Optimal step size for estimating Δ_C in different financial derivatives

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Summary

In this document we will focus on a market where the stock price assumed to follow a log-normal diffusion process on the interval $[0, T]$:

$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma \mathrm{d}W_t \right), \tag{1}$$

and we will discuss the precision of the best Monte-Carlo model (with fix $N = 5000$ paths) for two derivatives which behaves very differently. The comparison will be made analytically and for numerically on a concrete parameter set:

$$r = 0.01, S_0 = 1, \sigma = 0.4, T = 0.25\mathrm{Y}. \tag{2}$$

The document can be divided in six main section. The first section defines the payoff for digital call and for call option. In the next section we calculate the main quantities for these option in the analytical framework. The third section pictures that how well can the analytical values of the above quantities approximated for the two derivatives. The section *Bias and Variance* and section *Mean Square Error and optimal* ϵ help to understand the limit of the numerical precision for a given N and ϵ.

Derivatives

The derivatives are financial contracts, set between two or more parties, that derive their value from an underlying asset, group of assets, or benchmark. They can be defined by though the payoff of the contract.

Lets introduce the digital call (or binary option) which pays 1 if the stock price is higher that strike K at expiration time, otherwise its payoff is zero:

$$F^{\mathrm{Digital}}(S_T) = \Theta(S_T - K) = \begin{cases} 0 & \text{if } S_T \leq K \\ 1 & \text{if } S_T > K \end{cases}, \tag{3}$$

and European call option which gives the right for the option buyer to buy a stock at a specified price K at maturity:

$$F^{\mathrm{Option}}(S_T) = (S_T - K)^+ = \max(S_T - K, 0). \tag{4}$$

Black-Scholes (Analytical) Model

In the Black-Scholes model the stock price S_T at time $T > T_0 = 0$ follows a lognormal distribution with

$$\mathbf{E}[S_T] = S_0 e^{rt}, \tag{5a}$$

and

$$\mathbf{Var}[S_T] = \mathbf{E}[S_T]^2 \left(e^{\sigma^2 T} - 1 \right), \tag{5b}$$

where r is the drift and σ is the volatility parameter.

Lets define

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(\frac{1}{2}\sigma^2 + r \right) T}{\sigma \sqrt{T}} \tag{6a}$$

and

$$d_2 = d_1 - \sigma \sqrt{t} \tag{6b}$$

quantities which will appear in several formulas, sometimes as:

$$\frac{\partial d_i}{\partial S_0} = \frac{\partial}{\partial S_0}\left[\frac{\ln S_0}{\sigma\sqrt{T}} + \text{const.}\right] = \frac{1}{S_0\sigma\sqrt{T}}\,. \tag{7}$$

Alternatively, the stock price at maturity is a random variable, and as a consequence of Eq. 1, it follows a lognormal distribution with

$$\mathbf{E}\left[\ln S_T\right] = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T\,, \tag{8a}$$

and

$$\mathbf{Var}\left[\ln S_T\right] = \sigma^2 T\,. \tag{8b}$$

With a standardization (or Z-score normalization) its mean and the standard deviation can be transformed to 0 and 1, i.e.,

$$Z := \frac{\ln S_T - E\left[\ln S_T\right]}{\sqrt{\mathrm{Var}\left[\ln S_T\right]}} \tag{9}$$

is a standard normal variable with expected value 0 and standard deviation 1 ($Z \sim N(0,1)$).

Gaussian Distribution

As we seen in the previous section, the Gaussian distribution has a central role in Black-Scholes world which *Probability Density Function* (PDF) is defined by

$$n(x;\sigma,\mu) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,, \tag{10}$$

and for standard normal distribution

$$n(x) = n(x;\sigma=1,\mu=0) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,. \tag{11}$$

The *Cumulative Distribution Function* (CDF) of the standard normal distribution is denoted by N and defined by the following integral

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}}\,\mathrm{d}s\,. \tag{12}$$

By the *Fundamental Theorem of Calculus* we can verify that

$$\frac{\mathrm{d}}{\mathrm{d}x}N(x) = n(x)\,, \tag{13}$$

and with direct differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}n(x) = -xn(x)\,. \tag{14}$$

In few steps, using the linearity of the integral, it also can be see that

$$N(x) = 1 - N(-x)\,. \tag{15}$$

Price

The fair price of the contract is equal with discounted expectation value of the payoff on the risk neutral world:

$$C = \mathbf{E}[\mathrm{DF}(T) \cdot F(S_T)|S_0]\,, \tag{16}$$

where

$$\mathrm{DF}(T) = e^{-rT}\,. \tag{17}$$

For **digital call**: The digital call option has a non-zero value if $S_T > K$ or equivalently if $\ln S_T > \ln K$. Because of the constant 1 payoff (Θ-function) for get price of the digital option it is enough to get the probability of when $\ln S_T > \ln K$ (with the discount factor):

$$C_{\mathrm{BS}}^{\mathrm{Digital}} = \mathrm{DF}(T) \cdot \mathbf{E}[I_{S_T>K}]\,. \tag{18}$$

One can get the probability by using that the solution of Eq. 1 is well known:

$$S_T = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}\varepsilon} \tag{19}$$

with $\varepsilon \sim N(0,1)$. The condition $S_T \geq K$ is satisfied if

$$\begin{aligned} S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}\varepsilon} &\geq K \\ \varepsilon &\geq -d_2\,. \end{aligned} \tag{20}$$

Using this, the required probability can be expressed by the CDF of the Normal distribution $N(x)$, as

$$\mathbf{E}[I_{S_T>K}] = \int_{-\infty}^{\infty} I_{S_T>K}\, n(y)\mathrm{d}y = \int_{-d_2}^{\infty} n(y)\mathrm{d}y = 1 - N(-d_2) \equiv N(d_2)\,, \tag{21}$$

where we used the even parity of $n(y)$ and Eq. 15. Thus, inline with [3], we get

$$C_{\mathrm{BS}}^{\mathrm{Digital}} = e^{-rT} N(d_2)\,. \tag{22}$$

In our case:

$$\mathrm{C}_{\mathrm{BS}}^{\mathrm{Digital}}(S_0 = 1 = K, \sigma = 40\%, r = 1\%, T = 0.25) = 0.46398\,, \tag{23}$$

it is verified with the attached python code (task 1) and also inline with [2].

