

Gravitation and Cosmology Notes

April 15, 2016

Chapter 1

Special Relativity

The axioms of Special Relativity are:

1. The laws are invariant under change of inertial reference frames.
2. In inertial reference frames, there is an absolute speed of signal propagation $c = 1$.

1.1 Lorentz Transformations

Let x^μ, x'^μ be two coordinate systems and $c = 1$. For now we work in the classical vacuum and here experiments¹ point us to the fact that light travels at the absolute speed c . Suppose A, B are spacetime events representing emission and absorption of light. By the second postulate, the distance between these two points is the same in both reference frames, and in particular, the quantity

$$(\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 = 0 = (\Delta x'^0)^2 - (\Delta x'^1)^2 - (\Delta x'^2)^2 - (\Delta x'^3)^2.$$

This leads us to the definition of the **proper time**:

$$d\tau^2 := dt^2 - d\mathbf{x}^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta.$$

The notation $dx^\alpha dx^\beta$, in the mathematical sense, is a simple tensor $dx^\alpha \otimes dx^\beta$ and *not* the symmetric tensor. Using the Einstein summation convention and the fact that our metrics are always *symmetric*, this means that the contraction $\eta_{\alpha\beta} dx^\alpha dx^\beta = \eta_{00} dx^0 \otimes dx^0 + \eta_{10} dx^1 \otimes dx^0 + \eta_{01} dx^0 \otimes dx^1 + \dots = \eta_{00} dx^0 \otimes dx^0 + \eta_{10}(dx^1 \otimes dx^0 + dx^0 \otimes dx^1) + \dots$. Now we return back to proper time and see how far we can run with the concept.

Proposition 1.1.1. Let $\mathcal{L} \subseteq \text{GL}_{\mathbb{R}}(3, 1)$ denote the subgroup of linear operators that fix proper time: $d\tau'^2 = d\tau^2$. Then the set \mathcal{L} can be described concretely:

$$\mathcal{L} = \{\Lambda \mid \Lambda_\gamma^\alpha \Lambda_\delta^\beta \eta_{\alpha\beta} = \eta_{\gamma\delta}\}.$$

Proof. The proof of this is a straightforward and we follow Weinberg's derivation. Take a linear transformation $x'^\alpha = \Lambda_\beta^\alpha x^\beta$. Then $d\tau^2 = d\tau'^2$ equivalent to:

$$\begin{aligned} -\eta_{\beta\delta} dx^\beta dx^\delta &= -\eta_{\alpha\gamma} dx'^\alpha dx'^\gamma \\ &= -\eta_{\alpha\gamma} (\Lambda_\beta^\alpha dx^\beta) (\Lambda_\delta^\gamma dx^\delta) \end{aligned}$$

Therefore, $\eta_{\beta\delta} = \eta_{\alpha\gamma} \Lambda_\beta^\alpha \Lambda_\delta^\gamma$ as was desired. □

¹Is there a better explanation?

Remark 1.1.1. An important remark is in order. The above proposition emphasized *linear* transformations. In fact it is possible to show that the set of all nonsingular analytic coordinate transformations (ones which are invertible and locally given by a convergent power series) coincides with affine transformations of the form $x' = \Lambda x + a$, with $\Lambda \in \mathcal{L}$. To show this it suffices to show the second derivative of the coordinate transformation is identically zero, which implies that all higher order derivatives also vanish. Following the steps of the above proof, $x \rightarrow x'$ preserves proper time then

$$\eta_{\beta\delta} = \eta_{\alpha\gamma} \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \frac{\partial x'^{\gamma}}{\partial x^{\delta}}.$$

Taking a second derivative we get:

$$\begin{aligned} 0 &= \eta_{\alpha\gamma} \frac{\partial^2 x'^{\alpha}}{\partial x^{\varepsilon} \partial x^{\beta}} \frac{\partial x'^{\gamma}}{\partial x^{\delta}} + \eta_{\alpha\gamma} \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \frac{\partial^2 x'^{\gamma}}{\partial x^{\varepsilon} \partial x^{\delta}} \\ &= \eta_{\alpha\gamma} x'^{\alpha}_{;\varepsilon\beta} x'^{\gamma}_{;\delta} + \eta_{\alpha\gamma} x'^{\alpha}_{;\beta} x'^{\gamma}_{;\varepsilon\delta} \end{aligned}$$

Now we do a **magical** transformation. We add this equation with ε and β swapped, and subtract this equation but with ε and δ swapped. This will give:

$$0 = 2\eta_{\alpha\gamma} x'^{\alpha}_{;\varepsilon\beta} x'^{\gamma}_{;\delta}$$

Since η is invertible and so is $x'^{\gamma}_{;\delta}$ by assumption, it follows that $x'^{\alpha}_{;\varepsilon\beta} = 0$ for any $\alpha, \beta, \varepsilon$.

Remark 1.1.2. In the last remark, we took $d\tau^2$ invariant. If we only require that the equation $d\tau = 0$ (that is, the light-cone) to be invariant then we would have obtained the conformal group. **Check this and describe this and the last remark geometrically.**