1 Relativistic Quantum Mechanics

1.1 Summary

1. Quantum Mechanics: Axioms of QM: Hilbert spaces, rays, measurements.
2. Symmetries
   - How does one relate the states that $\mathscr{O}$ measures with the states that $\mathscr{O}'$ measures?
   - Wigner's theorem: "For any probability-preserving transformation, is either unitary and linear, or antiunitary and antilinear."
   - Trivial symmetry, $U = \mathbb{1}$. Continuity: symmetries in the path-connected component of $\mathbb{1}$ are all unitary and linear.
   - Infinitesimally close to $\mathbb{1}$: "generators" $\iff$ "observables".
   - Representations, projective representations, superselection rules.
   - Connected Lie groups and applications: Lie algebras and Abelian Lie groups.
3. Quantum Lorentz Transformations: Unitary representations of Lorentz Group.
   - Define: (in)homogeneous, orthochronous.
   - Exercise: when working on the Lie algebra level, why is it okay to work with the homogeneous Lorentz group?
4. Poincaré algebra: the section where we do Lie algebra computations.
   - Work infinitesimally away from identity, staying inside of the homogeneous Lorentz group.
   - Compute commutators.
5. One Particle States: Representations of inhomogeneous Lorentz group (ie. Poincaré group).
   - Reps of Poincaré $\leftrightarrow$ Reps of $\mathrm{SL}(2, \mathbb{C})$
   - Orbit Method!
   - Normalization of free particle states $N(p) = \sqrt{\frac{k^0}{p^0}}$.
   - Proof Mass Positive Definite, Proof Mass Zero (Helicity).
6. Space Inversion and Time Reversal
   - What are the operators corresponding to $\mathscr{P}, \mathscr{T}$ which act on the Hilbert space?
   - If so, what are the properties of $\boldsymbol{P}, \boldsymbol{T}$?
   - Casework: $M > 0, M = 0$ with $\boldsymbol{P}, \boldsymbol{T}$.
7. Projective Representations
   - When are projective reps necessary? Answer: We can set the phase $\phi = \phi(T, \bar{T}) \to 0$ if the there are no *central charges* in the Lie algebra and the group is simply connected.
   - Define central charges, prove theorem (see appendix 2).
   - That is: one algebraic condition, one topological condition. Investigate both to show that inhomogeneous Lorentz group it is *not* simply connected!
   - Discuss: intrinsic projective representations, superselection rules, spin.
8. Appendix A: Wigner's Theorem (see Dan Freed's article)
9. Appendix B: Group Operators and Homotopy Classes (for Projective Reps)
10. Appendix C: Inversions and Degenerate Multiplets
    - "Inversions might act in more complicated ways on degenerate multiplets of one-particle states"
    - Discuss the cases for both $\boldsymbol{P}$ and $\boldsymbol{T}$.

1.2 Symmetries

Before getting started we need to do a digression on quantum foundations. This will establish the need for theorems like Wigner's theorem on unitary and antiunitary operators, projective representations, etc. *Physical states* will be restricted to rays in a separable Hilbert space, so our configuration space is given by the projective Hilbert space: $\mathbb{P}\mathscr{H} = \mathscr{H} \setminus \{0\}/\sim$. One way to work with this projective Hilbert space is to choose representatives from each ray.

Probabilities of certain measurement processes are given by amplitudes of projections:

$$\Pr[\Psi \to \Psi_n] = |\langle \Psi_n | \Psi \rangle|^2.$$

Two observers may measure the same physical system in different ways using the same Hilbert space. The transformations that take one observer's ray to the other must preserve probabilities:

$$\Pr[\mathscr{R} \to \mathscr{R}_n] = \Pr[\mathscr{R}' \to \mathscr{R}'_n].$$

Theorem 1.1 (Wigner 1930). *Any probability preserving transformation on a projective Hilbert space can be lifted to either a unitary and linear operator or an antiunitary and conjugate-linear operator on the original Hilbert space.*

Proof. (Sketch following Weinberg). Given a transformation of projective Hilbert space $T\colon \mathbb{P}\mathscr{H} \to \mathbb{P}\mathscr{H}$, we must construct an operator $U\colon \mathscr{H} \to \mathscr{H}$. We choose an orthonormal basis Ψ_n on $\mathcal{H}$ and we scale the image of this basis by phases to ensure the result. More precisely, the key step in the proof is to define an new basis $\{\Upsilon_k = \frac{1}{\sqrt{2}}(\Psi_k + \Psi_1)\}_{k>1} \cup \{\Upsilon_1 = \Psi_1\}$ and choosing $U\Upsilon_k = \frac{1}{\sqrt{2}}(U\Psi_k + U\Psi_1)$. This provides a constraint: if $\Psi = \sum_{n=1}^{\infty} C_n \Psi_n$ maps to $U\Psi = \sum_{n=1}^{\infty} C'_n U\Psi_n$ then:

$$|C_n|^2 = |C'_n|^2, \qquad\qquad |C_k + C_1|^2 = |C'_k + C'_1|^2.$$

These two equations imply $C_k/C_1 = C'_k/C'_1$ or $C_k/C_1 = (C'_k/C'_1)^*$ which corresponds to unitary or antiunitary. After choosing $C_1 = C'_1$ or $C_1 = C'^*_1$, the proof proceeds to verify that the operator U constructed in this way is well-defined, unitary/antiunitary, and linear/conjugate-linear. □

1.2.1 Representations

Let G be a group of symmetries of our physical system. Mathematically, we ask that this is a probability-preserving group action on $\mathbb{P}\mathscr{H}$. In particular, any element $g \in G$ induces a transformation on the projective Hilbert space which, using Wigner's theorem 1.1, corresponds to an operator on the Hilbert space $U(g)\colon \mathscr{H} \to \mathscr{H}$. However, this assignment $U\colon G \to \mathrm{GL}(\mathscr{H})$ is not necessarily a unitary or antiunitary representation; but rather a **projective representation**. Indeed, fix a norm-one basis element Ψ_n and consider the action of $U_1U_2 \coloneqq U(g_1)U(g_2)$ and $U_{12} \coloneqq U(g_1g_2)$. Since the operators U_1, U_2, U_{12} by construction descend to operators on $\mathbb{P}\mathscr{H}$, this means that

$$U_1U_2\Psi_n = e^{i\phi_n(T_1,T_2)}U_{12}\Psi_n.$$

There is a subtlety involved in this: does the phase depend on the choice of state Ψ_n? If we are able to prepare superpositions of states $\Psi + \Phi$ for any Ψ and Φ then it's easy to show that the phase in the projective representation is independent of the basis element Ψ_n. However, the ability to construct superpositions may be prohibited by the choice of group that we pick. This restriction is what is known as **superselection rules** [?]. If it turns out that we are indeed able to make a certain linear combination that is not allowed by superselection rules, this simply means that we need to slightly enlarge our group. See the following section on projective representations for more details.

If the group is Abelian, which is the case for space-time translations or rotations around a fixed axis, then every unitary transformation is given by the exponential:

$$U(T(\theta)) = U(T(\theta/N))^N = \lim_{N\to\infty}\left[1 + \frac{it_a\theta^a}{N}\right]^N = e^{it_a\theta^a}.$$

1.3 Projective Representations (Weinberg Vol I §2.7)

1.3.1 Introduction

Recall that all of our representations will be unitary.