For **call option**: The European call option has a non-zero also if $S_T > K$, but know instead of 1 the payoff is $S_T - K$, i.e.,

$$C_{\mathrm{BS}}^{\mathrm{Option}} = \mathrm{DF}(T) \cdot \mathbf{E}[(S_T - K)^+ I_{S_T>K}] = e^{-rT}\mathbf{E}[S_T I_{S_T>K}] - K e^{-rT}\mathbf{E}[I_{S_T>K}]\,. \tag{24}$$

The last term is calculated in Eq. 21, and the first term can be handled similarly:

$$\begin{aligned} \mathbf{E}[S_T I_{S_T>K}] &= \int_{-d_2}^{\infty} S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}y} n(y)\mathrm{d}y = \frac{S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{\sigma\sqrt{T}y-\frac{y^2}{2}}\mathrm{d}y = \\ &= \frac{S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{T})^2+\frac{1}{2}\sigma^2 T}\mathrm{d}y = \\ &= \frac{S_0 e^{rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{T})^2}\mathrm{d}y\,. \end{aligned} \tag{25}$$

Using the even parity of the intergral

$$\mathbf{E}[S_T I_{S_T>K}] \; = \; \frac{S_0 e^{rT}}{\sqrt{2\pi}} \int\limits_{-\infty}^{d_2} e^{-\frac{1}{2}(y-\sigma\sqrt{T})^2} \mathrm{d}y \,. \tag{26}$$

Lets replace the integral variable y to $y' = y + \sigma\sqrt{T}$. Then $\mathrm{d}y = \mathrm{d}y'$, $y'_{\max} = y_{\max} + \sigma\sqrt{T} = d_2 + \sigma\sqrt{T} \equiv d_1$, and $y_{\min} = -\infty = y_{\min}$, so

$$\mathbf{E}[S_T I_{S_T>K}] \; = \; \frac{S_0 e^{rT}}{\sqrt{2\pi}} \int\limits_{-\infty}^{d_1} e^{-\frac{1}{2}y'^2} \mathrm{d}y' = S_0 e^{rT} N(d_1) \,. \tag{27}$$

Thus, inline with [4], we get

$$\mathrm{C}_{\mathrm{BS}}^{\mathrm{Option}} = S_0 N(d_1) - K e^{-rT} N(d_2) \tag{28}$$

In our case:

$$\mathrm{C}_{\mathrm{BS}}^{\mathrm{Option}}(S_0 = 1 = K, \sigma = 40\%, r = 1\%, T = 0.25) = 0.08081 \,, \tag{29}$$

it is verified with the attached python code (task 1) and also inline with [2, 5].

Greeks

Greeks are the quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. The name is used because the most common of these sensitivities are denoted by Greek letters (as are some other finance measures). Collectively these have also been called the risk sensitivities, or risk measures.

Delta

Delta is one of the main first-order risk measures, it measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. Delta is the first derivative of the value C of the option with respect to the underlying instrument's price S_0,

$$\delta = \frac{\partial C}{\partial S_0} \,. \tag{30}$$

For **digital call**:

$$\delta_{\mathrm{BS}}^{\mathrm{Digital}} = \frac{\partial C_{\mathrm{BS}}^{\mathrm{Digital}}}{\partial S_0} = \frac{\partial}{\partial S_0} e^{-rT} N(d_2) = e^{-rT} \cdot \frac{\partial}{\partial d_2} N(d_2) \cdot \frac{\partial d_2}{\partial S_0} \,, \tag{31}$$

Using Eq. 13 and Eq. 7, inline with [3], we get

$$\delta_{\mathrm{BS}}^{\mathrm{Digital}} = e^{-rT} \frac{n(d_2)}{S_0 \sigma \sqrt{T}} \,. \tag{32}$$

In our case:

$$\delta_{\mathrm{BS}}^{\mathrm{Digital}}(S_0 = 1 = K, \sigma = 40\%, r = 1\%, T = 0.25) = 1.98213 \,, \tag{33}$$

it is verified with the attached python code (task 1) and also inline with [2].

For **call option**:

$$\begin{aligned}
\delta_{\mathrm{BS}}^{\mathrm{Option}} &= \frac{\partial C_{\mathrm{BS}}^{\mathrm{Option}}}{\partial S_0} = \frac{\partial}{\partial S_0}\left[S_0 N(d_1) - Ke^{-rT}N(d_2)\right] = \\
&= N(d_1) + S_0 \cdot \frac{\partial}{\partial d_1}N(d_1)\cdot\frac{\partial d_1}{\partial S_0} - Ke^{-rT}\cdot\frac{\partial N(d_2)}{\partial d_2}\cdot\frac{\partial d_2}{\partial S_0} = \\
&= N(d_1) + \left[S_0 n(d_1) - Ke^{-rt}n(d_2)\right]\frac{\partial d_i}{\partial S_0} = \\
&= N(d_1)\,, \qquad (34)
\end{aligned}$$

which match with the formula in [4]. The only tricky part was to show that:

$$\begin{aligned}
S_0 n(d_1) - Ke^{-rt}n(d_2) &\sim \frac{S_0}{K} - e^{-rt}\frac{n(d_2)}{n(d_1)} = \frac{S_0}{K} - e^{-rT-\frac{1}{2}d_2^2+d_1^2} = \\
&= \frac{S_0}{K} - e^{-rT-\frac{1}{2}(d_2-d_1)(d_2+d_1)} = \frac{S_0}{K} - e^{\ln\frac{S_0}{K}} \\
&= 0\,. \qquad (35)
\end{aligned}$$

