Gravitational radiation in General Relativity

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Abstract

In this thesis the gravitational waves in General Relativity are studied as a consequence of applying the metric perturbation theory to the Einstein field equations for gravitation. We begin by introducing the consept of linearized theory of gravity and gauge transformations after which we show how the gravitational plane waves emerge from such a theory. Fixing the gauge properly leaves us the two degrees of freedom of a gravitational wave that are interpreted as two independent polarization states. We study the effects of a passing gravitational wave and furthermore turn our interest into the sources and emission of gravitational radiation, taking a simple a binary star as an example. Finally we study the energy carried by a gravitational wave and justify our examination with observed results.

Tiivistelmä

Tässä työssä tutkitaan gravitaatioaaltoja Yleisessä suhteellisuusteoriassa soveltaen metristä häiriöteoriaa Einsteinin kenttäyhtälöihin. Aluksi esitetään linearisoitu gravitaatioteoria ja sen mittamuunnos, joiden avulla osoitetaan kuinka kenttäyhtälöt voidaan palauttaa aaltoyhtälöksi. Aaltoyhtälön ratkaisuksi otetaan tasoaalto, jolla voidaan mittamuunnosta käyttäen osoittaa olevan kaksi riippumatonta polarisaatioastetta. Tämän jälkeen tutkitaan gravitaatioaallon vaikutuksia materiaan sekä aaltojen lähteitä. Esimerkkinä otetaan yksinkertainen kaksoistähti ja lopuksi verrataan teorian antamia tuloksia todellisiin mittaustuloksiin.

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"Gravity is a contributing factor in nearly 73 percent of all accidents involving falling objects."

– Dave Barry (1947–)

1 Introduction

In 1916 Einstein published his famous article "*The Foundations of General Theory of Relativity*" in which he presented a geometrical theory of gravitation. The basic playground behind this theory is a four-dimensional spacetime, the structure of which is coded in the metric tensor $g_{\mu\nu}$. The spacetime is then curved by matter and energy according to Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor constructed from the metric tensor and $T_{\mu\nu}$ is the stress-energy tensor of matter and energy under consideration. The curvature of spacetime in turn determines the trajectories of free bodies.

Linearized theory of gravitation is a way of studying weak gravitational fields, such as solar system or other very large scale structures except for example collapsing stars or black holes. In linearized theory one considers a backgound spacetime with small perturbations, or 'ripples', in it as an approximation of spacetime. In this approach one soon comes up with a linear wave equation for the metric perturbations. This was first noticed by Einstein and it is treated as a manifestation of gravitational waves, or gravitational radiation, propagating light-speed in vacuum.

These waves are of course highly studied and indirect evidence of the existence of them has been observed as the decaying orbital period of several binary stars, the binary pulsar PSR 1913+16 discovered by Hulse and Taylor in 1974 being the most famous of them. Direct evidence, i.e. observable effects of a passing gravitational wave into matter, are not yet been detected, but high expectations are on the future detectors with increased accuracy. In this thesis we try to understand the basic theory of gravitational waves, briefly introduced above, in full detail.

1.1 Conventions

A generic metric tensor is denoted by $g_{\mu\nu}$ with signature $(-1,1,1,1)$. The Minkowskian metric tensor is $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. We use Greek letters $\mu, \nu, \ldots \in \{0,1,2,3\}$ to represent the spacetime indeces of four-vectors in Minkowski space and Latin letters $i,j,k,\ldots \in \{1,2,3\}$ the purely spatial indeces of ordinary three-vectors. The Einstein summation convention holds for both Latin and Greek indeces. We use the notation

$$\partial_\mu \phi \equiv \frac{\partial \phi}{\partial x^\mu} \tag{1.2}$$

and

$$A_{(\mu} B_{\nu)} \equiv \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu). \tag{1.3}$$

1.2 Units

We use the geometrized unit system, in which $G = c = 1$. This implies $1 = c = 2{,}998 \cdot 10^8 \,\frac{\mathrm{m}}{\mathrm{s}}$ and $1 = G = 6{,}67 \cdot 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kgs}^2}$, from which we get [mass] = [lenght] = [time].

2 Linearized gravity

In this section we represent the basics of linearized theory of gravitation and linearized field equations which form the backbone of the theory of gravitational waves. We also consider the gauge transformations in this formalism, that is coordinate transformations that do not affect the 'smallness' of the metric perturbations. Finally we introduce the Lorenz gauge in which the gravitational waves should manifest themselves.

2.1 Perturbed metric

We start by considering a weak gravitational field, composed into a Minkowskian background metric $\eta = \mathrm{diag}(-1,1,1,1)$ and a small perturbation $|h_{\mu\nu}| \ll 1$, such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{2.1}$$

Here 'smallness' of $|h_{\mu\nu}|$ means that it is possible to find a coordinate system where equation (2.1) holds. To first order in $h_{\mu\nu}$ we then have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{2.2}$$

where $h^{\mu\nu} \equiv \eta^{\nu\alpha}\eta^{\mu\beta}h_{\alpha\beta}$. We shall interpret the small perturbation as a symmetric tensor field *propagating* through a flat background spacetime, and we can use $\eta_{\mu\nu}$ to lower and raise indeces. The perturbation is symmetric because the metric tensor is. Since the background spacetime is flat, we can use ordinary partial derivative instead of covariant derivate. We construct the linearized theory such that whenever we need to substitute (2.1), we keep only the terms which are linear in $h_{\mu\nu}$.

Calculating the Christoffel symbols to first order in $h_{\mu\nu}$ yields

$$\begin{aligned} \Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}g^{\alpha\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \\ &\approx \frac{1}{2}\eta^{\alpha\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}), \end{aligned} \tag{2.3}$$

from which we can see that the product of two Christoffel symbols is of the second order in perturbation. Substituting this one finds the Riemann tensor to be consistently, all indeces down,

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= g_{\mu\lambda}\left(\partial_\rho\Gamma^{\lambda}_{\nu\sigma} - \partial_\sigma\Gamma^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\rho\beta}\Gamma^{\beta}_{\nu\sigma} - \Gamma^{\lambda}_{\sigma\beta}\Gamma^{\beta}_{\mu\nu}\right) \\ &\approx \eta_{\mu\lambda}\left(\partial_\rho\Gamma^{\lambda}_{\nu\sigma} - \partial_\sigma\Gamma^{\lambda}_{\nu\rho}\right) \\ &\approx \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho}\right), \end{aligned} \tag{2.4}$$

and furthermore one can solve the Ricci tensor to be

$$R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma} = \frac{1}{2}\left(\partial_\mu\partial_\nu h^{\mu}{}_{\sigma} + \partial_\sigma\partial_\mu h_{\nu}{}^{\mu} - \Box h_{\nu\sigma} - \partial_\sigma\partial_\nu h\right), \tag{2.5}$$

where $\Box \equiv \partial_\mu\partial^\mu = -\frac{\partial^2}{\partial t^2} + \nabla^2$ is the d'Alembertian in Minkowski space and $h \equiv h^{\mu}{}_{\mu}$ is the trace of the perturbation. From the Ricci tensor one can contract again to obtain the

Ricci scalar

$$\begin{aligned} R &= \frac{1}{2}\left(\partial_\mu\partial^\nu h^\mu{}_\nu + \partial_\nu\partial_\mu h^{\nu\mu} - \Box h^\nu{}_\nu - \partial_\nu\partial^\nu h\right) \\ &= \partial_\mu\partial_\nu h^{\mu\nu} - \Box h. \end{aligned} \tag{2.6}$$

Plugging in the above results to the Einstein field equations (1.1) gives the field equations for linearized gravity

$$\frac{1}{2}\left(\partial_\sigma\partial_\mu h^\sigma{}_\nu + \partial_\nu\partial_\sigma h^\sigma{}_\mu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h\right) = 8\pi T_{\mu\nu}. \tag{2.7}$$

Keeping in mind that the perturbation is small, the above equations are reasonable only when $|T_{\mu\nu}| \ll 1$. To make the weak field equations (2.7) look less messy, we define the *trace-reversed perturbation*

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{2.8}$$

which contains exactly the same information as $h_{\mu\nu}$, yet it is trace-reversed[1]

$$\bar{h} \equiv \bar{h}^\mu{}_\mu = h - 2h = -h. \tag{2.9}$$

Substituting the inverse of (2.8) to field equations yields

$$\frac{1}{2}\left(\partial_\mu\partial_\sigma\bar{h}^\sigma{}_\nu + \partial_\nu\partial_\sigma\bar{h}^\sigma{}_\mu - \Box\bar{h}_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma}\right) = 8\pi T_{\mu\nu}, \tag{2.10}$$

which has only 4 terms on the left hand side and later on we shall see that this becomes solvable when we fix a coordinate system. But first we have to study which kind of coordinates are suitable for (2.1) to hold.

2.2 Gauge transformation

In this section we study the problem of gauge invariance in the linearized theory. Since the background metric is flat, both the metric tensor $\eta_{\mu\nu}$ and the perturbation $h_{\mu\nu}$ are invariant under Lorentz transformations, so for example when we consider the motion of particles we can move to the rest frame of the particle without losing the 'smallness' of the perturbation. But what about general coordinate transformations? There clearly are many ways to choose the coordinate system such that (2.1) holds, though we are not allowed to choose the coordinates *arbitrary* since the perturbations are not necessarily small in every frame. This means that we should be able to make some gauge transformation (special coordinate transformations) such that the linearized theory doesn't brake.

We begin by introducing a diffeomorphism $\phi : M_b \to M_p$ from the flat background spacetime M_b to the physical spacetime M_p. Keep in mind that the manifolds M_b,M_p are exactly the same, they only contain different tensor fields. Thus the physical weak field metric $g_{\mu\nu}$ in some point of M_p, obeying the Einstein's equations, can be pulled back by ϕ to the background spacetime M_b such that the metric perturbation can be thought to

[1] We shall see that in a specific gauge, called transverse-traceless gauge the trace of the perturbation vanishes and thus $\bar{h}_{\mu\nu}$ and h_{mn} are equal.

be the difference between flat spacetime and the actual physical spacetime, a tensor in the background manifold, i.e.

$$h_{\mu\nu} = (\phi^* g)_{\mu\nu} - \eta_{\mu\nu}, \tag{2.11}$$

where $\phi^* g$ denotes the pull-back $(g \circ \phi)$. Now the components $h_{\mu\nu}$ in this expression are small only for *some* ϕ, so we consider only those which do satisfy this condition. As we stated, ϕ is still not unique since there clearly are many coordinate systems where $h_{\mu\nu}$ *is* small and it really doesn't matter which coordinates of those we choose since the only requirement is that the linearized therory holds. In some coodinates though the calculations become more simple, so we want to specify *which kind* of transformations we are able to make.

