Gaussian Channel

Information Theory

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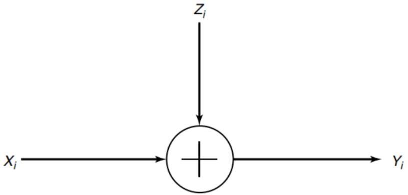
Overview

Introduction

- The Gaussian channel is a time-discrete channel characterized by the input relationship at time i

$$Y_i = X_i + Z_i, \quad \mathcal{N}(0, N)$$

where Z_i's are i.i.d random variable which are assumed indipendent of the signal X_i



- If the noise vairance is zero or the input is uncostrained, the capacity if the channel is infinity.

Introduction

- The most common limitation on the input is the power constraint, hence we assume an average power constraint.
  For any trasmitted codeword $(x_1, x_2, \ldots, x_n)$ over the channel, it requires that

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P.$$

- For example, assume that we want to sent one binary digit over the channel for each use of it. Given the power constraint, the best solution is to send one of two levels, $+\sqrt{P}$ or $-\sqrt{P}$.
- The receiver observes at the corresponding Y and tries to decide which of the two level was sent.

Introduction

- Assuming that both levels are equally likely and choosing the optimum decoding rule is to decide that $+\sqrt{P}$ was sent if $Y > 0$ and $-\sqrt{P}$ was sent if $Y < 0$, we can evaluate the probability of error with such a decoding schema:

$$\begin{aligned} P_e =& \frac{1}{2} \Pr\left(Y < 0 \mid X = +\sqrt{P}\right) + \frac{1}{2} \Pr\left(Y > 0 \mid X = -\sqrt{P}\right) \\ =& \frac{1}{2} \Pr\left(Z < -\sqrt{P}\right) + \frac{1}{2} \Pr\left(Z > \sqrt{P}\right) \\ =& \Pr\left(Z > \sqrt{P}\right) = 1 - \Phi\left(\sqrt{\frac{P}{N}}\right) \end{aligned}$$

where $\Phi(x)$ is. the comulative normal function $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt$.

Information Capacity

- We define the information capacity of Gaussian channel as the maximum of the mutual information between the input and output over all distributions on the input that satisfy the average power constraint:

$$C = \max_{f(x):E[X^2]\leq P} I(X;Y)$$

where P is the power constraint.

- We can observe that

$$\begin{aligned} I(X;Y) =& h(Y) - h(Y \mid X) \\ =& h(Y) - h(X+Z \mid X) \\ =& h(Y) - h(Z \mid X) \\ =& h(Y) - h(Z) \end{aligned}$$

being Z indipendent if X and the average does not effect the entropy.

Information Capacity

- We know that $h(Z) = \frac{1}{2}\log 2\pi e N$; moreover

$$E[Y^2] = E[(X+Z)^2] = E[X^2] + 2E[X]E[Z] + E[Z^2] = P + N$$

- As a consequence, the entropy of Y is boudned by $\frac{1}{2}\log 2\pi e(P+N)$, implying that

$$I(X;Y) = h(Y) - h(Z) \leq \frac{1}{2}\log 2\pi e(P+N) - \frac{1}{2}\log 2\pi e N = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

- Hence, the information capacity of the Gaussian channel is

$$C = \max_{E[X^2]\leq P} I(X;Y) = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

and the maximum is attained when $X \sim \mathcal{N}(0, P)$.

Information Capacity

Definition

An (M, n) code for a Gaussian channel with power constraint P is characterized by:

1. An encoding function

$$x : \{1, 2, \ldots, M\} \to \mathcal{X}^n,$$

   where M the number of messages to deliver, yielding codewords $x^n(1), x_2^n, \ldots, x^n(M)$, satisfying the power constraint, i.e., $\sum_{i=1}^n x_i^2(w) \le nP, w = 1, 2, \ldots, M$.

2. A decoding function

$$g = \mathcal{Y}^n \to \{1, 2, \ldots, M\}$$

Information Capacity: Shannon's Second Theorem

- A rate R is said to be achievable for a Gaussian channel with a power constraint P if there exists a sequence of $(2^{nR}, n)$ codes with codewords satisfying the power constraint such that the maximal probability of error λ_n tends to zero.
- The capacity of the channel is the supremum of the achievable rates.

Theorem

The capacity of a Gaussian channel with power constraint P ans noise variance N is

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N}\right) \quad \textit{bits per trasmission.}$$

Bullet Points

- Lorem ipsum dolor sit amet, consectetur adipiscing elit
- Aliquam blandit faucibus nisi, sit amet dapibus enim tempus eu
- Nulla commodo, erat quis gravida posuere, elit lacus lobortis est, quis porttitor odio mauris at libero
- Nam cursus est eget velit posuere pellentesque
- Vestibulum faucibus velit a augue condimentum quis convallis nulla gravida

Blocks of Highlighted Text

In this slide, some important text will be highlighted because it's important. Please, don't abuse it.

Block

Sample text

Alertblock

Sample text in red box

Examples

Sample text in green box. The title of the block is "Examples".

Multiple Columns

Heading

1. Statement
2. Explanation
3. Example

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Integer lectus nisl, ultricies in feugiat rutrum, porttitor sit amet augue. Aliquam ut tortor mauris. Sed volutpat ante purus, quis accumsan dolor.

Table

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table: Table caption

Theorem

Theorem (Mass–energy equivalence)

$E = mc^2$

Figure

Uncomment the code on this slide to include your own image from the same directory as the template .TeX file.

Citation

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2012].

References

Smith, J. (2012).
Title of the publication.
Journal Name, 12(3):45–678.

The End