MATH1002: Assignment 2

SID: 510415022

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Student ID: 510415022
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Location: F07.03.355.Carslaw Building Carslaw Seminar Room 355

1. Let:

$$A = \begin{bmatrix} -2 & 2 & 1 & -5 \\ 5 & -3 & -1 & 10 \\ 3 & -1 & 2 & 7 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = E_3.$$

(a) Compute $B = E_1E_2E_3E_4E_5A$.

Solution:
Each one of the elementary matrices E_1, E_2, E_3, E_4 and E_5 corresponds to a specific ERO:

- E_1 is the elementary matrix which corresponds to the ERO $R_3 \mapsto R_3 + R_2$.
- E_2 corresponds to the ERO $R_3 \mapsto R_3 - 3R_1$.
- $E_3 = E_5$ corresponds to the ERO $R_2 \mapsto R_2 + 2R_1$.
- E_4 corresponds to the ERO $R_1 \leftrightarrow R_2$.

Hence we can compute $B = E_1E_2E_3E_4E_5A$ by applying each of these transformations to A in order:

$$A = \begin{bmatrix} -2 & 2 & 1 & -5 \\ 5 & -3 & -1 & 10 \\ 3 & -1 & 2 & 7 \end{bmatrix}$$

$$E_5(A) = \begin{bmatrix} -2 & 2 & 1 & -5 \\ 1 & 1 & 1 & 0 \\ 3 & -1 & 2 & 7 \end{bmatrix}$$

$$E_4(E_5A) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 2 & 1 & -5 \\ 3 & -1 & 2 & 7 \end{bmatrix}$$

$$E_3(E_4E_5A) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & -5 \\ 3 & -1 & 2 & 7 \end{bmatrix}$$

$$E_2(E_3E_4E_5A) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & -5 \\ 0 & -4 & -1 & 7 \end{bmatrix}$$

$$E_1(E_2E_3E_4E_5A) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & -5 \\ 0 & 0 & 2 & 2 \end{bmatrix} = B$$

(b) By using the previous part or otherwise, compose an augmented matrix for the following system of linear equations and reduce it to row echelon form:

$$\begin{cases} -2x + 2y + z = -5 \\ 5x - 3y - z = 10 \\ 3x - y + 2z = 7 \end{cases}$$

Solution:

Consider the augmented matrix $C = \left[\begin{array}{ccc|c} -2 & 2 & 1 & -5 \\ 5 & -3 & -1 & 10 \\ 3 & -1 & 2 & 7 \end{array}\right]$.

Note that the matrix B from part (a) is the result of transforming matrix A into row echelon form. As matrix C has the same entries as matrix A, we will apply the same EROs E_1, E_2, E_3, E_4, and E_5 in order to reduce matrix C into row echelon form.

$$C = \left[\begin{array}{ccc|c} -2 & 2 & 1 & -5 \\ 5 & -3 & -1 & 10 \\ 3 & -1 & 2 & 7 \end{array}\right]$$

$$\xmapsto[R_2 \mapsto R_2 + 2R_1]{E_5} \left[\begin{array}{ccc|c} -2 & 2 & 1 & -5 \\ 1 & 1 & 1 & 0 \\ 3 & -1 & 2 & 7 \end{array}\right]$$

$$\xmapsto[R_1 \leftrightarrow R_2]{E_4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 2 & 1 & -5 \\ 3 & -1 & 2 & 7 \end{array}\right]$$

$$\xmapsto[R_2 \mapsto R_2 + 2R_1]{E_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & -5 \\ 3 & -1 & 2 & 7 \end{array}\right]$$

$$\xmapsto[R_3 \mapsto R_3 - 3R_1]{E_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & -5 \\ 0 & -4 & -1 & 7 \end{array}\right]$$

$$\xmapsto[R_3 \mapsto R_3 + R_2]{E_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & -5 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

which is in row echelon form.

2. (a) Let $A = \begin{bmatrix} 2 & 10 & 5 \\ 3 & -9 & 16 \\ 1 & -2 & 5 \end{bmatrix}$.

i. Find vectors $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ such that:

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

The vector equations are in the form $A\mathbf{w} = \mathbf{b}$ where $\mathbf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Solving for $\mathbf{w}$:

$$\begin{aligned} A\mathbf{w} &= \mathbf{b} \\ A^{-1}A\mathbf{w} &= A^{-1}\mathbf{b} \\ \mathbf{w} &= A^{-1}\mathbf{b} \end{aligned}$$

Hence we must find A^{-1}, which will then allow us to find $\mathbf{w}$ for each $\mathbf{b}$.

