CS 2300 : Discrete Computational Structures
Fall Semester, 2024
HOMEWORK ASSIGNMENT #6
Due Date: Wednesday, Oct 23

Suggested Reading: Rosen Section 5.1 - 5.2; Lehman et al. Chapter 5.1 - 5.3

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

For Problems 1-5, prove the statements by mathematical induction. Clearly state your basis step and prove it. What is your inductive hypothesis? Prove the inductive step and show clearly where you used the inductive hypothesis (assumption).

1. [**6 Pts**] $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2))^2$, for all positive integers n.
2. [**6 Pts**] $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$, for all positive integers n.
3. [**6 Pts**] $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$, for all non-negative integers n.
4. [**6 Pts**] $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 3 - \frac{1}{n}$, for all integers $n \geq 2$.
5. [**6 Pts**] 3 divides $n^3 + 2n$, for all positive integers n.
6. [**8 Pts**] $3^n < n!$ for all integers $n \geq 7$. In addition, show that the statement is false for all positive integers $n < 7$.
7. [**12 Pts**] Let $P(n)$ be the statement that n-cent postage can be formed using just 3-cent and 10-cent stamps. Prove that $P(n)$ is true for all $n \geq 18$, using the steps below.

   (a) First, prove $P(n)$ by regular induction. State your basis step and inductive step clearly and prove them.

   Hint: Argue that if $k \geq 18$, k-cent postage must have at least 3 3-cent stamps or at least 2 10-cent stamps. Then use these as your cases for the inductive step.

   (b) Now, prove $P(n)$ by strong induction. Again, state and prove your basis step and inductive step. Your basis step should have multiple cases.

8. **[6 Pts] Extra Credit** Suppose $P(n)$ is true for an infinite number of positive integers n. Also, suppose that $P(k+1) \to P(k)$ for all positive integers k. Now, prove that $P(n)$ is true for all positive integers. This is the *reverse induction principle.*

For more practice, you are encouraged to work on other problems in Rosen Sections 5.1 and 5.2 and in LLM Chapter 5.1 - 5.3.

1.) **Base Case:** $n = 1$**:**
$1^3 = 1^2(1+1)^2/4$
$1 = 1(2)^2/4$
$1 = 1(4)/4$
$1 = 4/4$
$1 = 1$

$n = 2$
$1^3 + 2^3 = 2^2(2+1)^2/4$
$9 = 4(3)^2/4$
$9 = (3)^2$
$9 = 9$

Inductive Hypothesis:
$n = k$
$1^3 + 2^3 + ...k^3 = k^2(k+1)^2/4$

$n = k+1$
$1^3 + 2^3 + ...k^3 + (k+1)^3 = (k+1)^2([k+1]+1)^2/4$
$((k^2(k+1)^2)/4) + (4(k+1)^3)/4) = (k+1)^2([k+1]+1)^2/4$
$k^2(k+1)^2 + 4(k+1)^3 = (k+1)^2(k+2)^2$
$(k+1)^2(k^2 + 4(k+1)) = (k+1)^2(k+2)^2$
$(k+1)^2(k + 4k + 4) = (k+1)^2(k+2)^2$
$(k+1)^2(k+2)(k+2) = (k+1)^2(k+2)^2$
CONCLUSION:$(k+1)^2(k+2)^2 = (k+1)^2(k+2)^2$

2.) **Base Case: n = 1:**
$1 \cdot 1! + 2 \cdot 2! + ...n \cdot n! = (n+1)! - 1$

Inductive Hypothesis:
n = k
$1 \cdot 1! + 2 \cdot 2! + ... + k \cdot k! = (k+1)! - 1$

$n = k + 1:$
$1 \cdot 1! + 2 \cdot 2! + ... + k \cdot k! + (k+1) \cdot (k+1)! = (k+1)! + (k+1)(k+1)! - 1$
Conductive Step: $1 \cdot 1! + 2 \cdot 2! + ... + k \cdot k! + (k+1) \cdot (k+1)! = (k+1)!(1 + (k+1))(k+1) - 1$
$1 \cdot 1! + 2 \cdot 2! + ... + k \cdot k! + (k+1) \cdot (k+1)! = (k+2)! - 1$
CONCLUSION: $1 \cdot 1! + 2 \cdot 2! + ... k \cdot k! = (k+1)! - 1$ **holds for all positive numbers**