The idea is that we introduce an arbitrary vector field ξ^α which generates a one-parameter family of diffeomorphisms $\Psi_\epsilon : M_b \to M_p$ such that the background coordinates x^α transform by

$$x^\alpha \xrightarrow{\Psi} x'^\alpha = x^\alpha + \epsilon \xi^\alpha, \tag{2.12}$$

where we take the parameter $\epsilon \ll 1$ and the primed coordinates denote the coordinates on M_p. The diffeomorphism is nothing but moving along the flow of ξ^α. From this we get the corresponding Jacobian to be

$$\frac{\partial x^{\alpha'}}{\partial x^\mu} = \delta^\alpha_\mu + \epsilon \partial_\mu \xi^a, \tag{2.13}$$

so that for example an arbitrary vector field A^μ transforms by

$$A^\mu \xrightarrow{\Psi} A'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} A^\alpha = A^\mu + \epsilon A^\alpha \partial_\alpha \xi^\mu. \tag{2.14}$$

Since Ψ is infinitesimal, so is also the composition $(\phi \circ \Psi)$, and thus using equation (2.11) we can write

$$\begin{aligned} h^{(\epsilon)}_{\mu\nu} &= [(\phi \circ \Psi)^* g]_{\mu\nu} - \eta_{\mu\nu} \\ &= [\Psi^*(\phi^* g)]_{\mu\nu} - \eta_{\mu\nu} \\ &= [\Psi^*(\eta + h)]_{\mu\nu} - \eta_{\mu\nu} \\ &= (\Psi^* h)_{\mu\nu} + (\Psi^* \eta)_{\mu\nu} - \eta_{\mu\nu}, \end{aligned} \tag{2.15}$$

where the superscript (ϵ) tells that the tensor belongs to a family of perturbations suitable for the linearized theory to hold. Now to lowest order in ϵ and $h_{\mu\nu}$, we have

$$\begin{aligned} (\Psi^* h(x))_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} h_{\alpha\beta}(x + \epsilon\xi) \\ &= (\delta^\alpha_\mu - \epsilon \partial_\mu \xi^\alpha)(\delta^\beta_\nu - \epsilon \partial_\nu \xi^\beta) h_{\alpha\beta}(x + \epsilon\xi) \\ &\approx h_{\mu\nu}(x), \end{aligned} \tag{2.16}$$

where in the second line we have used the inverse of (2.13) (since Ψ^* pulls the coordinates back) and in the last line we have used a Taylor expansion

$$h_{\mu\nu}(x + \epsilon\xi) \approx h_{\mu\nu}(x) + \mathcal{O}(\epsilon h_{\mu\nu}), \tag{2.17}$$

where $\mathcal{O}(\epsilon h_{\mu\nu})$ means that the corrections would be second order since both the perturbation and ϵ are small. Modifying (2.15) a little more, we get

$$h_{\mu\nu}^{(\epsilon)} = h_{\mu\nu} + \epsilon \left(\frac{(\Psi^* \eta)_{\mu\nu} - \eta_{\mu\nu}}{\epsilon} \right), \tag{2.18}$$

and as stated in [1, p.434], the latter term in parentheses is the Lie derivative of $\eta_{\mu\nu}$ along the vector field ξ:

$$\mathcal{L}_\xi \eta_{\mu\nu} \equiv \lim_{\epsilon \to 0} \frac{\Delta_\epsilon \eta_{\mu\nu}}{\epsilon} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2\nabla_{(\mu}\xi_{\nu)}. \tag{2.19}$$

Since the background metric is flat, the covariant derivative is just the normal partial derivative and thus we result *the gauge transformation of linearized theory*

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + 2\epsilon \partial_{(\mu}\xi_{\nu)}, \tag{2.20}$$

which specifies which kind of transformations we are allowed to make in order to keep the perturbations small. In further discussion we set $\epsilon = 1$ and think the vector field ξ^μ itself as being small.

3 Gravitational waves in vacuum

In this section we study how the gravitational waves rise from the field equations after fixing the gauge properly, and then we study the properties of these waves in the vacuum. We encounter the two independent polarization states and take a look on how a passing wave affects on bodies.

3.1 Plane wave solution and transverse-traceless gauge

We begin by deducing the gauge transformation for the trace-reversed perturbation by substituting (2.20) to (2.8) and get

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\alpha \xi^\alpha, \tag{3.1}$$

from which we get

$$\begin{aligned} \partial_\mu \bar{h}^{\mu\nu} \to \partial_\mu \bar{h}'^{\mu\nu} &= \partial_\mu \bar{h}^{\mu\nu} + \partial_\mu \partial^\mu \xi^\nu + \partial_\mu \partial^\nu \xi^\mu - \eta^{\mu\nu}\partial_\mu \partial_\alpha \xi^\alpha \\ &= \partial_\mu \bar{h}^{\mu\nu} + \Box \xi^\nu, \end{aligned} \tag{3.2}$$

and by fixing the gauge such that

$$\Box \xi^\nu = -\partial_\mu \bar{h}^{\mu\nu}, \tag{3.3}$$

we get

$$\partial_\mu \bar{h}^{\mu\nu} = 0. \tag{3.4}$$

This is called *the Lorentz condition* and (3.3) is called *the Lorentz gauge.* In Lorentz gauge the Einstein field equations (2.10) become a wave equation for each component

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{3.5}$$

and in vacuum the stress-energy tensor vanishes and (3.5) is simply

$$\Box \bar{h}_{\mu\nu} = 0. \tag{3.6}$$

A standard guess for a solution to the wave equation (3.6) is the plane wave

$$\bar{h}_{\mu\nu} = C_{\mu\nu} \Re(e^{ik_\alpha x^\alpha}), \tag{3.7}$$

where k_α is a constant wave vector with $k_0 = \omega =$ frequency of the wave. $C_{\mu\nu}$ is a constant tensor, and since $\bar{h}_{\mu\nu}$ is symmetric, so is $C_{\mu\nu}$. We call $C_{\mu\nu}$ the amplitude. The real plane wave is of course some combination of cosines and sines, or a cosine $C_{\mu\nu} \cos(k_\alpha x^\alpha + \delta)$, where δ is just some constant phase, but since we are only interested on the oscillating character of the wave, we can choose the wave to be cosine with $\delta = 0$ and use the complex exponential $e^{ik_\alpha x^\alpha}$ to ease the calculations and remember that in order to obtain the physical result we must take the real part $\Re(e^{ik_\alpha x^\alpha})$ in the end. From now on we drop the $\Re$-sign and keep in mind the discussion above. Substituting the ansatz to the wave equation yields

$$\begin{aligned} \Box \bar{h}_{\mu\nu} &= \eta^{\alpha\beta} \partial_\alpha \partial_\beta \left[C_{\mu\nu} e^{ik_\sigma x^\sigma} \right] \\ &= -C_{\mu\nu} k_\alpha k^\alpha e^{ik_\sigma x^\sigma} \\ &= -k_\alpha k^\alpha \bar{h}_{\mu\nu}. \end{aligned} \tag{3.8}$$

We assumme that $\bar{h}_{\mu\nu} \neq 0$ for at least some of the components, so that for a non-trivial solution we must have

$$k_\alpha k^\alpha = 0, \tag{3.9}$$

i.e. $\mathbf{k} = (\omega, k_1, k_2, k_3)$ is a null vector, or "light-like", that is

$$\omega = \pm \sqrt{(k_1)^2 + (k_2)^2 + (k_3)^2}. \tag{3.10}$$

The interpretation for this is that the wave propagates at light-speed. The Lorentz condition holds, so

$$0 = \partial_\mu \bar{h}^{\mu\nu} = \partial_\mu \left[C^{\mu\nu} e^{ik_\sigma x^\sigma} \right] = ik_\mu C^{\mu\nu} e^{ik_\sigma x^\sigma}, \tag{3.11}$$

from which we get the transversality relation

$$k_\mu C^{\mu\nu} = 0, \tag{3.12}$$

which means that the amplitude tensor is transverse to the wave vector.

Since the perturbation $\bar{h}_{\mu\nu}$ is a symmetric 4×4 tensor, we know that it has ten independent components, but since (3.12) gives 4 independent restrictions, we are left with only 6 six degrees of freedom. If we look at the Lorentz gauge condition, we notice that it does not fully fix the gauge field ξ. We shall impose a more concrete gauge transformation, see ref. [2],

$$\xi^\mu = iB^\mu e^{ik_\alpha x^\alpha}, \tag{3.13}$$

which actually preserves the Lorentz condition: if $k_\mu C^{\mu\nu} = 0$ and $k_\mu k^\mu = 0$, and if $B^\mu \neq 0$ we have

$$\Box\xi^\mu = -iB^\mu k_\alpha k^\alpha e^{ik_\sigma x^\sigma} = 0 = -ik_\mu C^{\mu\nu} e^{ik_\sigma x^\sigma} = -\partial_\mu \bar{h}^{\mu\nu}. \tag{3.14}$$

We are still allowed to choose the components B^μ freely. Now the gauge transformation for $\bar{h}_{\mu\nu}$ induces a transformation law for the amplitude tensor $C_{\mu\nu}$ as follows

$$\begin{aligned} \bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} &= \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\alpha \xi^\alpha \\ \Rightarrow C_{\mu\nu} \to C'_{\mu\nu} &= C_{\mu\nu} - B_\mu k_\nu - B_\nu k_\mu + B^\alpha k_\alpha \eta_{\mu\nu}. \end{aligned} \tag{3.15}$$

From this we can see in particular how the trace of $C_{\mu\nu}$ changes:

$$\begin{aligned} C^\mu{}_\mu \to C'^\mu{}_\mu &= C^\mu{}_\mu - B^\mu k_\mu - B_\mu k^\mu + 4B^\mu k_\mu \\ &= C^\mu{}_\mu + 2B^\mu k_\mu. \end{aligned} \tag{3.16}$$

