Gravitation and Cosmology Notes

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Chapter 1

Special Relativity

The axioms of Special Relativity are:

1. The laws are invariant under change of inertial reference frames.
2. In inertial reference frames, there is an absolute speed of signal propogation, $c = 1$.

1.1 Lorentz Transformations

Let x^μ, x'^μ be two coordinate systems and $c = 1$. For now we work in the classical vacuum and here experiments[1] point us to the fact that light travels at the absolute speed c. Suppose A, B are spacetime events representing emmission and absorption of light. By the second postulate, the distance between these two points is the same in both reference frames, and in particular, the quantity

$$(\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 = 0 = (\Delta x'^0)^2 - (\Delta x'^1)^2 - (\Delta x'^2)^2 - (\Delta x'^3)^2.$$

This leads us to the definition of the **proper time**:

$$d\tau^2 := dt^2 - d\boldsymbol{x}^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta.$$

The notation $dx^\alpha dx^\beta$, in the mathematical sense, is a simple tensor $dx^\alpha \otimes dx^\beta$ and *not* the symmetric tensor. Using the Einstein summation convention and the fact that our metrics are always *symmetric*, this means that the contraction $\eta_{\alpha\beta} dx^\alpha dx^\beta = \eta_{00} dx^0 \otimes dx^0 + \eta_{10} dx^1 \otimes dx^0 + \eta_{01} dx^0 \otimes dx^1 + \cdots = \eta_{00} dx^0 \otimes dx^0 + \eta_{10}(dx^1 \otimes dx^0 + dx^0 \otimes dx^1) + \cdots$. Now we return back to proper time and see how far we can run with the concept.

Proposition 1.1.1. Let $\mathscr{L} \subseteq \mathrm{GL}_{\mathbb{R}}(3, 1)$ denote the subgroup of linear operators that fix proper time: $d\tau'^2 = d\tau^2$. Then the set $\mathscr{L}$ can be described concretely:

$$\mathscr{L} = \{\Lambda \mid \Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta_{\alpha\beta} = \eta_{\beta\delta}\}.$$

Proof. The proof of this is a straightforward and we follow Weinberg's derivation. Take a linear transformation $x'^\alpha = \Lambda^\alpha_\beta x^\beta$. Then $d\tau^2 = d\tau'^2$ equivalent to:

$$\begin{aligned} -\eta_{\beta\delta} dx^\beta dx^\delta &= -\eta_{\alpha\gamma} dx'^\alpha dx'^\gamma \\ &= -\eta_{\alpha\gamma} (\Lambda^\alpha_\beta dx^\beta)(\Lambda^\gamma_\delta dx^\delta) \end{aligned}$$

Therefore, $\eta_{\beta\delta} = \eta_{\alpha\gamma} \Lambda^\alpha_\beta \Lambda^\gamma_\delta$ as was desired. $\square$

[1] Is there a better explanation?

Remark 1.1.1. An important remark is in order. The above proposition emphasized *linear* transformations. In fact it is possible to show that the set of all nonsingular analytic coordinate transformations (ones which are invertible and locally given by a convergent power series) coincides with affine transformations of the form $x' = \Lambda x + a$, with $\Lambda \in \mathscr{L}$. To show this it suffices to show the second derivative of the coordinate transformation is identically zero, which implies that all higher order derivatives also vanish. Following the steps of the above proof, $x \to x'$ preserves proper time then

$$\eta_{\beta\delta} = \eta_{\alpha\gamma} \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \frac{\partial x'^{\gamma}}{\partial x^{\delta}}.$$

Taking a second derivative we get:

$$\begin{aligned} 0 &= \eta_{\alpha\gamma} \frac{\partial^2 x'^{\alpha}}{\partial x^{\varepsilon} \partial x^{\beta}} \frac{\partial x'^{\gamma}}{\partial x^{\delta}} + \eta_{\alpha\gamma} \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \frac{\partial^2 x'^{\gamma}}{\partial x^{\varepsilon} \partial x^{\delta}} \\ &= \eta_{\alpha\gamma} x'^{\alpha}{}_{;\varepsilon\beta} x'^{\gamma}{}_{;\delta} + \eta_{\alpha\gamma} x'^{\alpha}{}_{;\beta} x'^{\gamma}{}_{;\varepsilon\delta} \end{aligned}$$

