Student ID: 510415022
Tutorial Time: 10:00 to 11:00 (10-11am), Monday
Location: F07.03.355.Carslaw Building Carslaw Seminar Room 355
Lecturers: Haotian Wu and James Parkinson

Question 1
Let $f(x, y) = x^2 \sin y + 3e^{xy} + x$.

(a) Find $f_x(x, y)$ and $f_y(x, y)$, and hence compute $f_x(2, 0)$ and $f_y(2, 0)$.

(b) Find the tangent plane to the surface $z = f(x, y)$ at the point $(x, y, z) = (2, 0, 5)$.

(a) We are to find the partial derivatives of f with respect to x and y.

$$f_x(x, y) = \frac{\partial f}{\partial x} = 2x \sin y + 3ye^{xy} + 1 \tag{1}$$
$$f_y(x, y) = \frac{\partial f}{\partial y} = x^2 \cos y + 3xe^{xy} \tag{2}$$

Now we can evaluate these partial derivatives at the specified points.

$$\begin{aligned} f_x(2, 0) &= 2(2) \cdot \sin 0 + 3(0)e^{2\cdot 0} + 1 && (3)\\ &= 0 + 0 + 1 && (4)\\ &= 1 && (5)\\ f_y(2, 0) &= 2^2 \cos 0 + 3(2)e^{2\cdot 0} && (6)\\ &= 4 + 6 \cdot 1 && (7)\\ &= 10 && (8) \end{aligned}$$

So $f_x(2, 0) = 1$ and $f_y(2, 0) = 10$.

(b) Note that both partial derivatives found in equations 1 and 2 are continuous functions. Therefore, a tangent plane to the surface $z = f(x, y)$ at any point $P(x_0, y_0, z_0)$ can be found using the equation

$$z - z_0 = f_x(x_0,\, y_0)(x - x_0) + f_y(x_0,\, y_0)(y - y_0) \tag{9}$$

Hence, an equation of the tangent plane to the surface $z = f(x, y)$ at the point $(2, 0, 5)$ is

$$\begin{aligned} z - 5 &= 1 \cdot (x - 2) + 10 \cdot (y - 0) && (10)\\ z - 5 &= x - 2 + 10y && (11)\\ \implies z &= x + 10y + 3 && (12) \end{aligned}$$

Question 2

(a) Sketch the graph of the function $f(x, y) = 9(x-1)^2 + y^2$ and describe this surface. Your sketch must be by hand, and not computer generated.

(b) Let

$$g(x, y) = \frac{x + y + 1}{x - y}, \tag{13}$$

where x and y are real variables.

(i) Find the natural domain D of g, and compute the corresponding range of g.

(ii) Sketch, on a single xy-plane, the level curves of $g(x, y)$ of heights $z = -1$, $z = 0$, and $z = 1$. Clearly label the three level curves on your sketch.

(a) Let us consider some projections of the function onto 2D-planes.

First, let us fix $x = 1$. Then, $f(x, y) = y^2$. This implies that at $x = 1$ on the yz-plane, the curve traces out an upwards parabola with vertex at $P(1, 0)$. Combined with the fact that the range of f is all positive values of z (as both terms in the function are definitely positive), this shows that the resulting curve originates at this point P and is never negative in the z direction.

Now let us fix $z = c$ to examine the projection of the curve on the xy-plane.

$$9(x-1)^2 + y^2 = c \tag{14}$$

$$\implies \frac{9(x-1)^2}{c} + \frac{y^2}{c} = 1 \tag{15}$$

$$\implies \frac{(x-1)^2}{\left(\frac{\sqrt{c}}{3}\right)^2} + \frac{y^2}{\sqrt{c}^2} = 1 \tag{16}$$

Equation 16 is now in the familiar form of the equation of an ellipse. This curve has been sketched in Figure 1. This shows us that as z increases, the ellipse will grow outwards.

Combining these two projections, we can satisfactorily sketch the basic shape of the curve given by $z = f(x, y)$. This is shown in Figure 3.

(b) Let

$$g(x,y) = \frac{x+y+1}{x-y}. \tag{13}$$

(i) As the denominator cannot evaluate to 0, the natural domain D of g is $x \neq y$. i.e.