Consider the augmented matrix $\left[\begin{array}{c|c} A & I_3 \end{array}\right]$:

$$\left[\begin{array}{c|c} A & I_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 10 & 5 & 1 & 0 & 0 \\ 3 & -9 & 16 & 0 & 1 & 0 \\ 1 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xmapsto[R_1 \leftrightarrow R_3]{} \left[\begin{array}{ccc|ccc} 1 & -2 & 5 & 0 & 0 & 1 \\ 3 & -9 & 16 & 0 & 1 & 0 \\ 2 & 10 & 5 & 1 & 0 & 0 \end{array}\right]$$

$$\xmapsto[R_2 \mapsto R_2 - 3R_1]{R_3 \mapsto R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 5 & 0 & 0 & 1 \\ 0 & -3 & 1 & 0 & 1 & -3 \\ 0 & 14 & -5 & 1 & 0 & -2 \end{array}\right]$$

$$\xmapsto[R_2 \mapsto -\frac{1}{3}R_2]{} \left[\begin{array}{ccc|ccc} 1 & -2 & 5 & 0 & 0 & 1 \\ 0 & 1 & -1/3 & 0 & -1/3 & 1 \\ 0 & 14 & -5 & 1 & 0 & -2 \end{array}\right]$$

$$\xmapsto[R_3 \mapsto R_3 - 14R_2]{} \left[\begin{array}{ccc|ccc} 1 & -2 & 5 & 0 & 0 & 1 \\ 0 & 1 & -1/3 & 0 & -1/3 & 1 \\ 0 & 0 & -1/3 & 1 & 14/3 & -16 \end{array}\right]$$

$$\xmapsto[R_3 \mapsto -3R_3]{} \left[\begin{array}{ccc|ccc} 1 & -2 & 5 & 0 & 0 & 1 \\ 0 & 1 & -1/3 & 0 & -1/3 & 1 \\ 0 & 0 & 1 & -3 & -14 & 48 \end{array}\right]$$

$$\xmapsto[R_1 \mapsto R_1 + 2R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 13/3 & 0 & -2/3 & 3 \\ 0 & 1 & -1/3 & 0 & -1/3 & 1 \\ 0 & 0 & 1 & -3 & -14 & 48 \end{array}\right]$$

$$\xmapsto[R_1 \mapsto R_1 - \frac{13}{3}R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13 & 60 & -205 \\ 0 & 1 & -1/3 & 0 & -1/3 & 1 \\ 0 & 0 & 1 & -3 & -14 & 48 \end{array}\right]$$

$$\xmapsto[R_2 \mapsto R_2 + \frac{1}{3}R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13 & 60 & -205 \\ 0 & 1 & 0 & -1 & -5 & 17 \\ 0 & 0 & 1 & -3 & -14 & 48 \end{array}\right]$$

As the LHS of the augmented matrix is now in reduced row echelon form, theory tells us that the RHS is A^{-1}. i.e.

$$A^{-1} = \begin{bmatrix} 13 & 60 & -205 \\ -1 & -5 & 17 \\ -3 & -14 & 48 \end{bmatrix}$$

Now, for $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 13 & 60 & -205 \\ -1 & -5 & 17 \\ -3 & -14 & 48 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 13 \\ -1 \\ -3 \end{bmatrix}$$

For $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$:

$$\mathbf{y} = A^{-1}\mathbf{b} = \begin{bmatrix} 13 & 60 & -205 \\ -1 & -5 & 17 \\ -3 & -14 & 48 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 60 \\ -5 \\ -14 \end{bmatrix}$$

For $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$:

$$\mathbf{z} = A^{-1}\mathbf{b} = \begin{bmatrix} 13 & 60 & -205 \\ -1 & -5 & 17 \\ -3 & -14 & 48 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} -205 \\ 17 \\ 48 \end{bmatrix}$$

ii. With help of the previous part or otherwise find the vector $\mathbf{t}$ such that $A\mathbf{t} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where a, b, c are real parameters. Your answer will depend on these parameters.