If we set

$$B^\mu k_\mu = -\frac{1}{2} C^\mu{}_\mu, \tag{3.17}$$

we have

$$C^\mu{}_\mu = 0, \tag{3.18}$$

i.e. $C_{\mu\nu}$ is traceless. Furthermore, since $\eta_{0i} = 0$, we get

$$\begin{aligned} C_{0i} \to C'_{0i} &= C_{0i} - B_i k_0 - B_0 k_i + B^\alpha k_\alpha \eta_{0i} \\ &= C_{0i} - B_i k_0 - B_0 k_i, \end{aligned} \tag{3.19}$$

so we can choose

$$B_i k_0 + B_0 k_i = C_{0i} \tag{3.20}$$

and have

$$C_{0i} = 0. \tag{3.21}$$

The orthogonality relation (3.12) for the $\nu = 0$ component gives

$$\omega C^{00} + k_i \underbrace{C^{i0}}_{=0} = 0, \tag{3.22}$$

so for $\omega \neq 0$ we have $C^{00} = 0$. Combining all the restrictions for $C_{\mu\nu}$ we have

$$\begin{cases} C_{0\nu} = 0 & (\textbf{purely spatial}) \\ \eta^{\mu\nu} C_{\mu\nu} = 0 & (\textbf{traceless}) \\ k_\mu C^{\mu\nu} = 0 & (\textbf{transverse}). \end{cases} \tag{3.23}$$

The equations (3.20) and (3.17) now fully fix the gauge parameter B^μ. This gauge is called the *transverse-traceless gauge* and is characterized by the equations

$$\begin{cases} \bar{h}_{0\nu} = 0 \\ \eta^{\mu\nu} \bar{h}_{\mu\nu} = 0 \\ \partial_\mu \bar{h}^{\mu\nu} = 0. \end{cases} \tag{3.24}$$

3.2 Polarizations of a gravitational wave

After all the gauge fixing the metric perturbation $\bar{h}_{\mu\nu}$ is left with only $10 - 4 - 4 = 2$ degrees of freedom, the first 4 restrictions coming from the Lorentz condition (3.4) and the next 4 coming from completely fixing the gauge. These degrees of freedom are interpreted as the two *polarizations* of a plane gravitational wave. Let us now choose the gravitational wave to propagate in the z-direction, so that

$$\mathbf{k} = (\omega, 0, 0, k_3), \tag{3.25}$$

and since $\mathbf{k}$ is null, we have $\omega = k_3 \neq 0$. Equations (3.23) now give

$$\begin{aligned} &k^\mu C_{\mu\nu} = \omega C_{0\nu} + k^1 C_{1\nu} + k^2 C_{2\nu} + \omega C_{3\nu} = 0 \\ &\Rightarrow C_{3\nu} = 0, \end{aligned}$$

so that the only nonzero components of $C_{\mu\nu}$ are $C_{11}, C_{12}, C_{21}, C_{22}$. From symmetry we get that $C_{12} = C_{21}$ and since $C_{\mu\nu}$ is traceless, we have $C_{11} = -C_{22}$. Let's rename the components for future convenience as follows: $C_{11} = h_+$ and $C_{12} = h_\times$, so that the amplitude tensor is now simply

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.26}$$

3.3 Observable effects caused by gravitational wave

In previous section we studied the basic properties of a plane gravitational wave, so next we take a look on how the passing wave could be detected and which kind of effects it has on the bodies and trajectories. First we study how the wave affects on the trajectory of a freely falling body, which shows out to be a rather bad way of approach, since the bodies moving in curved space don't really 'feel' the curvature. However, since the gravitational wave causes nonzero curvature, we can think the wave to be as like a propagating bunch of tidal forces which affects on the proper distances of particles and causes the geodesics of particles accelerate towards each other.

3.3.1 Geodesic equation

The geodesic equation for a freely falling particle is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0, \tag{3.27}$$

see [1] p.106. For a slowly moving particle we have $\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll 1$, moreover for slow particle we can identify $\tau = t$, so that we can neglect the 3-velocity terms in the term involving the connection coefficient in equation (3.27). This is also justified by the fact that the Christoffel symbols are first order in $h_{\mu\nu}$, so that the terms containing products of the

Christoffel symbols and 3-velocity would be roughly third order in perturbation. The geodesic equation thus reduces to

$$\frac{d^2x^\mu}{dt^2} \approx -\Gamma^\mu_{00}\frac{\mathrm{d}x^0}{\mathrm{d}t}\frac{\mathrm{d}x^0}{\mathrm{d}t} \approx -\Gamma^\mu_{00}, \tag{3.28}$$

where we have according to equation (2.3)

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\lambda}\left(\partial_0 h_{\lambda 0} + \partial_0 h_{0\lambda} - \partial_\lambda h_{00}\right) = \frac{1}{2}\eta^{\mu\lambda}\left(2\partial_0 h_{\lambda 0} - \partial_\lambda h_{00}\right). \tag{3.29}$$

In transverse-traceless gauge we have then, for the i-components

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00} = -\frac{1}{2}\left(2\partial_0 h_{i0} - \partial_i h_{00}\right) = 0, \tag{3.30}$$

i.e. the coordinates of a particle initially at rest at transverse-traceless coordinates will not change when a gravitational wave passes. It means that the transverse-traceless coordinates are *comoving* with freely falling particles, see page 5 in [3]. However, the location of a particle in a specific coordinate system is clearly not a coordinate invariant, so the above result does not fully unveil the effects of a passing gravitational wave, see [4] p.64. Next we will take a look on the *separation* of particles in the presence of a gravitational wave.

3.3.2 Geodesic deviation

In a flat spacetime the geodesics of particles are straight lines and initially parallel lines stay parallel forever. However, in curved space initially parallel geodesics accelerate relative to each other and the acceleration is proportional to the curvature. A gravitational wave causes ripples in the Minkowski metric and as we shall see, a nonzero curvature tensor, so that the effect of a passing wave could manifest itself as an oscillation of the separation of two bodies.

Consider now two freely falling test bodies, close to each other, such that the velocity of both bodies is $U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ and the separation of the bodies is characterized by the vector S^μ. The covariant derivative of S^μ along a geodesic is

$$\frac{D}{d\tau}S^\mu = \frac{\partial S^\mu}{\partial \tau} + \Gamma^\mu_{\alpha\beta}S^\alpha U^\beta \tag{3.31}$$

and the relative acceleration of two test bodies towards each other is given by the *geodesic deviation equation* (see ref. [1] p.144)

$$\frac{D^2}{d\tau^2}S^\mu = R^\mu{}_{\nu\rho\sigma}U^\nu U^\rho S^\sigma. \tag{3.32}$$

Let's assume that the bodies are moving slowly, so that the four-velocity is $\mathbf{U} = (1{,}0{,}0{,}0) + \mathcal{O}(h_{\mu\nu})$. The Riemann tensor in weak field is already of the first order in $h_{\mu\nu}$, so that we can neglect the first order corrections in U^μ. Therefore the right-hand side calculated to first order is non zero only for the components $R^\mu{}_{00\sigma}$. From (2.4) we get

$$\begin{aligned} R_{\mu 00\sigma} &= \frac{1}{2}\left(\partial_0\partial_0 h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{00} - \partial_0\partial_\mu h_{0\sigma} - \partial_\sigma\partial_0 h_{\mu 0}\right), \\ &= \frac{1}{2}\left(\partial_0\partial_0 \bar{h}_{\mu\sigma} + \partial_\sigma\partial_\mu \bar{h}_{00} - \partial_0\partial_\mu \bar{h}_{0\sigma} - \partial_\sigma\partial_0 \bar{h}_{\mu 0}\right) \\ &\quad + \frac{1}{4}\left(\eta_{\mu\sigma}\partial_\rho\partial_\nu h + \eta_{\nu\rho}\partial_\sigma\partial_\mu h - \eta_{\nu\sigma}\partial_\rho\partial_\mu h - \eta_{\mu\rho}\partial_\sigma\partial_\nu h\right) \end{aligned} \tag{3.33}$$

but when moving into transverse-traceless gauge we have $\bar{h}_{0\nu} = \bar{h}_{\nu 0} = 0$ and $h = 0$, so that we are left only with

$$R_{\mu 00\sigma} = \frac{1}{2}\partial_0\partial_0\bar{h}_{\mu\sigma} = \frac{1}{2}\frac{\partial^2}{\partial t^2}\bar{h}_{\mu\sigma}. \tag{3.34}$$

This shows that the metric perturbation really is a physically meaningful quantity since it causes a nonzero curvature which can not be removed by a further gauge transformation [5].

Let's next look at the left hand side of (3.32). Because we are assuming slowly moving particles we can identify $t = \tau$. We can assume the spacetime to be flat in enough big region so we can take

$$\frac{D^2}{dt^2}S^\mu = \frac{\partial^2 S^\mu}{\partial t^2}. \tag{3.35}$$

The geodesic deviation equation thus becomes, to first order

$$\frac{\partial^2}{\partial t^2}S^\mu = \frac{1}{2}S^\sigma\frac{\partial^2}{\partial t^2}\bar{h}^\mu{}_\sigma = \frac{1}{2}S^\sigma\frac{\partial^2}{\partial t^2}(\eta^{\alpha\mu}\bar{h}_{\alpha\sigma}). \tag{3.36}$$

The two degrees of freedom, or polarizations, h_+ and $h_\times$ characterize the gravitational wave and equation (3.36) tells how they affect the physical separation of nearby spatial points. Of course in realistic situation, as in detectors, there will be all kinds of background disturbance affecting the test bodies, but the result we obtained tells only on how a *pure* gravitational wave would affect. The effort to filter the background noise from the actual object of interest is then another story. We however focus now only on solving equation (3.36) to see what is the true influence of a passing gravitational wave.