In our case:

$$\delta_{\mathrm{BS}}^{\mathrm{Option}}(S_0 = 1 = K, \sigma = 40\%, r = 1\%, T = 0.25) = 0.54479\,, \qquad (36)$$

it is verified with the attached python code (task 1) and also inline with [2, 5].

Gamma

Gamma is one of the main second-order risk measures, it measures the rate of change in the delta with respect to changes in the underlying price. Gamma is the second derivative of the value function with respect to the underlying price,

$$\Gamma = \frac{\partial}{\partial S_0}\delta = \frac{\partial^2}{\partial S_0^2}C\,. \qquad (37)$$

For **digital call**:

$$\begin{aligned}
\Gamma_{\mathrm{BS}}^{\mathrm{Digital}} &= \frac{\partial}{\partial S_0}\delta_{\mathrm{BS}}^{\mathrm{Digital}} = e^{-rT}\left[-d_2 n(d_2)\frac{1}{(S_0\sigma\sqrt{T})^2} - n(d_2)\frac{1}{S_0^2\sigma\sqrt{T}}\right] = \\
&= -\frac{e^{-rT}}{S_0^2\sigma\sqrt{T}}n(d_2)\left[\frac{d_2}{\sigma\sqrt{T}} + 1\right] = -\frac{e^{-rT}}{S_0^2\sigma\sqrt{T}}n(d_2)\cdot\frac{d_1}{\sigma\sqrt{T}} \\
&= -e^{-rT}\frac{d_1 n(d_2)}{S_0^2\sigma^2 T}\,, \qquad (38)
\end{aligned}$$

compere it with [3].

In our case:

$$\Gamma_{\mathrm{BS}}^{\mathrm{Digital}}(S_0 = 1 = K, \sigma = 40\%, r = 1\%, T = 0.25) = -1.11495\,, \qquad (39)$$

it is verified with the attached python code (task 1) and also inline with [2].

For **call option**:

$$\Gamma_{\mathrm{BS}}^{\mathrm{Option}} = \frac{\partial}{\partial S_0}\delta_{\mathrm{BS}}^{\mathrm{Option}} = \frac{\partial}{\partial S_0}N(d_1) = \frac{n(d_1)}{S_0\sigma\sqrt{T}}\,, \qquad (40)$$

checked with [4].

In our case:

$$\Gamma_{\mathrm{BS}}^{\mathrm{Option}}(S_0 = 1 = K, \sigma = 40\%, r = 1\%, T = 0.25) = 1.98213\,, \qquad (41)$$

it is verified with the attached python code (task 1) and also inline with [2, 5].

Monte-Carlo (Numerical) Model

Models based on probabilistic Monte Carlo (MC) method are used in financial industry to price and risk manage complex high dimensional derivative instruments.

The common consequence of the arbitrage free pricing theory is that value of a derivative instrument is equal to expectation (integral) with respect to some probability measure over an underlying market scenario space.

In the case of Monte-Carlo methods, instead of computing a high-dimensional integral one simulates sufficiently big set of independent paths of the underlying process and replaces integration by the additive averaging, making use of the law of large numbers. Thus, it relies on repeated random sampling to obtain numerical results. The underlying concept is to use randomness to solve problems that might be deterministic in principle.

Assuming a log-normal diffusion process for stock price on time interval, see in Eq. 1, S_T would follow a log-normal distribution, but instead of directly using the known probability function we will use a random sampling from that distribution by generating S_T as

$$S_T^{(i)} = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}\varepsilon^{(i)}}\,, \tag{42}$$

where $\varepsilon^{(i)}$ sampled from a normal distribution, i.e. $\varepsilon^{(i)} \sim N(0,1)$. It can be shown that $S_T^{(i)}$ which is generated by Eq. 42 follows log-normal distribution with parameters in Eq. 8.

Thus, for each simulation one need to generate N (number of paths) normal variable $\{\varepsilon^{(i)}\}_{i=1\ldots N}$ which will define a set of $\{S_T^{(i)}\}_{i=1\ldots N}$ and by them the requested quantities can be calculated.

Because of the linearity of Eq. 1 , one can use the same set of $\{\varepsilon^{(i)}\}_{i=1\ldots N}$ for get the distribution of the spot bumped value of the stock price:

$$\tilde{S}_T^{(i)} = \tilde{S}_0 \frac{S_T^{(i)}}{S_0}\,. \tag{43}$$

Note that, the studied payoffs are path independent, i.e., no intermediate value of S_t ($T_0 < t < T$) will play a role in the pricing, thus it is enough to generate $S_T^{(i)}$ like it is in Eq. 42.