Solution:

This vector equation $A\mathbf{t} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the same form as $A\mathbf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ from part i.

Hence:

$$\begin{aligned} \mathbf{t} &= A^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} 13 & 60 & -205 \\ -1 & -5 & 17 \\ -3 & -14 & 48 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} 13a + 60b - 205c \\ -a - 5b + 17c \\ -3a - 14b + 48c \end{bmatrix} \end{aligned}$$

(b) Let $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} z \\ t \end{bmatrix}$ be two vectors in $\mathbb{R}^2$ considered as 2×1 matrices. Show that for all values of x, y, z, t the matrix $\mathbf{u}\mathbf{v}^T$ is not invertible.

Solution:

The transpose of a matrix is the matrix flipped over its diagonal.

Hence $\mathbf{v} = \begin{bmatrix} z \\ t \end{bmatrix} \implies \mathbf{v}^T = \begin{bmatrix} z & t \end{bmatrix}$. Thus:

$$\begin{aligned} \mathbf{u}\mathbf{v}^T &= \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} z & t \end{bmatrix} \\ &= \begin{bmatrix} xz & tx \\ yz & ty \end{bmatrix} \end{aligned}$$

Now $\mathbf{u}\mathbf{v}^T$ is invertible $\iff \det\left(\mathbf{u}\mathbf{v}^T\right) \neq 0$.

$$\begin{aligned} \det\left(\mathbf{u}\mathbf{v}^T\right) &= ad - bc \\ &= (xz)(ty) - (tx)(yz) \\ &= txyz - txyz \\ &= 0 \\ &\implies \mathbf{u}\mathbf{v}^T \text{is not invertible.} \end{aligned}$$

Hence for all values of x, y, z, t the matrix $\mathbf{u}\mathbf{v}^T$ is not invertible.

(c) Let A be an $n \times n$ matrix such that $A^2 + A$ is invertible. Show that A is invertible too.

Solution:

Proof 1. To show that A is invertible, we must show that $\det(A) \neq 0$.

We know $A^2 + A$ is invertible.

$$\begin{aligned}
&\implies \det\left(A^2 + A\right) \neq 0 \\
&\implies \det(A(A+1)) \neq 0 \\
&\implies \det(A)\det(A+1) \neq 0 \\
&\implies \det(A) \neq 0 \quad \text{as required.}
\end{aligned}$$

Proof 2. Suppose B is the inverse of $A^2 + A$. That is, suppose B is the matrix such that $(A^2 + A) \cdot B = I$ where I is the identity matrix. Then,

$$\begin{aligned}
&(A^2 + A) \cdot B = I \\
\implies &A(A+1) \cdot B = I \\
\implies &A \cdot (A+1)B = I \\
\implies &A \,\text{is invertible with inverse}\, (A+1)B
\end{aligned}$$

So A is invertible too.

3. Let $A = \begin{bmatrix} 4 & 5 \\ -1 & a \end{bmatrix}$.

(a) Find all values of the parameter a such that A has an eigenvalue 1.

Solution:
The characteristic polynomial of A is:

$$\begin{aligned}
\det(A - \lambda I_2) &= \det \begin{bmatrix} 4-\lambda & 5 \\ -1 & a-\lambda \end{bmatrix} \\
&= (4-\lambda)(a-\lambda) + 5
\end{aligned}$$

For the matrix A to have eigenvalue 1, this characteristic polynomial must have root $\lambda = 1$. That is, $\lambda = 1$ when $\det(A - \lambda I_2) = 0$. Thus,

$$\begin{aligned}
(4-1)(a-\lambda) + 5 &= 0 \\
3(a-1) + 5 &= 0 \\
3a - 3 + 5 &= 0 \\
3a + 2 &= 0 \\
\implies a &= -\frac{2}{3}
\end{aligned}$$

Hence for the matrix A to have eigenvalue 1, $a = -\frac{2}{3}$.

(b) For the value a from the previous part, find the other eigenvalue c of A.

