# Mathematical Follies: Elegant Solutions to Hard Series, Sums Limits

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**Abstract**

In this article series, I present a curated set of challenging problems on **series, sums, and limits**, solved with an emphasis on clarity and elegance. Each solution is developed step by step, highlighting the key ideas—transformations, comparisons, bounding arguments, and strategic substitutions—rather than relying on brute-force computation. Along the way, I point out common pitfalls, explain why standard approaches fail, and show how to recognize the right tool for each problem. The goal is twofold: to provide complete, rigorous solutions and to help readers build intuition and technique for tackling advanced exercises efficiently and confidently.

# Contents

# 1   Limits

## Solution of limits

## 1.1

**Find the solution**

$$\lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$$

**Solution**   Since $\sin^2 x = 1 - \cos^2 x$, we have

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}} &= \lim_{x \to 0} \frac{1 - \cos^2 x}{\sqrt{2} - \sqrt{1 + \cos x}} \\
&= \lim_{x \to 0} \left( \frac{1 - \cos^2 x}{\sqrt{2} - \sqrt{1 + \cos x}} \right) \left( \frac{\sqrt{2} + \sqrt{1 + \cos x}}{\sqrt{2} + \sqrt{1 + \cos x}} \right) \\
&= \lim_{x \to 0} \frac{(1 - \cos^2 x)(\sqrt{2} + \sqrt{1 + \cos x})}{2 - 1 - \cos x} \\
&= \lim_{x \to 0} \frac{(1 - \cos^2 x)(\sqrt{2} + \sqrt{1 + \cos x})}{1 - \cos x} \\
&= \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)(\sqrt{2} + \sqrt{1 + \cos x})}{1 - \cos x} \\
&= \lim_{x \to 0} (1 + \cos x)(\sqrt{2} + \sqrt{1 + \cos x}) \\
&= (1 + \cos 0)(\sqrt{2} + \sqrt{1 + \cos 0}) \\
&= (1 + 1)(2\sqrt{2}) \\
&= 4\sqrt{2}
\end{aligned}
$$

# 2   Series

## Solutions of the series

## 2.1

**Find the sum**

$$\sum_{n=0}^{20} \frac{\binom{20}{n}}{n + 1}$$

**Solution**

$$\sum_{n=0}^{20} \frac{\binom{20}{n}}{n + 1} = \sum_{n=0}^{20} \binom{20}{n} \frac{1}{n + 1}$$

It's possible to see that

$$\int_0^1 x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{1}{n+1}$$

Then, we can write

$$\sum_{n=0}^{20} \binom{20}{n} \frac{1}{n+1} = \sum_{n=0}^{20} \binom{20}{n} \int_0^1 x^n dx =$$

$$= \int_0^1 \sum_{n=0}^{20} \binom{20}{n} x^n dx$$

The binomial expansion is

$$\sum_{n=0}^{20} \binom{20}{n} x^n = \binom{20}{0} x^0 + \binom{20}{1} x + \binom{20}{2} x^2 + \cdots + \binom{20}{20} x^{20} = (1+x)^{20}$$

Then

$$\int_0^1 \sum_{n=0}^{20} \binom{20}{n} x^n dx = \int_0^1 (1+x)^{20} dx = \left[\frac{(1+x)^{20+1}}{20+1}\right]_0^1$$

$$= \left[\frac{(1+x)^{21}}{21}\right]_0^1 = \frac{(1+1)^{21}}{21} - \frac{(1+0)^{21}}{21} =$$

$$= \frac{2^{21}}{21} - \frac{1^{21}}{21} = \frac{2^{21}}{21} - \frac{1}{21} =$$

$$= \frac{2^{21} - 1}{21}$$

So the sum is

$$\sum_{n=0}^{20} \frac{\binom{20}{n}}{n+1} = \frac{2^{21} - 1}{21}$$

## 2.2

**Find the sum**

$$\sum_{n=0}^{\infty} \frac{n^2 + n + 1}{(n+1)!}$$

**Solution**

$$\sum_{n=0}^{\infty} \frac{n^2 + n + 1}{(n+1)!} = \sum_{n=1}^{\infty} \frac{n(n+1) + 1}{(n+1)!} =$$

$$= \sum_{n=1}^{\infty} \left[ \frac{n(n+1)}{(n+1)!} + \frac{1}{(n+1)!} \right] =$$

$$= \sum_{n=1}^{\infty} \left[ \frac{n(n+1)}{(n+1)n(n-1)!} + \frac{1}{(n+1)!} \right] =$$

$$= \sum_{n=1}^{\infty} \left[ \frac{1}{(n-1)!} + \frac{1}{(n+1)!} \right] =$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!}$$

The first sum, can be write

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

Remember that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \implies e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} = e^1$$

The second sum, can be write

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e - 1 - \frac{1}{1!} = e - 2$$

Then,

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} = e + e - 2 = 2e - 2$$

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# 3 Integrals

## Solution of integrals

## 3.1

**Solve the intergral**

$$\int_1^2 x^{x^2+1} \left(2 \ln x + 1\right) dx$$

**Solution**

$$\int_1^2 x^{x^2+1} \left(2 \ln x + 1\right) dx = \int_1^2 x^{x^2} x \left(2 \ln x + 1\right)$$

Let $u = x^{x^2}$. That implies

$$
\begin{aligned}
u = x^{x^2} &\implies \ln u = \ln\left(x^{x^2}\right) \\
&\implies \ln u = x^2 \ln x \\
&\implies \frac{1}{u} du = \left(2x \ln x + x^2 \frac{1}{x}\right) dx \\
&\implies \frac{1}{u} du = \left(2x \ln x + x\right) dx \\
&\implies \frac{1}{u} du = x \left(2 \ln x + 1\right) dx
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_1^2 x^{x^2} x \left(2 \ln x + 1\right) &= \int_1^{16} u \frac{1}{u} du \\
&= \int_1^{16} 1 du \\
&= [u]_1^{16} \\
&= 16 - 1 = 15
\end{aligned}
$$