Mathematical Follies: Elegant Solutions to Hard Series, Sums Limits

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Abstract

In this article series, I present a curated set of challenging problems on **limits, sums, and integrals**, solved with an emphasis on clarity and elegance. Each solution is developed step by step, highlighting the key ideas—transformations, comparisons, bounding arguments, and strategic substitutions—rather than relying on brute-force computation. Along the way, I point out common pitfalls, explain why standard approaches fail, and show how to recognize the right tool for each problem. The goal is twofold: to provide complete, rigorous solutions and to help readers build intuition and technique for tackling advanced exercises efficiently and confidently.

Contents

1 Limits **2**

1.1 2

2 Series **3**

2.1 3

2.2 4

2.3 4

3 Integrals **7**

3.1 7

3.2 7

1 Limits

Solution of limits

1.1

Find the solution

$$\lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}$$

Solution Since $\sin^2 x = 1-\cos^2 x$, we have

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}} &= \lim_{x\to 0} \frac{1-\cos^2 x}{\sqrt{2}-\sqrt{1+\cos x}} \\
&= \lim_{x\to 0} \left(\frac{1-\cos^2 x}{\sqrt{2}-\sqrt{1+\cos x}}\right)\left(\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}\right) \\
&= \lim_{x\to 0} \frac{(1-\cos^2 x)(\sqrt{2}+\sqrt{1+\cos x})}{2-1-\cos x} \\
&= \lim_{x\to 0} \frac{(1-\cos^2 x)(\sqrt{2}+\sqrt{1+\cos x})}{1-\cos x} \\
&= \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)(\sqrt{2}+\sqrt{1+\cos x})}{1-\cos x} \\
&= \lim_{x\to 0} (1+\cos x)(\sqrt{2}+\sqrt{1+\cos x}) \\
&= (1+\cos 0)(\sqrt{2}+\sqrt{1+\cos 0}) \\
&= (1+1)(2\sqrt{2}) \\
&= 4\sqrt{2}
\end{aligned}$$

2 Series

Solutions of the series

2.1

Find the sum

$$\sum_{n=0}^{20} \frac{\binom{20}{n}}{n+1}$$

Solution

$$\sum_{n=0}^{20} \frac{\binom{20}{n}}{n+1} = \sum_{n=0}^{20} \binom{20}{n} \frac{1}{n+1}$$

It's possible to see that

$$\int_0^1 x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{1}{n+1}$$

Then, we can write

$$\begin{aligned}\sum_{n=0}^{20} \binom{20}{n} \frac{1}{n+1} &= \sum_{n=0}^{20} \binom{20}{n} \int_0^1 x^n dx = \\ &= \int_0^1 \sum_{n=0}^{20} \binom{20}{n} x^n dx\end{aligned}$$

The binomial expansion is

$$\sum_{n=0}^{20} \binom{20}{n} x^n = \binom{20}{0} x^0 + \binom{20}{1} x + \binom{20}{2} x^2 + \cdots + \binom{20}{20} x^{20} = (1+x)^{20}$$

Then

$$\begin{aligned}\int_0^1 \sum_{n=0}^{20} \binom{20}{n} x^n dx = \int_0^1 (1+x)^{20} dx &= \left[\frac{(1+x)^{20+1}}{20+1}\right]_0^1 \\ &= \left[\frac{(1+x)^{21}}{21}\right]_0^1 = \frac{(1+1)^{21}}{21} - \frac{(1+0)^{21}}{21} = \\ &= \frac{2^{21}}{21} - \frac{1^{21}}{21} = \frac{2^{21}}{21} - \frac{1}{21} = \\ &= \frac{2^{21}-1}{21}\end{aligned}$$

So the sum is

$$\sum_{n=0}^{20} \frac{\binom{20}{n}}{n+1} = \frac{2^{21}-1}{21}$$

2.2

Find the sum

$$\sum_{n=0}^{\infty} \frac{n^2+n+1}{(n+1)!}$$

Solution

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{n^2+n+1}{(n+1)!} &= \sum_{n=1}^{\infty} \frac{n(n+1)+1}{(n+1)!} = \\
&= \sum_{n=1}^{\infty} \left[\frac{n(n+1)}{(n+1)!} + \frac{1}{(n+1)!}\right] = \\
&= \sum_{n=1}^{\infty} \left[\frac{n(n+1)}{(n+1)n(n-1)!} + \frac{1}{(n+1)!}\right] = \\
&= \sum_{n=1}^{\infty} \left[\frac{1}{(n-1)!} + \frac{1}{(n+1)!}\right] = \\
&= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!}
\end{aligned}$$