3.) **Base Case:** $n = 0$**:**
$2(-7)^0 = (1 - (-7)^0 + 1)/4$
$2(1) = 1 - (-7)/4$
$2 = 1 + 7/4$
$2 = 8/4$
$2 = 2$

Inductive Hypothesis: n = k:
$2 - 2 \cdot 7 + 2 \cdot 72 - .... + 2(-7)^k = 41 - (-7)^{k+1}$

$n = k + 1$
$2 - 2 \cdot 7 + 2 \cdot 72 - .... + 2(-7)^k + 2(-7)^{k+1} = 41 - (-7)^{k+2}$

Inductive Step:
$2 - 2 \cdot 7 + 2 \cdot 72 - .... + 2(-7)^k = (1 - (-7)^k + 1)/4$ STARTING ASSUMPTION
$2 - 2 \cdot 7 + 2 \cdot 72 - .... + 2(-7)^k + 2(-7)^{k+1} = ((1 - (-7)^k + 1)/4) + 2(-7)^k$
ADD $2(-7)^{k+1}$ TO BOTH SIDES $(1 - (-7)^{k+1}/4) + 2(-7)^{k+1} = (1 - (-7)^{k+1} + 8(-7)^{k+1}/4)$ SIMPLIFYING THE RIGHT SIDE
$(1 + 7(-7)^{k+1}/4) = (1 + 7(-7)(-7)^{k+1}/4)$ COMBINE LIKE TERMS
$1 - 7[(-7)(-7)k]/4$ FACTOR OUT A -7
$1 - 7(-7)^{k+1}/4 = 1 - 7[(-7)(-7)k]/4$ SIMPLIFY
CONCLUSION:$1 - 7(-7)^{k+1}/4 = 1 - (-7) \cdot 7^{k+1}/4$

4.) **Base Case: n = 2:**
$1 + (1/4) = (4/4) + (1/4) = (5/4)$ LEFT HAND SIDE WHEN n = 2
$3 - (1/2) = (6/2) - (1/2) = (5/2)$ RIGHT HAND SIDE WHEN n = 2
$(5/4) < (5/2)$

Inductive Hypothesis: $k \geq 2$**:**
$1 + (1/4) + (1/9) + ... + (1/k^2) < 3 - (1/k)$

Inductive Step: n = k+1:
$1 + (1/4) + (1/9) + ... + (1/k^2) + 1/(k+1)^2 < 3 - (1/k+1)$
$1 + (1/4) + (1/9) + ... + (1/k^2) < 3 - (1/k)$
$1 + (1/4) + (1/9) + ... + (1/k^2) + 1/(k+1)^2 < 3 - (1/k) + 1/(k+1)^2$ ADDING $1/(k+1)^2$ TO BOTH SIDES
$3 - 1/(k+1) = (k+1) - k/k(k+1) = 1/k(k+1)$ SIMPLIFY THE RIGHT SIDE
$1/k - (1/k+1) = (k+1) - k/k(k+1) = 1/k(k+1)$
CONCLUSION:$1/(k+1)^2 < 1/k(k+1)$

5.) **Base Case: n=1:**
$n^3 + 2n$
$1^3 + 2(1)$
$1 + 2$
$= 3$

Inductive Hypothesis: 3 **divides** $k^3 + 2k$
Which would mean that some m exists: $k^3 + 2k = 3m$

Inductive Step: n=k+1
3 divides $(k+1)^3 + 2(k+1)$
$(k+1)^3 + 2(k+1)$ EXPANDING:
$(k+1)^3 = k^3 + 3k^2 + 3k + 1$
$2(k+1) = 2k + 2$
$(k+1)^3 + 2(k+1) = (k^3 + 3k^2 + 3k + 1) + (2k + 2)$
$= k^3 + 3k^2 + 5k + 3$ ADDING THE EXPRESSIONS TOGETHER
$(k+1)^3 + 2(k+1) = (k^3 + 2k) + (3k^2 + 3k + 3)$
$(k+1)^3 + 2(k+1) = 3m + (3k^2 + 3k + 3)$
$= 3(m + k^2 + k + 1)$ SUBSTITUTE IN THE INTEGER m
CONCLUSION: 3 is a factor of $3(m+k^2+k+1)$ **which shows that** $(k+1)^3 + 2(k+1)$ **is divisible by 3**

