“Logistic Difference Equation”

Team #1

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Introduction to LDE

review article

Simple mathematical models with very complicated dynamics

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First-order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications. This is an interpretive review of them.

Logistic Difference Equation

- The logistic difference equation is a mathematical model used to describe the growth of a population over time.
- Also called logistic map.

- Discrete-time mapping version of population dynamics analogous to the logistic differential equation.
- Dynamics of overlapping-generation populations (e.g. insects) are often more appropriately expressed by difference equations.

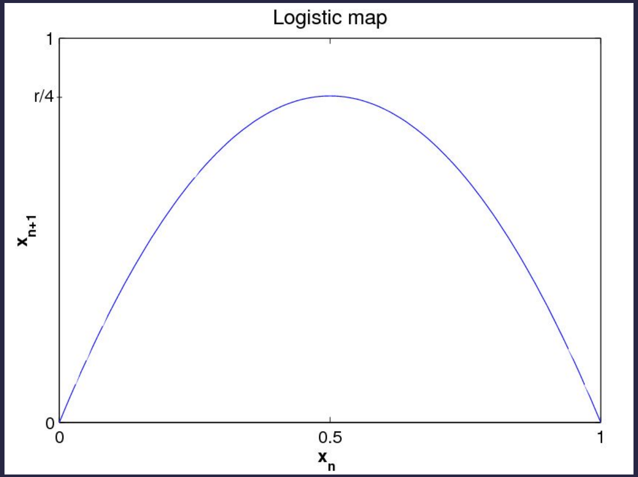


Fig. 1: Logistic *difference* equation population growth curve

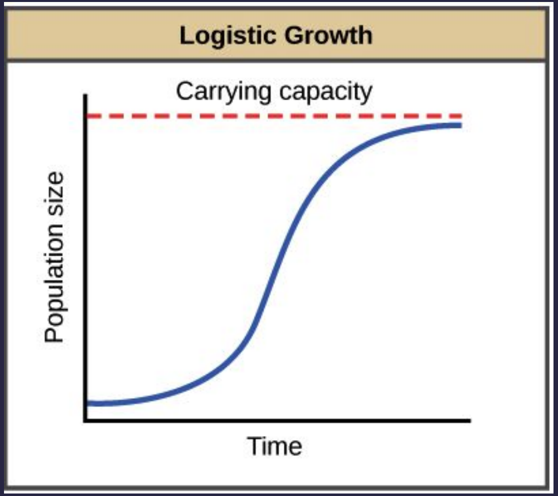


Fig. 2: Logistic *differential* equation sigmoidal-curve of population growth

Mathematically, the logistic map is written as:

Growth rate

$$x_{n+1} = r x_n (1 - x_n)$$

Pop. next generation

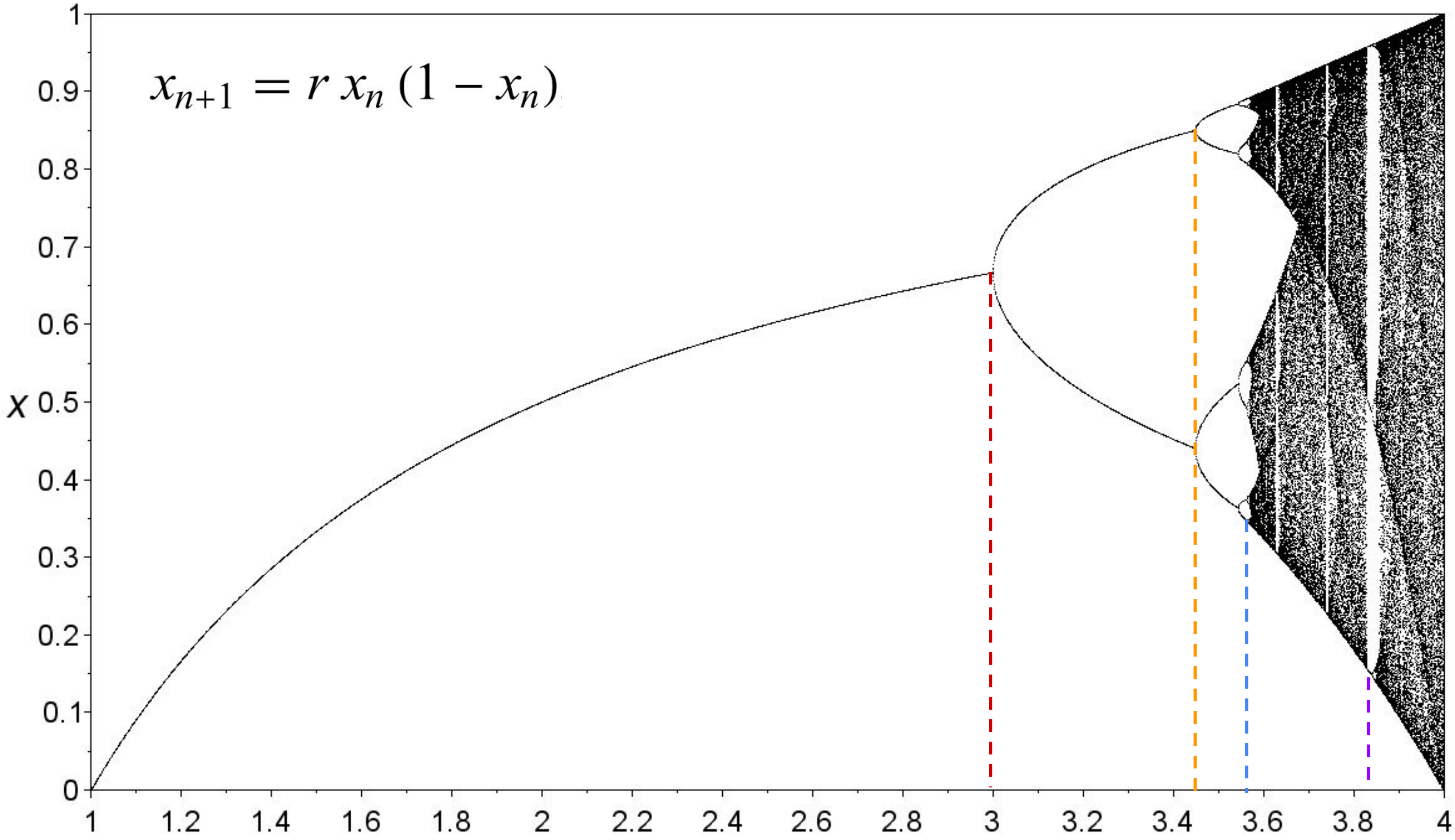
Pop. current generation

where, x is population size, r is the intrinsic growth rate and $1 \leq r \leq 4$