Price

The price of a derivative is the expected value of the payoff, see Eq. 16. Following the the idea of the Monte-Carlo method:

$$\overline{C}(N) = \frac{1}{N}\sum_{i=1}^{N} \mathrm{DF}(T)\cdot F\left(S_T^{(i)}\right) \tag{44}$$

can be used as an estimator for the value for the expectation C. Using the theorem of *Law of large numbers* it can be shown that it is an unbaised estimator,

$$\lim_{N\to\infty} \overline{C}(N) = C\,. \tag{45}$$

In the attached python code (task 2), we can see that for $N = 5000$ we get a good price match for both payoff:

$$\overline{C}^{\mathrm{Digital}}(N=5000) = 0.45646 \tag{46a}$$

$$\overline{C}^{\mathrm{Option}}(N=5000) = 0.08357\,, \tag{46b}$$

compared to Eq. 23 and Eq. 28.

Sensitivity (Central approximation)

In mathematical finance, the *sensitivities* (called "Greeks" or "deltas") are the quantities representing the sensitivity of the price of derivatives such as options to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. The sensitivities are significant part of information needed for quantifying and reducing risk.

Three basic types are commonly considered: forward, backward, and central finite differences. Now, we will focus on the central one, because this is the most numerically stable.

Delta

Delta is a first order derivative which is in the central finite difference approximation is the following:

$$\Delta_C(N, \epsilon) = \frac{\overline{C}(N, S_0 + \epsilon) - \overline{C}(N, S_0 - \epsilon)}{2\epsilon} \tag{47}$$

In the attached python code (task 2), we can see that for $N = 5000$ and $\epsilon = 0.001$ we get a good delta match for both payoff:

$$\Delta_C^{\text{Digital}}(N = 5000, \epsilon = 0.001) = 1.79551 \tag{48a}$$

$$\Delta_C^{\text{Option}}(N = 5000, \epsilon = 0.001) = 0.53354 \tag{48b}$$

compared to Eq. 33 and Eq. 36.

Gamma

Gamma is a second order derivative which is in the central finite difference approximation is the following:

$$\overline{\Gamma}_C(N, \epsilon) = \frac{\overline{C}(N, S_0 + \epsilon) - 2\overline{C}(N, S_0) + \overline{C}(N, S_0 - \epsilon)}{\epsilon^2} \tag{49}$$

In the attached python code (task 2), by increasing the number of paths to $N = 50,000,000$ we get a good gamma match call option but a very inaccurate for the digital payoff:

$$\overline{\Gamma}_C^{\text{Digital}}(N = 5000, \epsilon = 0.001) = 9.45633 \tag{50a}$$

$$\overline{\Gamma}_C^{\text{Option}}(N = 5000, \epsilon = 0.001) = 1.97644 \tag{50b}$$

compare to Eq. 39 and 41.

Bias and Variance

Bias is the amount that a model's differs (theoretically) from the target value, e.g. for the finite value of ϵ the Monte-Carlo delta (Δ_C) is not equal (even for $N \to \infty$) with the exact Black-Scholes delta (δ_{BS}).

While the variance is describes how much a estimator (random variable) differs from its expected value.

Note that, since bias is a theoretical difference of the estimator and the real quantity it does not depend on N, but the variance is based on a single set of random variables $\{\varepsilon^{(i)}\}$, consequently it will depend on N.

Bias

Lets define the bias of the delta estimator (defined in Eq. 47) as:

$$\mathrm{Bias}(\Delta_{\mathrm{C}}) = E\left[\Delta_{\mathrm{C}}(N,\epsilon) - \delta_{\mathrm{BS}}\right] = \frac{C_{\mathrm{BS}}(d_2(S_0+\epsilon)) - C_{\mathrm{BS}}(d_2(S_0-\epsilon))}{2\epsilon} - \delta_{\mathrm{BS}}\,, \tag{51}$$

where we used the fact that $\overline{C}(N)$ is an unbaised of C (see Eq. 45). If C_{BS} is infinitely differentiable then we can apply the Taylor-expansion:

$$\begin{aligned} C_{\mathrm{BS}}(d_2(S_0 \pm \epsilon)) &= C_{\mathrm{BS}}(d_2(S_0)) \pm \frac{1}{1!}\frac{\partial}{\partial S_0} C_{\mathrm{BS}}(d_2(S_0)) \cdot \epsilon + \frac{1}{2!}\frac{\partial^2}{\partial S_0^2} C_{\mathrm{BS}}(d_2(S_0)) \cdot \epsilon^2 \\ &\pm \frac{1}{3!}\frac{\partial^3}{\partial S_0^3} C_{\mathrm{BS}}(d_2(S_0)) \cdot \epsilon^3 + O(\epsilon^4)\,. \end{aligned} \tag{52}$$

Using the definitions in Eq. 30 and Eq. 37 we end up with the following:

$$\mathrm{Bias}(\Delta_C) = \frac{1}{3!}\frac{\partial}{\partial S_0}\Gamma_{\mathrm{BS}} \cdot \epsilon^2 + O(\epsilon^4)\,. \tag{53}$$

From the derivation steps of the formula for the bias (in Eq. 53) one can take the following important notices:

- as long as C is infinitely differentiable the formula can be used,
- it is independent of N (since it is a theoretical limit),
- its leading order scales by ϵ^2 (independently of the payoff).