The first sum, can be write

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

Remember that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \implies e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} = e^1$$

The second sum, can be write

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e - 1 - \frac{1}{1!} = e - 2$$

Then,

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} = e + e - 2 = 2e - 2$$

2.3

Find the sum

$$\sum_{n=0}^{\infty} \frac{n}{(n+1)! + n!}$$

Solution

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{n}{(n+1)!+n!} &= \sum_{n=0}^{\infty} \frac{n}{(n+1)\,n!+n!} \\
&= \sum_{n=0}^{\infty} \frac{n}{n!\,(n+1+1)} \\
&= \sum_{n=0}^{\infty} \frac{n+2-2}{n!\,(n+2)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{2}{n!\,(n+2)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{2\,(n+1)}{(n+2)\,(n+1)\,n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{2n+2}{(n+2)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{2n+4-2}{(n+2)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{2(n+2)-2}{(n+2)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{2(n+2)}{(n+2)!} + \sum_{n=0}^{\infty} \frac{2}{(n+2)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{\infty} \frac{2}{(n+1)!} + \sum_{n=0}^{\infty} \frac{2}{(n+2)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} - 2\sum_{n=0}^{\infty} \frac{1}{(n+1)!} + 2\sum_{n=0}^{\infty} \frac{1}{(n+2)!}
\end{aligned}
$$

We know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \implies e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

So we have that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{x}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e^1 = e$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e^1 - 1 = e - 1$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+2)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e^1 - 1 = e - 1 - \frac{1}{1!} = e - 2$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n!}-2\sum_{n=0}^{\infty}\frac{1}{(n+1)!}+2\sum_{n=0}^{\infty}\frac{1}{(n+2)!} &= e-2(e-1)+2(e-2)\\ &= e-2e+2+2e-4=\\ &= e-2\end{aligned}$$

3 Integrals

Solution of integrals

3.1

Solve the intergral

$$\int_1^2 x^{x^2+1}\left(2\ln x+1\right)dx$$

Solution

$$\int_1^2 x^{x^2+1}\left(2\ln x+1\right)dx = \int_1^2 x^{x^2}x\left(2\ln x+1\right)$$

Let $u = x^{x^2}$. That implies

$$\begin{aligned} u = x^{x^2} &\implies \ln u = \ln\left(x^{x^2}\right) \\ &\implies \ln u = x^2\ln x \\ &\implies \frac{1}{u}du = \left(2x\ln x + x^2\frac{1}{x}\right)dx \\ &\implies \frac{1}{u}du = \left(2x\ln x + x\right)dx \\ &\implies \frac{1}{u}du = x\left(2\ln x+1\right)dx \end{aligned}$$

Then,

$$\begin{aligned} \int_1^2 x^{x^2}x\left(2\ln x+1\right) &= \int_1^{16} u\frac{1}{u}du \\ &= \int_1^{16} 1du \\ &= [u]_1^{16} \\ &= 16-1=15 \end{aligned}$$

3.2

Solve the intergral

$$\int \ln\left(1+\sqrt{x}\right)dx$$

Solution Considering

$$u = \ln\left(1+\sqrt{x}\right) \qquad dv = dx$$

That implies

$$du = \frac{1}{1+\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}dx \qquad v = x$$

So

$$\int \ln\left(1+\sqrt{x}\right)dx = x\ln\left(1+\sqrt{x}\right) - \int \frac{x}{\left(1+\sqrt{x}\right)2\sqrt{x}}dx$$

Let

$$t = \sqrt{x} \implies dt = \frac{1}{2\sqrt{x}}dx$$

$$\begin{aligned}
x\ln\left(1+\sqrt{x}\right) - \int \frac{x}{\left(1+\sqrt{x}\right)2\sqrt{x}}dx =& x\ln\left(1+\sqrt{x}\right) - \int \frac{t^2}{1+t}dt \\
=& x\ln\left(1+\sqrt{x}\right) - \int \left(t-1+\frac{1}{t+1}\right)dt \\
=& x\ln\left(1+\sqrt{x}\right) - \frac{1}{2}t^2 + t - \ln|t+1| + c \\
=& x\ln\left(1+\sqrt{x}\right) - \frac{1}{2}x + \sqrt{x} - \ln\left(\sqrt{x}+1\right) + c \\
=& \left(x-1\right)x\ln\left(1+\sqrt{x}\right) - \frac{x}{2} + \sqrt{x} + c
\end{aligned}$$

References

[1] Chris McMullen. *Essential Calculus Skills Practice workbook with full solutions.* Zishka Publishing, 2023.