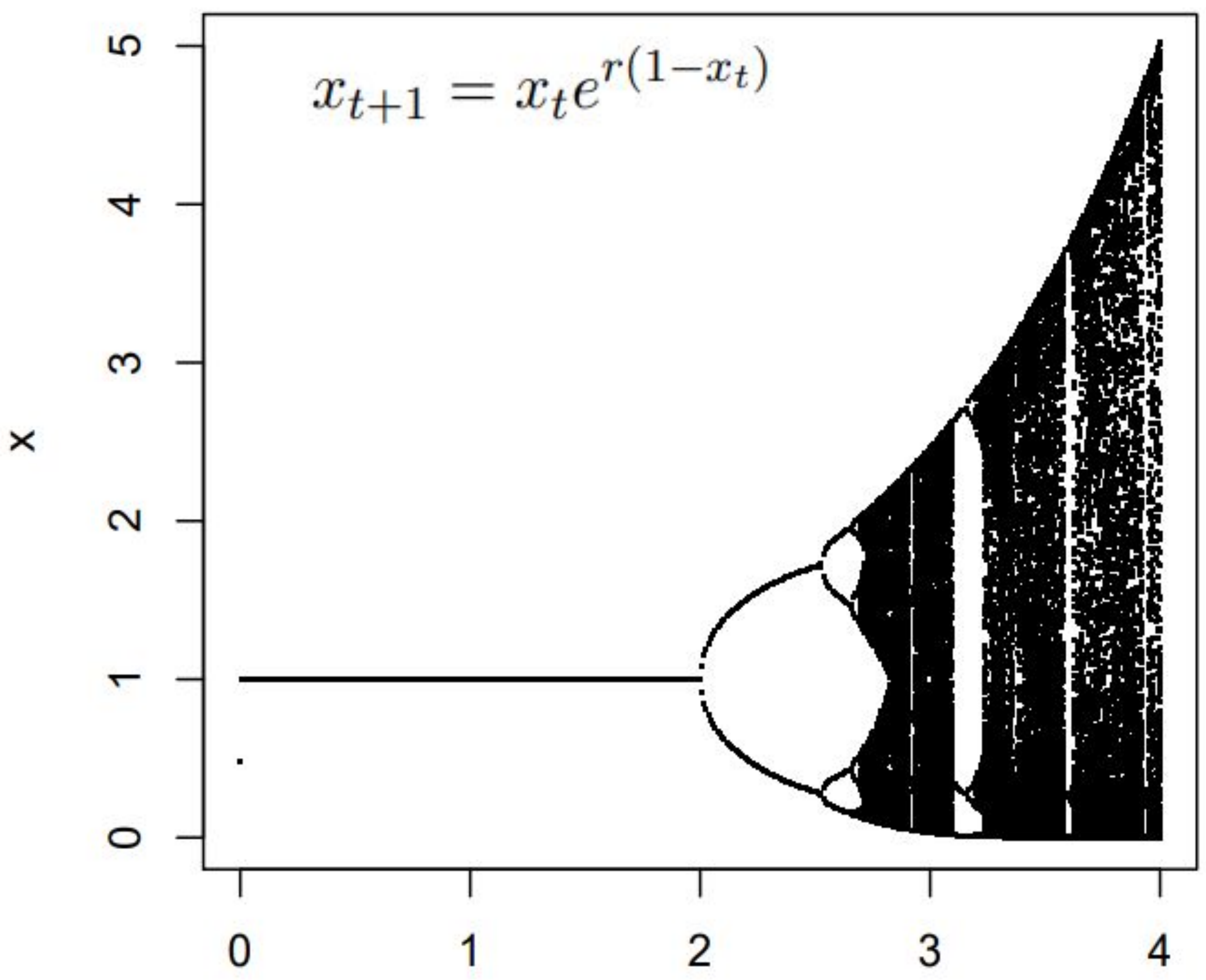
- The population size of the next generation $x(n + 1)$ is expressed as a function of the population size in the current generation $x(n)$.
- $x(n)$ is a number between 0 and 1, which represents the ratio of existing population to the maximum possible population
- This equation is intended to capture two effects: reproduction and starvation (density dependent mortality).

$$x_{t+1} = x_t e^{r(1-x_t)}$$

Bifurcation Diagram and Analysis of LDE



$x_{n+1} = r\,x_n\,(1 - x_n)$
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4


$x_{t+1} = x_t e^{r(1-x_t)}$
x
0
1
2
3
4
5
0
1
2
3
4

Bifurcation - The Feigenbaum constant

r_1	r_2	r_3	r_4	r_5	...	r_∞
3,0	3,449949...	3,54409...	3,568759...	3,569692..	...	3,5699457...

In 1975 Feigenbaum discovered that a sequence of values in bifurcations occur following a regular pattern in second-order maps. It converges geometrically when expressed in terms of the distance between bifurcations. This constant is 4.669. Finding the constant is as easy as:

$$\lim_{k\to\infty} \frac{r_k - r_{k-1}}{r_{k+1} - r_k} = \delta = 4{,}669209\ldots$$

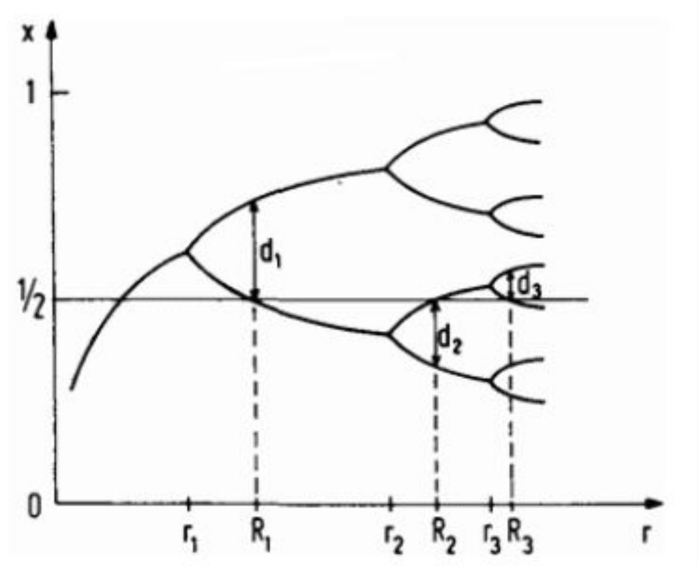


Figure 5: The Feigenbaum constant calculation

Periodicity in the Bifurcation

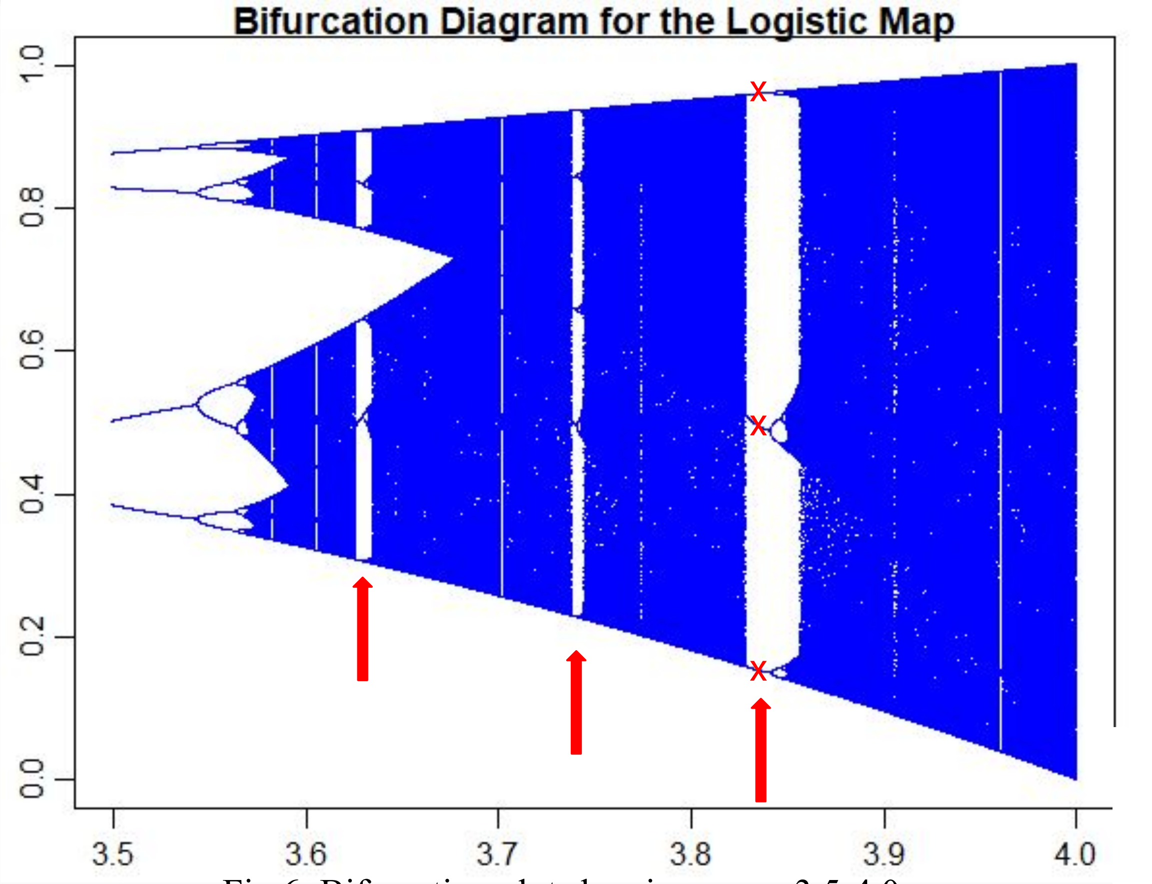
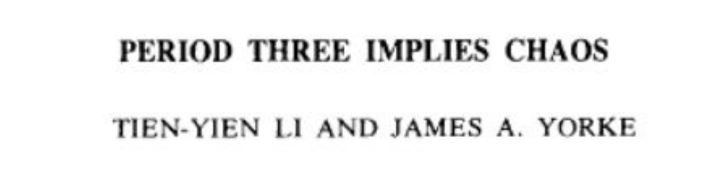


Fig 6: Bifurcation plot showing range 3.5-4.0

“Periodicity windows” appear in the region parameters from 3.6 to 3.85 where **periods 6, 5 and 3** can be observed. In the given figure these regions are denoted with arrows.