Definition 1. A **projective representation** of a group G is a map $\rho : G \to U(V)$ such that

$$\rho(T_1T_2) = e^{i\phi(T_1,T_2)}\rho(T_1)\rho(T_2). \tag{1}$$

Imposing associativity, $\rho(T_1)(\ \rho(T_2)\rho(T_3)\ ) = (\ \rho(T_1)\rho(T_2)\ )\rho(T_3)$ gives a necessary condition for the phase ϕ to satisfy.

$$\begin{aligned}\rho(T_1T_2T_3) &= e^{i\phi(T_1,T_2T_3)}\rho(T_1)\rho(T_2T_3) = e^{i\phi(T_1,T_2T_3)+i\phi(T_2,T_3)}\rho(T_1)\rho(T_2)\rho(T_3)\\ &= e^{i\phi(T_1T_2,T_3)}\rho(T_1T_2)\rho(T_3) = e^{i\phi(T_1T_2,T_3)+i\phi(T_1,T_2)}\rho(T_1)\rho(T_2)\rho(T_3).\end{aligned}$$

This gives the condition:

$$\phi(T_1, T_2T_3) + \phi(T_2, T_3) \equiv \phi(T_1T_2, T_3) + \phi(T_1, T_2) \pmod{2\pi} \tag{2}$$

The following example and definition is to make precise the meaning of equivalent representations.

Example 1 (Trivial phase)**.** If the phase $\alpha(T_1, T_2)$ is defined by $\alpha(T_1, T_2) = f(T_1T_2) - f(T_1) - f(T_2)$, then this α satisfies (2). This is called a trivial phase because, by multiplying the projective representation $\rho(T)$ by $e^{i\alpha(T)}$ we obtain linear representation $\tilde{\rho} = \rho \cdot e^{i\alpha}$, such that $\tilde{\rho}(T_1T_2) = \tilde{\rho}(T_1)\tilde{\rho}(T_2)$.

Definition 2. Denote by $\mathrm{Map}(G \times G, \mathbb{R}/2\pi\mathbb{Z})$ the set of all phases $\phi : (T_1, T_2) \mapsto [0, 2\pi]$. A **2-cocycle** is an element of the group $\mathrm{Map}(G \times G, \mathbb{R}/2\pi\mathbb{Z})/\sim$ where $\phi \sim \psi$ if $\phi - \psi$ has the form of Example 1. The trivial 2-cocycle is given by the equivalence class containing the $\phi \equiv 0$ phase.

Remark. At this point we shall make a deal with the devil, but only once in this section: let us identify $\mathfrak{g}$ with G. This can be done in a neighbourhood of the identity using the exponential map, but our deal will consist applying Taylor's formula naïvely. We do this only for clarity's sake.

Let $\vec{\theta} = (\theta^1, \dots) \equiv \{\theta^a\}$ parameterize the group elements $T(\theta)$ near the identity. We must express the following statements on the level of the Lie algebra: group multiplication, representation, phase. Applying Taylor's formula, we find constants $f^a_{\ bc}$,

$$\rho(T(\theta)) = \mathbb{1} + i\theta^a \mathfrak{t}_a + \frac{1}{2}\theta^b\theta^c\mathfrak{t}_{bc} + \cdots \tag{3}$$

$$f^a(\theta, \bar{\theta}) = \theta^a + \bar{\theta}^a + f^a_{\ bc}\theta^b\bar{\theta}^c + \cdots \tag{4}$$

$$\phi(\theta, \bar{\theta}) = \phi_{bc}\theta^b\bar{\theta}^c + \cdots \tag{5}$$

Let's substitute these three equations to get the infinitesimal equivalent of $\rho(T_1)\rho(T_2) = e^{i\phi(T_1,T_2)}\rho(T_1T_2)$:

$$\rho(T(\theta))\rho(T(\bar{\theta})) = e^{i\phi(T(\theta),T(\bar{\theta}))}\rho(T_{f(\theta,\bar{\theta})}) \tag{6}$$

$$(\mathbb{1} + i\theta^a\mathfrak{t}_a + \frac{1}{2}\theta^b\theta^c\mathfrak{t}_{bc})(\mathbb{1} + i\bar{\theta}^a\mathfrak{t}_a + \frac{1}{2}\bar{\theta}^b\bar{\theta}^c\mathfrak{t}_{bc}) = (1 + i\phi_{bc}\theta^b\bar{\theta}^c)(\mathbb{1} + if^a\mathfrak{t}_a + \frac{1}{2}f^bf^c\mathfrak{t}_{bc}) \tag{7}$$

$$= (1 + i\phi_{bc}\theta^b\bar{\theta}^c)\Big[\mathbb{1} + i(\theta^a + \bar{\theta}^a + f^a_{\ bc}\theta^b\bar{\theta}^c)\mathfrak{t}_a \tag{8}$$

$$+ \frac{1}{2}(\theta^b + \bar{\theta}^b)(\theta^c + \bar{\theta}^c)\mathfrak{t}_{bc}\Big] \tag{9}$$

The $\theta, \bar{\theta}, \theta\theta, \bar{\theta}\bar{\theta}$ terms are equal on both sides, up to second order. Let's look at the $\theta\bar{\theta}$ terms on both sides:

$$LHS_{\theta\bar{\theta}} = (-\mathfrak{t}_b\mathfrak{t}_c)\theta^b\bar{\theta}^c \tag{10}$$

$$RHS_{\theta\bar{\theta}} = (if^a_{\ bc}\mathfrak{t}_a + \mathfrak{t}_{bc} + i\phi_{bc}\mathbb{1})\theta^b\bar{\theta}^c \tag{11}$$

$$\implies \quad \mathfrak{t}_{bc} = -\mathfrak{t}_b\mathfrak{t}_c - if^a_{\ bc} - i\phi_{bc}\mathbb{1} \tag{12}$$

By our choice of $\mathfrak{t}_{bc}$ being symmetric with respect to b, c, we have an additional relation $\mathfrak{t}_{bc} = \mathfrak{t}_{cb}$. Rearranging we easily get:

$$[\mathfrak{t}_b, \mathfrak{t}_c] = -i(f^a{}_{bc} - f^a{}_{cb})\mathfrak{t}_a + i(\phi_{bc} - \phi_c b) \tag{13}$$
$$= iC^a{}_{bc}\mathfrak{t}^a + iC_{bc}\mathbb{1} \tag{14}$$

Definition 3. When we have a linear representation, ie. $\phi(T_1, T_2) = 0$ for all T_1, T_2, then $C^a{}_{bc}$ completely define the Lie algebra and are called the **structure constants**. When ϕ is non-zero, then the C_{bc}'s are called the **central charges**.

1.3.2 Sufficient Conditions: Turning Projective Representations into Affine Representations

The key relation that we have come to is $[\mathfrak{t}_b, \mathfrak{t}_c] = iC^a{}_{bc}\mathfrak{t}^a + iC_{bc}\mathbb{1}$. A necessary condition of the generators of a Lie algebra to satisfy is the Jacobi identity. These will lead to a pair of equations. Combining this with a topological condition we shall arrive at a theorem which gives sufficient conditions which allow making projective representations into linear representations.