Let's look at the effects of the independent polarizations separately, beginning by setting $\mathbf{h}_\times = \mathbf{0}$. We call this a *+-polarized wave.* A general gravitational wave would of course be a mixture of +- and ×-polarizations. From (3.36) we get now two equations

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}S^1 = \dfrac{1}{2}S^\sigma\dfrac{\partial^2}{\partial t^2}(\eta^{\alpha 1}\bar{h}_{\alpha\sigma}) = \dfrac{1}{2}S^1\dfrac{\partial^2}{\partial t^2}(h_+e^{ik_\sigma x^\sigma}) \\ \dfrac{\partial^2}{\partial t^2}S^2 = \dfrac{1}{2}S^\sigma\dfrac{\partial^2}{\partial t^2}(\eta^{\alpha 2}\bar{h}_{\alpha\sigma}) = -\dfrac{1}{2}S^2\dfrac{\partial^2}{\partial t^2}(h_+e^{ik_\sigma x^\sigma}), \end{cases} \tag{3.37}$$

where the second equality in the first equation follows since $\eta^{\alpha 1} \neq 0$ only if $\alpha = 1$ and $\bar{h}_{1\sigma} \neq 0$ only if $\sigma = 1$, and similarly to $\mu = 2$. We solve this by assuming that the change in S^μ is linear in $h_{\mu\nu}$. The simplest non trivial guess would then be (see ref. [1] p.297)

$$\begin{cases} S^1 = (1 + h_+\Re(e^{ik_\sigma x^\sigma}))S^1(0) = [1 + h_+\cos(k_\sigma x^\sigma)]S^1(0) \\ S^2 = (1 - h_+\Re(e^{ik_\sigma x^\sigma}))S^2(0) = [1 - h_+\cos(k_\sigma x^\sigma)]S^2(0). \end{cases} \tag{3.38}$$

To first order this really is a solution:

$$\frac{\partial^2}{\partial t^2}S^1 = \frac{1}{2}S^1(0)\frac{\partial^2}{\partial t^2}(h_+e^{ik_\sigma x^\sigma}) = \frac{1}{2}S^1\frac{\partial^2}{\partial t^2}(h_+e^{ik_\sigma x^\sigma}) + \mathcal{O}(h_{\mu\nu})^2. \tag{3.39}$$

From our solution we can see that a purely +-polarized gravitational wave causes particles initially separated in x-direction to oscillate in x-direction, similarly to particles

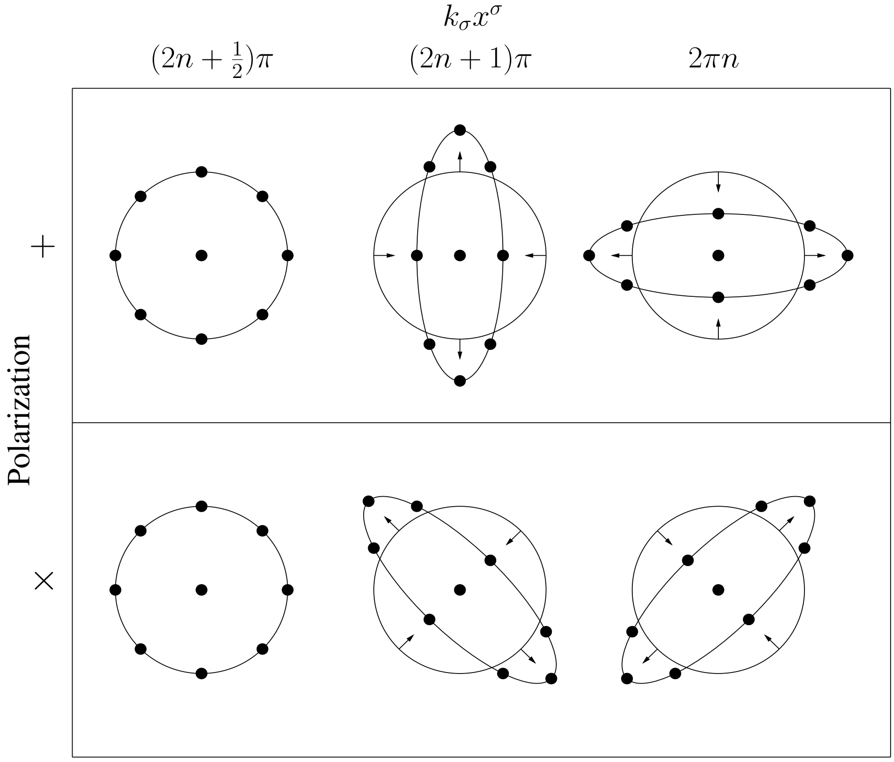

Figure 1: Effects of +- and ×-polarized waves on a ring of particles at rest

separated in y-direction will oscillate in y-direction, though there is a phase shift of π between x- and y-directional oscillation. This can be demonstrated by studying a ring of particles at rest, see [4] and [6], the idea is illustrated in figure 1. For a purely $\times$-*polarized* wave we set $\mathbf{h}_+ = \mathbf{0}$ and then the geodesic deviation equation (3.36) gives

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}S^1 = \dfrac{1}{2}S^2\dfrac{\partial^2}{\partial t^2}(h_\times e^{ik_\sigma x^\sigma}) \\ \dfrac{\partial^2}{\partial t^2}S^2 = \dfrac{1}{2}S^1\dfrac{\partial^2}{\partial t^2}(h_\times e^{ik_\sigma x^\sigma}), \end{cases} \tag{3.40}$$

and we can solve this in a similar manner as we did with +-polarized wave, yielding

$$\begin{cases} S^1 = S^1(0) + h_\times \cos(k_\sigma x^\sigma)S^2(0) \\ S^2 = S^2(0) + h_\times \cos(k_\sigma x^\sigma)S^1(0). \end{cases} \tag{3.41}$$

This solution gives slightly different behaviour as you can see, since the objects initially separated in the x-direction oscillate in the y-direction and so on, see figure 1.

4 Production of gravitational waves

Here we follow the footsteps of Carrol [1] and Wald [5] and try to understand which kind of sources could produce gravitational radiation and what is the relation between the wave and the energy-momentum of the source.

4.1 Using the Green's function

To see how gravitational waves could be produced in nature we have to study the full wave equation (3.5) with nonzero stress-energy tensor, presented in the Lorentz gauge:

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{4.1}$$

A standard method for solving the wave equation with a source is to use the Green's function

$$\Box_x G(x^\alpha - y^\alpha) = \delta^{(4)}(x^\alpha - y^\alpha), \tag{4.2}$$

where $\Box_x = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}$ and coordinates x^α represent the "point of observation" and y^α represent the "point of emission". Then the general solution can be written as

$$\bar{h}_{\mu\nu}(x^\alpha) = -16\pi \int G(x^\alpha - y^\alpha) T_{\mu\nu}(y^\alpha)\mathrm{d}^4 y, \tag{4.3}$$

and the problem boils down to solving the Green's function. Clearly (4.3) is a solution as one can see after substituting it back to (4.1). The background metric is flat so there is no need for a factor $\sqrt{-g}$ in the integral. We are interested in wave solutions $\bar{h}_{\mu\nu}(x^\alpha)$ generated by a source in a past event y^α. This kind of a solution is given by the retarded Green's function, see appendix A:

$$G(x^\alpha - y^\alpha) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\delta\left[|\mathbf{x}-\mathbf{y}| - (x^0 - y^0)\right]\theta(x^0 - y^0), \tag{4.4}$$

where $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{y} = (y^1, y^2, y^3)$ are the spatial 3-vectors, the norm $|\mathbf{x}-\mathbf{y}|$ is just the euclidean norm and θ is the Heaviside function

$$\theta(x^0 - y^0) = \begin{cases} 1, & \text{when } x^0 > y^0 \\ 0, & \text{elsewhere.} \end{cases} \tag{4.5}$$

Plugging (4.4) into (4.3) yields

$$\begin{aligned} \bar{h}_{\mu\nu}(x^0, \mathbf{x}) &= 16\pi \int \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\delta\left[|\mathbf{x}-\mathbf{y}| - (x^0 - y^0)\right]\theta(x^0 - y^0) T_{\mu\nu}(y^\alpha)\mathrm{d}^4 y \\ &= 4\int \frac{1}{|\mathbf{x}-\mathbf{y}|} T_{\mu\nu}(x^0 - |\mathbf{x}-\mathbf{y}|, \mathbf{y})\theta(|\mathbf{x}-\mathbf{y}|)\mathrm{d}^3 y \end{aligned}$$

where we have done the y^0 integration by using the delta function. Let's denote $x^0 \equiv t$, and since $|\mathbf{x}-\mathbf{y}| \geq 0$, we have $\theta(|\mathbf{x}-\mathbf{y}|) = 1$, so that we get

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4\int \frac{1}{|\mathbf{x}-\mathbf{y}|} T_{\mu\nu}(t - |\mathbf{x}-\mathbf{y}|, \mathbf{y})\mathrm{d}^3 y. \tag{4.6}$$

This is a general solution which we start to unravel in the next section after performing a Fourier transformation and some approximations. But even now one can intepret the solution: the perturbation in spacetime point $(t, \mathbf{x})$ is an outcome of the sum of energy-momentum in the past light-cone at a point $(t - |\mathbf{x}-\mathbf{y}|, \mathbf{y})$.

4.2 Quadrupole formula

The Fourier transformation of a function $f(t, \mathbf{x})$ with respect to t is defined as

$$\tilde{f}(\omega,\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} f(t, \mathbf{x}) \mathrm{d}t, \tag{4.7}$$

and the inverse transformation is

$$f(t, \mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int e^{-i\omega t} \tilde{f}(\omega,\mathbf{x}) \mathrm{d}\omega. \tag{4.8}$$

Now let's perform a Fourier transformation to our general solution (4.6) and obtain

$$\begin{aligned} \tilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x}) &= \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} \bar{h}_{\mu\nu}(t, \mathbf{x}) \mathrm{d}t \\ &= \frac{4}{\sqrt{2\pi}} \iint \mathrm{d}t\, \mathrm{d}^3 y\, e^{i\omega t} \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|}. \end{aligned} \tag{4.9}$$

If we make a change of variables $t_r = t - |\mathbf{x} - \mathbf{y}|$ (t_r is called the *retarded time*), we get

$$\begin{aligned} \tilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x}) &= \frac{4}{\sqrt{2\pi}} \iint \mathrm{d}t_r\, \mathrm{d}^3 y\, e^{i\omega t_r} e^{i\omega|\mathbf{x}-\mathbf{y}|} \frac{T_{\mu\nu}(t_r, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \\ &= 4 \int \mathrm{d}^3 y\, e^{i\omega|\mathbf{x}-\mathbf{y}|} \frac{\tilde{T}_{\mu\nu}(\omega, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|}, \end{aligned} \tag{4.10}$$

where the second line is just the definition of Fourier transformation. The Fourier transformation of the Lorentz condition (3.4) gives now

$$\begin{aligned} \tfrac{1}{\sqrt{2\pi}} \textstyle\int e^{i\omega t} \partial_\mu \bar{h}^{\mu\nu}(t, \mathbf{x}) \mathrm{d}t &= 0 \\ \Rightarrow \quad \tfrac{1}{\sqrt{2\pi}} \textstyle\int e^{i\omega t} \partial_0 \bar{h}^{0\nu}(t, \mathbf{x}) \mathrm{d}t &= \tfrac{1}{\sqrt{2\pi}} \textstyle\int e^{i\omega t} \partial_i \bar{h}^{i\nu}(t, \mathbf{x}) \mathrm{d}t \\ i\omega \tilde{\bar{h}}^{0\nu}(\omega, \mathbf{x}) &= \partial_i \tilde{\bar{h}}^{i\nu}(\omega, \mathbf{x}), \end{aligned} \tag{4.11}$$

where in the last line we have just used the property of the Fourier transform for a derivative of a function, which is easy to show by partial integration. Now that we have the identity above, we only need to consider the space-like components of (4.10) and use (4.11) to obtain $\tilde{\bar{h}}^{0\nu}$. But before we move on let's make some facilitating assumptions of the physical situation:

- The source is isolated and far away, i.e. the source is located in a constant distance $|\mathbf{x} - \mathbf{y}| \equiv R$ away from the observer and $r \ll R$, where r is the diameter of the source.