In Li and Yorke’s paper (1975), it is noted that if a system has a period 3, then the system exhibits chaotic behavior. Period 3 denoted in the figure with x.



PERIOD THREE IMPLIES CHAOS

TIEN-YIEN LI AND JAMES A. YORKE

Values of Interest of Growth parameter (r)

Bifurcation points:

Period-doubling bifurcations occur at **r = 3.0, 3.45, 3.54, 3.564, 3.569** (approximately) etc.

At r ~3.56995 is the onset of chaos, at the end of the period-doubling cascade.

Chaotic onset regions of interest:

r = 3.75, 3.829, 3.85, 3.9

Time Series Plots of LDE

For different values of growth parameter (r)

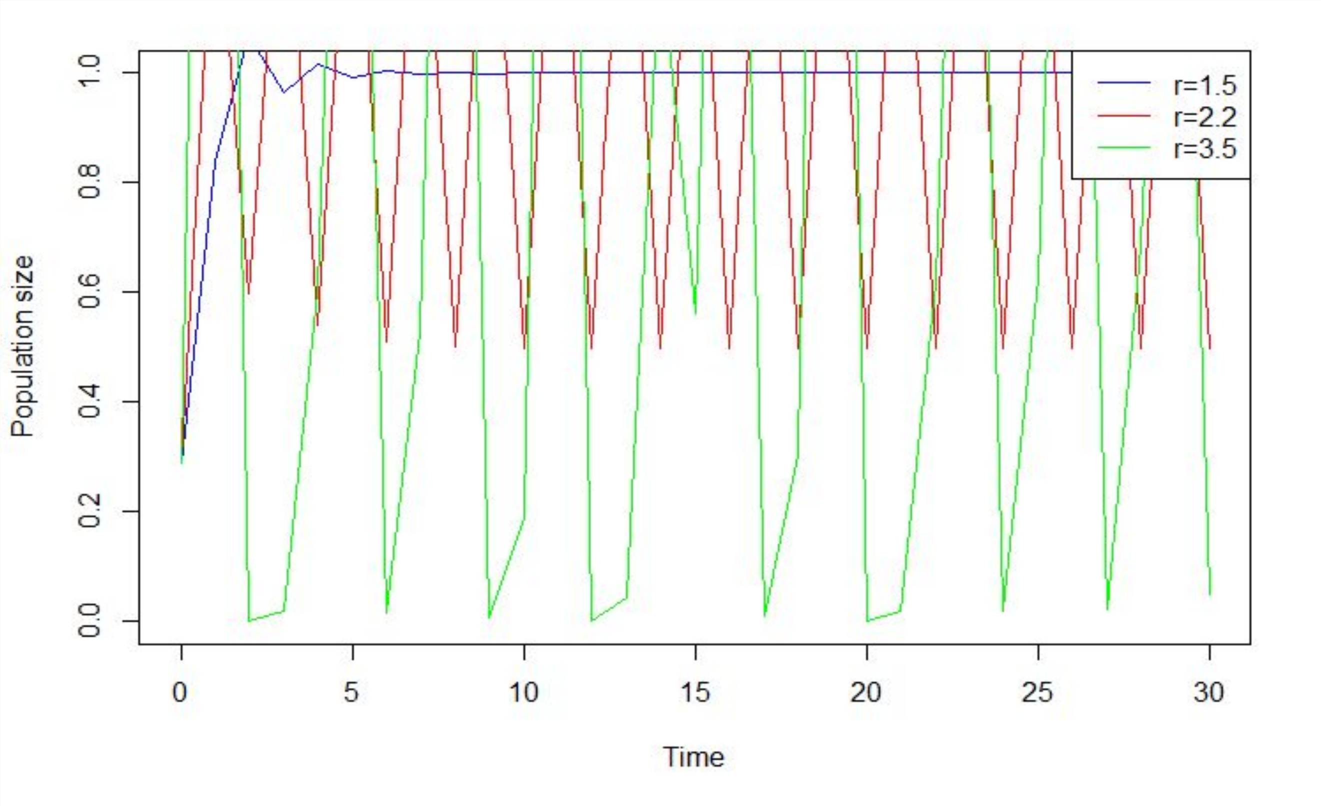
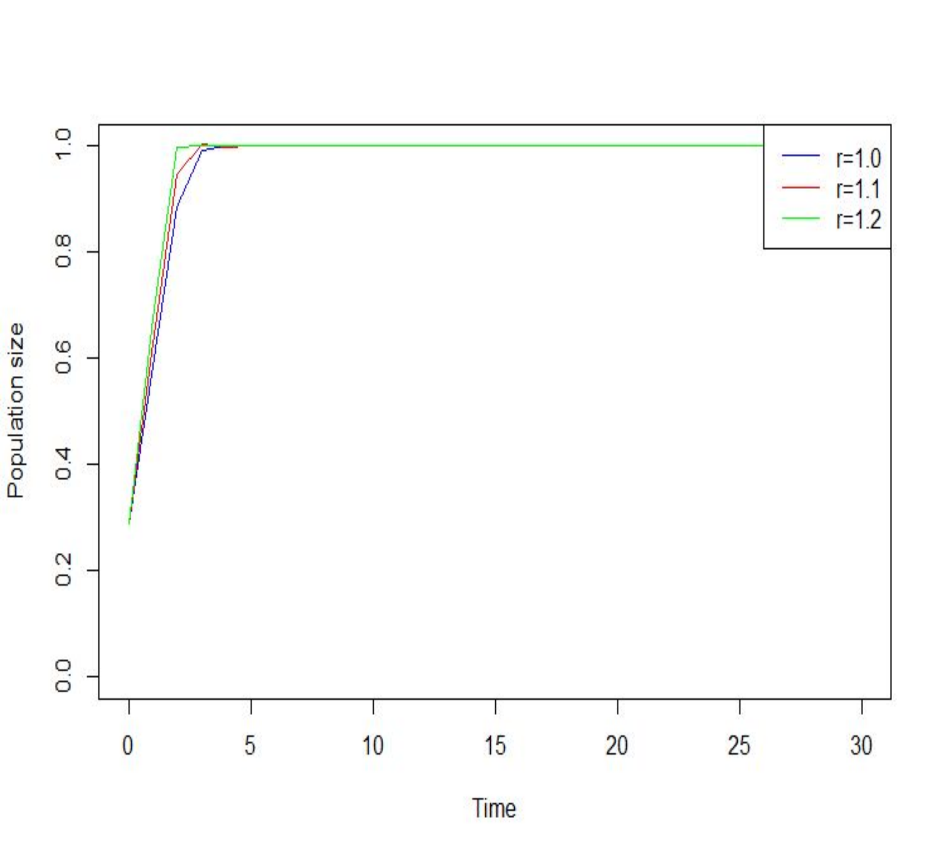


Fig 7: Population size growth at different r values for 30 generations

The obtained time series plots can be analyzed to identify the different types of behavior.

For example, a stable equilibrium is indicated by a plateau at value r=1.5, while a somewhat periodic behavior is indicated by a repeating pattern, albeit the chaotic behavior is indicated by a seemingly random pattern.

