Rule of Total Probability

Rule of Total Probability

Suppose the events A_1, A_2, …, A_k are mutually exclusive and exhaustive, i.e., exactly one of these events will occur and they cover the entire sample space.

For any event B, the events (A_1 and B), (A_2 and B), …, (A_k and B) are mutually exclusive, and hence

$$P(B) = P(A_1 \text{ and } B) + P(A_2 \text{ and } B) + \ldots + P(A_k \text{ and } B)$$

Using the multiplication rule,

$$P(B) = P(B|A_1)*P(A_1) + P(B|A_2)*P(A_2) + \ldots + P(B|A_k)*P(A_k)$$

$$P(B) = \sum_{j=1}^{k} P(B|A_j) * P(A_j)$$

Rule of Total Probability (Cont'd)

Example: In an university, 60% are undergraduate students, 35% are graduate students, and 5% are postdocs. 55% of undergraduates are female, 15% of graduate students are female, and 10% of postdocs are female.

Type	Percentage of college students	Percentage females
Undergraduate	60	55
Graduate	35	15
Postdoc	5	10
	100%	

Event B = Selected student is a female
Event A1 = Selected student is an undergraduate
Event A2 = Selected student is a graduate
Event A3 = Selected student is a postdoc

A1, A2, and A3 are mutually exclusive and exhaustive

P(B) = P(A1 and B) + P(A2 and B) + P(A3 and B)

P(B) = P(B|A1)*P(A1) + P(B|A2)*P(A2) + P(B|A3)*P(A3)
= 0.55*0.60 + 0.15*0.35 + 0.10*0.05
= 0.3875

P(A1) = 0.60	P(B|A1) = 0.55
P(A2) = 0.35	P(B|A2) = 0.15
P(A3) = 0.05	P(B|A3) = 0.10

With a probability of 0.3875, a randomly selected student is a female

Bayes' Theorem

Bayes' Theorem:

Suppose the events A_1, A_2, …, A_k are mutually exclusive and exhaustive. Let B be any event.

Given

Prior probabilities: $P(A_1)$, $P(A_2)$, …, $P(A_k)$, and

Conditional probabilities: $P(B|A_1)$, $P(B|A_2)$, …, $P(B|A_k)$

Determine

Posterior probabilities: $P(A_1|B)$, $P(A_2|B)$, …, $P(A_k|B)$

$$P(A_i|B) = \frac{P(A_i \; and \; B)}{P(B)} = \frac{P(B|A_i) * P(A_i)}{P(B)}$$

$$\boxed{P(A_i|B) = \frac{P(B|A_i) * P(A_i)}{\sum_{j=1}^{k} P(B|A_j) * P(A_j)}}$$

Bayes' Theorem (Cont'd)

Example: In an university, 60% are undergraduate students, 35% are graduate students, and 5% are postdocs. 55% of undergraduates are female, 15% of graduate students are female, and 10% of postdocs are female.

Type	Percentage of college students	Percentage females
Undergraduate	60	55
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Postdoc	5	10
	100%	

Event B = Selected student is a female
Event A1 = Selected student is an undergraduate
Event A2 = Selected student is a graduate
Event A3 = Selected student is a postdoc

$$P(B) = P(B|A1)*P(A1) + P(B|A2)*P(A2) + P(B|A3)*P(A3)$$
$$= 0.55*0.60 + 0.15*0.35 + 0.10*0.05$$
$$= 0.3875$$

P(A1) = 0.60	P(B|A1) = 0.55
P(A2) = 0.35	P(B|A2) = 0.15
P(A3) = 0.05	P(B|A3) = 0.10

$P(A1|B) = P(B|A1)*P(A1)/P(B) = 0.55*0.60/0.3875 = 0.85$
$P(A2|B) = P(B|A2)*P(A2)/P(B) = 0.15*0.35/0.3875 = 0.14$
$P(A3|B) = P(B|A3)*P(A3)/P(B) = 0.10*0.05/0.3875 = 0.01$

With a probability of 0.85, a randomly selected female student is an Undergraduate.