Let us denote *cyc* to be the cyclic permutation of (a, b, c) in that order. The Jacobi identity reads:

$$\begin{aligned}
[\mathfrak{t}_a, [\mathfrak{t}_b, \mathfrak{t}_c]] + cyc &= 0 \\
[\mathfrak{t}_a, iC^d{}_{bc}\mathfrak{t}_d + iC_{bc}\mathbb{1}] + cyc &= 0 \\
iC^d{}_{bc}[\mathfrak{t}_a, \mathfrak{t}_d] + iC_{bc}[\mathfrak{t}_a, \mathbb{1}] + cyc &= 0 \\
iC^d{}_{bc}[\mathfrak{t}_a, \mathfrak{t}_d] + cyc &= 0 \\
iC^d{}_{bc}(iC^e{}_{ad}\mathfrak{t}_e + iC_{ad}\mathbb{1}) + cyc &= 0
\end{aligned}$$

The coefficient of $\mathfrak{t}_e$ and $\mathbb{1}$ must be zero, which gives the following two relations:

$$\begin{aligned}
0 &= C^d{}_{bc}C^e{}_{ad} + cyc = C^d{}_{bc}C^e{}_{ad} + C^d{}_{ca}C^e{}_{bd} + C^d{}_{ab}C^e{}_{cd} \\
0 &= C^d{}_{bc}C{}_{ad} + cyc = C^d{}_{bc}C_{ad} + C^d{}_{ca}C_{bd} + C^d{}_{ab}C_{cd}
\end{aligned}$$

There is an immediate way to reduce the second equation to the first: set $C_{ab} = C^e{}_{ab}\phi_e$ for some choice of ϕ_e. In particular, shifting $\mathfrak{t}_a \to \tilde{\mathfrak{t}}_a = \mathfrak{t}_a + \phi_a\mathbb{1}$ we get:

$$[\tilde{\mathfrak{t}}_a, \tilde{\mathfrak{t}}_b] = [\mathfrak{t}_b + \phi_b\mathbb{1}, \mathfrak{t}_c + \phi_c\mathbb{1}] = [\mathfrak{t}_b, \mathfrak{t}_c] = iC^a{}_{bc}\mathfrak{t}_a + iC^a_{bc}\phi_a\mathbb{1} = iC^a{}_{bc}\tilde{\mathfrak{t}}_a$$

Theorem 1.2. *Let $\rho : G \to U(V)$ be a projective representation with $\dim G < \infty$. If there is a choice of generators $\{\mathfrak{t}_a\} \subseteq \mathfrak{u}(V)$ that may be extended to a Lie-algebra basis on $\mathfrak{u}(V)$, that is the $\{\mathfrak{t}_a\}$ do not give rise to central charges, and if G is simply connected, then we may "choose phases of the operators to obtain a linear representation."*

Proof. Weinberg's proof is clear. Here is an outline of his exposition. What do we want? A linear representation. How will we get one? By brute force: writing down a differential equation that the 'representation operators' must satisfy, finding the properties of such operators, show that they form a linear representation. It's quite fun actually. □

Big Question: If we are given a projective transformation of the inhomogeneous orthochronous Lorentz group, can we redefine the operators to make a *linear* representation? In other words, can we apply the above theorem to the Lorentz group?

We need to check the algebraic hypothesis and the topological hypothesis: it will turn out that the topological condition fails.

1. (Topological Condition)

   (a) $\mathbb{R}^{3,1} \xrightarrow{\cong} \mathfrak{u}(2, \mathbb{C})$

$$V^\mu \mapsto v = V^\mu\sigma_\mu = \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}$$

(b) Let $\lambda \in \mathfrak{gl}(2,\mathbb{C})$, with $|\det \lambda| = 1$.

$$v \to \lambda v \lambda^\dagger \qquad\qquad V^\mu \sigma_\mu \to \lambda V^\mu \sigma_\mu \lambda^\dagger = (\Lambda^\mu_\nu(\lambda) V^\nu)\sigma_\mu$$

where there is a unique $\Lambda(\lambda)$ since $V^\mu V_\mu = -\det v$ and since conjugation by λ (having $|\det \lambda| = 1$) perserves the inner product.

(c) Property: $\Lambda(\lambda)\Lambda(\lambda') = \Lambda(\lambda\lambda')$.

(d) Property: If $\lambda' = e^{i\theta}\lambda$, then conjugation by λ and λ' is equivalent. Therefore we have an induced action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{R}^{3,1}$.

(e) Property: if $\lambda \in \mathrm{SL}_2(\mathbb{C})$, then $-\lambda \in \mathrm{SL}_2(\mathbb{C})$ and produces the same Lorentz transformation.

Proposition 1. *$\mathscr{L} \cong \mathrm{SL}_2(\mathbb{C})/\mathbb{Z}_2$, where $\mathscr{L}$ is the homogeneous Lorentz group.*

(f) Topology of $\mathrm{SL}_2(\mathbb{C})$: Polar Decomposition. For any $\lambda \in \mathrm{SL}_2(\mathbb{C})$, there exist unique $u \in U(2)$, $h \in i\mathfrak{u}(2) = \mathrm{Hermitian}(2)$, such that $\lambda = ue^h$. Exercise: since $\lambda \in \mathrm{SL}_2(\mathbb{C})$ show that:

$$u \in \mathrm{SU}(2), \qquad \mathrm{Tr}\, h = 0.$$

Traceless, Hermitian matrices can be written uniquely as $n \cdot \vec{\sigma}$ where $\vec{\sigma}$ are the Pauli matrices; $\mathrm{SU}(2) \cong S^3$, the 3-sphere. Therefore:

$$\begin{aligned} \mathrm{SL}_2(\mathbb{C}) &\xrightarrow{homeo.} \mathbb{R}^3 \times S^3 \\ \text{homogeneous Lorentz group} &\xrightarrow{homeo.} \mathbb{R}^3 \times (S^3/\mathbb{Z}^2) \cong \mathbb{R}^3 \times \mathbb{R}\mathrm{P}^2 \\ \text{inhomogeneous Lorentz group} &\xrightarrow{homeo.} \mathbb{R}^4 \times \mathbb{R}^3 \times (S^3/\mathbb{Z}^2) \cong \mathbb{R}^7 \times \mathbb{R}\mathrm{P}^2 \end{aligned}$$

2. (Algebraic Condition)

Fun fact (only applicable to subgroup of inhom. Lorentz group, the $\{J^{\mu\nu}\}'s$):

Proposition 2. *A semi-simple Lie algebra allows no central charges. (Remember: central charges are properties of the Lie algebra of the target space of a representation.)*

This proposition does ***NOT*** apply to the entire Lorentz group.

Combining the topological description of the inhomogeneous Lorentz group and the proof of the theorem we may conclude that the phase can *almost* be eliminated. In fact, since the fundamental group, $\pi_1(\text{inhom. Lorentz group}) = \mathbb{Z}_2$, we may conclude that "because the double loop that goes twice from 1 to Λ to $\Lambda\bar{\Lambda}$ and then back to 1 *can* be contracted to a point, so we must have

$$\left[U(\Lambda)U(\bar{\Lambda})U^{-1}(\Lambda\bar{\Lambda})\right]^2 = \mathbb{1},$$

and hence the phase $e^{i\phi(\Lambda,\bar{\Lambda})} = \pm 1$. Similarly for the inhomogeneous group we get the relation:

$$U(\Lambda, a)U(\bar{\Lambda}, \bar{a}) = \pm U(\Lambda\bar{\Lambda}, \Lambda a + \bar{a})." \tag{15}$$

Definition 4. [1] Fix a unitary (possibly projective) representation $U : \mathscr{L} \to U(V)$. If $U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)$ for all $\Lambda_i \in \mathscr{L}$ then the set of vectors $v \in V$ are called integer-spin states. Otherwise, if there exist Λ_1, Λ_2 for which $U(\Lambda_1)U(\Lambda_2) = -U(\Lambda_1\Lambda_2)$, then the $v \in V$ are called a half-integer states.