Solution:
Taking $a = -\dfrac{2}{3}$, we have $A = \begin{bmatrix} 4 & 5 \\ -1 & -\frac{2}{3} \end{bmatrix}$. The characteristic polynomial of A is therefore

$$\det(A - \lambda I_2) = \det \begin{bmatrix} 4-\lambda & 5 \\ -1 & -\frac{2}{3} - \lambda \end{bmatrix} \tag{1}$$

$$= (4-\lambda)\left(-\frac{2}{3} - \lambda\right) + 5 \tag{2}$$

To find all possible eigenvalues of A, we must find the roots of this equation. That is, we must solve $\det(A - \lambda I_2) = 0$ for all possible values of λ. Setting equation (2) equal to 0 and expanding:

$$\begin{aligned} -\frac{8}{3} - 4\lambda + \frac{2}{3}\lambda + \lambda^2 + 5 &= 0 \\ \lambda^2 - \frac{10}{3}\lambda + \frac{7}{3} &= 0 \\ \implies 3\lambda^2 - 10\lambda + 7 &= 0 \end{aligned}$$

From the quadratic formula:

$$\lambda = \frac{10 \pm \sqrt{(-10)^2 - 4 \cdot 3 \cdot 7}}{2 \cdot 3}$$

$$\lambda = \frac{10 \pm \sqrt{16}}{6}$$

$$\lambda = \frac{10 \pm 4}{6}$$

$$\lambda = \frac{14}{6} \qquad \text{or} \qquad \lambda = \frac{6}{6}$$

$$\implies \lambda = \frac{7}{3} \qquad \text{or} \qquad \lambda = 1$$

As $\lambda = 1$ was given in part a), the other eigenvalue c of A is $c = \dfrac{7}{3}$.

(c) For the value a from the previous parts, find corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_c$. If there are many such eigenectors, choose one answer for both of them.

Solution:

In order to find the 1−eigenvector for $A = \begin{bmatrix} 4 & 5 \\ -1 & -\frac{2}{3} \end{bmatrix}$ we must find $\mathbf{v}_1$ which satisfies

$$\begin{aligned} A\mathbf{v}_1 &= 1 \cdot \mathbf{v}_1 & (3)\\ A\mathbf{v}_1 &= I\mathbf{v}_1 & (4)\\ A\mathbf{v}_1 - I\mathbf{v}_1 &= \mathbf{0} & (5)\\ \implies (A - I)\mathbf{v}_1 &= \mathbf{0} & (6) \end{aligned}$$

This is equivalent to:

$$\begin{bmatrix} 4-1 & 5 \\ -1 & -\frac{2}{3}-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -1 & -5/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\equiv \left[\begin{array}{cc|c} 3 & 5 & 0 \\ -1 & -5/3 & 0 \end{array}\right]$$

$$\xmapsto[R_1 \leftrightarrow R_2]{} \left[\begin{array}{cc|c} -1 & -5/3 & 0 \\ 3 & 5 & 0 \end{array}\right]$$

$$\xmapsto[R_2 \mapsto R_2 + 3R_1]{} \left[\begin{array}{cc|c} -1 & -5/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

As there are no leading 1's in R_2, y is a free variable.

Let $y = t \in \mathbb{R}$. Then from R_1 we have

$$-x - \frac{5}{3}y = 0 \implies x = -\frac{5}{3}t$$

So the general solution to equation (6) is

$$\mathbf{v}_1 = \begin{cases} x = -\frac{5}{3}t \\ y = t \end{cases}$$

where $t \in \mathbb{R}$ and $t \neq 0$. One specific eigenvector occurs when $t = 1 \implies \mathbf{v}_1 = \begin{bmatrix} -\frac{5}{3} \\ 1 \end{bmatrix}$.