Now let's try smaller and higher r values…



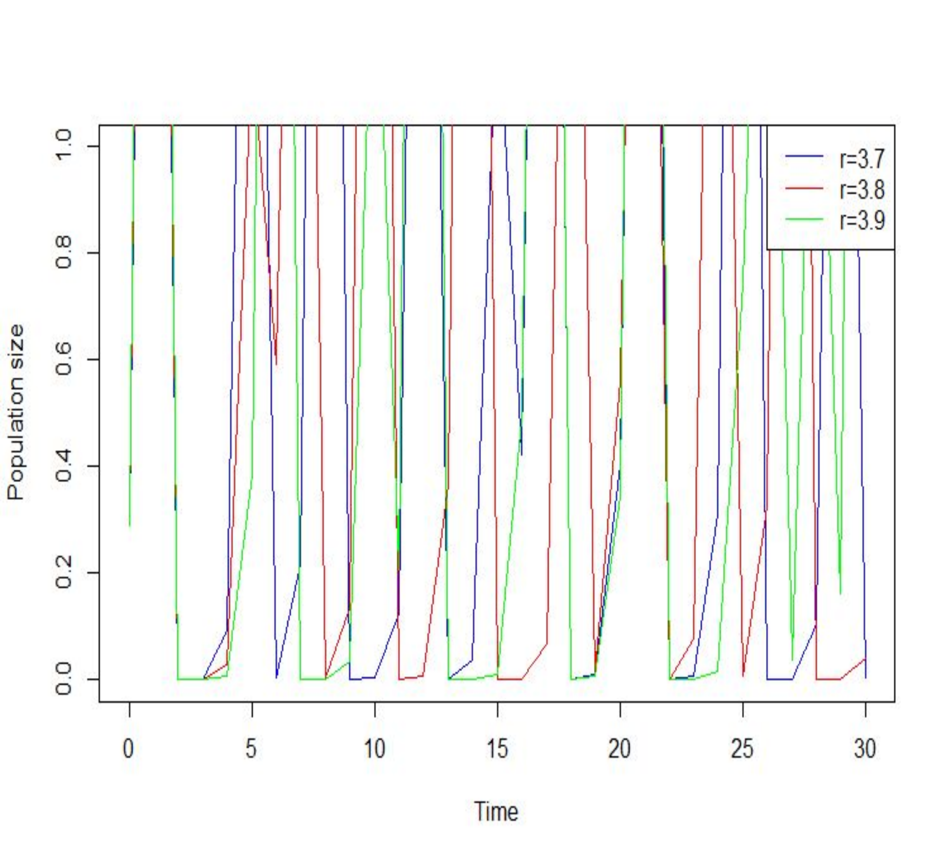


Fig 8: Population size growth at different r values (3.7-3.9) for 30 generations

Plots for Values of Interest

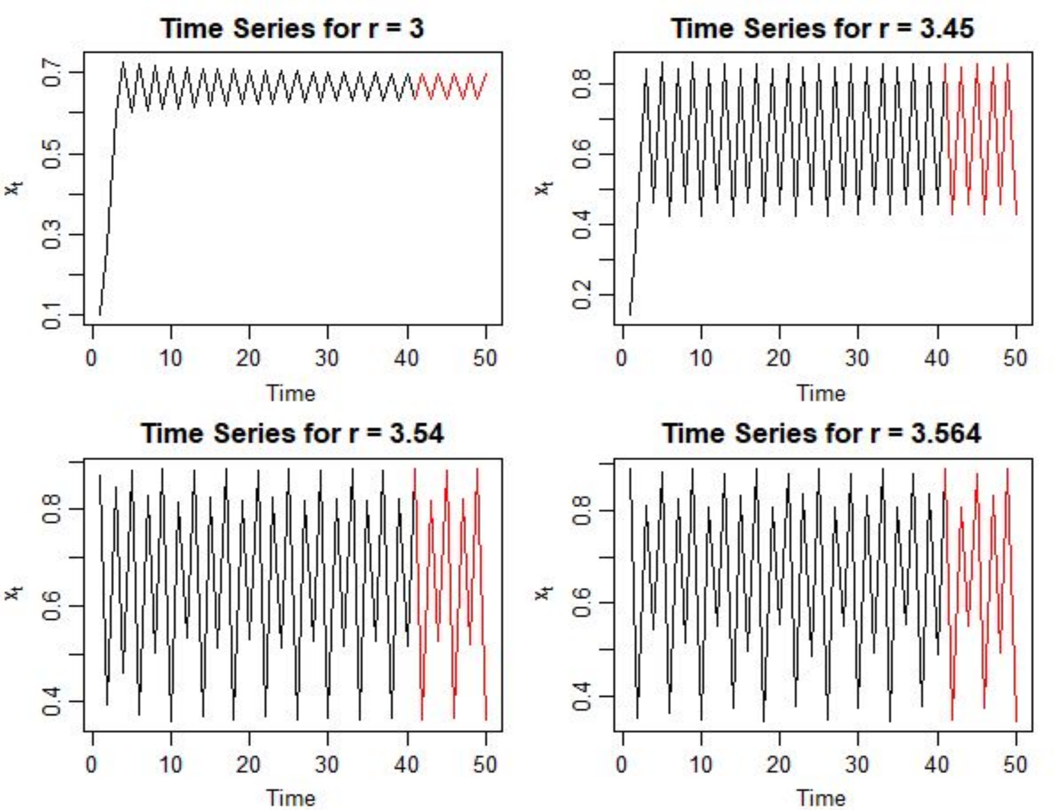


Fig 9: Time series plots for interesting values of r=3 to r=3.564. The red section is added for clarity to represent 10 iterations, from a 50 of total on the x-axis.

Plots for Values of Interest

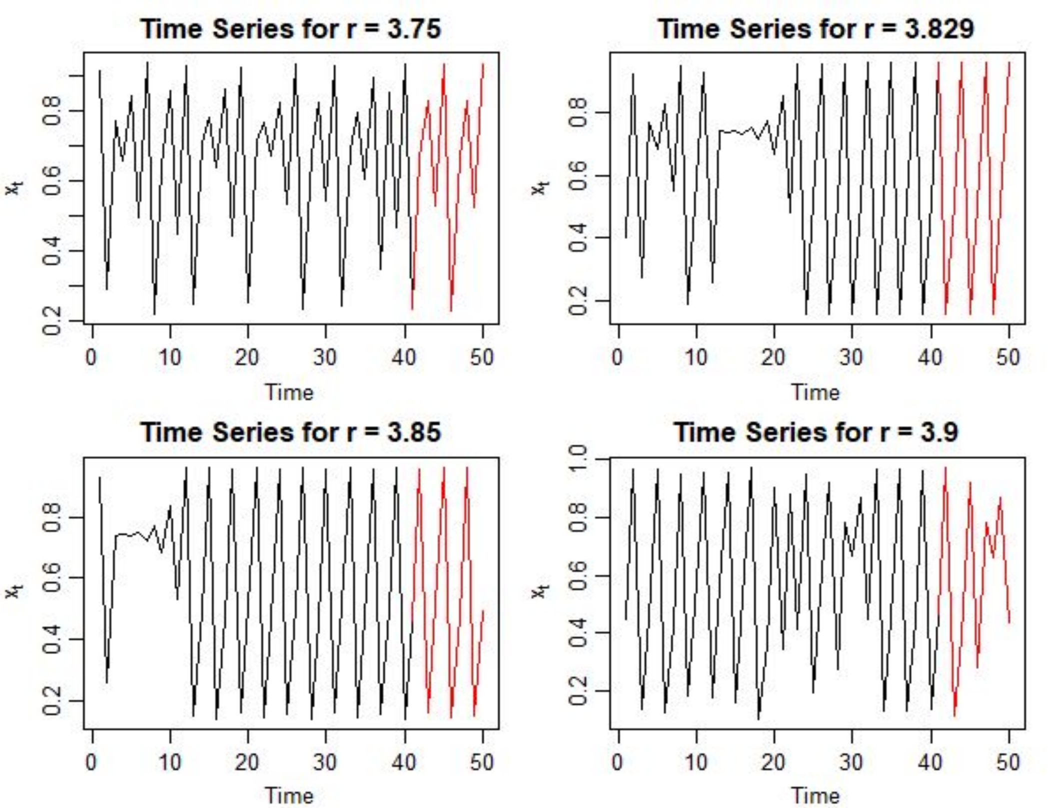


Fig 10: Time series plots for interesting values of r=3.75 to r=3.9. The red section is added for clarity to represent 10 iterations, from a 50 of total on the x-axis.

Histograms of time series

The distributions of x values in the period 3.85 is a direct point in time on the bifurcation diagram at the given r: after the initial period 3 divergence, chaos ensues. This step was repeated for values 3.75 - 3.95, increasing by 0.05 in the next slide.