Remark. In fact, in the half-integer case, we know exactly which Λ_1, Λ_2 are problematic: they are precisely the ones for which the path $1 \to \Lambda_1 \to \Lambda_1\Lambda_2 \to 1$ is not contractible to a point. For concreteness, suppose you have a *massless* state with angular momentum (helicity) σ in the z-direction. After applying a rotation about the z-axis, the state will pick up a phase of $e^{2\pi i\sigma}$ (check). This distinguishes integer and half-integer σ.

[1] Check this: perhaps this interpretation is not correct.

Remark. The restriction (15) imposes the superselection rule: "we cannot mix integer and half-integer states." Let's spell this out in full: suppose ψ_0 is an integer-state whereas $\psi_{1/2}$ is half-integer-state.

$$\pm U_{12}(\psi_0 + \psi_{1/2}) = U_1 U_2(\psi_0 + \psi_{1/2}) = U_1 U_2 \psi_0 + U_1 U_2 \psi_{1/2} = U_{12}\psi_0 - U_{12}\psi_{1/2} = U_{12}(\psi_0 - \psi_{1/2})$$

Applying U_{12}^{-1} to both sides we obtain from the leftmost equal to the rightmost expression: $\pm(\psi_0 + \psi_{1/2}) = \psi_0 - \psi_{1/2}$. This means either $\psi_{1/2} = 0$ or $\psi_0 = 0$, both of which contradict the fact that these should represent physical states.

1.3.3 Covering spaces

We have seen that $\mathrm{SL}_2(\mathbb{C}) \to \mathscr{L}$ is a 2-1 covering map.

Question: What should we call the Lorentz group?

1. Can we be bold and *replace* what we call the Lorentz group ($\mathrm{SL}_2(\mathbb{C})/\mathbb{Z}_2$) with $\mathrm{SL}_2(\mathbb{C})$?
2. If we do this, what do we gain and what do we lose?

Our conventional Lorentz group, $\mathrm{SL}_2(\mathbb{C})/\mathbb{Z}_2 \cong \mathrm{SO}^+(3,1)$, has two features relevant for this discussion:

1. $\mathrm{SO}^+(3,1)$ admits "intrinsic" projective representations.
2. Topological structure implies there are two superselection sectors.

The group $\mathrm{SL}_2(\mathbb{C})$, however, does not admit projective representations because it is simply connected and semi-simple. The nonexistence of projective representations means that $\mathrm{SL}_2(\mathbb{C})$ is not able to impose superselection rules. However, the representations (apart from being all non-projective) of $\mathrm{SL}_2(\mathbb{C})$ are equivalent to the original $\mathrm{SL}_2(\mathbb{C})/\mathbb{Z}_2$: which means that the *physical consequences* are the same modulo superselection.

To summarize, the only caveat of saying that the Lorentz group is $\mathrm{SL}_2(\mathbb{C})$ is that we cannot conclude whether or not "we can prepare physical systems in linear combinations of states of integer and half-integer spin, but only that the observed Lorentz invariance ($\mathrm{SL}_2(\mathbb{C})$) of nature cannot be used to show that such superpositions are impossible.

This procedure applies to any Lie group G that admits central charges or is not simply connected. If it has central charges, enlarge the group;[2] if it is not simply connected, replace G with the covering space. However, when we do this replacement we must always remember: we will not be able to conclude whether or not all linear combinations are allowed.

To quote Weinberg: "In short, the issue of superselection rules is a bit of a red herring; *it may or may not be possible to prepare physical systems in arbitrary superpositions of states, but one cannot settle the question by reference to symmetry principles, because whatever one thinks the symmetry group of nature may be, there is always another group whose consequences are identical except for the absence of superselection rules.*

1.4 Lorentz Transformations

In this subsection the relativistic part gets incorporated into our quantum mechanical framework. The important equations are highlighted but most of the discussion surrounding them is suppressed.

The equation that preserves proper time is given by:

$$\eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\alpha\beta} dx'^\alpha dx'^\beta. \tag{16}$$

Lorentz transformations are a subgroup of the general linear group $\mathrm{GL}(\mathbb{R}^4)$ preserving proper time. Rewriting (16) as $\eta_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} = \eta_{\mu\nu}$, we get that:

$$\eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu}.$$

Taking the determinant we get $(\mathrm{Det}\,\Lambda)^2 = 1$. Another restriction comes from considering the Λ_0^0 component: $(\Lambda_0^0)^2 = 1 + \sum_{i=1}^3 \Lambda_i^0 \Lambda_i^0$ which means $\Lambda_0^0 \geq 1$ or $\Lambda_0^0 \leq -1$. This analysis shows there are at least **four connected components** of the Lorentz group. The $\mathrm{Det}\,\Lambda = 1, \Lambda_0^0 \geq 1$ is called the proper orthochronous Lorentz group.

[2] this has not yet been shown in these notes

Theorem 1.3. *Let $\mathscr{P}$ denote a parity transformation: ${\mathscr{P}^0}_0 = 1, {\mathscr{P}^i}_i = -1$ for $i = 1, 2, 3$ and let $\mathscr{T}$ denote a time-reversal: ${\mathscr{T}^0}_0 = -1$ and ${\mathscr{T}^i}_i = 1$ for $i = 1, 2, 3$. Then the Lorentz group is generated by the proper, orthochronous Lorentz group and $\mathscr{P}, \mathscr{T}, \mathscr{P} \circ \mathscr{T}$.*

1.4.1 Poincaré Algebra

The **Poincaré group** is an extension of the Lorentz group to include translations: $x'^\mu = {\Lambda^\mu}_\nu x^\nu + a^\mu$. To derive the Lie algebra associated with this group we will consider an infinitesimal transformation:

$$ {\Lambda^\mu}_\nu = \delta^\mu_\nu + {\omega^\mu}_\nu, \qquad\qquad a^\mu = \epsilon^\mu, $$

with ω, ϵ infinitesimal parameters. Imposing that this infinitesimal transformation preserves proper-time, we obtain:

$$ \begin{aligned} \eta_{\rho\sigma} &= \eta_{\mu\nu}({\delta^\mu}_\rho + {\omega^\mu}_\rho)({\delta^\nu}_\sigma + {\omega^\nu}_\sigma) \\ &= \eta_{\sigma\rho} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + O(\omega^2) \end{aligned} $$

Which means that ω is an antisymmetric 2-tensor in 4 dimensions. The Poincaré group is a symmetry of space-time and so if we suppose our quantum theory is also invariant under this group then by the discussion at the beginning of section 1.2 we must have a unitary operator corresponding to this infinitesimal transformation:

$$ U(\mathbb{1} + \omega, \epsilon) = \mathbb{1} + \frac{1}{2} i\omega_{\rho\sigma} J^{\rho\sigma} - i\epsilon_\mu P^\mu. $$

Unitarity implies that each component of J and P are Hermitian, and antisymmetry of $\omega_{\rho\sigma}$ implies antisymmetry of $J^{\rho\sigma}$. The sign in front of $i\epsilon_\mu P^\mu$ is a convention.

