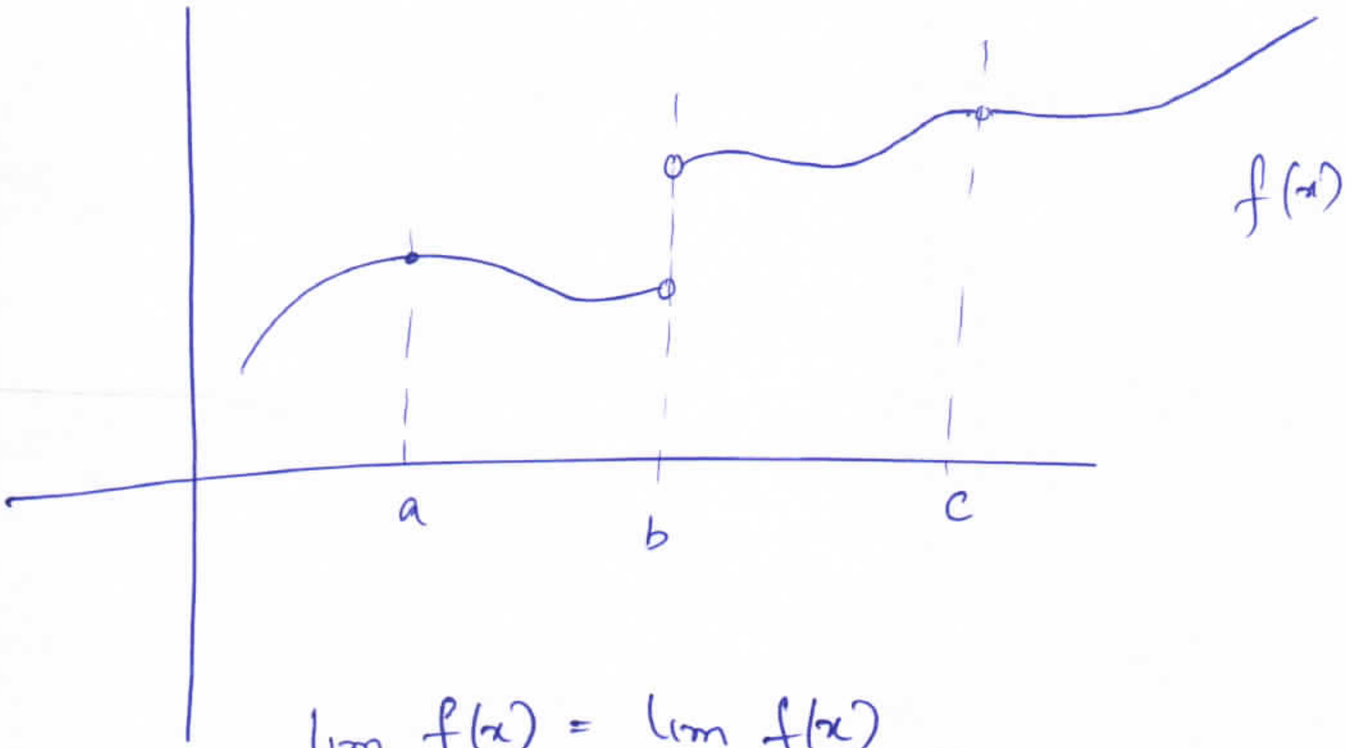
CONTINUITY

↓

NO BREAK



$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

$$\lim_{x\to b^-} f(x) \neq \lim_{x\to b^+} f(x)$$

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$$

If at $x=a$ L.H.L $\neq$ R.H.L $\Longrightarrow$ NOT CONTINUOS At $x=a$

CONDITION for CONTINUITY

A function $f(x)$ is said to be continuous at $x=a$

if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

L.H.L = R.H.L = function value at $x=a$.