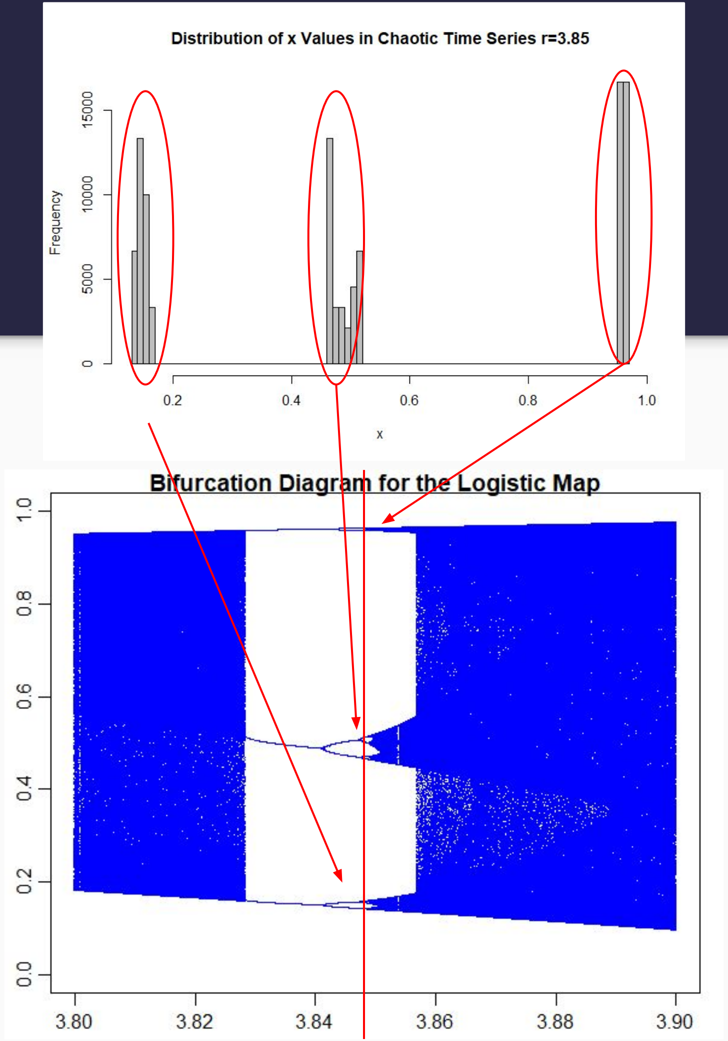
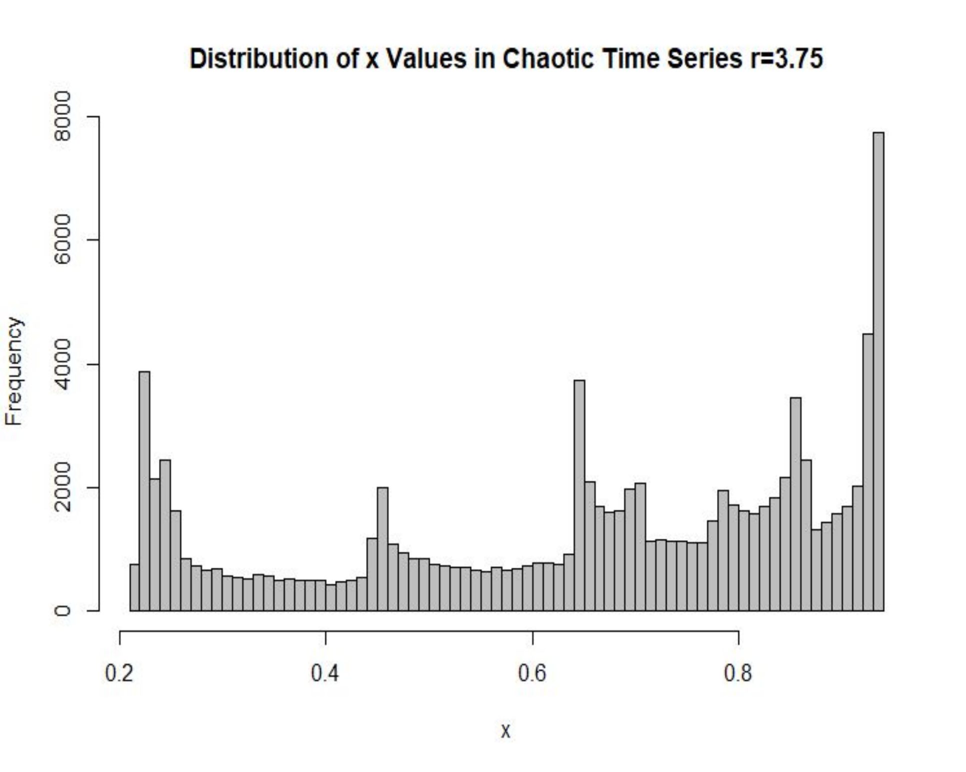


Fig. 11: Chaotic time series in the distribution of x values; r = 3.75 to r = 3.95 increasing by 0.05



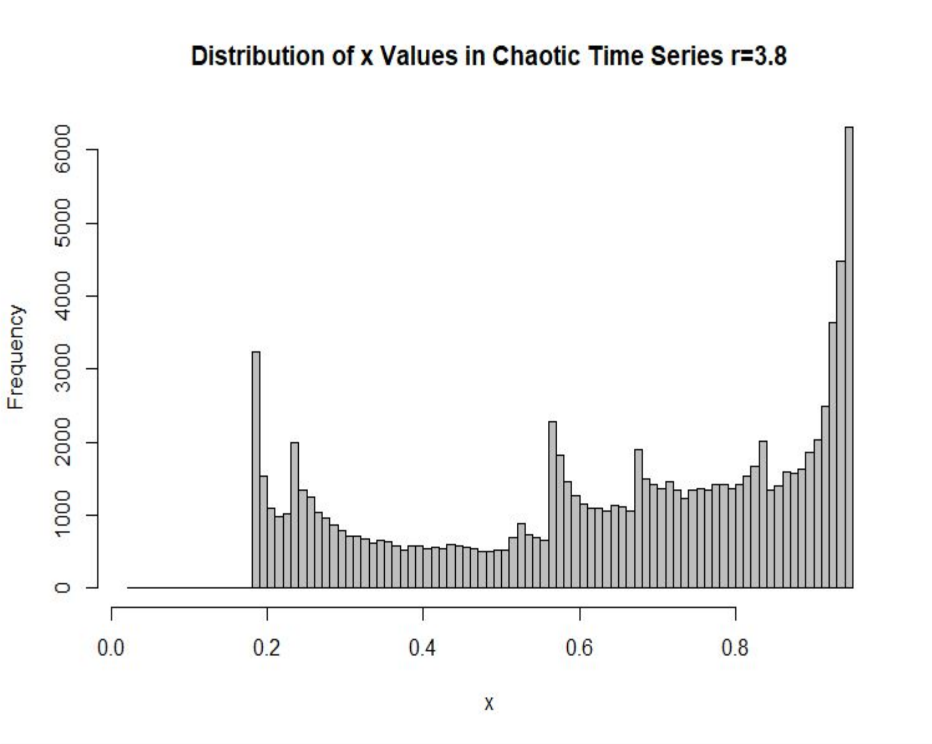
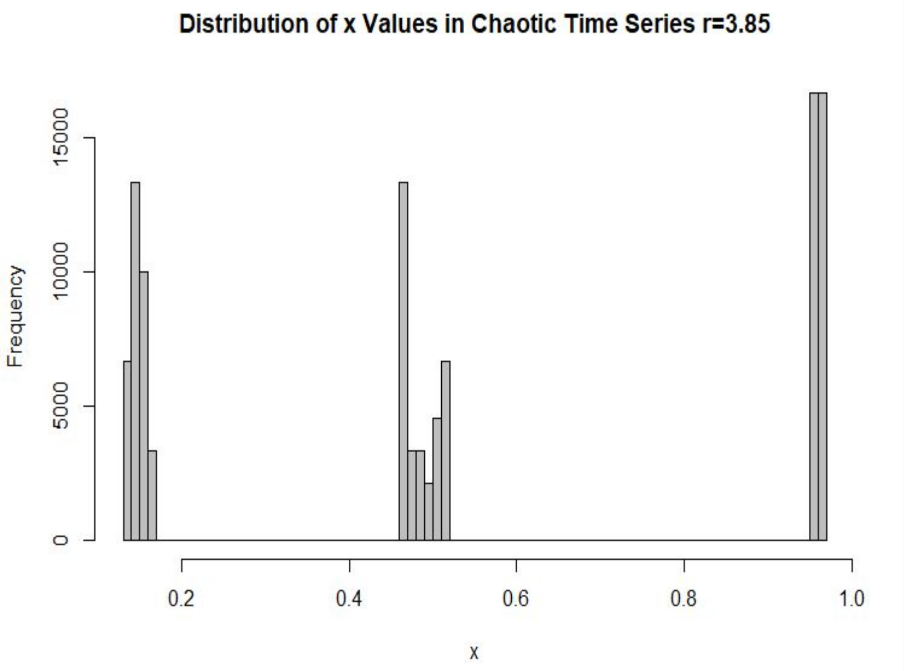