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \neq y\} \tag{17}$$

Furthermore, there are no restrictions on what g can evaluate to, and so the corresponding range of g is all real z. To show this (i.e. to show that the range of $g(x,y)$ is $\mathbb{R}$), we need to show that for each $c \in \mathbb{R}$ there exists $(x,y) : g(x,y) = c$. To this end, consider the following sequence of equations:

$$c = \frac{x+y+1}{x-y} \tag{18}$$

$$cx - cy = x + y + 1 \tag{19}$$

$$x(c-1) - y(c+1) = 1 \tag{20}$$

After rearranging equation 20 we find that

$$x = \frac{1+y(c+1)}{c-1} \qquad \text{or} \qquad y = \frac{x(c-1)-1}{a+1} \tag{21}$$

Therefore, for *any* value of c (i.e. $c \in \mathbb{R}$), there is a corresponding value of x and y. To make this clearer, it is simple enough to fix a value for one of the variables in the above equations and then solve for the other. From this it is evident that the range for $g(x,y)$ is all real values.

(ii) We will first calculate the corresponding relationship for each level curve.

For $z = -1$,

$$-1 = \frac{x+y+1}{x-y} \tag{22}$$

$$-x + y = x + y + 1 \tag{23}$$

$$2x = -1 \tag{24}$$

$$\implies x = -\frac{1}{2} \tag{25}$$

For $z = 0$,

$$0 = \frac{x+y+1}{x-y} \tag{26}$$

$$x + y + 1 = 0 \tag{27}$$

$$\implies y = -x - 1 \tag{28}$$

For $z = 1$,

$$1 = \frac{x+y+1}{x-y} \tag{29}$$

$$x - y = x + y + 1 \tag{30}$$

$$2y = -1 \tag{31}$$

$$\implies y = -\frac{1}{2} \tag{32}$$

Now we plot these functions and relations on a single plane. This has been shown in Figure 2.

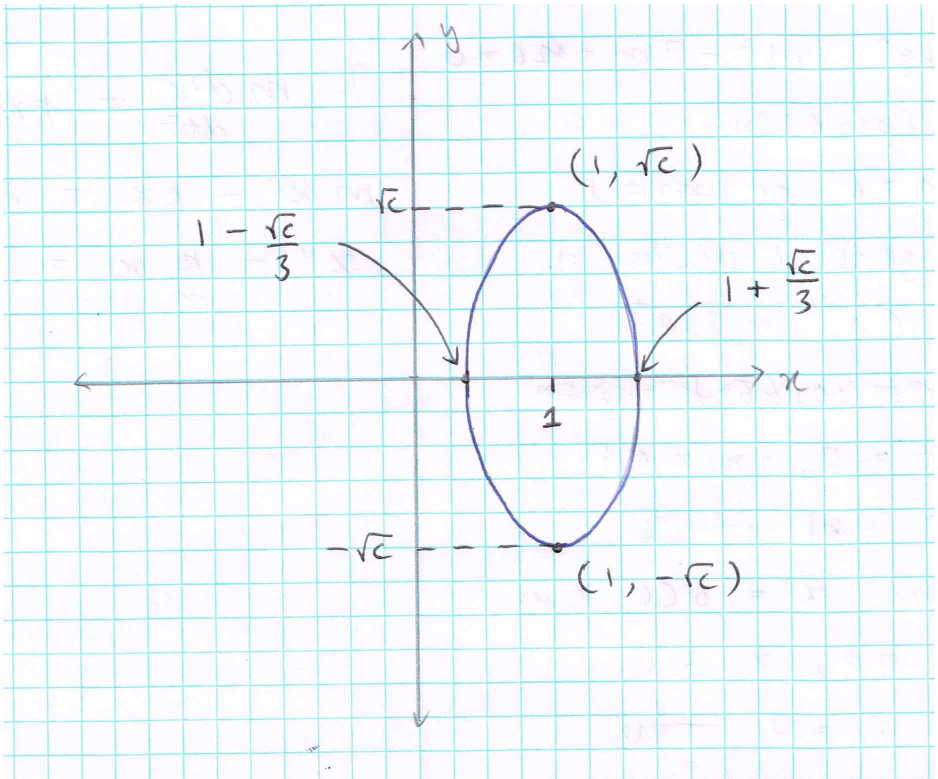

Figure 1: The ellipse formed when projecting f onto the xy-plane.