- The frequency of the emitted wave is small, i.e. $\omega = \frac{c}{\lambda_{\mathrm{GW}}} \sim \frac{v}{r} \ll 1$, where λ_{GW} is the wavelength of a gravitational wave and v is the the speed of the deformations (motion of different parts) of the source. This is true if the internal motions of the source are slow compared to the speed of light, i.e. $v/c \ll 1$. This is of course equivalent to saying that the wavelength of an emitted wave is much larger than the diameter of the source: $\lambda_{\mathrm{GW}} \sim \frac{c}{v} r \gg r$.

Under these assumptions we can write $\frac{e^{i\omega|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} = \frac{e^{i\omega R}}{R}$ so that we have

$$\tilde{\bar{h}}_{ij}(\omega, R) = 4\frac{e^{i\omega R}}{R}\int \mathrm{d}^3y\, \tilde{T}_{ij}(\omega, \mathbf{y}). \tag{4.12}$$

Let's have a look at the integral. By using the product rule we get that

$$\begin{aligned}\int \mathrm{d}^3y\, \partial_k(y^i\tilde{T}^{kj}) &= \int \mathrm{d}^3y\, \partial_k(y^i)\tilde{T}^{kj} + \int \mathrm{d}^3y\, y^i\partial_k(\tilde{T}^{kj}), \\ &= \int \mathrm{d}^3y\, \tilde{T}^{ij} + \int \mathrm{d}^3y\, y^i\partial_k(\tilde{T}^{kj}),\end{aligned} \tag{4.13}$$

where the left-hand side is a surface integral, which vanishes by Stokes' theorem. Thus we have

$$\int \mathrm{d}^3y\, \tilde{T}^{ij} = -\int \mathrm{d}^3y\, y^i\partial_k(\tilde{T}^{kj}). \tag{4.14}$$

Similarly as we did in (4.11), we can find that in Fourier-space $\partial_\alpha T^{\alpha\nu}(t, \mathbf{x}) = 0$ becomes

$$\partial_k\tilde{T}^{k\nu} = i\omega\tilde{T}^{0\nu}, \tag{4.15}$$

and substituting this to the (ij)-component version of (4.14) we get

$$\begin{aligned}\int \mathrm{d}^3y\, \tilde{T}^{ij} &= i\omega\int \mathrm{d}^3y\, y^i\tilde{T}^{0j} \\ &= \frac{1}{2}i\omega\int (y^i\tilde{T}^{0j} + y^j\tilde{T}^{0i})\mathrm{d}^3y\,.\end{aligned} \tag{4.16}$$

Now if use again the product rule we get that

$$\int \mathrm{d}^3y\, \partial_l(y^iy^j\tilde{T}^{0l}) = \int \mathrm{d}^3y\, (y^j\tilde{T}^{0j} + y^i\tilde{T}^{0j} + y^iy^j\underbrace{\partial_l(\tilde{T}^{0l})}_{=i\omega\tilde{T}^{00}}), \tag{4.17}$$

where in the last term we have again used (4.15), and where the left-hand side again vanishes due to the Stokes' theorem. Finally, after substitution we get

$$\int \mathrm{d}^3y\, \tilde{T}^{ij} = \frac{-\omega^2}{2}\int \mathrm{d}^3y\, y^iy^j\tilde{T}^{00}. \tag{4.18}$$

Next we define the *quadrupole moment tensor*[2]

$$I^{ij}(\omega) \equiv \int \mathrm{d}^3y\, y^iy^jT^{00}(\omega, \mathbf{y}), \tag{4.19}$$

where T^{00} is the total energy density, so that in Fourier space we have

$$\tilde{\bar{h}}_{ij}(\omega, R) = -2\omega^2\frac{e^{i\omega R}}{R}\int \mathrm{d}^3y\, y^iy^j\tilde{T}^{00}(\omega, \mathbf{y}) = -2\omega^2\frac{e^{i\omega R}}{R}\tilde{I}^{ij}(\omega). \tag{4.20}$$

[2]In contrast, the dominant source of electromagnetic radiation comes from the changing electric *dipole*-moment. For discussion of the term *quadrupole* and the quadrupole nature of gravitational waves, see ref. [6] page 975.

Taking the inverse Fourier transformation yields

$$\begin{aligned}\bar{h}_{ij}(t,R) &= \frac{-2}{R}\int \mathrm{d}\omega\,\omega^2 \underbrace{e^{-i\omega t}e^{i\omega R}}_{=e^{-i\omega t_R}}\tilde{I}_{ij}(\omega) \\ &= \frac{-2}{R}\frac{d^2 I_{ij}}{dt^2}\Big|_{t_R},\end{aligned} \tag{4.21}$$

where the derivative of quadrupole moment is evaluated at the retarded time $t_R = t - R$.

4.3 Gravitational radiation of a simple binary system

Birkhoff's theorem, ref. [1] p.197, states that every spherically symmetric solution to Einstein's equation is static and asymptotically flat, and the exterior solution where the stress-energy tensor vanishes is the Schwarzschild metric. The fact that the solution is necessarily static can be intepreted so that *spherically symmetric sources do not emit gravitational radiation*, even if the source itself is radially pulsating (for example a collapsing star).

In order to take some grasp of the emission of gravitational radiation we consider a simple example of two stars, A and B, with masses M, orbiting each other in the xy-plane. This kind of system is called a binary star. Here we are going to cheat a little, since we treat the motion of the stars to be circular as in the Newtonian theory, allthough these kind of orbits clearly are not geodesics in the flat background spacetime. If we take the stars to be at distance R apart of their common center of mass and take the inward pointing gravitational force to be equal to the centrifugal force, we get

$$\frac{M^2}{(2R)^2} = \frac{Mv^2}{R}, \tag{4.22}$$

where v is the tangential speed. From this we get

$$v = \sqrt{\frac{M}{4R}}, \tag{4.23}$$

and since the time taken to travel one orbit is

$$T = \frac{2\pi R}{v}, \tag{4.24}$$

we get the angular frequency to be

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{M}{4R^3}}. \tag{4.25}$$

Now the paths of the stars can be written in terms of the frequency:

$$\begin{cases} A: & x_A = R\cos\omega t, \quad\;\;\, y_A = R\sin\omega t \\ B: & x_B = -R\cos\omega t, \quad y_B = -R\sin\omega t. \end{cases} \tag{4.26}$$

The energy-density of the system is now (if we assume the stars to be point-like)

$$\begin{aligned}T^{00}(t,\mathbf{x}) =& M\delta(z)\Big[\delta(x - R\cos\omega t)\delta(y - R\sin\omega t) \\ &+ \delta(x + R\cos\omega t)\delta(y + R\sin\omega t)\Big],\end{aligned} \tag{4.27}$$

so the quadrupole moment (4.19) is straightforward to calculate with the delta functions. For example the (11)-component is

$$\begin{aligned} I^{11} &= \int x^2 T^{00}(t, \mathbf{x}) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\ &= 2MR^2 \cos^2 \omega t = MR^2(1 + \cos 2\omega t), \end{aligned} \tag{4.28}$$

and similarly the other components are

$$\begin{aligned} I^{22} = &\ 2MR^2 \sin^2 \omega t = MR^2(1 - \cos 2\omega t) \\ I^{21} = I^{12} = &\ 2MR^2 \cos \omega t \sin \omega t = MR^2 \sin 2\omega t \\ I^{i3} = &\ 0. \end{aligned} \tag{4.29}$$

Straightforwardly we can calculate the second time derivatives of the nonzero components of the quadrupole moment and get

$$\ddot{I}^{11} = -4MR^2\omega^2 \cos 2\omega t \tag{4.30}$$

$$\ddot{I}^{22} = 4MR^2\omega^2 \cos 2\omega t \tag{4.31}$$

$$\ddot{I}^{12} = \ddot{I}^{21} = -4MR^2\omega^2 \sin 2\omega t, \tag{4.32}$$

$$\tag{4.33}$$

so that we can write the spatial components of our gravitational wave in this simple example as a matrix

$$\bar{h}_{\mu\nu}(t) = \omega^2 R^2 \frac{8M}{r} \begin{pmatrix} -\cos 2\omega t_R & -\sin 2\omega t_R & 0 \\ -\sin 2\omega t_R & \cos 2\omega t_R & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.34}$$

where r represents the distance from the observer to the binary star and the timelike components of the wave can be derived from the Lorentz gauge condition $\partial_\mu \bar{h}^{\mu\nu} = 0$. The wave has frequency 2ω, which is not surprising since this is the frequency with which the binary system returns to identical configuration.

5 Energy loss by a radiating system

It is well known that waves familiar from our everyday life, e.g. electromagnetic waves or waves in water, are able to carry away energy and momentum off the source. Thus we would assume that also sources radiating gravitational waves would also lose their energy, and for example in the binary star example the energy loss would be detected as decreasing angular momentum. Decreasing angular momentum would be observed as a decaying orbital period and lead the stars spiraling towards each other. This is of course what has been observed and that is at least an indirect evidence of the existence of gravitational waves and a test of the accuracy of General Relativity and other theories of gravitation [7].

Before we start practical calculation, it is appropriate to think about some philosophical issues first. First of all, what we would like to have is some kind of stress-energy tensor for the perturbation $h_{\mu\nu}$ but in linearized gravity this seems to be difficult. We don't yet have tools to tell what how much energy the wave is carrying in the vacuum since the stress-energy tensor $T^{\mu\nu}$ vanishes. So, in order to have some information of the energy of the wave, we probably need to take into account some higher-order terms in the Einstein equation, see ref. [1]. Secondly, to speak on behalf the higher-order terms of the perturbation, what we have seen in other fields of physics is that the stress-energy tensor of some field strength, for example the of electromagnetic tensor $F^{\mu\nu}$, is always quadratic (second order in the field) or higher:

$$T^{\mu\nu}_{EM} = \frac{1}{\mu_0}\left(F^{\mu\alpha}g_{\alpha\beta}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}F_{\delta\nu}F^{\delta\nu}\right). \tag{5.1}$$

So it seems that our linearized theory is not enough for us to build a stress-energy tensor for the pertubation $h_{\mu\nu}$.