6.) **Base Case: n = 7:**
$3^7 = 2187 and 7! = 5040$
Since 2187 ¡ 5040, $3^n < n! for n = 7$

Inductive Hypothesis: $3^n < n! for n = k$
$3^k < k!$

Inductive Step:
n = k +1 $3^{k+1} = 3(3^k)$
$3^k < k!$
So then $3^{k+1} < 3(k!)$
Since $3 < k + 1$ for all $k \geq 8$, If we replace 3 with (k+1) then inequality will not be affected. $3^{k+1} < (k + 1)(k!)$
$3^{k+1} < (k + 1)!$
$3^{k+1} <$ (k+1)! is in the form $3^n < n!$ **CONCLUSION:** $3^n < n!$ **is true for n = k+1**

7.)
a.) **Base Case:**
Show that $P(n)$ is true for some starting value $n \geq 18n$ Check $n = 18$, $n = 19$, and $n = 20$
$n = 18$: We can use six 3-cent stamps. Thus, 18 cents can be formed: $18 = 63$
$n = 19$: We can use three 3-cent stamps and one 10-cent stamp. Thus, 19 cents can be formed: $19 = 33 + 110$
$n = 20$: We can use two 10-cent stamps. Thus, 20 cents can be formed: $20 = 210$

Therefore, $P(18), P(19)$, and $P(20)$ are all true.

Inductive Step:
Assume that $P(k)P(k)P(k)$ is true for some arbitrary $k \geq 20$
Inductive Step:
Show that $P(k + 1)P(k + 1)P(k + 1)$ is also true
Case 1: $k \geq 18$ **has at least one 3-cent stamp**
If k-cent postage uses at least one 3-cent stamp, we can replace one 3-cent stamp with a 10-cent stamp to form $(k + 3)$-cent postage. Since $k \geq 18$ we know from the inductive hypothesis that k-cent postage can be formed, and by replacing a 3-cent stamp, $(k + 3)$-cent postage can also be formed.
Case 2: k**-cent postage uses at least two 10-cent stamps**
If k-cent postage uses at least two 10-cent stamps, we can remove one 10-cent stamp and replace it with three 3-cent stamps to form $(k + 1)$-cent postage. Therefore, we can form $(k + 1)$-cent postage using this transformation.
Conclusion: Thus, by either case, $P(k{+}1)$ **is true. P(n) is true for all** $n \geq 18$

b.)
Base Case: Base cases for $n = 18, n = 19$, and $n = 20$
$n = 18$: We can use six 3-cent stamps: $18 = 63$
$n = 19$: We can use three 3-cent stamps and one 10-cent stamp: $19 = 33 + 110$
$n = 20$: We can use two 10-cent stamps: $20 = 210$

Therefore, the Base Case holds for $n = 18, n = 19$, and $n = 20$

Inductive Hypothesis:
Assume that $P(m)$ is true for all m such that $18 \leq m \leq k$

Inductive Step:
Prove that$P(k + 1)$ is true
Since $P(m)$ is true for all m such that $18 \leq m \leq k$, we know that $(k2)$-cent postage can be formed using 3-cent and 10-cent stamps
Form$(k+1)$-cent postage, we can add a 3-cent stamp to the $(k2)$-cent postage. Therefore, $P(k + 1)$ is true. **CONCLUSION:** $P(n)$ **is true for all** $n \geq 18$

8.) **Staring point going backwards:**
$P(n)$ is true for an infinite number of positive integers n so there exists at least one n_0 such that $P(n0)$ is true

We let m be the smallest integer such that $P(m)$ is true. We know that there are infinitely many true $P(n)$ statements so m must exist and be a finite positive integer

Implication Step:
$P(k + 1)$ for all k
Applying the reverse since $P(m)$ is the smallest such that $P(m)$ is true
From $P(m)$ we can infer that $P(m1)$ is true
From $P(m1)$ we can infer that $P(m2)$ is true
The process continues moving backwards until we reach $P(1)$

CONCLUSION: By reverse induction, $P(n)$ **is true for all** $n \leq m$ **where**m **is the smallest integer such that** $P(m)$ **is true.**