Now we do a **magical** transformation. We add this equation with ε and β swapped, and subtract this equation but with ε and δ swapped. This will give:

$$0 = 2\eta_{\alpha\gamma} x'^{\alpha}{}_{;\varepsilon\beta} x'^{\gamma}{}_{;\delta}$$

Since η is invertible and so is $x'^{\gamma}{}_{;\delta}$ by assumption, it follows that $x'^{\alpha}{}_{;\varepsilon\beta} = 0$ for any $\alpha, \beta, \varepsilon$.

Remark 1.1.2. In the last remark, we took $d\tau^2$ invariant. If we only require that the equation $d\tau = 0$ (that is, the light-cone) to be invariant then we would have obtained the conformal group. Check this and desribe this and the last remark geometrically.

Here are some of the commonly used names associated with different types of Lorentz transformations.

- **Inhomogenous Lorentz Group/Poincaré Group** $\{x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\beta} + a^{\alpha} \mid \Lambda \in \mathscr{L}\}$
- **Homogenous Lorentz Group:** $\{x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\beta} \mid \Lambda \in \mathscr{L}\}$
- **Proper** (in)homogenous Lorentz groups: $\Lambda^0_0 \geq 1$; $\det\Lambda = 1$.

Using $\eta_{\alpha\beta} \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} = \eta_{\gamma\delta}$ with $\gamma = \delta = 0$ we have:

$$(\Lambda^0_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i_0)^2 \geq 1 \text{ and } (\det\,\Lambda)^2 = 1$$

This means that the *proper* Lorentz groups are the connected components of the identity $\Lambda^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$. There are Lorentz transformations that involve *space inversion*: $\det\,\Lambda = -1, \Lambda^0_0 \geq 1$; and that involve *time reversal*: $\det\,\Lambda = -1, \Lambda^0_0 \leq 1$. What does Weinberg mean when he says that space inversion is not an exact symmetry of nature and why is time reversal suspected not to be one as well?

1.1.1 Boosts

One may wonder what are the differences between the Galilean group and the Poincaré group. The former can be recovered by taking $\Lambda^0_0 = 1$ and $\Lambda^0_i = \Lambda^i_0 = 0$. The Poincaré enjoy the extra feature of **boosts**. Suppose there are two observers O and O' watching a particle move. O sees it at rest, while O' sees it moving away at $\boldsymbol{v}$. Then, since $d\boldsymbol{x} = 0$:

$$\begin{aligned} dx'^{i} &= \Lambda^i_0 dt \\ dt' &= \Lambda^0_0 dt \end{aligned}$$

Dividing dx'^i/dt' gives $\boldsymbol{v}'$:

$$\Lambda^i_0 = \boldsymbol{v}\Lambda^0_0. \tag{1.1}$$

In the defining relation for Λ, $\eta_{\alpha\beta}\Lambda^\alpha_\gamma\Lambda^\beta_\delta = \eta_{\gamma\delta}$, we can set $\gamma = \delta = 0$ and obtain:

$$-1 = -(\Lambda^0_0)^2 + \sum_{i=1}^{3}(\Lambda^i_0)^2. \tag{1.2}$$

Plugging in the first relation, (1.1), into equation (1.2), solving for Λ^μ_0 describing the boost:

$$\Lambda^0_0 = \gamma \;\text{ and }\; \Lambda^i_0 = \gamma v_i. \tag{1.3}$$

A convenient choice, although not the only one, for $\Lambda^i{}_j$ is the following:

$$\Lambda^i{}_j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{\boldsymbol{v}^2} \tag{1.4}$$

$$\Lambda^0{}_j = \gamma v_j \tag{1.5}$$

1.2 Time Dilation