1.5 Poincaré Algebra

The transformation properties of $J^{\mu\nu}$ and P^μ are obtained by considering the equation:

$$ U(\Lambda, a)U(1 + \omega, \epsilon)U(\Lambda, a)^{-1} = U(\Lambda', a'). $$

Applying a taylor expansion to both the LHS and RHS we should arrive at a transformation law for each $J^{\mu\nu}$ and P^μ. Doing this also shows that μ, ν are tensor indices.

The commutation relations follow from taking $\Lambda = \mathbb{1} + \omega$.

1.5.1 Low-Velocity Limit

1.6 One-Particle States

The Poincaré algebra has the following commutation relation: $[P^\mu, P^\nu] = 0$. This, in particular, shows that $[H, P^\mu] = [P^0, P^\mu] = 0$ giving us good quantum numbers p^μ so we shall label our basis states as:

$$ |p, \sigma\rangle . $$

A priori we don't know if σ is a discrete index or a continuous one, however we take *as part of the definition of a one-particle state* that σ is a discrete index.

However, to define a one-particle state let's not use $|p\sigma\rangle$, ie. just a single vector inside some Hilbert space. Let us remember that *the state of the particle depends on the observer measuring the particle*! Therefore we shall identify the vectors

$$ |p\sigma\rangle \sim U(\Lambda)\,|p\sigma\rangle = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda)\,|p\sigma'\rangle\,, $$

where the notation $|\alpha\rangle \sim |\beta\rangle$ means "we identify $|\alpha\rangle$ with $|\beta\rangle$." This motivates:

Definition 5. A **one-particle state** is a finite-dimensional irreducible representation of the Lorentz group.

1.7 Classification of Finite Dimensional Irreducible Representations of the Lorentz Group

Our goal in this section will be to *build* representations of the homogeneous orthochronous Lorentz group which we shall denote $\mathscr{L}$. We will do this explicitly and diligently. A rough outline of the procedure is as follows. First, we notice there are a lot of possible basis vectors for the Hilbert space $\mathscr{H}$. In fact, for each $p \in \mathbb{R}^{3,1}$ there are many *basis* vectors $|p\sigma\rangle \in \mathscr{H}$. Second, to write down a representation we must describe the action of each $\Lambda \in \mathscr{L}$, or more precisely $U(\Lambda) \in U(\mathscr{H})$, on each $|p\sigma\rangle$. To accomplish this mighty task we will be even mightier: first, we will choose only a small subset of the basis vectors and describe the action of $\mathscr{L}$; second, we'll extend our work to all other basis vectors.

Part One: Simplification Let's push ourselves to the inside of the light-cone and restrict to $p^2 \leq 0$. Here, there are two invariant functions for $\mathscr{L}$, the *homogeneous, orthochronous* group (draw a picture of the light-cone to convince yourself):

$$f(p) = p^2 = \eta_{\mu\nu} p^\mu p^\nu \qquad\qquad g(p) = \operatorname{sgn}(p^0)$$

The pictorial proof makes one more result apparent: the action of $\mathscr{L}$ partitions the inner-light-cone into orbits. Each orbit can be classified/described physically. There are three relevant parts that are of interest to us, two of which are non-trivial:

- Massive orbit: $\mathscr{L} \cdot (0, 0, 0, M), M > 0$
- Massless orbit: $\mathscr{L} \cdot (0, 0, \kappa, \kappa), \kappa > 0$
- Vacuum: $\mathscr{L} \cdot (0, 0, 0, 0)$

The notation $\mathscr{L} \cdot p$ denotes the set $\{\Lambda p\}_{\Lambda \in \mathscr{L}}$, or in words, the orbit of p under the action of $\mathscr{L}$.

For each orbit, we may define a representative $k \in \mathbb{R}^{3,1}$ from which we can generate any other vector in the orbit. In particular, we define the map $L_k(p) \in \mathscr{L}$ that takes k to p, or briefly: $p = L(p)k$. These k's will be our building blocks for our representation. How do we recover the basis vectors $|p\sigma\rangle$ from $|k\sigma\rangle$? Define the state $|p\sigma\rangle$ by:

$$|p\sigma\rangle := N(p) U(L(p)) |k\sigma\rangle \,,$$

where $N(p)$ is a phase factor that is important, but will be determined later.

Part Two: Induction Now we make a little jump, to show that if we have a representation for the subspace spanned by $\{|k\sigma\rangle\}_\sigma$ then we can *induce* a representation for the subspace corresponding to the orbit of k: $\{|p\sigma\rangle \mid p \in (\mathscr{L} \cdot k), \sigma\}$.

$$\begin{aligned}
U(\Lambda) |p\sigma\rangle &= N(p) U(\Lambda) U(L_p) |k\sigma\rangle \\
&= N(p) U(L_{\Lambda p}) U(L_{\Lambda p}^{-1}) U(\Lambda) U(L_p) |k\sigma\rangle \\
&= N(p) U(L_{\Lambda p}) U\Big(\underbrace{L_{\Lambda p}^{-1} \cdot \Lambda \cdot L_p}_{W(\Lambda,p)}\Big) |k\sigma\rangle \\
&= N(p) U(L_{\Lambda p}) \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p)) |k\sigma'\rangle \\
&= N(p) \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p)) \frac{|\Lambda p, \sigma'\rangle}{N(\Lambda p)} \\
&= \left(\frac{N(p)}{N(\Lambda p)}\right) \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p)) |\Lambda p, \sigma'\rangle
\end{aligned}$$

Here is a quick explanation of the calculation. The second equality is just inserting an identity. The third equality combines the Lorentz transformations so that $W(\Lambda, p)$ is a Lorentz transformation that fixes k (easy exercise). The fourth line is the application of the representation on the $\{|k\sigma\rangle\}_\sigma$ space:

$$U(W) |k\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W) |k\sigma\rangle \,.$$

The fifth and sixth equalities are rearrangements and simplifications. To be precise, the representation that we use in the fourth line is a representation of the stabiliser group of k:

$$\mathrm{Stab}(k) = \{W \in \mathscr{L} \mid Wk = k\} \subseteq \mathscr{L}.$$

In physics, we call this the "little group". Notice that there is a different little group depending on which representative k we are dealing with.

1.7.1 Representations of the Massive Little Group

There are three orbits of interest and thus three possible little groups: massive $k = (0, 0, 0, M)$, massless $k = (0, 0, \kappa, \kappa)$, vacuum. The vacuum case is straightforward so let's consider the other two cases.

The little group is the stabiliser of k. In the massive case, these are by inspection $\mathrm{SO}(3) \subseteq \mathscr{L}$ the rotation matrices sitting inside the homogeneous orthochronous Lorentz group. The representations of SO(3) can be labelled by $j = 0, \frac{1}{2}, 1, \ldots$; let's denote them as $D^{(j)}_{\sigma'\sigma}(R)$ where $R \in \mathrm{SO}(3)$. The induced representation is given by:

$$U(\Lambda)\,|p\sigma\rangle = \frac{N(p)}{N(\Lambda p)} \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W_{\Lambda,p})\,|\Lambda p, \sigma'\rangle$$

where $W(\Lambda, p) = L^{-1}_{\Lambda p} \cdot \Lambda \cdot L_p$ can be computed explicitly for any p in the same orbit as $k = (0, 0, 0, M)$. Let's not devote the next two sections to the massless case.