5.1 Second order Einstein equations

Justified by the discussion above, our next move is to study the Einstein's vacuum equation to second order in $h_{\mu\nu}$. We interpret this such that the first order perturbations propagate freely in the vacuum and in the process act as a source for the second order perturbations. First we expand the metric tensor and Ricci tensor to second order. By second order we mean that a quantity is of the order $[h_{\mu\nu}]^2$, keeping in mind that $h_{\mu\nu} \ll 1$. We denote by a superscript $^{(1)}$ the terms that are first order in perturbation, i.e. $h^{(1)}_{\mu\nu}$ and $R^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu}[h^{(1)}]$, and by a superscript $^{(2)}$ the second order terms, these are $h^{(2)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$, so that we can write

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} \tag{5.2}$$

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}. \tag{5.3}$$

Here we work in vacuum, so that

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta^{\alpha\beta}R_{\alpha\beta}\eta_{\mu\nu} = 0, \tag{5.4}$$

which implies that $R_{\mu\nu} = 0$. The zeroth-order Ricci tensor is simply $R^{(0)}_{\mu\nu} = 0$ since the background metric is flat. The first order vacuum equation is

$$R^{(1)}_{\mu\nu}[h^{(1)}] = 0, \tag{5.5}$$

as it is in linearized theory. This equation determines the 'envelope' wave, the one which we have been considering in the previous sections. The second order perturbation is determined by the equation[3]

$$R^{(2)}_{\mu\nu} = R^{(1)}_{\mu\nu}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}] = 0. \tag{5.6}$$

[3]Here it is sufficient to emphasize that $h^{(1)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$ are totally different entities, only the order of magnitude of $(h^{(1)}_{\mu\nu})^2$ is the same as $h^{(2)}_{\mu\nu}$. Hence we can write $R^{(2)}_{\mu\nu} = R^{(1)}_{\mu\nu}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}]$, where $R^{(1)}_{\mu\nu}[h^{(2)}]$ is the Ricci tensor calculated to first order in $h^{(2)}_{\mu\nu}$ (thus it is of second order) and $R^{(2)}_{\mu\nu}[h^{(1)}]$ is only the quadratic terms of the Ricci tensor calculated to second order in $h^{(1)}_{\mu\nu}$ (the whole tensor minus the linear terms).

Now we can write the vacuum equation (5.4) equivalently as

$$R^{(2)}_{\mu\nu} - \frac{1}{2}\eta^{\alpha\beta}R^{(2)}_{\alpha\beta}\eta_{\mu\nu} = 0, \tag{5.7}$$

from which we get

$$R^{(1)}_{\mu\nu}[h^{(2)}] - \frac{1}{2}\eta^{\alpha\beta}R^{(1)}_{\alpha\beta}[h^{(2)}]\eta_{\mu\nu} = 8\pi \underbrace{\left[-\frac{1}{8\pi}\left(R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta^{\alpha\beta}R^{(2)}_{\alpha\beta}[h^{(1)}]\eta_{\mu\nu}\right)\right]}_{\doteq t_{\mu\nu}}, \tag{5.8}$$

where we have done our trick: the above equation now looks as an Einstein's equation in presence of a nonzero 'stress-energy tensor' $t_{\mu\nu}$ which we defined in terms of the first order perturbation. We call $t_{\mu\nu}$ as an *effective stress energy tensor* for a gravitational wave. We can justify this definition as follows:

- $t_{\mu\nu}$ is symmetric since the perturbation is symmetric and the Ricci tensor constructed of it is symmetric
- $t_{\mu\nu}$ is purely quadratic in $h_{\mu\nu}$ as we suggested that it should be in the introduction
- $\partial_\mu t^{\mu\nu} = 0$, i.e. the flat space-version of the energy-momentum conservation is valid since $\partial^\mu G^{(1)}_{\mu\nu}[h^{(2)}] = 0$.

As stated in Carrol (2003), $t_{\mu\nu}$ is not a tensor in the full nonlinear theory of General Relativity. It is not even gauge invariant in our gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + 2\epsilon\partial_{(\mu}\xi_{\nu)}$ discussed in section 2.2. Also, it is poorly defined to speak of gravitational energy and momentum of a specific point in spacetime. This is a consequence of the ability to choose locally the coordinates such that the Christoffel symbols ('the gravitational fields') vanish, and thus vanishes the 'local gravitational energy and momentum', so we really can't tell in which part of the wave the energy is carried, see ref. [6] p.467.

In order to get some grasp of the energy carried by a gravitational wave, we need to integrate $t_{\mu\nu}$ (we omit here the fact that $t_{\mu\nu}$ is non-tensorial) over some region of spacetime which contains several wavelenghts of gravitational radiation, and take the average. We denote this operation by angle brackets $\langle \cdots \rangle$. Since the brackets involve integration over a spacetime volume, all the terms containing derivatives vanish due to Stoke's theorem: $\langle \partial_\mu(X)\rangle = 0$. This provides also a way to integrate by parts inside the brackets: $\langle(\partial_\mu A)B\rangle = -\langle A(\partial_\mu B)\rangle$. We use these averaging brackets to calculate the stress-energy tensor defined in (5.8) and hope to get a gauge invariant result. We make here an assumption that the stress-energy of an average wave is the average of the effects of a second order perturbations over a sufficiently large region of spacetime

$$t_{\mu\nu}[\langle h^{(1)}\rangle] \equiv \langle G^{(2)}_{\mu\nu}[h^{(1)}]\rangle. \tag{5.9}$$

From here on, to keep expressions simpler, we drop the superscripts since we are anyway only considering the first-order perturbation. The quadratic part of the Ricci tensor in terms of the first-order perturbations is now

$$\begin{aligned} R^{(2)}_{\mu\nu} &= \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} + \frac{1}{4}(\partial_\mu h_{\rho\sigma})\partial_\nu h^{\rho\sigma} + (\partial^\sigma h^\rho{}_\nu)\partial_{[\sigma}h_{\rho]\mu} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} \\ &\quad + \frac{1}{2}\partial_\sigma(h^{\rho\sigma}\partial_\rho h_{\mu\nu}) - \frac{1}{4}(\partial_\rho h_{\mu\nu})\partial^\rho h - (\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h)\partial_{(\mu}h_{\nu)\rho}, \end{aligned} \tag{5.10}$$

and if one approves the efforts of Carrol (2003) and Misner *et al* (1973) in calculating the energy-momentum pseudotensor for (5.10), one gets

$$t_{\mu\nu} = \frac{1}{32\pi}\Big\langle(\partial_\mu h_{\rho\sigma})(\partial_\nu h^{\rho\sigma}) - \frac{1}{2}(\partial_\mu h)(\partial_\nu h) - (\partial_\rho h^{\rho\sigma})(\partial_\mu h_{\nu\sigma}) - (\partial_\rho h^{\rho\sigma})(\partial_\nu h_{\mu\sigma})\Big\rangle, \quad (5.11)$$

which really is gauge invariant. When moving into transverse-traceless gauge (denote this by TT) we have $h^{\mathrm{TT}}_{0\nu} = 0$, $h^{\mathrm{TT}} = 0$ and $\partial_\mu h^{\mu\nu}_{\mathrm{TT}} = 0$, so that we get

$$t_{\mu\nu} = \frac{1}{32\pi}\Big\langle(\partial_\mu h^{\mathrm{TT}}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{\mathrm{TT}})\Big\rangle. \quad (5.12)$$

Notice that since $h^{\mathrm{TT}}_{\mu\nu}$ is purely spatial, so is $t_{\mu\nu}$.

5.2 Power radiated by a slow-motion source

Now that we have an expression for the stress-energy carried by a gravitational wave, we can start contemplating the energy loss of a source due to gravitational radiation. We consider a case in which the gravitational waves are generated by a source according to the quadrupole formula (4.21). What we would like to have is the power output of a source so that we could calculate the change in energy. The $t^{0\mu}$ component of a stress-energy tensor represents the energy flowing through a surface orthogonal to x^μ per unit time, i.e. (remembering that really we need to take the average over several wavelenghts)

$$\frac{\mathrm{d}E}{\mathrm{d}S^\mu \mathrm{d}t} = t_{0\mu}. \quad (5.13)$$

We can define the power P (or luminosity) as the energy flux integrated over a 2-sphere S^2 at distance r as

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \int_{S^2} \frac{\mathrm{d}E}{\mathrm{d}S^\mu \mathrm{d}t}\mathrm{d}S^\mu = \int_{S^2} t_{0\mu}\mathrm{d}S^\mu = \int_{S^2} t_{0\mu} n^\mu r^2 \mathrm{d}\Omega\,, \quad (5.14)$$

where n^μ is the unit vector normal to S^2. We would like to apply here (5.12) and there again apply the quadrupole formula (4.21). The quadrupole formula in transverse-traceless gauge implies that we need to somehow consider only the transverse-traceless part of the quadrupole moment. Since we are looking at the power output through a sphere surrounding the source, we want to project out the components of I_{ij} that are tangent to the sphere and consider only those. Also we need to eliminate the trace of I_{ij}. Choosing the TT-gauge is of course justified since we are interested of the energy carried by a gravitational wave in vacuum far away of the source. We introduce a spatial projection tensor

$$P_{ij} \equiv \delta_{ij} - n_i n_j, \quad (5.15)$$

which projects tensor components into a surface orthogonal to the unit vector n^i. It is a projection since

$$P_{ij}P_{jk} = \delta_{ik} - n_i n_k - n_i n_k + n_i \underbrace{n_j n_j}_{=1} n_k = P_{ik}, \quad (5.16)$$

and it is transverse to the normal vector:

$$n^i P_{ij} = 0. \tag{5.17}$$

From this we can construct a transverse-traceless projector

$$P_{ijkl} \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \tag{5.18}$$

which can be used to make any symmetric second rank tensor to be transverse and traceless. It has the following properties:

- it is a projection since $P_{ijkl}P^{klmn} = P_{ij}{}^{mn}$,
- it is transverse since $n^i P_{ijkl} = n^j P_{ijkl} = n^k P_{ijkl} = n^l P_{ijkl} = 0$,
- it is traceless since $\delta^{ij}P_{ijkl} = \delta^{kl}P_{ijkl} = 0$.

We can furthermore introduce the *reduced quadrupole moment*

$$J_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}I_{kl}, \tag{5.19}$$

which is simply the traceless part of I_{ij}. Thus we have

$$\begin{cases} P_{ijkl}h^{kl} = P_{ijkl}\bar{h}^{kl} = h_{ij}^{\mathrm{TT}} \\ P_{ijkl}J^{kl} = P_{ijkl}I^{kl} = I_{ij}^{\mathrm{TT}}, \end{cases} \tag{5.20}$$

so that the quadrupole formula reads

$$h_{ij}^{\mathrm{TT}} = \bar{h}_{ij}^{\mathrm{TT}} = \frac{2}{r}\frac{d^2 J_{ij}^{\mathrm{TT}}}{dt^2}\bigg|_{t_R}. \tag{5.21}$$

Henceforth we leave the evaluation at $t_R = t - r$ implicit.