CASES WHERE FUNCTION IS NOT CONTINUOS

at $x = a$

i) If $\lim_{x \to a^-} f(x)$ & $\lim_{x \to a^+} f(x)$ exists but are unequal.

$$L.H.L \neq R.H.L$$

ii) If $\lim_{x \to a^-} f(x)$ & $\lim_{x \to a^+} f(x)$ exists and are equal

BUT not equal to $f(a)$

$$L.H.L = R.H.L \neq f(a)$$

iii) $f(a)$ is not defined.

iv) Atleast one of the limits (L.H.L, R.H.L) does not exist

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

At critical points. (where function is changing value)

At $x = 0$

L.H.L. $= -0$

R.H.L $= +0$

$f(0) = 0$

$\because$ L.H.L $=$ R.H.L $= f(0)$

$\Rightarrow$ continuos at $x=0$

$$f(x) = [x] = \begin{cases} \vdots & \\ -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ \vdots & \end{cases}$$

$x = 0$

$L.H.L = -1 \neq R.H.L = 0$

$\Rightarrow$ NOT CONTINUOS At $x = 0$

$f(x) = [x]$ is not continuous at $x = a$, $a \in \mathbb{Z}$

Check Continuity of

1) If $f(x) = \begin{cases} 5 + 3\dfrac{\tan x}{x} & x \neq 0 \\ 8 & x = 0 \end{cases}$

At $x = 0$

$L.H.L = R.H.L = 5+3 = 8 = f(0) = 8$

continuos at $x = 0$

2) If $f(x) = \dfrac{|x|}{x} = \begin{cases} 1. & x > 0 \\ -1 & x < 0 \\ \text{Undefined.} & x = 0 \end{cases}$

$f(0)$ is not defined

$\Rightarrow$ not continuos at $x = 0$

TYPES OF DISCONTINUITY

(I) DISCONTINUITY OF FIRST KIND.

If L·H·L & R·H·L exist & are finite

(a) L·H·L $\neq$ R·H·L

(NON-REMOVABLE)

(b) L·H·L = R·H·L $\neq f(a)$

(REMOVABLE)

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

At $x = 0$

L·H·L $= -1 \neq$ R·H·L $= 1$

$$f(x) = \begin{cases} 4-x & 0 < x < 2 \\ x & 2 < x < 4 \end{cases}$$

At $x = 2$

L·H·L $= 4-2 = 2 =$ R·H·L $= 2$

$f(2)$ is not defined.

We can remove the discontinuity by defining $f(2) = 2$

$$f(x) = \begin{cases} 4-x & 0 < x < 2 \\ 2 & x = 2 \\ x & 2 < x < 4 \end{cases}$$

(II) DISCONTINUITY OF SECOND KIND.

If either L·H·L or R·H·L or both does not exist.

(a) Infinite Discontinuity.

at $x = a$

$\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ be infinite

L·H·L R·H·L (∞ or $-\infty$)

(b) Oscillatory Discontinuity.

$f(x)$ oscillates finitely or infinitely as $x \longrightarrow a$.

$f(x) = x \sin \frac{1}{x}$

$$\lim_{x \to 0} f(x)$$

$$= 0$$

$f(x) = \sin \frac{1}{x}$

$$\lim_{x \to 0} f(x) \Rightarrow \lim_{x \to 0} \sin \frac{1}{x}$$

Q1 Test the continuity of

$$f(x) = \begin{cases} 4-x^2 & x \le 0 \\ x-5 & 0 < x \le 1 \\ 4x^2-9 & 1 < x < 2 \\ 3x+4 & x \geq 2 \end{cases}$$

$x = 0$

L·H·L
$f(0^-) = 4-0 = 4$

R·H·L
$f(0^+) = 0-5 = -5$

L·H·L $\neq$ R·H·L
Discontinuos.
(I) Non-Removable

$x = 1$

L·H·L
$f(1^-) = 1-5 = -4$

R·H·L
$f(1^+) = 4(1)^2-9 = -5$

L·H·L $\neq$ R·H·L
Discontinuos
(I) Non-Removable

$x = 2$

L·H·L
$f(2^-) = 4(2)^2-9 = 7$

R·H·L
$f(2^+) = 3(2)+4 = 10$

L·H·L $\neq$ R·H·L
Discontinuos
(I) Non-Removable

Q2 $$f(x) = \begin{cases} ax^2 + 9x - 5 & x < 1 \\ b & x = 1 \\ (x+3)(2x-a) & x > 1 \end{cases}$$

If $f(x)$ is continuos everywhere
find a, b

$x = 1$

$f(1^-) = f(1^+) = f(1)$

$a(1)^2 + 9(1) - 5 = (1+3)(2(1)-a) = b$

$a + 4 = 8 - 4a = b$

$a = 4/5 \qquad b = 24/5$

Q3 $f(x) = \lim_{n \to \infty} \dfrac{\ln(2+x) - x^{2n} \sin x}{1+x^{2n}}$

Check continuity at $x = 1$

$$f(1) = \lim_{n \to \infty} \frac{\ln(2+1) - 1 \times \sin 1}{1+1}$$

$x \to 1^-$ $x^{2n} \to 0$
$n \to \infty$

$$= \frac{\ln 3 - \sin 1}{2}$$

L.H.L

$f(1^-)$

$$= \frac{\ln 3 - 0}{1+0} = \ln 3$$

R.H.L

$$f(1^+) \lim_{n \to \infty} \frac{\dfrac{\ln(2+x)}{x^{2n}} - \sin x}{\dfrac{1}{x^{2n}} + 1}$$