Digital Call

In order to get the bias we need to differentiate the analytical $\Gamma_{\mathrm{BS}}^{\mathrm{Digital}}$ which is defined in Eq. 38,

$$-e^{rT}\frac{\partial}{\partial S_0}\Gamma_{\mathrm{BS}}^{\mathrm{Digital}} = \frac{\partial}{\partial S_0}\left(\frac{d_1 n(d_2)}{S_0^2\sigma^2 T}\right)\,. \tag{54}$$

Using Eq. 7 and Eq. 14

$$\begin{aligned} -e^{rT}\frac{\partial}{\partial S_0}\Gamma_{\mathrm{BS}}^{\mathrm{Digital}} &= \left(-\frac{2}{S_0^3}\right)\frac{d_1 n(d_2)}{\sigma^2 T} + \left(\frac{1}{S_0\sigma\sqrt{T}}\right)\frac{n(d_2)}{S_0^2\sigma^2 T} + \left(-\frac{d_2 n(d_2)}{S_0\sigma\sqrt{T}}\right)\frac{d_1}{S_0^2\sigma^2 T} = \\ &= \frac{n(d_2)}{S_0^3\sigma^2 T}\left(-2d_1 + \frac{1}{\sigma\sqrt{T}} - \frac{d_1 d_2}{\sigma\sqrt{T}}\right)\,. \end{aligned} \tag{55}$$

Thus, the bias is:

$$\begin{aligned} \mathrm{Bias}(\Delta_C^{\mathrm{Digital}}) &= \frac{1}{3!}\frac{\partial}{\partial S_0}\Gamma_{\mathrm{BS}}^{\mathrm{Digital}} \cdot \epsilon^2 = \frac{1}{6} \cdot e^{-rT}\frac{n(d_2)}{S_0^3\sigma^2 T}\left(2d_1 - \frac{1}{\sigma\sqrt{T}} + \frac{d_1 d_2}{\sigma\sqrt{T}}\right) \cdot \epsilon^2 \equiv \\ &\equiv \frac{1}{6S_0^2\sigma\sqrt{T}}\delta_{\mathrm{BS}}^{\mathrm{Digital}}\left(2d_1 - \frac{1}{\sigma\sqrt{T}} + \frac{d_1 d_2}{\sigma\sqrt{T}}\right) \cdot \epsilon^2 \end{aligned} \tag{56}$$

Call Option

In order to get the bias we need to differentiate the analytical $\Gamma_{\text{BS}}^{\text{Option}}$ which is defined in Eq. 40,

$$\frac{\partial}{\partial S_0}\Gamma_{\text{BS}}^{\text{Option}} = \frac{\partial}{\partial S_0}\left(\frac{n(d_1)}{S_0\sigma\sqrt{T}}\right). \tag{57}$$

Using Eq. 7 and Eq. 14

$$\begin{aligned}\frac{\partial}{\partial S_0}\Gamma_{\text{BS}}^{\text{Option}} &= \left(-\frac{1}{S_0^2}\right)\frac{n(d_1)}{\sigma\sqrt{T}} + \left(-\frac{d_1 n(d_1)}{S_0\sigma\sqrt{T}}\right)\frac{1}{S_0\sigma\sqrt{T}} = \\ &= -\frac{n(d_1)}{S_0^2\sigma\sqrt{T}}\left(1+\frac{d_1}{\sigma\sqrt{T}}\right).\end{aligned} \tag{58}$$

Thus, the bias is:

$$\text{Bias}(\Delta_C^{\text{Option}}) = \frac{1}{3!}\frac{\partial}{\partial S_0}\Gamma_{\text{BS}}^{\text{Option}}\cdot\epsilon^2 = -\frac{1}{6}\frac{n(d_1)}{S_0^2\sigma\sqrt{T}}\left(1+\frac{d_1}{\sigma\sqrt{T}}\right)\cdot\epsilon^2. \tag{59}$$

Variance

Lets define the variance of the delta estimator (defined in Eq. 47) as:

$$\mathbf{Var}(\Delta_C) = \mathbf{Var}\left(\frac{\overline{C}(N,S_0+\epsilon)-\overline{C}(N,S_0-\epsilon)}{2\epsilon}\right) \equiv \frac{1}{4\epsilon^2}\mathbf{Var}\left(\overline{C}(N,S_0+\epsilon)-\overline{C}(N,S_0-\epsilon)\right). \tag{60}$$

By construction, see Eq. 42, $\{S_T^i\}_{i=1\ldots N}$ (or $\{F(S_T^{(i)})\}_{i=1\ldots N}$) are independent, so that

$$e^{2rT}\mathbf{Var}\left(\overline{C}(N,S_0+\epsilon)-\overline{C}(N,S_0-\epsilon)\right) = \frac{1}{N}\mathbf{Var}\left(F(S_0+\epsilon)-F(S_0-\epsilon)\right), \tag{61}$$

where

$$\begin{aligned}\mathbf{Var}\left(F(S_0+\epsilon)-F(S_0-\epsilon)\right) &= \mathbf{E}\left[\left(F(S_0+\epsilon)-F(S_0-\epsilon)\right)^2\right] + \\ &+ \mathbf{E}\left[F(S_0+\epsilon)-F(S_0-\epsilon)\right]^2.\end{aligned} \tag{62}$$

From the above formula one can see that the satisfaction of the *uniform Lipschitz condition* plays a determinative role. We say that a function $f: X \to Y$ follows the uniform Lipschitz condition if $\exists M$ with

$$\frac{1}{M}|x_2-x_1| \le |f(x_2)-f(x_1)| \le M|x_2-x_1| \tag{63}$$

for all $x_{,1}\, x_2 \in X$.