1.7.2 The Massless Little Group: Warm-Up

The little group for the massless vector $(0, 0, \kappa, \kappa)$ is isomorphic to $ISO(2) \cong \mathbb{R}^2 \rtimes \mathrm{SO}(2)$. The proof of this is a bit more involved, but since the result is so interesting let's reproduce it.

Proposition 3. Let $k = (0, 0, 1, 1) \in \mathbb{R}^{3,1}$. If we denote the stabiliser of k by $\mathrm{Stab}(k) \subseteq \mathscr{L}$, then $\mathrm{Stab}(k) \cong ISO(2)$.

Proof. (Weinberg) Fix $W \in \mathrm{Stab}(k)$. Choose $t = (0, 0, 0, 1)$ to be a time-like vector. Then Wt has the following two properties:

$$(Wt)^2 = t^2 = -1 \qquad\qquad (Wt) \cdot k = t \cdot k = -1.$$

Since both k and t have simple forms, it is straightforward to check that Wt has the form:

$$Wt = (\alpha, \beta, \zeta, 1 + \zeta), \quad \text{with} \quad \zeta = (\alpha^2 + \beta^2)/2$$

In particular, W acting on t is equal to the action of a simple Lorentz transformation $S(\alpha, \beta)$:

$$Wt = S(\alpha, \beta)t = \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1 - \zeta & \zeta \\ \alpha & \beta & -\zeta & 1 + \zeta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Finally, this means that $S(\alpha, \beta)^{-1} W = R(\theta)$ is a rotation keeping the t axis invariant. Rearranging we have the decomposition:

$$W(\theta, \alpha, \beta) = S_{\alpha,\beta} R_\theta.$$

By brute force multiplication, we can look for Abelian subgroups, $S_{\alpha,\beta} S_{\alpha',\beta'} = S_{\alpha+\alpha',\beta+\beta'}$ and $R_\theta R_{\theta'} = R(\theta + \theta')$ and also for an isomorphism with $ISO(2)$: $R_\theta S_{\alpha,\beta} R_\theta^{-1} = S(\alpha \cos\theta + \beta \sin\theta, -\alpha \sin\theta + \beta \cos\theta)$. The last equality shows what an observer $\mathscr{O}'$ (who is rotated by an angle θ from $\mathscr{O}$) would measure if $\mathscr{O}$ moved an object by a vector (α, β). $\square$

Remark. Since $ISO(2)$ contains Abelian subgroups, $ISO(2)$ is not semi-simple.

1.7.3 The Massless Little Group

Let's compute $W(\Lambda, p)$. Infinitesimally $W(\theta, \alpha, \beta) = \mathbb{1} + \omega$ where

$$\omega = \begin{pmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ -\alpha & -\beta & 0 & 0 \end{pmatrix} \implies U(W_{\theta,\alpha,\beta}) = \mathbb{1} + i\alpha A + i\beta B + i\theta J_3$$

where $A = J_2 + K_1$, and $B = -J_1 + K_2$. The algebra leads to a surprise. The commutator $[A, B] = 0$, and $[H, A] = [H, B] = 0$ which means that we can introduce new quantum numbers: a, b corresponding to A, B. However by applying a rotation by θ we get new distinct states which are also eigenstates labelled by the same k-vector.

Input from experimental observations: massless particles are not observed to have a continuous symmetry. We must conclude that $|k\sigma\rangle$ must vanish under A and B. The only remaining operator for which we may distinguish $|k\sigma\rangle$ is J_3 and so we write:

$$J_3 |k\sigma\rangle = \sigma |k\sigma\rangle .$$

The value σ is the magnitude of the angular momentum in the direction of $k = (0, 0, 1, 1)$, or more precisely in the $\hat{\boldsymbol{k}}$ direction. This is called the **helicity** of the massless particle.

1.7.4 Representations of the Massless Little Group

We know, from the Abelian subgroup structure, $U(S_{\alpha,\beta}) = e^{i\alpha A + i\beta B}$ and $U(R_\theta) = e^{i\theta J_3}$. Therefore for any $W = S_{\alpha,\beta} R_\theta$ we may use the definition of the representation $U(W) = U(S_{\alpha,\beta})U(R_\theta)$ so that by acting on $|k\sigma\rangle$ we get: $U(W)|k\sigma\rangle = e^{i\theta\sigma}|k\sigma\rangle$. Remember: $A|k\sigma\rangle = B|k\sigma\rangle = 0$. Therefore:

$$U(W) |k\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W) |k\sigma\rangle , \qquad D_{\sigma'\sigma}(W) = e^{i\theta\sigma}\delta_{\sigma'\sigma}$$

Inducing this action to the subspace corresponding to the entire orbit, $\mathscr{L} \cdot k$, we get:

$$U(\Lambda) |p\sigma\rangle = \frac{N(p)}{N(\Lambda p)} e^{i\sigma\theta(\Lambda,p)} |\Lambda p, \sigma\rangle \qquad W(\Lambda, p) \equiv L^{-1}_{\Lambda p} \Lambda L_p \equiv S_{\alpha(\Lambda,p),\beta(\Lambda,p)} R_{\theta(\Lambda,p)}$$

Remark. The part of the little group parameterized by α, β will be responsible for electromagnetic gauge invariance. Stay tuned, don't change the channel!

Remark. As we saw in our discussion of projective representations, the fact that the Lorentz group has $\pi_1(\mathscr{L}) = \mathbb{Z}_2$ imposes the constraint that $\sigma \in \{\dots, -\frac{1}{2}, 0, \frac{1}{2}, 1, \dots\}$. The statment is that $\mathbb{1} \to \Lambda = e^{i(2\pi)J_3} \to \mathbb{1}$ picks up a phase $e^{2\pi i\sigma}$ for a state $|k\sigma\rangle$. Doing this twice would pick up $e^{4\pi i\sigma}$. However, the path $\mathbb{1} \to \Lambda \to \mathbb{1} \to \Lambda \to \mathbb{1}$ is contractible to a point, which means that $e^{4\pi i\sigma} = 1$ and so $4\pi\sigma \equiv 0 \pmod{2\pi}$ or that σ is an integer or half-integer.

Remark. Spatial inversion, as we shall see, connects states of helicity that differ by a sign (-1). Electromagnetism and gravitation both are spatially symmetric. Electroweak theory is not spatially symmetric (cf. nuclear beta decay).

$$\text{photons} : \sigma = \pm 1 \qquad \text{neutrinos} : \sigma = +1/2$$
$$\text{gravitons} : \sigma = \pm 2 \qquad \text{antineutrinos} : \sigma = -1/2$$

1.7.5 Polarization

Proposition 4. Helicity is Lorentz invariant.

Proof. This follows from the induced action: $U(\Lambda)|p\sigma\rangle = \frac{N(p)}{N(\Lambda p)} e^{i\sigma\theta(\Lambda,p)} |\Lambda p, \sigma\rangle$. □

The above proposition does not mean that the state is the same. A general one-photon state is a superposition:

$$|p; \alpha\rangle = \alpha_+ |p; +1\rangle + \alpha_- |p; -1\rangle$$

We say the photon is

- **elliptically polarized** if $|\alpha_\pm| \neq 0$ and $|\alpha_+| \neq |\alpha_-|$,
- **circularly polarized** if only one of $\alpha_+ = 0$ or $\alpha_- = 0$,
- **linearly polarized** if $|\alpha_+| = |\alpha_-|$.

The relative phase may change as we boost in certain directions.

