MATH1002: Assignment 1

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Tutorial Time: 15:00 to 16:00 (3-4pm), Wednesday
Location: F07.03.355.Carslaw Building Carslaw Seminar Room 355
Tutor's Name:

1. Consider the following two vectors in $\mathbb{R}^3$: $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x+1 \\ x-2 \\ 2x-1 \end{bmatrix}$

   (a) Calculate $\mathbf{u} \cdot \mathbf{v}$ in terms of x.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (1)(x+1) + (2)(x-2) + (3)(2x-1) \\ &= x + 1 + 2x - 4 + 6x - 3 \\ &= 9x - 6 \end{aligned}$$

   (b) Using your answer from part (a), find all values of x for which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

   For $\mathbf{u}$ and $\mathbf{v}$ to be orthogonal,

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 0 \\ 9x - 6 &= 0 \\ x &= \frac{6}{9} = \frac{2}{3} \end{aligned}$$

(c) Calculate $\mathbf{u} \times \mathbf{v}$ in terms of x. (We do not assume here that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal as in part (b).)

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

$$= \begin{bmatrix} (4x-2) - (3x-6) \\ (3x+3) - (2x-1) \\ (x-2) - (2x+2) \end{bmatrix}$$

$$= \begin{bmatrix} x+4 \\ x+4 \\ -x-4 \end{bmatrix}$$

$$= (x+4) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

(d) Use your answer from part (a) to calculate the projection of the vector $\mathbf{v}$ onto $\mathbf{u}$. Your answer should be a vector whose components are expressions in terms of x.

$$proj_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

$$\text{Now, from part a)}, \quad \mathbf{u} \cdot \mathbf{v} = 9x - 6$$

$$\begin{aligned} \text{And,} \quad \mathbf{u} \cdot \mathbf{u} &= 1^1 + 2^2 + 3^2 \\ &= 1 + 4 + 9 \\ &= 14 \\ \implies proj_{\mathbf{u}}\mathbf{v} &= \frac{9x-6}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \frac{3(3x-2)}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{aligned}$$

2. Suppose that $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are two vectors with the following properties:

   - $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
   - $\mathbf{a} \cdot \mathbf{b} = 3$
   - The angle between $\mathbf{a}$ and $\mathbf{b}$ is $45°$

   (a) Find the length of $\mathbf{a}$.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

$$\implies \|\mathbf{a}\| = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\| \cos\theta}$$

$$\begin{aligned} \text{Now,} \quad & \mathbf{a} \cdot \mathbf{b} = 3 \\ \text{And,} \quad & \|\mathbf{b}\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2} \end{aligned}$$

$$\begin{aligned} \therefore \|\mathbf{a}\| &= \frac{3}{\sqrt{2} \cos 45°} \\ &= \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{1} \\ &= 3 \end{aligned}$$

(b) Suppose in addition that $\mathbf{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x = 1$. What are all possible values of y and z?

$$\mathbf{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ y \\ z \end{bmatrix}$$

$$\text{Since } \ \mathbf{a} \cdot \mathbf{b} = 3,$$

$$\implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ y \\ z \end{bmatrix} = 3$$

$$\begin{aligned} 1 + 0 + z &= 3 \\ z &= 2 \end{aligned}$$

$$\therefore \mathbf{a} = \begin{bmatrix} 1 \\ y \\ 2 \end{bmatrix}$$

$$\text{Since } \ \|\mathbf{a}\| = 3,$$

$$\begin{aligned} \implies \sqrt{1^2 + y^2 + 2^2} &= 3 \\ 1 + y^2 + 4 &= 9 \\ y^2 &= 4 \\ y &= \pm 2 \end{aligned}$$

So $z = 2$ and $y = \pm 2$.

3. Consider the line ℓ in $\mathbb{R}^2$ with normal vector $\mathbf{n} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and passing through the point $P = (2, 4)$.

(a) Write down equations in normal form and general form for this line.

Let $\mathbf{n} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

A line in normal form is of the form $\mathbf{x} \cdot \mathbf{n} = \mathbf{p} \cdot \mathbf{n}$.

$$\implies \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = -2 + 12$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 10$$

To find the general form, simplify the LHS:

$$-x + 3y = 10$$

(b) Use the general form to find parametric equations for ℓ, and then write down a vector equation for ℓ as well.

From the general form we have $-x + 3y = 10$. Solving for any variable, in this case we choose x, we get $x = 3y - 10$. Now, let $y = \lambda$, $\lambda \in \mathbb{R}$. Then:

$$\begin{cases} x = 3\lambda - 10 \\ y = \lambda \end{cases}$$

where $\lambda \in \mathbb{R}$.
A vector equation from these parametric equations would then simply be:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

where $\lambda \in \mathbb{R}$.

(c) Hence or otherwise write down a direction vector for ℓ.

A vector equation in $\mathbb{R}^2$ is of the form: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ where $t \in \mathbb{R}^2$.

Hence, looking at the vector equation from part 3b), we can see that the direction vector for the line ℓ will be $d = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.