Similarly, in order to find the $\frac{7}{3}$−eigenvector for $A = \begin{bmatrix} 4 & 5 \\ -1 & -\frac{2}{3} \end{bmatrix}$ we must find $\mathbf{v}_c$ which satisfies

$$A\mathbf{v}_c = \frac{7}{3} \cdot \mathbf{v}_c \tag{7}$$

$$A\mathbf{v}_c = \frac{7}{3} I\mathbf{v}_c \tag{8}$$

$$A\mathbf{v}_c - \frac{7}{3} I\mathbf{v}_c = \mathbf{0} \tag{9}$$

$$\implies (A - \frac{7}{3}I)\mathbf{v}_1 = \mathbf{0} \tag{10}$$

This is equivalent to:

$$\begin{bmatrix} 4 - \frac{7}{3} & 5 \\ -1 & -\frac{2}{3} - \frac{7}{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{5}{3} & 5 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\equiv \left[\begin{array}{cc|c} \frac{5}{3} & 5 & 0 \\ -1 & -3 & 0 \end{array}\right]$$

$$\xmapsto[R_1 \leftrightarrow R_2]{} \left[\begin{array}{cc|c} -1 & -3 & 0 \\ \frac{5}{3} & 5 & 0 \end{array}\right]$$

$$\xmapsto[R_2 + \frac{5}{3}R_1]{} \left[\begin{array}{cc|c} -1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

As there are no leading 1's in R_2, y is a free variable.
Let $y = \mu \in \mathbb{R}$. Then from R_1 we have

$$-x - 3y = 0 \implies x = -3\mu$$

So the general solution to equation (10) is

$$\mathbf{v}_c = \begin{cases} x = -3\mu \\ y = \mu \end{cases}$$

where $\mu \in \mathbb{R}$ and $\mu \neq 0$. One specific eigenvector occurs when $t = 1 \implies \mathbf{v}_c = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

(d) Let a be the same as in the previous part. Compute $\mathbf{v} = 3\mathbf{v}_1 + 5\mathbf{v}_c$, where $\mathbf{v}_1$ and $\mathbf{v}_c$ are taken from the previous part of the question. Then compute $A^{10}\mathbf{v}$.

Solution:

$$
\begin{aligned}
\mathbf{v} &= 3\mathbf{v}_1 + 5\mathbf{v}_c \\
&= 3\begin{bmatrix} -\frac{5}{3} \\ 1 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -5 \\ 3 \end{bmatrix} + \begin{bmatrix} -15 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} -20 \\ 8 \end{bmatrix}
\end{aligned}
$$

To compute $A^{10}\mathbf{v}$ we shall first compute A^{10}.

To do this, consider:

- the matrix $A = \begin{bmatrix} 4 & 5 \\ -1 & -\frac{2}{3} \end{bmatrix}$
- the diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ where $\lambda_1, \lambda_2 \in \mathbb{R}$ are the eigenvalues of the matrix A.
- the matrix $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ where $\mathbf{v}_1, \mathbf{v}_2$ are the eigenvectors corresponding to the eigenvalues λ_1 and λ_2 respectively.
- the matrix P^{-1} which is the inverse of matrix P.

Hence we recover:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{3} \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{5}{3} & -3 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 1 & 3 \\ -1 & -\frac{5}{3} \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 & 3 \\ -1 & -\frac{5}{3} \end{bmatrix}$$

Now, it can be shown in general that for any matrices A, D, P and P^{-1} of the type listed above, $P^{-1}AP = D$. Here it is shown for this particular example:

$$\begin{aligned}
\text{LHS} &= \frac{3}{4}\begin{bmatrix} 1 & 3 \\ -1 & -\frac{5}{3} \end{bmatrix}\begin{bmatrix} 4 & 5 \\ -1 & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} -\frac{5}{3} & -3 \\ 1 & 1 \end{bmatrix} \\
&= \frac{3}{4}\begin{bmatrix} 4-3 & 5-2 \\ -4+\frac{5}{3} & -5+\frac{10}{9} \end{bmatrix}\begin{bmatrix} -\frac{5}{3} & -3 \\ 1 & 1 \end{bmatrix} \\
&= \frac{3}{4}\begin{bmatrix} 1 & 3 \\ -\frac{7}{3} & -\frac{35}{9} \end{bmatrix}\begin{bmatrix} -\frac{5}{3} & -3 \\ 1 & 1 \end{bmatrix} \\
&= \frac{3}{4}\begin{bmatrix} -\frac{5}{3}+3 & -3+3 \\ \frac{35}{9}-\frac{35}{9} & 7-\frac{35}{9} \end{bmatrix} \\
&= \frac{3}{4}\begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{28}{9} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{3} \end{bmatrix} \\
&= D = \text{RHS}
\end{aligned}$$