1.8 Space Inversion and Time Reversal

> **The Fight for a True Representation**
>
> Can we enforce $U(\Lambda, a)U(\Lambda', a') = U(\Lambda\Lambda', \Lambda a' + a)$ for the entire Poincaré group, including the cases when parity and time reversal $\mathscr{P}$ and $\mathscr{T}$ are allowed?

Suppose there are operators $\mathrm{P} = U(\mathscr{P}, 0), \mathrm{T} = U(\mathscr{T}, 0)$ satisfying

$$\mathrm{P}\, U(\Lambda, a)\, \mathrm{P}^{-1} = U(\mathscr{P}\Lambda\mathscr{P}^{-1}, \mathscr{P}a) \tag{17}$$

$$\mathrm{T}\, U(\Lambda, a)\, \mathrm{T}^{-1} = U(\mathscr{T}\Lambda\mathscr{T}^{-1}, \mathscr{T}a) \tag{18}$$

Nuclear beta decay shows that the first of these is only approximate and the second seems also approximate (PRL 13 138, 1964). Still, suppose P and T do exist satisfying (17) and (18). Applying (17), (18) to the infinitesimal operators of the Poincaré group we get:

$$\mathrm{P}\, iH\, \mathrm{P}^{-1} = iH, \qquad\qquad \mathrm{T}\, iH\, \mathrm{T}^{-1} = -iH.$$

From these two equations it immediately follows that P and T must be respectively unitary and linear, and antiunitary and antilinear.

How do P and T transform one-particle states? It is useful to collect the commutation relations:

$$\mathrm{P}\, \boldsymbol{J}\, \mathrm{P}^{-1} = +\boldsymbol{J} \tag{19}$$

$$\mathrm{P}\, \boldsymbol{K}\, \mathrm{P}^{-1} = -\boldsymbol{K} \tag{20}$$

$$\mathrm{P}\, \boldsymbol{P}\, \mathrm{P}^{-1} = -\boldsymbol{P} \tag{21}$$

$$\mathrm{P}\, H\, \mathrm{P}^{-1} = H \tag{22}$$

$$\mathrm{T}\, \boldsymbol{J}\, \mathrm{T}^{-1} = -\boldsymbol{J} \tag{23}$$

$$\mathrm{T}\, \boldsymbol{K}\, \mathrm{T}^{-1} = +\boldsymbol{K} \tag{24}$$

$$\mathrm{T}\, \boldsymbol{P}\, \mathrm{T}^{-1} = -\boldsymbol{P} \tag{25}$$

$$\mathrm{T}\, H\, \mathrm{T}^{-1} = H \tag{26}$$

Parity, $M > 0$. $|k\sigma\rangle$ are eigenvectors of $\boldsymbol{P}, H, J_3$ with eigenvalues $(0, M, \sigma)$. Applying P: takes $0 \to -0$, $M \to M$, $\sigma \to \sigma$; thus $\mathrm{P}\,|k\sigma\rangle = \eta_\sigma\,|k\sigma\rangle$ where $\eta_\sigma \in U(1)$ is a phase. By applying $J_\pm = J_1 \pm iJ_2$, we can show that η is independent of σ. For finite spatial momentum states, using $\mathscr{P}L(p)\mathscr{P}^{-1} = L(\mathscr{P}p)$ we get:

$$\mathrm{P}\,|p,\sigma\rangle = \eta\,|\mathscr{P}p,\sigma\rangle\,.$$

Time Reversal, $M > 0$. $|k\sigma\rangle$ are eigenvectors of $\boldsymbol{P}, H, J_3$ with eigenvalues $(0, M, \sigma)$. Applying T: takes $0 \to -0$, $M \to M$, $\sigma \to -\sigma$; thus $\mathrm{T}\,|k,\sigma\rangle = \zeta_\sigma\,|k,-\sigma\rangle$ where ζ_σ is a phase. For finite spatial momentum states,

$$\mathrm{T}\,|p,\sigma\rangle = \zeta(-1)^{j-\sigma}\,|\mathscr{P}p,-\sigma\rangle\,.$$

Parity, $M = 0$. Consider $|k\sigma\rangle$ with eigenvalue $k = (0,0,\kappa,\kappa)$ under P^μ and eigenvalue σ under J_3. Applying P, yields $k \to (\mathscr{P}k) = (0,0,-\kappa,\kappa)$, and eigenvalue σ under J_3. The helicity $\boldsymbol{J}\cdot\hat{k}$ changes $-\sigma$ (since now it is antiparallel, if it originally was parallel). After some analysis we get:

$$\mathrm{P}\,|p,\sigma\rangle = \eta_\sigma e^{\mp i\pi\sigma}\,|\mathscr{P}p,\sigma\rangle\,,$$

check Weinberg for the details in the sign. They intrinsically come from the fact that $\pi_1(\text{Poincaré}) = \mathbb{Z}_2$.

Time Reversal, $M = 0$. Consider $|k\sigma\rangle$ with eigenvalue $k = (0,0,\kappa,\kappa)$ under P^μ and eigenvalue σ under J_3. Applying T, yields $k \to (\mathscr{P}k) = (0,0,-\kappa,\kappa)$, and eigenvalue $-\sigma$ under J_3. The helicity $\boldsymbol{J}\cdot\hat{k}$ does not change σ. After some analysis we get: $\mathrm{T}\,|p,\sigma\rangle = \zeta_\sigma e^{\pm i\pi\sigma}\,|\mathscr{P}p,\sigma\rangle$.

1.8.1 T^2

Working the action of T^2 on $|p\sigma\rangle$ yields:

$$\mathrm{T}^2\,|p\sigma\rangle = e^{\mp 2i\pi\sigma}\,|p\sigma\rangle \equiv (-1)^{2|\sigma|}\,|p\sigma\rangle$$

If a state contains an odd-number of particles with half-integer spin there is an overall change of sign: $\mathrm{T}^2\,\Psi = -\Psi$. This is independent of the direction of spatial momentum and so even in the presence of interactions which only obey time-reversal this result is preserved.

If Ψ is an eigenstate, then $\mathrm{T}\,\Psi$ is also an eigenstate if $[H,\mathrm{T}] = 0$. Moreover, $\mathrm{T}\,\Psi \not\propto \Psi$ otherwise: $-\Psi = \mathrm{T}^2\,\Psi = \mathrm{T}(\alpha\Psi) = \alpha^*\,\mathrm{T}\,\Psi = |\alpha|^2\Psi$, which is a contradiction. This is called a **Kramers degeneracy**. If the system had rotational symmetry, then an odd-number of half-integer states means a total spin of $2j+1 = 2, 4, 6, \dots$, and that is an even degeneracy. Kramers result is that even when rotational invariance is broken, say by electric fields, there is still a two-fold degeneracy as long as the Hamiltonian is invariant under T.

This can rule out electric dipole moments: suppose there is an anomalous electric dipole moment of a particle, then in the presence of a static electric field, the particle would choose a preferred direction thus breaking the spin degeneracy – contradicting the Kramers degeneracy, or more precisely, time-reversal invariance.

1.9 Appendix: Projective Representations and Central Extensions

In this subsection, we shall understand where the terms 2-cocycle and central charge come from. Skipping this section will not

$$\text{Projective Rep} \longrightarrow \text{Phase } \phi \longrightarrow [\phi] \in H^2(G, U(1)) \longrightarrow U(1) \rtimes_{\tilde{\phi}} G$$

Let us quickly introduce/review the gadgets from group cohomology that will be useful to us. We shall start from cochains.