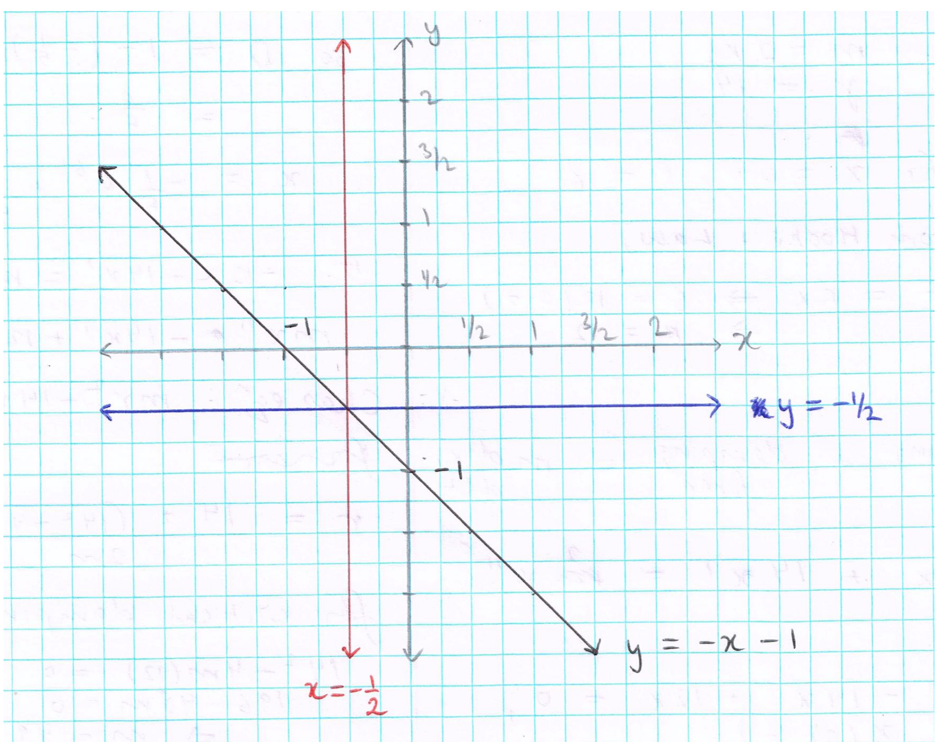

Figure 2: The level curves of g.