Now that we have the transverse-traceless –problem solved, we can jump back to the energy/power issue. First we are interested in $t_{0\mu}n^\mu$ which appears in the expression of the power. In spherical coordinates (t,r,θ,ϕ) we have

$$\mathbf{n} = (0,1,0,0) \tag{5.22}$$

i.e. it is the normal vector to a 2-sphere surrounding the source (and thus points in the direction of the wave propagation). Thus we have $t_{0\mu}n^\mu = t_{0r}$ and so, looking at the definition of $t_{\mu\nu}$, we need the following derivatives

$$\begin{cases} \partial_0 h_{ij}^{\mathrm{TT}} = \dfrac{2}{r}\dfrac{\mathrm{d}^3 J_{ij}^{\mathrm{TT}}}{\mathrm{d}t^3} \\ \partial_r h_{ij}^{\mathrm{TT}} = -\dfrac{2}{r}\dfrac{\mathrm{d}^3 J_{ij}^{\mathrm{TT}}}{\mathrm{d}t^3} - \dfrac{2}{r^2}\dfrac{\mathrm{d}^2 J_{ij}^{\mathrm{TT}}}{\mathrm{d}t^2} \approx -\dfrac{2}{r}\dfrac{\mathrm{d}^3 J_{ij}^{\mathrm{TT}}}{\mathrm{d}t^3}, \end{cases} \tag{5.23}$$

where we drop the $\mathcal{O}(\frac{1}{r^2})$ term since we are interest in the limit where $r \to \infty$. Thus we can write the meaningful energy-flux as

$$t_{0r} = \frac{1}{32\pi}\Big\langle(\partial_0 h_{ij}^{\mathrm{TT}})(\partial_r h^{ij}_{\mathrm{TT}})\Big\rangle = -\frac{1}{8\pi r^2}\Big\langle\Big(\frac{\mathrm{d}^3 J_{ij}^{\mathrm{TT}}}{\mathrm{d}t^3}\Big)\Big(\frac{\mathrm{d}^3 J^{ij}_{\mathrm{TT}}}{\mathrm{d}t^3}\Big)\Big\rangle. \tag{5.24}$$

Now we transform the above formula into a nontransverse-traceless form. For a tensor X_{ij} it is straightforward to show that

$$X_{ij}^{\mathrm{TT}} X_{\mathrm{TT}}^{ij} = X_{ij}X^{ij} + XX^{ij}n_i n_j - 2X_i{}^j X^{ik} n_j n_k + \frac{1}{2}X^{ij}X^{kl}n_i n_j n_k n_l - \frac{1}{2}X^2, \quad (5.25)$$

where $X = \delta^{ij}X_{ij}$ is the trace which vanishes for the reduced quadrupole moment, so that when applied to (5.24) we get

$$t_{0r} = -\frac{1}{8\pi r^2}\Big\langle \frac{\mathrm{d}^3 J_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 J^{ij}}{\mathrm{d}t^3} - 2\frac{\mathrm{d}^3 J_i{}^j}{\mathrm{d}t^3}\frac{\mathrm{d}^3 J^{ik}}{\mathrm{d}t^3}n_j n_k + \frac{1}{2}\frac{\mathrm{d}^3 J^{kl}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 J^{ij}}{\mathrm{d}t^3}n_i n_j n_k n_l\Big\rangle. \quad (5.26)$$

The power is thus

$$P = -\frac{1}{8\pi}\int_{S^2}\Big\langle \frac{\mathrm{d}^3 J_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 J^{ij}}{\mathrm{d}t^3} - 2\frac{\mathrm{d}^3 J_i{}^j}{\mathrm{d}t^3}\frac{\mathrm{d}^3 J^{ik}}{\mathrm{d}t^3}n_j n_k + \frac{1}{2}\frac{\mathrm{d}^3 J^{kl}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 J^{ij}}{\mathrm{d}t^3}n_i n_j n_k n_l\Big\rangle \mathrm{d}\Omega\,, \quad (5.27)$$

and since the quadrupole moments are defined as integrals over whole space (see (4.19)) they don't depend on angular coordinates θ, ϕ and thus they can be pulled out of the integrals. Since $n^i = x^i/r$ we are left with integrals

$$\begin{aligned} \int \mathrm{d}\Omega &= 4\pi \\ \int n_i n_j \mathrm{d}\Omega &= \frac{4\pi}{3}\delta_{ij} \\ \int n_i n_j n_k n_l \mathrm{d}\Omega &= \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \end{aligned} \quad (5.28)$$

which are easy to verify by using spherical coordinates

$$\begin{cases} x^1 &= r\cos\theta\sin\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\cos\phi \\ \mathrm{d}\Omega &= \sin\phi\, \mathrm{d}\theta\, \mathrm{d}\phi\,. \end{cases} \quad (5.29)$$

Finally after all calculations we have a nice compact expression for the power radiated by a slow-motion source far away

$$P = -\frac{1}{5}\Big\langle \frac{\mathrm{d}^3 J_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 J^{ij}}{\mathrm{d}t^3}\Big\rangle. \quad (5.30)$$

In next section we try to apply this.

5.3 Binary star revisited

In section (4.3) we calculated the quadrupole moment for a simple binary star. The reduced quadrupole moment is thus straightforward to calculate. For example

$$\begin{aligned} J_{11} &= I_{11} - \frac{1}{3}(I_{11} + I_{22} + I_{33}) \\ &= \frac{MR^2}{3}\left(1 + 3\cos 2\omega t_R\right), \end{aligned} \quad (5.31)$$

and similarly the other components, so that the reduced quadrupole moment can be written

$$J_{\mu\nu} = \frac{MR^2}{3} \begin{pmatrix} 1 + 3\cos 2\omega t_R & 3\sin 2\omega t_R & 0 \\ 3\sin 2\omega t_R & 1 - 3\cos 2\omega t_R & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{5.32}$$

and furthermore one can calculate the third time derivative to be

$$\frac{\mathrm{d}^3 J_{\mu\nu}}{\mathrm{d}t^3} = 8MR^2\omega^3 \begin{pmatrix} \sin 2\omega t_R & -\cos 2\omega t_R & 0 \\ -\cos 2\omega t_R & -\sin 2\omega t_R & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.33}$$

The power is thus simply

$$\begin{aligned} P &= -\frac{64}{5} M^2 R^4 \omega^6 \Big\langle \sin^2 2\omega t_R + \cos^2 2\omega t_R + \sin^2 2\omega t_R + \cos^2 2\omega t_R \Big\rangle \\ &= -\frac{128}{5} M^2 R^4 \omega^6 \end{aligned} \tag{5.34}$$

Now we would like to get some feeling of how we could compare the result above in actual measurements. Since the system loses energy in gravitational radiation, it would be most likely that the orbital period of the binary companions decreases in time. We start going towards this and first modify the expression of the power by using equation (4.25) and write $R^3 = \frac{M}{4\omega^2}$ in the second line to get

$$\begin{aligned} P &= -\frac{128}{5} \frac{M^3}{4\omega^2} \left(\frac{M}{4\omega^2}\right)^{1/3} \omega^6 \\ &= \frac{-128}{20\sqrt[3]{4}} (M\omega)^{10/3} \\ &\approx -4.0 (M\omega)^{10/3}. \end{aligned} \tag{5.35}$$

Next we calculate the Newtonian energy of the system under consideration, and find it to be

$$\begin{aligned} E &= \underbrace{\frac{1}{2} M\omega^2 R^2 + \frac{1}{2} M\omega^2 R^2}_{\text{kinetic energy}} \quad \underbrace{-\frac{M^2}{2R}}_{\text{potential energy}} \\ &= \frac{M}{R}\left(\omega^2 R^3 - \frac{1}{2}M\right) \\ &= \frac{M}{R}\left(\omega^2 \frac{M}{4\omega^2} - \frac{1}{2}M\right) \\ &= \frac{-M^2}{4R} \\ &= -\frac{1}{4^{2/3}} M^{5/3} \omega^{2/3} \end{aligned} \tag{5.36}$$

From here we can see clearly, that when the energy decreases, the absolute value $|E|$ gets larger and thus ω must increase. Increasing angular momentum means smaller orbital period, which was exactly what we assumed. Next observe that taking the logarithmic derivative gives some useful identities:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln E\right] = \frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}t}, \tag{5.37}$$

but on the other hand the right-hand side is, by using eq. (5.36),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln E\right] = \frac{2}{3\omega}\frac{\mathrm{d}\omega}{\mathrm{d}t}. \tag{5.38}$$

Furthermore, since $\omega = \frac{2\pi}{T}$, we get

$$\frac{2}{3\omega}\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{2}{3T}\frac{\mathrm{d}T}{\mathrm{d}t}. \tag{5.39}$$

Equating (5.37) with (5.39), and remembering that $\frac{\mathrm{d}E}{\mathrm{d}t} = P$, yields

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{3}{2E}PT = -\frac{3\pi P}{E\omega} \approx -95 \cdot \left(\frac{2\pi M}{T}\right)^{5/3}, \tag{5.40}$$

which is dimensionless in any system of units.

To make some estimate of the order of magnitude, let's take the binary components to have masses $M = 1{,}4M_\odot \approx 2{,}8 \cdot 10^{30}\mathrm{kg}$ (1,4 solar masses) and an orbital period $T = 7\mathrm{h}\,45\mathrm{min}$. In geometrized units these are

$$\begin{cases} M &= \frac{G}{c^2} \cdot 2{,}8 \cdot 10^{30}\mathrm{kg} \approx 2078\mathrm{m} \\ T &= c \cdot (7 \cdot 60 \cdot 60 + 45 \cdot 60)\mathrm{s} \approx 8{,}36 \cdot 10^{12}\mathrm{m}. \end{cases} \tag{5.41}$$

With these values we get

$$\frac{\mathrm{d}T}{\mathrm{d}t} \approx -2{,}0 \cdot 10^{-13}. \tag{5.42}$$

This can be converted to a meaningful quantity:

$$\frac{\mathrm{d}T}{\mathrm{d}t} \approx -2{,}0 \cdot 10^{-13} \cdot \frac{365 \cdot 60 \cdot 60\,\mathrm{s}}{1\,\mathrm{year}} \approx -2{,}7 \cdot 10^{-7}\frac{\mathrm{s}}{\mathrm{year}}. \tag{5.43}$$

The above calculation is of course highly idealized since the orbiting stars rarely have exactly the same mass, nor is their orbit exactly circular but rather elliptic. The measurement of such damping of the period is however possible since there have been observed such binary stars in which the other component is a pulsar, a rotating neutron star that emits electromagnetic radiation which can be detected as extremely accurate periodic pulses. In 1974 Hulse & Taylor discovered such a binary pulsar, named PSR B1913+16, and they were awarded with the Nobel Prize in 1993 of the discover. Over 30 years of measurements done by Weisberg & Taylor have given highly agreeable results with the predictions of General Relativity, see ref. [7]. Weisberg & Taylor measured the orbital derivative to be $\dot{T} = -(2{,}30 \pm 0{,}22) \cdot 10^{-12}$ while the prediction to such system is $\dot{T} = -(2{,}404 \pm 0{,}005) \cdot 10^{-12}$ which is a profound confirmation of the General theory of Relativity.