It is easy to see that for an European call option with $M=1$ the required inequities are satisfied, but for European digital option there is no such M. Consider the case where $S_0 = 1 = K$ and $x_\pm = 1 \pm \epsilon$ than; as $\epsilon \to 0$ the $|x_+ - x_-| \to 0$, but $|f(x_+)-f(x_-)| \to 1$. Thus for estimating the central finite differences we have case 2) where variance in Eq. 61 scales by $O(\varepsilon)$, but for call option we have case 3) where the variance scales by $O(\varepsilon^2)$, or equivalently one could expect that

$$\mathbf{Var}(\Delta_C^{\text{Digital}}) = \frac{\text{const.}}{N\cdot\epsilon}, \tag{64a}$$

$$\mathbf{Var}(\Delta_C^{\text{Option}}) = \frac{\text{const.}}{N}. \tag{64b}$$

Lets introduce the following notations (inline with Eq. 43)

$$\begin{aligned} S_T^+ &= (S_0+\epsilon)\frac{S_T}{S_0}\,, && (65a)\\ S_T^- &= (S_0-\epsilon)\frac{S_T}{S_0}\,, && (65b)\end{aligned}$$

and

$$\begin{aligned} d_i^+ &= d_i(S_0+\epsilon)\,, && (66a)\\ d_i^- &= d_i(S_0-\epsilon)\,. && (66b)\end{aligned}$$

Digital Call

The infinitesimal difference off the payoffs

$$F(S_0+\epsilon)-F(S_0-\epsilon)=\Theta(S_T^+-K)-\Theta(S_T^--K)=I_{S_T^+\geq K}-I_{S_T^-\geq K}\,, \tag{67}$$

using Eq. 20:

$$F(S_0+\epsilon)-F(S_0-\epsilon)=I_{\varepsilon\geq -d_2^+}-I_{\varepsilon\geq -d_2^-}\,. \tag{68}$$

It is easy to see that the above difference is positive, since

$$\begin{aligned} -d_2^- &> -d_2^+\\ \ln(S_0+\epsilon) &> \ln(S_0-\epsilon)\,. && (69)\end{aligned}$$

Thus,

$$F(S_0+\epsilon)-F(S_0-\epsilon)=I_{-d_2^-\geq\varepsilon\geq -d_2^+}\,. \tag{70}$$

The square of the infinitesimal difference off the payoffs is:

$$\left(F(S_0+\epsilon)-F(S_0-\epsilon)\right)^2=I^2_{S_T^+\geq K}+I^2_{S_T^-\geq K}-2I_{S_T^+\geq K}I_{S_T^-\geq K} \tag{71}$$

where in the second term $I_{S_T^-\geq K}$ is more strict as $I_{S_T^+\geq K}$, thus

$$\begin{aligned}\left(F(S_0+\epsilon)-F(S_0-\epsilon)\right)^2 &= I^2_{S_T^+\geq K}+I^2_{S_T^-\geq K}-2I_{S_T^-\geq K}\\ &= I_{S_T^+\geq K}+I_{S_T^-\geq K}-2I_{S_T^-\geq K}\\ &= I_{S_T^+\geq K}-I_{S_T^-\geq K}\\ &= F(S_0+\epsilon)-F(S_0-\epsilon)\,. && (72)\end{aligned}$$

Note that, we could directly use the fact that the square of a difference of two Heaviside-function is the absolute value of the difference. Since we have already show that the given difference is positive, so we would end-up with the same result.

Using Eq. 21

$$\mathbf{E}\left[F(S_0+\epsilon)-F(S_0-\epsilon)\right] = \int_{-d_2^+}^{-d_2^-} n(y)\mathrm{d}y=-\int_{-d_2^-}^{-d_2^+} n(y)\mathrm{d}y=\int_{d_2^-}^{d_2^+} n(y)\mathrm{d}y\,. \tag{73}$$

The rectangle rule let us approximate the above expression as,

$$\mathbf{E}\left[F(S_0+\epsilon)-F(S_0-\epsilon)\right]\simeq(d_2^+-d_2^-)n(d_2)\,, \tag{74}$$

where

$$
\begin{aligned}
d_2^+ - d_2^- &= d_1^+ - d_1^- = \frac{\ln \frac{S_0+\epsilon}{K} + \ldots}{\sigma\sqrt{T}} - \frac{\ln \frac{S_0-\epsilon}{K} + \ldots}{\sigma\sqrt{T}} = \\
&= \frac{1}{\sigma\sqrt{T}} \left(\ln(S_0+\epsilon) - \ln(S_0-\epsilon)\right) = \\
&= \frac{1}{\sigma\sqrt{T}} \left(\ln\left(S_0 \cdot (1+\epsilon/S_0)\right) - \ln\left(S_0 \cdot (1-\epsilon/S_0)\right)\right) \qquad (75) \\
&= \frac{1}{\sigma\sqrt{T}} \left(\ln(1+\epsilon/S_0) - (\ln(1-\epsilon/S_0))\right) . \qquad (76)
\end{aligned}
$$

For small value of x we can apply the $\ln(1 \pm x) \simeq \pm x$ approximation, thus

$$
d_2^+ - d_2^- \simeq= \frac{2\epsilon}{S_0\sigma\sqrt{T}} , \qquad (77)
$$

or

$$
\mathbf{E}\left[F(S_0+\epsilon) - F(S_0-\epsilon)\right] \simeq \frac{2\epsilon}{S_0\sigma\sqrt{T}} n(d_2) = 2\epsilon \frac{n(d_2)}{S_0\sigma\sqrt{T}} \equiv 2\epsilon e^{rT} \delta_{\mathrm{BS}}^{\mathrm{Digital}} , \qquad (78)
$$

and for digital option

$$
\begin{aligned}
\mathbf{Var}\left(F(S_0+\epsilon) - F(S_0-\epsilon)\right) &\simeq 2\epsilon e^{rT} \delta_{\mathrm{BS}}^{\mathrm{Digital}} - \left(2\epsilon e^{rT} \delta_{\mathrm{BS}}^{\mathrm{Digital}}\right)^2 = \\
&= 2 e^{rT} \delta_{\mathrm{BS}}^{\mathrm{Digital}} \cdot \epsilon + O(\epsilon^2) \qquad (79)
\end{aligned}
$$