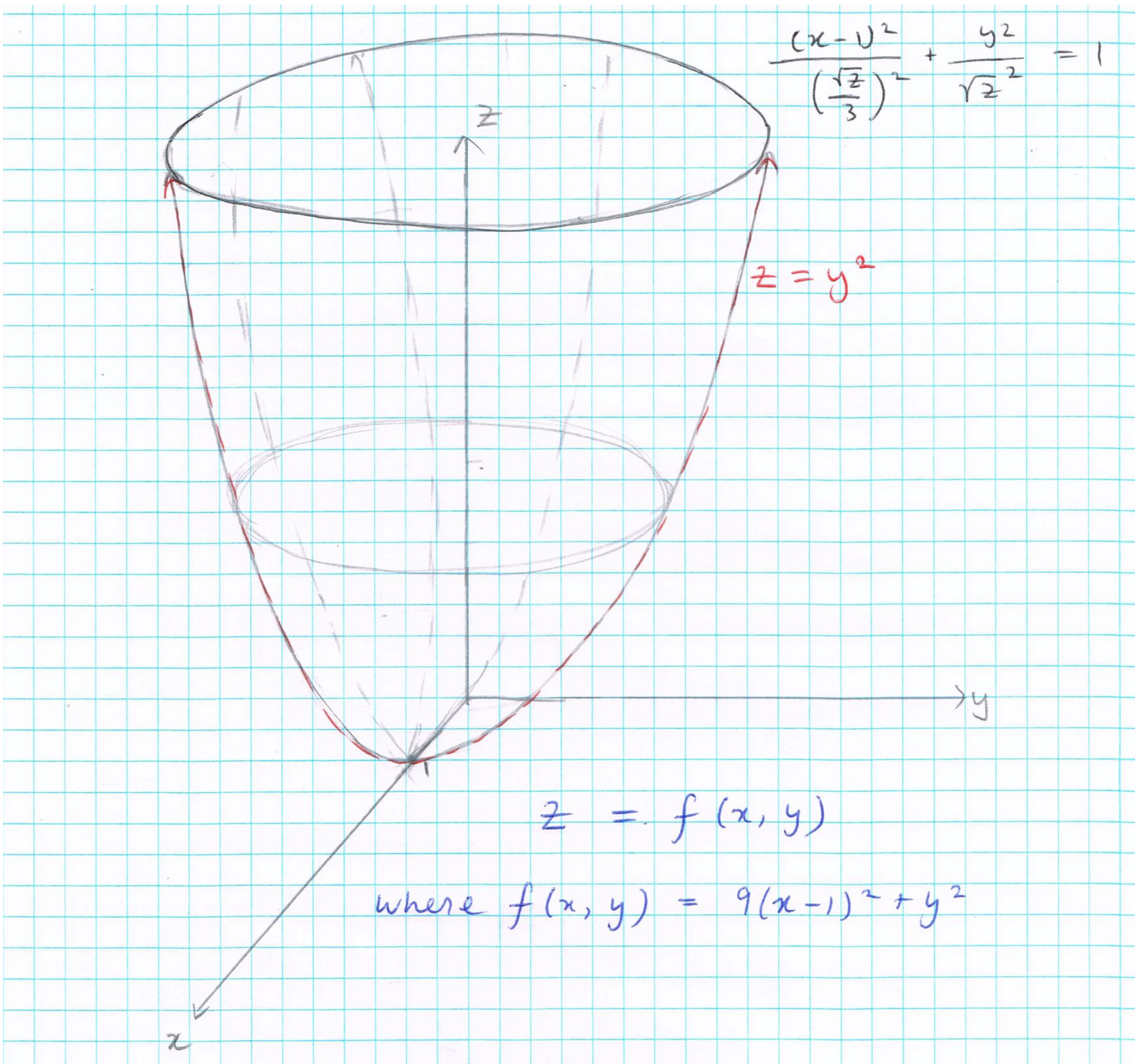

Figure 3: The curve $z = f(x, y) = 9(x-1)^2 + y^2$.

Question 3

Consider the following system of first order differential equations

$$\frac{dx}{dt} = 2x(t) - 3y(t) + 6e^{2t} \tag{33}$$

$$\frac{dy}{dt} = 2x(t) - 5y(t). \tag{34}$$

(a) Find a second order inhomogeneous linear differential equation satisfied by $x(t)$.

(b) Hence, or otherwise, find the general solution to the system of first order differential equations.

Solving this question requires the use of the superposition principles. These are defined below.

Theorem 1. The superposition principle for homogenous linear DEs:
If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the homogeneous linear differential equation

$$P_n(x)\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_1(x)\frac{dy}{dx} + P_0(x)y = 0,$$

then the general solution of the DE is given by the linear combination of y_1 and y_2

$$y(x) = Ay_1(x) + By_2(x)$$

for arbitrary constants A and B.

Theorem 2. The superposition principle for inhomogenous linear DEs:
If y_p is a particular solution to an inhomogeneous nth order linear DE

$$P_n(x)\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_1(x)\frac{dy}{dx} + P_0(x)y = F(x)$$

and y_h is the general solution to the complementary homogeneous equation

$$P_n(x)\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_1(x)\frac{dy}{dx} + P_0(x)y = 0$$

then $y = y_p + y_h$ is the general solution to the original inhomogeneous equation.

(a) We wish to eliminate $y(t)$ from equation 33 in order to find a second order inhomogeneous linear DE satisfied by $x(t)$ only.

