 0) \in R_2$

R_2 is symmetric if $(x, y) \in R_2$ implies $(y, x) \in R_2$ which means that $y = 5x$, then $x = 5y$
We can consider $y = 5x$. Then for $(y, x) \in R_2$to hold we would need $x = 5y$, which leads to $x = 5(5x) = 25x$. But this is only true when $x = 0$.
CONCLUSION: R_2 is not symmetric

R_2 is anti-symmetric if $(x, y) \in R_2$ and $(y, x) \in R_2$ imply that $x = y$
Suppose that $(x, y) \in R_2$ and $(y, x) \in R_2$, which would mean $y = 5x$ and $x = 5y$, this would lead to $x = 0$ and $y = 0$
CONCLUSION: $x = y$ which makes R_2 anti-symmetric

R_2 is transitive if $(x, y) \in R_2$ and $(y, z) \in R_2$ imply $(x, z) \in R_2$
We can say $(x, y) \in R_2$ and $(y, z) \in R_2$, which would mean that $y = 5x$ and $z = 5y$, and if we sub in $y = 5x$ into $z = 5y$, we would get that $z = 5(5x) = 25x$ which does not satisfy $z = 5x$ unless $x = 0$
CONCLUSION: R_2 is not transitive

3.) To show equivalence relation we must prove for all of reflexivity, symmetry, and transitivity
R_3 is reflexive if $(x, x) \in R3$ for all $x \in R$, which would mean that x = x - 5n for some integer n.
We can set $n = 0$ for all $x \in R$;
$x = x - 4 * 0 = x$
CONCLUSION: R_3 is reflexive

R_3 is symmetric if $(x, y) \in R_3$ implies $(y, x) \in R_3$ which means that $x = y - 4n$ for some integer n, then $y = x - 4m$ for some integer m

R_3 is transitive if $(x, y) \in R_3$ and $(y, z) \in R_3$ imply $(x, z) \in R_3$, which means if $x = y - 4n$ and $y = z - 4m$, then $x = z - 4k$ for some integer k
We can suppose that $(x, y) \in R_3$ so $x = y - 4n$ and $(y, z) \in R_3$ so that $y = z - 4n$ for some integers n and m.
We can then sub the second equation into the first: so that $x = (z - 4m) - 4n =$

$z - 4(m+n)$
Making $x = z - 4k$ where $k = m+n$ is an integer **CONCLUSION: R_3 is transitive**
OVERALL CONCLUSION: WE CAN SAY THAT R_3 IS A EQUIVALENCE RELATION

For the equivalence class of 2, which is denoted [2], consists of all real numbers y such that $(2, y) \in R_3$ meaning that $2 = y - 4n$ for some integer n
When solving for y we get that $y = 2 + 4n$, where n is an integer.
CONCLUSION: The equivalence class of 2 is [2] $= \{y \in R | y = 2 + 4n, n \in Z\}$

For the equivalence class of π, denoted $[\pi]$, it consists of all real numbers y such that $(2, y) \in R_3$ meaning $2 = y - 4n$ for some integer n
Solving for y we get that $y = \pi + 4n$, where n is an integer
CONCLUSION: The The equivalence class of π is $[\pi] = \{y \in R | y = \pi + 4n, n \in Z$

Describing all equivalence classes under R_3 by using set- builder notation it can be described as:
$[x] = \{y \in R | y \equiv x(mod4)\}$

4.) Partial order, is when R_4 is reflexive, anti-symmetric and transitive.

For R_4 to be reflexive, if $(x, x) \in R_4$ for all $x \in R+$, which would mean that $x/x \in Z$
We can say for any positive number x we have $x/x = 1$, and $1 \in Z$ **CONCLUSION: R_4 is reflexive**.

R_4 is anti-symmetric if $(x, y) \in R_4$ and $(y, x) \in R_4$, which implies that x = y meaning $x/y \in Z$ and $x/y \in Z$ then $x = y$.
This would imply that y is a multiple of x and x is a multiple of y
Multiplying $(y/x) * (x/y) = k * (1/k) = 1$
CONCLUSION: R_4 is anti-symmetric

A relation is transitive if $(x, y) \in R_4$ and $(y, z) \in R_4$ imply that $(x, z) \in R_4$, meaning if $y/x \in Z$ and $z/y \in Z$, then $x/x \in Z$