Substituting Eq. 79 back to Eq. 62, Eq. 61, and Eq. 60:

$$
\begin{aligned}
\mathbf{Var}(\Delta_C^{\mathrm{Digital}}) &\simeq \frac{1}{4\epsilon^2} \cdot \frac{e^{-2rT}}{N} \cdot 2 e^{rT} \delta_{\mathrm{BS}}^{\mathrm{Digital}} \epsilon \\
&= \frac{e^{-rT} \delta_{\mathrm{BS}}^{\mathrm{Digital}}}{2} \cdot (\epsilon N)^{-1} \qquad (80)
\end{aligned}
$$

Call Option

The infinitesimal difference off the payoffs

$$
\begin{aligned}
F(S_0+\epsilon) - F(S_0-\epsilon) &= \max(S_T^+ - K) - \max(S_T^- - K) = \\
&= (S_T^+ - K) I_{S_T^+ \geq K} - (S_T^- - K) I_{S_T^- \geq K} = \\
&= (S_T^+ - K) I_{\varepsilon \geq -d_2^+} - (S_T^- - K) I_{\varepsilon \geq -d_2^-} , \qquad (81)
\end{aligned}
$$

and now notice that:

$$
I_{\varepsilon \geq -d_2^+} = I_{\varepsilon \geq -d_2^-} + I_{-d_2^- > \varepsilon \geq -d_2^+} , \qquad (82)
$$

then

$$
F(S_0+\epsilon) - F(S_0-\epsilon) = (S_T^+ - S_T^-) I_{\varepsilon \geq -d_2^+} + (S_T^+ - K) I_{-d_2^- > \varepsilon \geq -d_2^+} , \qquad (83)
$$

where

$$
S_T^+ - S_T^- = 2\epsilon \frac{S_T}{S_0} \qquad (84a)
$$

and

$$
S_T^+ - K = \epsilon \frac{S_T}{S_0} + (S_T - K) . \qquad (84b)
$$

The square of the infinitesimal difference off the payoffs is:

$$
\begin{aligned}
(F(S_0+\epsilon)-F(S_0-\epsilon))^2 &= 4\epsilon^2\left(\frac{S_T}{S_0}\right)^2 I_{\varepsilon\geq -d_2^+}+ \\
&+ \left[\epsilon^2\left(\frac{S_T}{S_0}\right)^2+2\frac{S_T}{S_0}\epsilon(S_T-K)+(S_T-K)^2\right] I_{-d_2^->\varepsilon\geq -d_2^+} \qquad (85)
\end{aligned}
$$

where we used that $I_A^2=I_A$ and $I_AI_B=0$ if $A\cap B=\emptyset$.

Assuming that $K\simeq \mathbf{E}\left[S_T\right]$ we can significantly simplify the above formulas: The expectation of the infinitesimal difference is reducing to

$$
\begin{aligned}
\mathbf{E}\left[F(S_0+\epsilon)-F(S_0-\epsilon)\right] &\simeq \mathbf{E}\left[2\epsilon\frac{S_T}{S_0}I_{\varepsilon\geq -d_2^+}\right]+O(\epsilon^2)= \\
&= \frac{2\epsilon}{S_0}\mathbf{E}\left[S_TI_{\varepsilon\geq -d_2^+}\right]\equiv \\
&\equiv 2\epsilon e^{rT}N(d_1^+)= \\
&= 2e^{rT}N(d_1)\cdot\epsilon+O(\epsilon^2) \qquad (86)
\end{aligned}
$$

while the expected value of the square of the infinitesimal difference is becomes:

$$
\begin{aligned}
\mathbf{E}\left[(F(S_0+\epsilon)-F(S_0-\epsilon))^2\right] &\simeq \mathbf{E}\left[4\epsilon^2\left(\frac{S_T}{S_0}\right)^2 I_{\varepsilon\geq -d_2^+}\right]+O(\epsilon^3)= \\
&= \frac{4\epsilon^2}{S_0^2}\mathbf{E}\left[S_T^2I_{\varepsilon\geq -d_2^+}\right]= \\
&= \frac{4\epsilon^2}{S_0^2}\mathbf{E}\left[S_T^2I_{\varepsilon\geq -d_2}\right]+O(\epsilon^3)\equiv \\
&\equiv \frac{4\epsilon^2}{S_0^2}\mathbf{E}\left[S_T^2I_{S_T>K}\right]+O(\epsilon^3) \qquad (87)
\end{aligned}
$$

where

$$
\mathbf{E}[S_T^2I_{S_T>K}] = \int_{-d_2}^{\infty} S_0e^{2\left(r-\frac{1}{2}\sigma^2\right)T+2\sigma\sqrt{T}y}n(y)\mathrm{d}y = \frac{S_0^2e^{2\left(r-\frac{1}{2}\sigma^2\right)T}}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{1}{2}(y-2\sigma\sqrt{T})^2+2\sigma^2T}, \qquad (88)
$$

by introducing $y'=y+2\sigma\sqrt{T}$ we end up with the following:

$$
\mathbf{E}[S_T^2I_{S_T>K}] = S_0^2e^{2rT+\sigma^2T}N(d_1+\sigma\sqrt{T}), \qquad (89)
$$

Thus,

$$
\mathbf{E}\left[(F(S_0+\epsilon)-F(S_0-\epsilon))^2\right] = 4e^{2rT+\sigma^2T}N(d_1+\sigma\sqrt{T})\epsilon^2+O(\epsilon^3) \qquad (90)
$$