Fig. 11: Chaotic time series in the distribution of x values; r = 3.75 to r = 3.95 increasing by 0.05



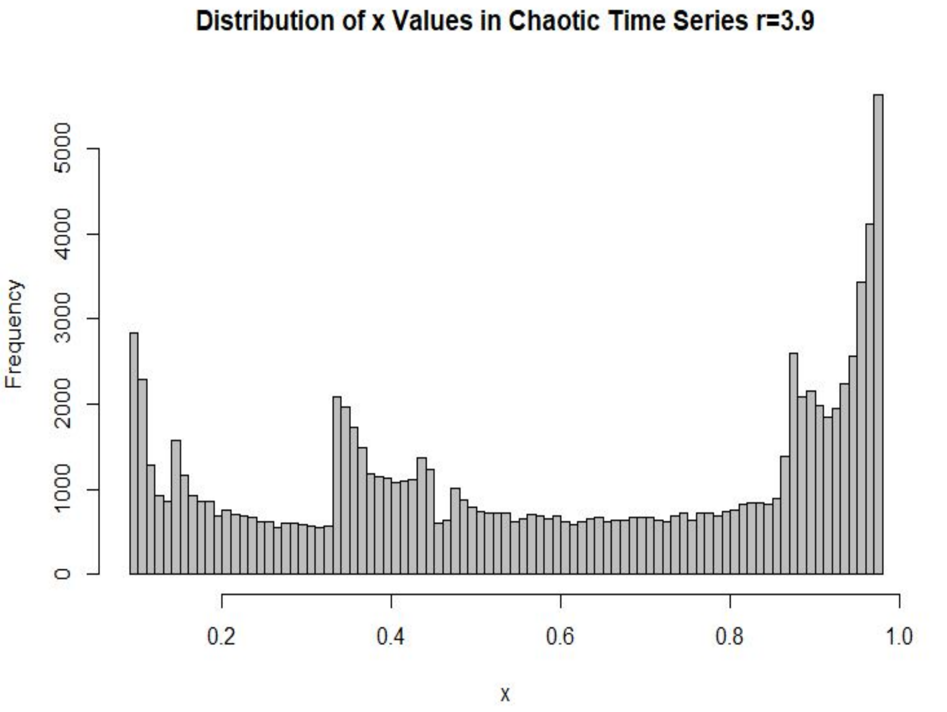
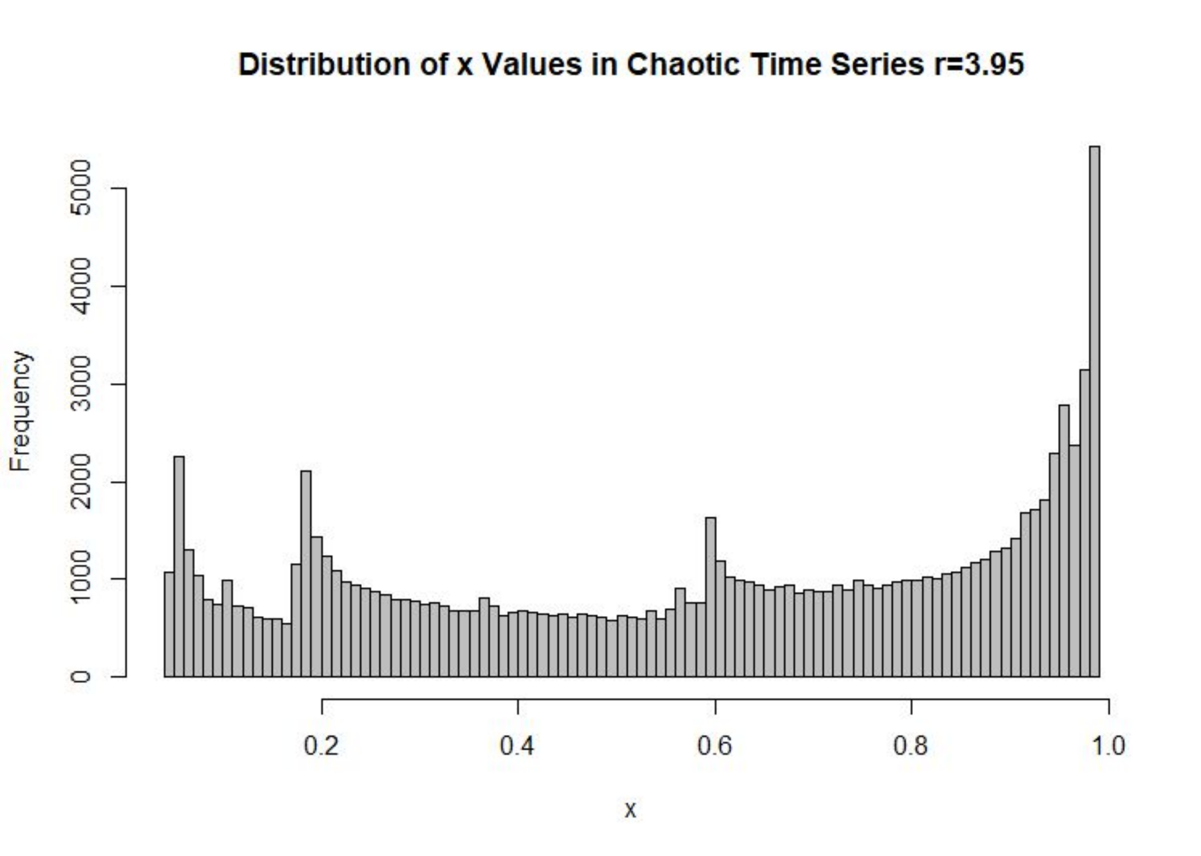
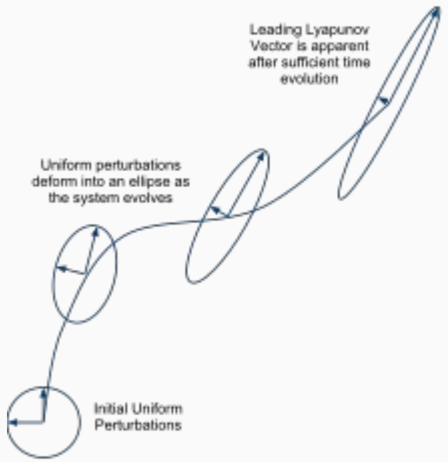


Fig. 11: Chaotic time series in the distribution of x values; r = 3.75 to r = 3.95 increasing by 0.05



Defining Chaos: The Lyapunov exponent

- **Chaotic behavior:** sensitivity to initial conditions and the presence of unpredictable, irregular patterns.
- **Non-chaotic regimes:** stability, predictability, and regular patterns.
- A quantitative measure of chaos: **Lyapunov exponent,** which measures the rate at which nearby trajectories in a system diverge over time.
- The Lyapunov exponent can be used to identify <u>the critical value of a system parameter,</u> such as the parameter r in the logistic map, at which chaotic behavior appears.