Definition 6. Let G be a group, and A be an Abelian group, which admits a right G-action. An **n-cochain** is a function $f : \overbrace{G \times \cdots \times G}^{n} \to A$. Let $C^n(G,A)$ refer to the set of all n-cochains, and also let's write $C^0(G,A) = A$. Define a map $d : C^n(G,A) \to C^{n+1}(G,A)$ that takes an n-chain f and returns an $n+1$ chain df defined by the formula:

$$df(g_1,\dots,g_{n+1}) = f(g_2,\dots,g_{n+1})\left(\prod_{i=1}^{n} f(g_1,..,g_{i-1},g_i g_{i+1},g_{i+2},..,g_{n+1})^{(-1)^i}\right)\left(f(g_1,\dots,g_n)^{(-1)^{n+1}}\cdot g_{n+1}\right)$$

Denote by

$$\begin{aligned} Z^n(G,A) &= \ker(d : C^n \to C^{n+1}), \quad \text{the } \textbf{n-cocycles}, \\ B^n(G,A) &= \operatorname{im}(d : C^{n-1} \to C^n), \quad \text{the } \textbf{n-coboundaries}, \\ H^n(G,A) &= Z^n(G,A)/B^n(G,A), \quad \text{the } \textbf{n}^{\textbf{th}}\textbf{-cohomology group}. \end{aligned}$$

Note that C^n, Z^n, B^n, and H^n are all groups with the addition endowed from C^n.

Example 2. If the group action is trivial, that is $x \cdot g = x$ for any $g \in G$ and $x \in A$, then

$$\begin{aligned} Z^2(G,A) &= \{f(x,y) | f_{y,z} f^{-1}_{xy,z} f_{x,yz} f^{-1}_{x,y} = 1\} \\ &= \{f(x,y) | f_{y,z} f_{x,yz} = f_{x,y} f_{xy,z}\} \end{aligned}$$

Compare this with the condition (equation 2) on the phase ϕ: $\phi(T_1, T_2T_3) + \phi(T_2,T_3) \equiv \phi(T_1T_2,T_3) + \phi(T_1,T_2)$ (mod 2π). The group A in this example is $\mathbb{R}/2\pi\mathbb{Z}$ with additive operation. This is where the term 2-cocycle comes from. What are the 2-coboundaries? For arbitrary G and A:

$$B^2(G,A) = \{f_\lambda(x,y) | f(x,y) = \lambda(y)\lambda(xy)^{-1}\lambda(x)^y\}$$

Comparing this with the "trivial phases" in example 1, we see these trivial phases are precisely the 2-coboundaries. The 2^{nd} cohomology group then the equivalence class of phases modulo the addition of "trivial phases" which is precisely what we have been studying above. Finally, since $\mathbb{R}/2\pi\mathbb{Z} \cong U(1)$ as Abelian groups, it makes no difference which Abelian group A we pick when speaking of $H^2(G, A)$. [3]

Now that we've seen that each projective representation gives rise to a cohomology class, how do these relate to Abelian central extensions?

Definition 7. Let G and A be groups, with A Abelian. We say that E is an **group extension of G by A** if there is a short exact sequence

$$1 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } G \longrightarrow 1\ .$$

Proposition 5. If $1 \to A \to E \to G \to 1$ is a group extension with a set-theoretic section $s : G \to E$ satisfying $\pi \circ s = \mathbb{1}_G$, then there is an action of both E and G on A by conjugation. Moreover A is normal in E.

Morally speaking, we should think of A and G sitting inside of E. We may abuse notation $i(a) \equiv a$ and $s(g) \equiv g$, but not forgetting that $\pi(a) = 0$ and $\pi(g) = g$. The E-action and G-action are obvious: $e \cdot a = eae^{-1}$ and $g \cdot a = gag^{-1}$. It is easy with this notation to check that these actions are well-defined. Here is the formal proof.

Proof. Let $a \in A, e \in E, g \in G$. Define $e \cdot a \in A$ be the unique element such that $i(e \cdot a) = e\, i(a)\, e^{-1}$. We may check that $e\, i(a)\, e^{-1}$ is indeed in $i(A)$, because $\pi(e\, i(a)\, e^{-1}) = \pi(e)[\pi \circ i(a)]\pi(e)^{-1} = 1$ and exactness at E shows that $e\, i(a)\, e^{-1} \in \ker \pi = i(A)$. We similarly get a G-action on A, by writing $g \cdot a = s(g)i(a)s(g)^{-1}$. $\square$

In fact, we do not need sections for this proof to go through. Exercise.

Definition 8. Let $1 \to A \to E \to G \to 1$ be an extension of groups. We say E is a **central extension** if the image of A lies in the centre of E. In particular, this means that the G-action on A is trivial, because G acts by conjugation and A commutes with every element.

$E \cong A \rtimes_f G$

Let us assume that $1 \to A \to E \to G \to 1$ is a central extension. Right now, we do not know what the structure of E is except that it's an extension. Our next goal is to show that E can be characterized by a single function $f : G \times G \to A$. Let us choose a set-theoretic section $s : G \to A$, which means that in general we have $s(g)s(h) \neq s(gh)$. To measure how poorly s fails to be a homomorphism we may introduce a function $f : G \times G \to A$ defined by:

$$s(g)s(h) = i(f_{g,h})s(gh).$$

We may now define the map from $\phi : A \times G \to E$ given by $(a, g) \mapsto i(a)s(g)$. This is a bijection, as one can easily check. Let us put the following group multiplication on $A \times G$:

$$(a_1, g_1) * (a_2, g_2) = (a_1(g_1 \cdot a_2)f_{g_1,g_2}, g_1g_2).$$

It is easy to check that this is well-defined and makes $A \times G$ into a group. In fact, it is chosen so that operation is exactly the one needed to make $\phi : A \times G \to E$ an isomorphism:

$$\begin{aligned} \phi(a_1, g_1) \cdot \phi(a_2, g_2) &= i(a_1)s(g_1)i(a_2)s(g_2) = i(a_1(g_1 \cdot a_2))s(g_1)s(g_2) = i(a_1(g_1 \cdot a_2))i(f_{g_1,g_2})\, s(g_1g_2) \\ &= (\, a_1(g_1 \cdot a_2)f_{g_1,g_2},\ s(g_1g_2)\,) \end{aligned}$$

Associativity of this multiplication imposes the following constraint on the function f. Remember, since E is a central extension, that the G-action on A is trivial:

$$\begin{aligned} [(a_1, g_1)(a_2, g_2)](a_3, g_3) &= (a_1, g_1)[(a_2, g_2)(a_3, g_3)] \\ (a_1a_2f_{g_1,g_2}, g_1g_2)(a_3, g_3) &= (a_1, g_1)(a_2a_3f_{g_2,g_3}, g_2g_3) \\ (a_1a_2a_3f_{g_1,g_2}f_{g_1g_2,g_3}, g_1g_2g_3) &= (a_1a_2a_3f_{g_2,g_3}f_{g_1,g_2g_3}, g_1g_2g_3) \\ \implies \qquad f(g_1, g_2)f(g_1g_2, g_3) &= f(g_2, g_3)f(g_1, g_2g_3), \end{aligned}$$

which says that f is a 2-cocycle, or in other words f defines an equivalence class $[f] \in H^2(G, A)$.

[3] Our G should be one of the Lorentz group, projective Lorentz group, or the image of the Lorentz group under the representation.