Substituting Eq. 86 and 90 back to Eq. 62, Eq. 61, and Eq. 60:

$$
\begin{aligned}
\mathbf{Var}(\Delta_C^{\text{Option}}) &\simeq \frac{1}{4\epsilon^2}\cdot\frac{e^{-2rT}}{N}\left[4e^{2rT+\sigma^2T}N(d_1+\sigma\sqrt{T})\epsilon^2-\left(2e^{rT}N(d_1)\cdot\epsilon\right)^2\right]= \\
&= \left(e^{\sigma^2T}N(d_1+\sigma\sqrt{T})-N(d_1)^2\right)\cdot N^{-1} \qquad (91)
\end{aligned}
$$

Mean Squared Error and optimal ϵ

In statistics, the mean squared error (MSE) of an estimator measures the average of the squares of the errors, i.e., it is the average squared difference between the estimated values and the actual value. For estimator Δ_C the true value (reference) is the delta from the Black-Scholes analytical model (δ_{BS}), thus

$$
\mathbf{MSE}(\Delta_C) = \mathbf{E}\left[(\Delta_C-\delta_{\text{BS}})^2\right]\equiv \mathbf{Bias}^2(\Delta_C)+\mathbf{Var}(\Delta_C). \qquad (92)
$$

Digital Call

Using the bias in Eq. 56 and the variance in Eq. 80 we get the following:

$$\begin{aligned}\mathbf{MSE}(\Delta_C^{\text{Digital}}) &= \left[\frac{1}{6S_0^2\sigma\sqrt{T}}\delta_{\text{BS}}^{\text{Digital}}\left(2d_1 - \frac{1}{\sigma\sqrt{T}} + \frac{d_1 d_2}{\sigma\sqrt{T}}\right)\cdot\epsilon^2\right]^2 + \\ &+ \frac{e^{-rT}\delta_{\text{BS}}^{\text{Digital}}}{2}\cdot(\epsilon N)^{-1}\,. \end{aligned} \tag{93}$$

Lets introduce

$$b_{\text{Digital}} = \frac{1}{6S_0^2\sigma\sqrt{T}}\left(2d_1 - \frac{1}{\sigma\sqrt{T}} + \frac{d_1 d_2}{\sigma\sqrt{T}}\right)\,, \tag{94a}$$

$$\sigma^2_{\text{Digital}} = \frac{e^{-rT}}{2\delta_{\text{BS}}^{\text{Digital}}}\,, \tag{94b}$$

then

$$\frac{\mathbf{MSE}(\Delta_C^{\text{Digital}})}{\left(\delta_{\text{BS}}^{\text{Digital}}\right)^2} = b^2_{\text{Digital}}\cdot\epsilon^4 + \sigma^2_{\text{Digital}}\cdot(\epsilon N)^{-1}\,. \tag{95}$$

The optimal $\epsilon_N^*(\text{Digital})$ can be found be requiring that **MSE** has an extreme value (minimum) at that point, thus:

$$\epsilon_N^*(\text{Digital}) = \left(\frac{\sigma^2}{4b^2}\right)^{1/5}\cdot N^{-\frac{1}{5}} \sim N^{-\frac{1}{5}} \tag{96}$$

and its optimal value

$$\frac{\mathbf{MSE}(\Delta_C^{\text{Digital}})(\epsilon_N^*(\text{Digital}))}{\left(\delta_{\text{BS}}^{\text{Digital}}\right)^2} = \left[b^2_{\text{Digital}}\cdot\left(\frac{\sigma^2}{4b^2}\right)^{4/5} + \sigma^2_{\text{Digital}}\left(\frac{\sigma^2}{4b^2}\right)^{-1/5}\right]\cdot N^{-\frac{4}{5}} \tag{97}$$

converging by $N^{-\frac{4}{5}}$.

Call Option

Using the bias in Eq. 59 and the variance in Eq. 91 we get the following:

$$\begin{aligned}\mathbf{MSE}(\Delta_C^{\text{Option}}) &= \left[-\frac{1}{6}\frac{n(d_1)}{S_0^2\sigma\sqrt{T}}\left(1 + \frac{d_1}{\sigma\sqrt{T}}\right)\cdot\epsilon^2\right]^2 + \\ &+ \left(e^{\sigma^2 T}N(d_1+\sigma\sqrt{T}) - N(d_1)^2\right)\cdot N^{-1} \end{aligned} \tag{98}$$

Lets introduce

$$b_{\text{Option}} = \frac{1}{6}\frac{n(d_1)}{S_0^2\sigma\sqrt{T}}\left(1 + \frac{d_1}{\sigma\sqrt{T}}\right)\,, \tag{99a}$$

$$\sigma^2_{\text{Option}} = \left(e^{\sigma^2 T}N(d_1+\sigma\sqrt{T}) - N(d_1)^2\right)\,, \tag{99b}$$

then

$$\mathbf{MSE}(\Delta_C^{\text{Option}}) = b^2_{\text{Option}}\epsilon^4 + \sigma^2_{\text{Option}}\cdot N^{-1}\,. \tag{100}$$

Since the second term in Eq. 100 is independent of ϵ, there is no competing terms and for get the optimal value we can do the $\epsilon \to 0$ limes.

References

[1] Model Risk Final - Code https://github.com/antsticky/MRM_final GitHub

[2] Black Scholes Calculator https://www.math.drexel.edu/~pg/fin/VanillaCalculator.html Drexel University

[3] Digital Options https://silo.tips/download/digital-options-and-d-1-d-2-p-int-e-r-kn-d-2-f Silo

[4] Black–Scholes model https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_model Wikipedia

[5] Online Black-Scholes Calculator http://www.deltaquants.com/calc-test DeltaQuants