Defining Chaotic Behaviour

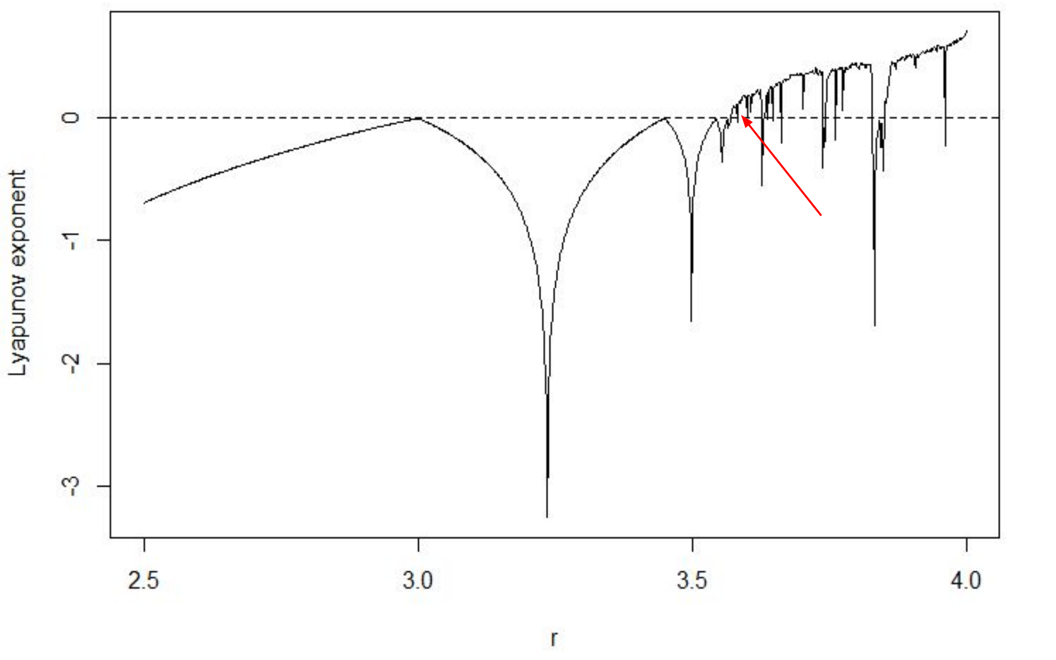


Fig 17: Lyapunov exponent as a quantitative measure for chaotic behavior

- When the Lyapunov exponent is positive, nearby trajectories in the system diverge exponentially, indicating chaotic behavior.
- The critical value of r at which the LDE **exhibits chaotic behavior is approximately 3.57**. This value can be determined by calculating the Lyapunov exponent at different values of r and identifying the value at which the exponent becomes positive.

The Logistic Difference Equation: Simulation Runs

A Spatial Extension of the LDE

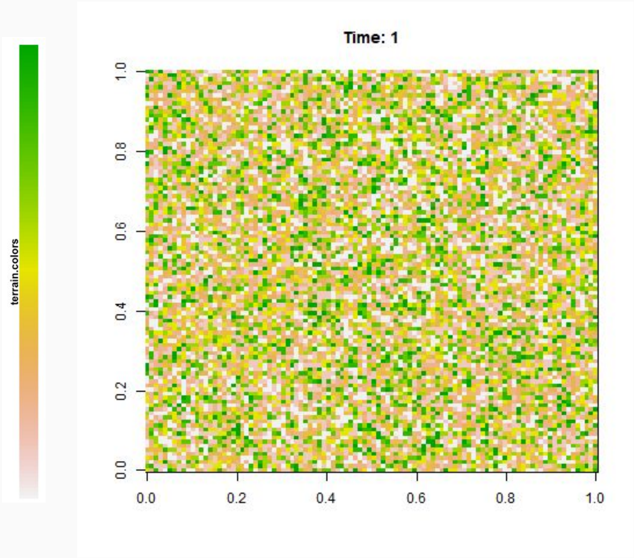


Fig 23. Size of the spatial grid = 100

The Lotka-Volterra model with logistic growth and diffusion:

Used to study the interactions between different species in a spatially structured environment.

1. Define growth rate, carrying capacity, death rate, migration rate and the size of spatial grid.
2. After initializing the grid, we define functions to calculate migration and another to simulate dynamics which updates itself using LDE dynamics.
3. Enforce carrying capacity - this will help us see how space affects populations.
4. Run simulation for 100 times.

A Spatial Extension of the LDE

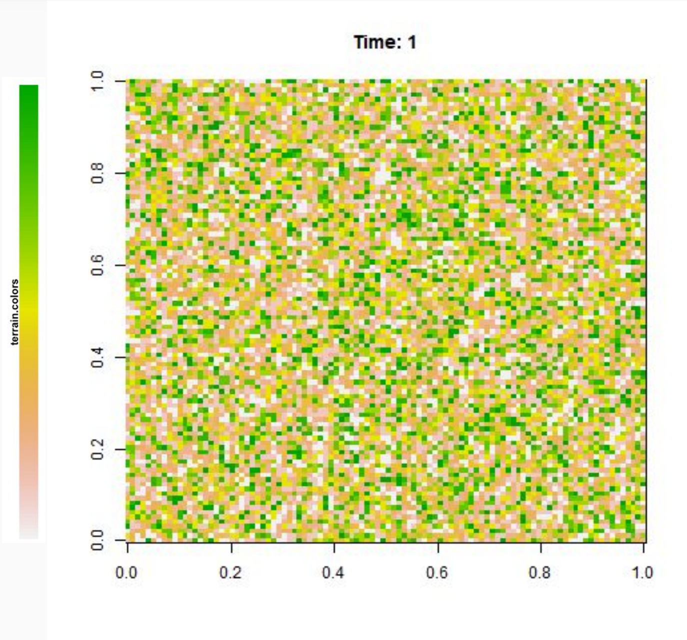


Fig 24. Size of the spatial grid = 100

- Each plot shows the spatial distribution of the population at a given time step, with the color representing the population density.
- As density changes over time as the individuals move and interact with each other and the environment.
- This type of model can be used to study various ecological phenomena.