As a matter of fact, in the binary pulsar PSR B1913+16 the masses of the binary components are approximately those that we used in our example above and also the orbital period is roughly the same. If we would have taken account the highly elliptic orbits of the stars, it would have given us a factor of 12 multiplying our estimate, which corresponds to an eccentricity of $e = 0{,}617$, see ref. [7] and [8]. Our final estimate would then be

$$\frac{\mathrm{d}T}{\mathrm{d}t} \approx -12 \cdot 2{,}0 \cdot 10^{-13} \approx 2{,}4 \cdot 10^{-12}, \tag{5.44}$$

which is already remarkably close to the measured value.

6 Final words

What we have seen is that linearized gravitational waves propagate at the speed of light and they have two independent polarization states. The waves carry energy and this has provided a way to detect them indirectly as a damping of the orbital period in a binary star. Also, we saw that the gravitational waves affect on the relative motion of test masses. This provides a way of detecting the waves directly, but none has yet been reported. The most sensitive ground-based detector LIGO – Laser Interferometer Gravitational-Wave Observatory – in Louisiana US, is now under upgrade after which it be operational by 2014 and it is anticipated to make wave detections a routine occurrence. Direct evidence of the waves could then offer a remarkable opportunity to observe the universe from an entirely new perspective.

A Derivation of retarded Green's function

In this Appendix we show how the explicit form of the retarded Green's function in eq. (4.4) is derived and follow the derivation of ref. [9]. We start from

$$\Box_x G(x^\alpha - y^\alpha) = \delta^{(4)}(x^\alpha - y^\alpha) \tag{A.1}$$

by first performing a Fourier transformation to both sides. The delta function in Fourier space is

$$\delta^{(4)}(x^\alpha - y^\alpha) = \frac{1}{(2\pi)^4}\int e^{ik_\mu(x^\mu - y^\mu)}\,\mathrm{d}^4k \tag{A.2}$$

and respectively

$$G(x^\alpha - y^\alpha) = \frac{1}{(2\pi)^4}\int e^{ik_\mu(x^\mu - y^\mu)}\tilde{G}(k)\,\mathrm{d}^4k. \tag{A.3}$$

Thus we get

$$\begin{aligned}\Box_x G(x^\alpha - y^\alpha) &= -\frac{1}{(2\pi)^4}\int k_\alpha k^\alpha e^{ik_\mu(x^\mu - y^\mu)}\tilde{G}(k)\,\mathrm{d}^4k \\ &\overset{(\mathrm{A.2})}{=} \frac{1}{(2\pi)^4}\int e^{ik_\mu(x^\mu - y^\mu)}\,\mathrm{d}^4k,\end{aligned} \tag{A.4}$$

so equating the integrands yields

$$\tilde{G}(k) = -\frac{1}{k_\alpha k^\alpha} = -\frac{1}{\omega^2 - \mathbf{k}^2}, \tag{A.5}$$

where $k_\alpha = (\omega, \mathbf{k})$. Now, when we substitute (A.5) back to (A.3), we get

$$\begin{aligned}G(x^\alpha - y^\alpha) &= -\frac{1}{(2\pi)^4}\int \frac{e^{ik_\mu(x^\mu - y^\mu)}}{\omega^2 - \mathbf{k}^2}\,\mathrm{d}^3k\mathrm{d}\omega \\ &= -\frac{1}{(2\pi)^4}\int \frac{e^{-i\omega(t-t')}e^{ik(x-y)\cos\theta}}{(\omega - k)(\omega + k)}\,\mathrm{d}^3k\mathrm{d}\omega.\end{aligned} \tag{A.6}$$

where we see that the integrand has poles at $\omega = \pm k$. Let's consider the ω-integral first and denote

$$I \equiv \int f(\omega)\mathrm{d}\omega \equiv \int \frac{e^{-i\omega(t-t')}}{(\omega - k)(\omega + k)}\mathrm{d}\omega \tag{A.7}$$

In order to make some sense of our solution we have to figure out how to deal with the singularities. We can set ω to have also complex values and go round the poles in complex ω-plane by using Cauchy theorem and residue theorem. But we need to figure out how to do so because there are many possibilities. We can start by thinking the physical restricitions, or boundary conditions:

- The solution $G(t, \mathbf{x}, t', \mathbf{y})$ must be such that for $t > t'$ it represent an *outgoing* wave that has been emitted from a spatial point $\mathbf{y}$ at time t' from a source that has been 'turned on' only for an infinitesimal moment.

- Before the moment of emission there is no propagating solution, i.e. $G(t,\mathbf{x},t',\mathbf{y}) = 0$ for $t < t'$.

Accordind to the Cauchy theorem, for a closed contour C which contains no singularites in complex plane, we have

$$\oint_C f(z)\,\mathrm{d}z = 0. \tag{A.8}$$

If the contour however closes in poles of $f(z)$, the residue theorem states that

$$\oint_C f(z) = 2\pi i \sum_i \mathrm{Res} f(z_i), \tag{A.9}$$

where z_i is a pole of $f(z)$. In order to get the poles off from the integration range, we can tilt them such that $k \to k \pm i\epsilon$, and in the end take $\epsilon \to 0$. The question is, should we choose $+i\epsilon$ or $-i\epsilon$ to get our boundary conditions satisfied? Regardless, the poles are displaced either above or below the real axis, so we can use the theorems stated above and choose the integral contour to be the semicircle in either upper or lower half-plane. Let's parametrize the semicircle by $\omega = Re^{i\phi}, R \to \infty, \phi \in [0, \pm\pi]$, the plus or minus sign depending in which half-plane the semicircle lies. This parametrization gives now

$$\mathrm{d}\omega = iRe^{i\phi}\mathrm{d}\phi,$$

so the norm of the integrand is now

$$\begin{aligned}
\Big|\frac{e^{-i\omega(t-t')}}{\omega^2 - (k \pm i\epsilon)^2}\Big| &= \Big|\frac{e^{-iRe^{i\phi}(t-t')}Re^{i\phi}}{R^2 e^{2i\phi} - (k \pm i\epsilon)^2}\Big| \\
&= \Big|\frac{Re^{-iR[\cos\phi + i\sin\phi](t-t')}}{R^2 e^{2i\phi} - (k \pm i\epsilon)^2}\Big| \\
&= \Big|\frac{Re^{-iR(t-t')\cos\phi}e^{R(t-t')\sin\phi}}{R^2 e^{2i\phi} - (k \pm i\epsilon)^2}\Big| \\
&= \Big|\frac{Re^{R(t-t')\sin\phi}}{R^2 e^{2i\phi} - (k \pm i\epsilon)^2}\Big| \\
&\leq \Big|\frac{Re^{R(t-t')\sin\phi}}{|R^2 e^{2i\phi}| - |(k \pm i\epsilon)^2|}\Big| \\
&= \Big|\frac{Re^{R(t-t')\sin\phi}}{R^2 - (k \pm i\epsilon)^2}\Big|
\end{aligned}$$

and since $\sin\phi < 0$ for $\phi \in [0, -\pi]$, the integral along the arc vanishes for $t \geq t'$ when $R \to \infty$. So for $t \geq t'$ we choose the contour to be the semicircle in the lower half plane, but since we want a non-trivial answer for this boundary condition, we need to have the poles *below* the real axis, i.e. choose $k \to k - i\epsilon$. Denote $(k - i\epsilon) = k_\epsilon$ for brevity. Thus we can use the residue theorem. For $t \leq t'$ we close the contour to the upper half-plane and use the Cauchy theorem so that the integral vanishes since the interior of the contour contains no poles. The residues are now

$$\begin{aligned}
\mathrm{Res}[f(\omega = k_\epsilon)] &= \lim_{\omega \to k_\epsilon} (\omega - k_\epsilon) \frac{e^{-i\omega(t-t')}}{(\omega - k_\epsilon)(\omega + k_\epsilon)} \\
&= \frac{e^{-ik_\epsilon(t-t')}}{2k_\epsilon}
\end{aligned} \tag{A.10}$$

and

$$\mathrm{Res}[f(\omega = -k_\epsilon)] = -\frac{e^{ik_\epsilon(t-t')}}{2k_\epsilon}, \tag{A.11}$$

so that we get

$$\begin{aligned}
\int_{-\infty}^{\infty} f(\omega)\, \mathrm{d}\omega &= \frac{\pi i}{k_\epsilon} \left[e^{-ik_\epsilon(t-t')} - e^{ik_\epsilon(t-t')} \right] \\
&= \frac{1}{k_\epsilon} \sin(k_\epsilon(t - t')),
\end{aligned} \tag{A.12}$$

and taking $\epsilon \to 0$, we get just

$$\int_{-\infty}^{\infty} f(\omega)\, \mathrm{d}\omega = \frac{1}{k} \sin(k(t - t')).$$

Thus we are left with

$$\begin{aligned}
G(x^\alpha - y^\alpha) &= -\frac{1}{(2\pi)^4} \int \frac{1}{k} e^{ik(x-y)\cos\theta} \sin(k(t - t'))\, \mathrm{d}^3k \\
&= \frac{1}{|x - y|} \int_0^\infty \sin(k(x - y)) \sin(k(t - t'))\, \mathrm{d}k,
\end{aligned} \tag{A.13}$$

which can be written

$$\begin{aligned}
G(x^\alpha - y^\alpha) &= \frac{1}{|x - y|} \int_{-\infty}^{\infty} \left(e^{i((t-t')-(x-y))k} - e^{i((t-t')+(x-y))k} \right) \mathrm{d}k \\
&= \frac{\delta((t - t') - (x - y))}{|x - y|}.
\end{aligned} \tag{A.14}$$

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