Differentiating equation 33 with respect to t,

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - 3\frac{dy}{dt} + 12e^{2t} \tag{35}$$

Putting equation 34 into 35,

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - 3(2x - 5y) + 12e^{2t} \tag{36}$$

$$= 2\frac{dx}{dt} - 6x + 15y + 12e^{2t} \tag{37}$$

To eliminate the y term still present, note that from equation 33 we have that

$$y = \frac{1}{3}\left[2x + 6e^{2t} - \frac{dx}{dt}\right]. \tag{38}$$

Thus, equation 37 becomes

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - 6x + 15\cdot\frac{1}{3}\left[2x + 6e^{2t} - \frac{dx}{dt}\right] + 12e^{2t} \tag{39}$$

$$= 2\frac{dx}{dt} - 6x + 10x + 30e^{2t} - 5\frac{dx}{dt} + 12e^{2t} \tag{40}$$

$$= -3\frac{dx}{dt} + 4x + 42e^{2t} \tag{41}$$

This leads us to the second order inhomogeneous linear differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 42e^{2t}. \tag{42}$$

(b) We must solve equation 42 for $x(t)$. This will then allow us to use equation 38 to find $y(t)$. To solve the inhomogeneous DE, consider it's complementary homogeneous equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 0. \tag{43}$$

Now, assume the solution is of the form

$$x = Ce^{mt} \tag{44}$$

$$\implies \frac{dx}{dt} = Cme^{mt} \tag{45}$$

$$\implies \frac{d^2x}{dt^2} = Cm^2e^{mt} \tag{46}$$

Then, from equation 43, we have that

$$Cm^2e^{mt} + 3Cme^{mt} - 4Ce^{mt} = 0 \tag{47}$$

$$\implies Ce^{mt}\left[m^2 + 3m - 4\right] = 0 \tag{48}$$

As $Ce^{mt} \neq 0$, equation 48 holds $\iff m^2 + 3m - 4 = 0$. That is, if and only if the DE's auxiliary equation equals to zero. Solving the quadratic leads to two possible solutions for m:

$$(m+4)(m-1) = 0 \tag{49}$$

$$\implies m = -4 \quad \text{or} \quad m = 1 \tag{50}$$

So the two solutions to the complementary DE are $x_1 = Ae^{-4t}$ and $x_2 = Be^t$ for $A, B \in \mathbb{R}$. Note that both of these solutions are linearly independent (i.e. $x_1 \neq \lambda x_2$ for any constant λ). Therefore, by Theorem 1, the general solution to the homogeneous DE is

$$x_h = Ae^{-4t} + Be^t, \qquad A, B \in \mathbb{R} \tag{51}$$

We have the general solution to the inhomogeneous DE's complementary homogeneous equation, x_h. Now, we must find some particular solution x_p to the original DE:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 42e^{2t} \tag{42}$$

As the RHS is an exponential function, we try a particular solution of the form

$$x_p = De^{2t} \tag{52}$$

$$\implies \frac{dx_p}{dt} = 2De^{2t} \tag{53}$$

$$\implies \frac{d^2x_p}{dt^2} = 4De^{2t} \tag{54}$$

Plugging these three equations into equation 42, we get that

$$4De^{2t} + 6De^{2t} - 4De^{2t} = 42e^{2t} \tag{55}$$

$$6D = 42 \tag{56}$$

$$\implies D = 7 \tag{57}$$

Therefore our particular solution to equation 42 is $x_p = 7e^{2t}$. Applying Theorem 2, the general solution to equation 42 is

$$x(t) = x_p + x_h = Ae^{-4t} + Be^t + 7e^{2t}, \qquad A, B \in \mathbb{R} \tag{58}$$

Now we use equation 38 to find an expression for $y(t)$. Noting that $\frac{dx}{dt} = -4Ae^{-4t} + Be^t + 14e^{2t}$, we have that

$$y = \frac{1}{3}\left[2 \cdot (Ae^{-4t} + Be^t + 7e^{2t}) + 6e^{2t} - (-4Ae^{-4t} + Be^t + 14e^{2t})\right] \tag{59}$$

$$= \frac{1}{3}\left[2Ae^{-4t} + 2Be^t + 14e^{2t} + 6e^{2t} + 4Ae^{-4t} - Be^t - 14e^{2t}\right] \tag{60}$$

$$= \frac{1}{3}\left[6Ae^{-4t} + Be^t + 6e^{2t}\right] \tag{61}$$

$$y(t) = 2Ae^{-4t} + \frac{1}{3}Be^t + 2e^{2t} \tag{62}$$

So the general solution to the system of first order differential equations is

$$\begin{cases} x(t) = Ae^{-4t} + Be^t + 7e^{2t} \\ y(t) = \frac{1}{3}\left[6Ae^{-4t} + Be^t + 6e^{2t}\right] \end{cases}$$

for $A, B \in \mathbb{R}$.