We can suppose that $(x, y) \in R_4$, so that $y/x = k$ for some integer k and $(y, z) \in R_4$, so that $z/y = m$ for some integer m
We can show that $x/z \in Z$ by $(z/x) = (z/y) * (y/x) = m * k$
CONCLUSION: Since both m and k are integers $z/x = mk$ is also an integer so R_4 is transitive.
OVERALL CONCLUSION: R_4 is partial order

5a.) To show that R_5 is a equivalence relation, we need to prove reflexivity, symmetry, and transitivity.
R_5 is reflexive if $((a, b), (a, b) \in R_5$ for all (a, b) meaning that $a/b = a/b$
We can see that any pair for (a, b) we have that $a/b = a/b$
CONCLUSION: R_5 is reflexive
R_5 is symmetric if $((a, b), (c, d)) \in R_5$ implies $((c, d), (a, b)) \in R_5$, which means that $a/b = c/d$, then $c/d = a/b$.
CONCLUSION: Since the ratios are symmetric, it would mean that R_5 is symmetric
R_5 is transitive if $((a, b), (c, d)) \in R_5$ and $((c, d), (e, f)) \in R_5$ which implies that $((a, b), (e, f)) \in R_5$, so that $a/b = c/d$ and $c/d = e/f$, then $a/b = e/g$
If we see that $a/b = c/d$ and $c/d = e/f$ then by transitive equality we have that $a/b = e/f$
CONCLUSION: R_5 is transitive
IT WOULD CONCLUDE THAT R_5 is an equivalence relation
5b.) We can choose that $f(a, b) = a/b$
We can show this if $a/b = d/c$ then $f(a, b) = f(c, d)$ and if $f(a, b) = f(c, d)$ then $a/b = d/c$ which means that $((a, b), (c, d)) \in R_5$ So $f(a, b) = a/b$ satisfies the condition given.
5c.) We can simply the equation to get $1/1 = d/c$ we would get that $d = c$
CONCLUSION: There for the class containing (1,1) is [(1,1)] = $\{(c, c)|c \in R+\}$ which means the two elements are equal
5d.) By using set builder notation we describe the equivalence classes as: $[(a, b)] = \{(c, d) \in R + xR + |c/d = a/b\}$

6a.) R_6 is reflexive because we can see that $f(0) = f(0)$ and $f(1) = f(1)$ is true for any function.
R_6 is symmetric because if $f(0) = g(0)$ and $f(1) = g(1)$, then $g(0) = f(0)$ and $g(1) = f(1)$.
R_6 is transitive because $f(0) = g(0), f(1) = g(1), g(0) = h(0)$and $g(1) = h(1)$, then f(0) = h(0) and $f(1) = h(1)$. When we combine then we get that R_6 is transitive

CONCLUDES that R_6 is an equivalence relation.

Describing the equivalence class we get that f consists of all functions that agree with f at 0 and 1 but can be different somewhere else, if we wanted to we can rewrite it as:

$[f] = \{g|g(0) = f(0)$ **and** $g(1) = f(1)$

6b.)For reflexivity if $(f, f) \in R_7$ for all functions f meaning there exists a constant $C \in Q+$ such that f(x) = Cf(x) for all $x \in Z$. R_7 is reflexive by taking in that $C = 1 \in Q+$ such that $f(x) = 1 * f(x)$ so that $(f, f) \in R_7$ Therefore R_7 is reflexive.

For symmetry if $(f, g) \in R_7$ implies $(g, f) \in R_7$ meaning there exists $C \in Q+$ such that $f(x) = Cg(x)$ then there must exist $D \in Q+$ such that $g(x) = Df(x)$. R_7 is symmetric if $f(x) = Cg(x)$, then $g(x) = 1/Cf(x)$ and since $C \in Q+$, $1/C \in Q+$ Therefore $(g, f) \in R_7$is symmetric

For transitivity if $(f, g) \in R_7$ and $(g, h) \in R_7$ imply $(f, h) \in R_7$, if there exists $C, D \in Q+$ such that $f(x) = Ch(x)$ and $g(x) = Dh(x)$ there must exist $E \in Q+$ such that $f(x) = Eh(x)$. R_7 is transitive when we have that $f(x) = Cg(x) = C * Dh(x)$, so $f(x) = (C * D)h(x)$. Since $C, D \in Q+$, which means that $(f, h) \in R_7$, Therefore R_7 is transitive.

CONCLUSION: After proving each it makes that R_7 is an equivalence relation

We can rewrite and describe the equivalence classes by saying $[f] = \{g|\exists C \in Q+, \forall x \in Z, f(x) = Cg(x)\}$