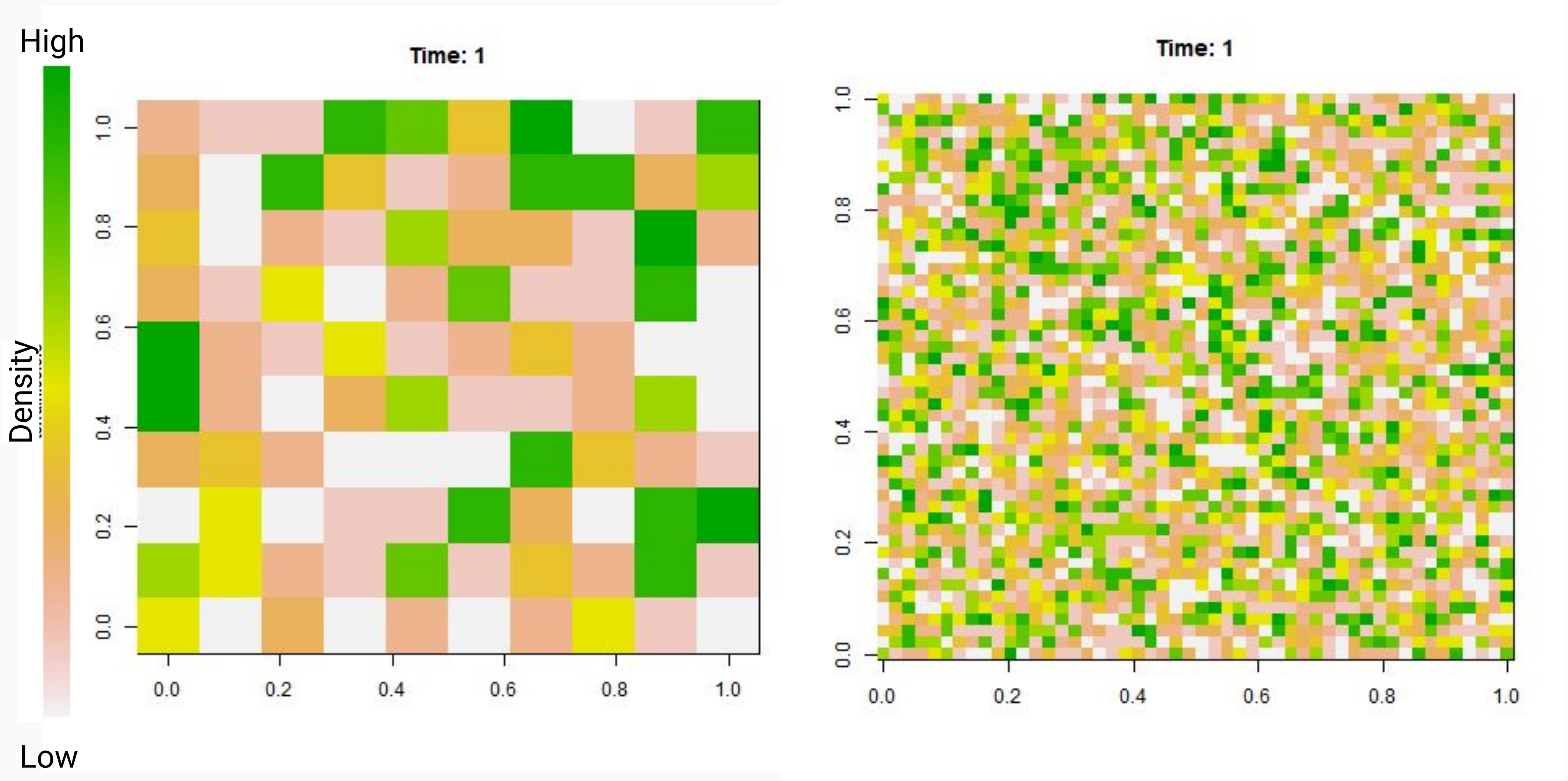


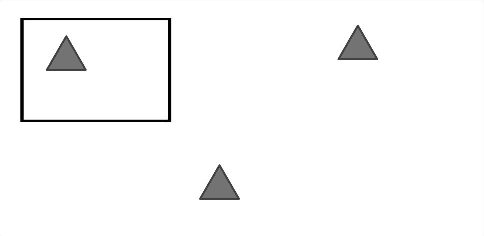
Fig. 25: Size of the spatial grid = 10 and 50

Question: Is it realistic?

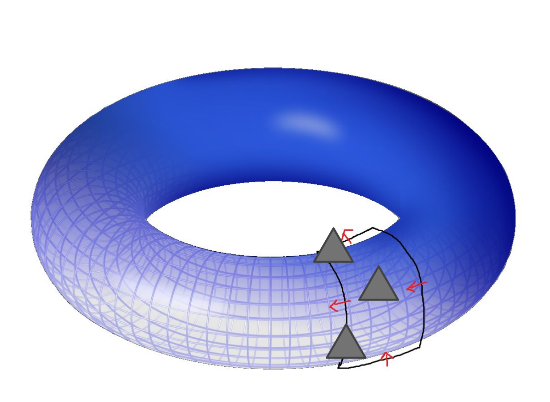
In the infinite space model, it is assumed that the **population density is constant** and **dispersed** throughout an infinite spatial domain. As a result, the system has <u>no corners or edges</u>, and population shifts can happen in any direction.

The population is thought to be restricted to a small, closed domain with a torus-like topology, according to the torus model. As a result, <u>movements of species who cross the boundary "wrap around"</u> to the other side of the domain.

This model assumes that the <u>population is constrained by the spatial structure of the domain</u>, and that there may be **spatial heterogeneities** that could affect the population dynamics. This makes for a more realistic model.



1. Assumes map exists on infinite space



2. Assumes that map exists on a torus-like plane

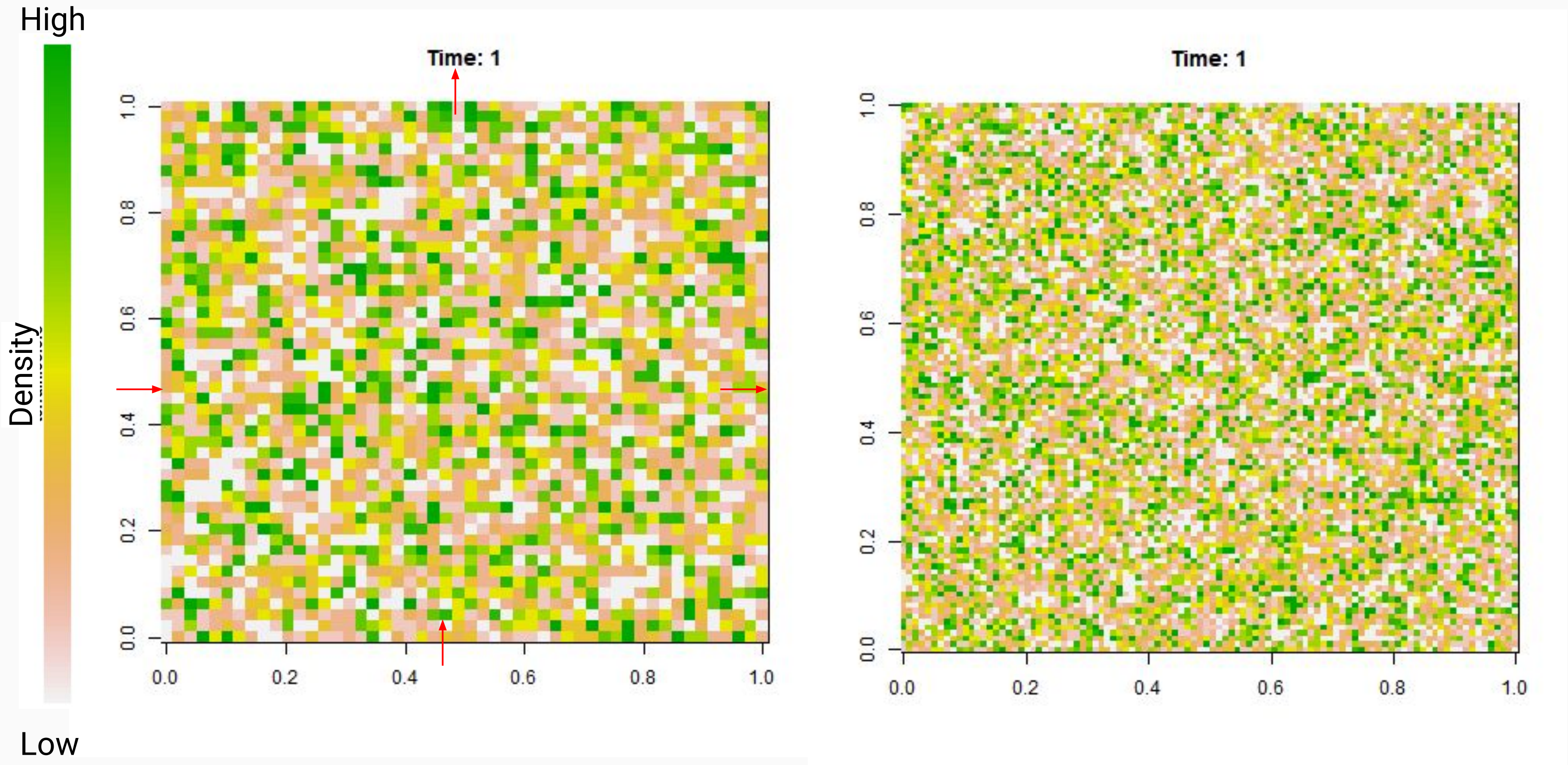


Fig. 26: Torus-like model, spatial grid = 50 and 100

Conclusions: LDE and Its Patterns

- The logistic difference equation is a **simple yet powerful** model that can capture complex behaviours in populations dynamics over time.

- The period-three implies chaos theorem suggests that the onset of chaos occurs at the boundary between the **period-three** and **period-four windows**.

- The logistic difference equation has unique traits: Feigenbaum constant, its bifurcation overlapping on Mandelbrot sets, Butterfly effect, periodicity, and other complex patterns.

- Logistic map is a <u>**deterministic model**</u> in the sense that if you run the model again for a different value it would still generate the same unpredictable aperiodic chaotic values. Hence, it is **not random**.

- The chaotic regime of the logistic equation exhibits **sensitive dependence on initial conditions**, which means that small differences in the initial values of x can lead to large differences in the long-term behavior of the system. This makes it difficult to predict the long-term fate of a population in chaotic conditions.

Conclusions: Defining Chaos & Spatial Models of the LDE

- **The Lyapunov exponent** was used to determine chaos in the LDE. It measures the average rate of exponential divergence or convergence of nearby trajectories in phase space. It

- The Lyapunov exponent can be used to quantify the predictability of a system and to estimate the level of uncertainty or error. In our graph, we saw a stark dip before the Lyapunov exponent was positive, leading us to believe that chaotic behaviour ensues in this deterministic system later on, approximately after ~3.65

- After simulation runs using infinite space and a **torus-like model** we found that the torus-like spatial model is *better suited to simulate real-life population changes*. The simulations also show that spatial heterogeneity can affect the dynamics of populations and lead to the emergence of spatial patterns.

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