

# Mathematical Follies: Elegant Solutions to Hard Series, Sums Limits

Antonio Sirignano

*Università degli Studi Federico II*

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## Abstract

In this article series, I present a curated set of challenging problems on **limits**, **sums**, and **integrals**, solved with an emphasis on clarity and elegance. Each solution is developed step by step, highlighting the key ideas—transformations, comparisons, bounding arguments, and strategic substitutions—rather than relying on brute-force computation. Along the way, I point out common pitfalls, explain why standard approaches fail, and show how to recognize the right tool for each problem. The goal is twofold: to provide complete, rigorous solutions and to help readers build intuition and technique for tackling advanced exercises efficiently and confidently.

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# 1    Limits

## Solution of limits

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### 1.1

**Find the solution**

$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$$

**Solution** Since  $\sin^2 x = 1 - \cos^2 x$ , we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}} &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\sqrt{2} - \sqrt{1 + \cos x}} \\ &= \lim_{x \rightarrow 0} \left( \frac{1 - \cos^2 x}{\sqrt{2} - \sqrt{1 + \cos x}} \right) \left( \frac{\sqrt{2} + \sqrt{1 + \cos x}}{\sqrt{2} + \sqrt{1 + \cos x}} \right) \\ &= \lim_{x \rightarrow 0} \frac{(1 - \cos^2 x)(\sqrt{2} + \sqrt{1 + \cos x})}{2 - 1 - \cos x} \\ &= \lim_{x \rightarrow 0} \frac{(1 - \cos^2 x)(\sqrt{2} + \sqrt{1 + \cos x})}{1 - \cos x} \\ &= \lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x)(\sqrt{2} + \sqrt{1 + \cos x})}{1 - \cos x} \\ &= \lim_{x \rightarrow 0} (1 + \cos x)(\sqrt{2} + \sqrt{1 + \cos x}) \\ &= (1 + \cos 0)(\sqrt{2} + \sqrt{1 + \cos 0}) \\ &= (1 + 1)(2\sqrt{2}) \\ &= 4\sqrt{2} \end{aligned}$$


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# 2    Series

## Solutions of the series

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### 2.1

**Find the sum**

$$\sum_{n=0}^{20} \frac{\binom{20}{n}}{n+1}$$

**Solution**

$$\sum_{n=0}^{20} \frac{\binom{20}{n}}{n+1} = \sum_{n=0}^{20} \binom{20}{n} \frac{1}{n+1}$$

It's possible to see that

$$\int_0^1 x^n dx = \left[ \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{1}{n+1}$$

Then, we can write

$$\begin{aligned} \sum_{n=0}^{20} \binom{20}{n} \frac{1}{n+1} &= \sum_{n=0}^{20} \binom{20}{n} \int_0^1 x^n dx = \\ &= \int_0^1 \sum_{n=0}^{20} \binom{20}{n} x^n dx \end{aligned}$$

The binomial expansion is

$$\sum_{n=0}^{20} \binom{20}{n} x^n = \binom{20}{0} x^0 + \binom{20}{1} x + \binom{20}{2} x^2 + \cdots + \binom{20}{20} x^{20} = (1+x)^{20}$$

Then

$$\begin{aligned} \int_0^1 \sum_{n=0}^{20} \binom{20}{n} x^n dx &= \int_0^1 (1+x)^{20} dx = \left[ \frac{(1+x)^{20+1}}{20+1} \right]_0^1 \\ &= \left[ \frac{(1+x)^{21}}{21} \right]_0^1 = \frac{(1+1)^{21}}{21} - \frac{(1+0)^{21}}{21} = \\ &= \frac{2^{21}}{21} - \frac{1^{21}}{21} = \frac{2^{21}}{21} - \frac{1}{21} = \\ &= \frac{2^{21} - 1}{21} \end{aligned}$$

So the sum is

$$\sum_{n=0}^{20} \frac{\binom{20}{n}}{n+1} = \frac{2^{21} - 1}{21}$$

## 2.2

### Find the sum

$$\sum_{n=0}^{\infty} \frac{n^2 + n + 1}{(n+1)!}$$

**Solution**

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{n^2 + n + 1}{(n+1)!} &= \sum_{n=1}^{\infty} \frac{n(n+1) + 1}{(n+1)!} = \\
&= \sum_{n=1}^{\infty} \left[ \frac{n(n+1)}{(n+1)!} + \frac{1}{(n+1)!} \right] = \\
&= \sum_{n=1}^{\infty} \left[ \frac{n(n+1)}{(n+1)n(n-1)!} + \frac{1}{(n+1)!} \right] = \\
&= \sum_{n=1}^{\infty} \left[ \frac{1}{(n-1)!} + \frac{1}{(n+1)!} \right] = \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!}
\end{aligned}$$

The first sum, can be write

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots$$

Remember that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \implies e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} = e^1$$

The second sum, can be write

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = e - 1 - \frac{1}{1!} = e - 2$$

Then,

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} = e + e - 2 = 2e - 2$$

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### 3 Integrals

#### Solution of integrals

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##### 3.1

**Solve the integral**

$$\int_1^2 x^{x^2+1} (2 \ln x + 1) dx$$

**Solution**

$$\int_1^2 x^{x^2+1} (2 \ln x + 1) dx = \int_1^2 x^{x^2} x (2 \ln x + 1)$$

Let  $u = x^{x^2}$ . That implies

$$\begin{aligned} u = x^{x^2} &\implies \ln u = \ln(x^{x^2}) \\ &\implies \ln u = x^2 \ln x \\ &\implies \frac{1}{u} du = \left(2x \ln x + x^2 \frac{1}{x}\right) dx \\ &\implies \frac{1}{u} du = (2x \ln x + x) dx \\ &\implies \frac{1}{u} du = x (2 \ln x + 1) dx \end{aligned}$$

Then,

$$\begin{aligned} \int_1^2 x^{x^2} x (2 \ln x + 1) dx &= \int_1^{16} u \frac{1}{u} du \\ &= \int_1^{16} 1 du \\ &= [u]_1^{16} \\ &= 16 - 1 = 15 \end{aligned}$$