Hence we can see that:

$$\begin{aligned}
P^{-1}AP &= D \\
\implies AP &= PD \\
\implies A &= PDP^{-1}
\end{aligned}$$

Note the following corollary of the above result:

$$A^2 = AA = PDP^{-1}PDP^{-1} = PDDP^{-1} = PD^2P^{-1} \tag{11}$$
$$A^3 = AA^2 = PDP^{-1}PD^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1} \tag{12}$$
$$\vdots \tag{13}$$
$$A^n = PD^nP^{-1} \tag{14}$$

And also note that as D is a diagonal matrix:

$$D^n = \begin{bmatrix} 1^n & 0 \\ 0 & \frac{7}{3}^n \end{bmatrix} \tag{15}$$

Hence, combining our results from equations (14) and (15), we find that:

$$\begin{aligned}
A^{10} &= PD^{10}P^{-1} \\
&= \begin{bmatrix} -\frac{5}{3} & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & \frac{7}{3}^{10} \end{bmatrix} \cdot \frac{3}{4}\begin{bmatrix} 1 & 3 \\ -1 & -\frac{5}{3} \end{bmatrix}
\end{aligned}$$

Letting $x = \frac{7}{3}$ for ease of calculation,

$$A^{10} = \frac{3}{4}\begin{bmatrix} -\frac{5}{3} & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & x^{10} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -\frac{5}{3} \end{bmatrix}$$

$$= \frac{3}{4}\begin{bmatrix} -\frac{5}{3}+0 & 0-3x^{10} \\ 1+0 & 0+x^{10} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -\frac{5}{3} \end{bmatrix}$$

$$= \frac{3}{4}\begin{bmatrix} -\frac{5}{3} & -3x^{10} \\ 1 & x^{10} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -\frac{5}{3} \end{bmatrix}$$

$$= \frac{3}{4}\begin{bmatrix} -\frac{5}{3}+3x^{10} & -5+5x^{10} \\ 1-x^{10} & 3-\frac{5}{3}x^{10} \end{bmatrix}$$

$$= \frac{3}{4}\begin{bmatrix} \frac{-5+9x^{10}}{3} & -5(1-x^{10}) \\ 1-x^{10} & \frac{9-5x^{10}}{3} \end{bmatrix}$$

Now calculating $A^{10}\mathbf{v}$:

$$A^{10}\mathbf{v} = \frac{3}{4}\begin{bmatrix} \frac{-5+9x^{10}}{3} & -5(1-x^{10}) \\ 1-x^{10} & \frac{9-5x^{10}}{3} \end{bmatrix}\begin{bmatrix} -20 \\ 8 \end{bmatrix}$$

$$= \frac{3}{4}\begin{bmatrix} -\frac{20}{3}(-5+9x^{10}) - 40(1-x^{10}) \\ -20(1-x^{10}) + \frac{8}{3}(9-5x^{10}) \end{bmatrix}$$

$$= \frac{3}{4}\begin{bmatrix} \frac{100}{3} - 60x^{10} - 40 + 40x^{10} \\ -20 + 20x^{10} + 24 - \frac{40}{3}x^{10} \end{bmatrix}$$

$$= \frac{3}{4}\begin{bmatrix} -\frac{20}{3} - 20x^{10} \\ 4 + \frac{20}{3}x^{10} \end{bmatrix}$$

$$= \frac{3}{4}\begin{bmatrix} \frac{4(-5-15x^{10})}{3} \\ \frac{4(3+5x^{10})}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -5-15x^{10} \\ 3+5x^{10} \end{bmatrix}$$

As $x = \frac{7}{3}$,

$$A^{10}\mathbf{v} = \begin{bmatrix} -5 - 15\left(\frac{7}{3}\right)^{10} \\ \\ 3 + 5\left(\frac{7}{3}\right)^{10} \end{bmatrix}$$

$$= \begin{bmatrix} -5 - 5\left(\frac{7^{10}}{3^9}\right) \\ \\ 3 + 5\left(\frac{7}{3}\right)^{10} \end{bmatrix}$$

$$A^{10}\mathbf{v} = \begin{bmatrix} -5\left(1 + \frac{7^{10}}{3^9}\right) \\ \\ 3 + 5\left(\frac{7}{3}\right)^{10} \end{bmatrix}$$