$x \to 1^+$ $x^{2n} \to \infty$
$n \to \infty$

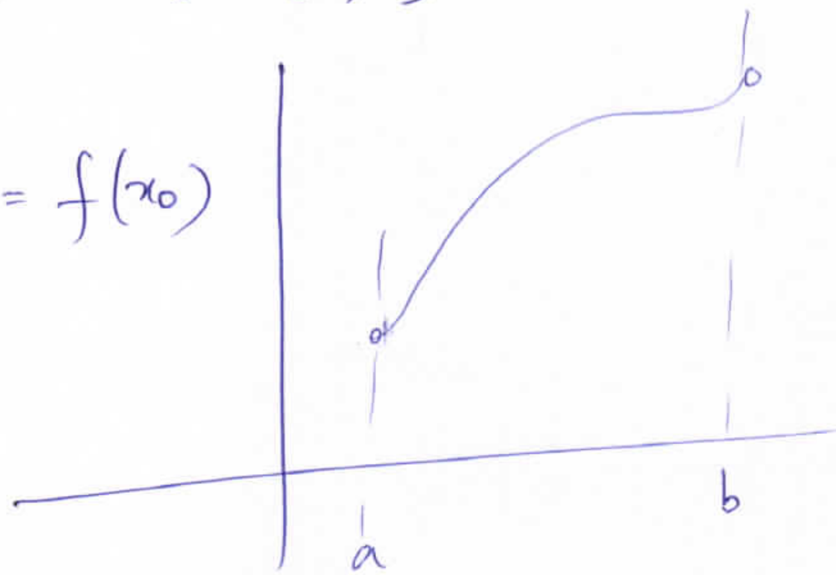
$$= -\sin 1$$

Discontinuos at $x = 1$

Non-Removable Type 1.

a) Suppose we have to check continuity of $f(x)$ in $x_0 \in (a, b)$

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$



b) Suppose we have to check continuity in $x_0 \in [a, b]$

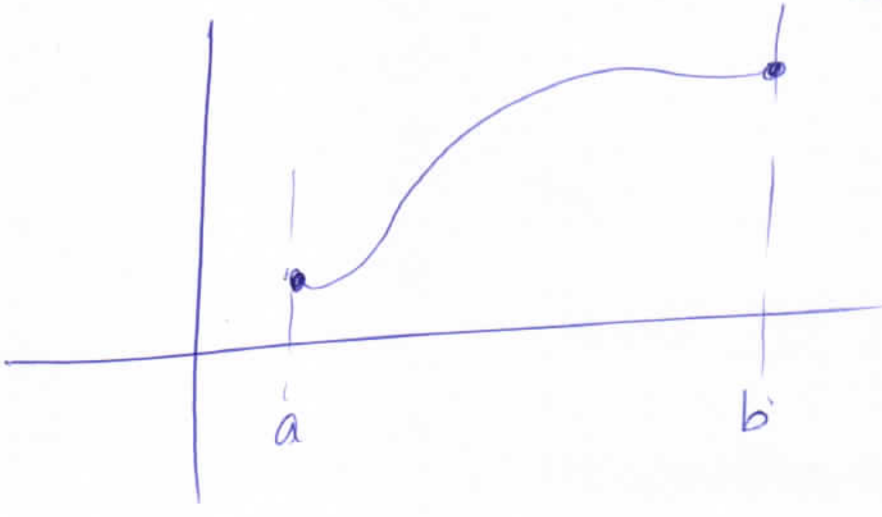
$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0) \qquad x_0 \in (a, b)$$

at $x_0 = a$

$f(a^+) = f(a) =$ finite

at $x_0 = b$.

$f(b^-) = f(b)$ = finite.



Properties of continuous functions

(1) All polynomials, exponential, logarithmic functions are continuous in its domain.

(2) If f & g are two continuous function at $x = a$

then $f \pm g$ is continuous at $x = a$

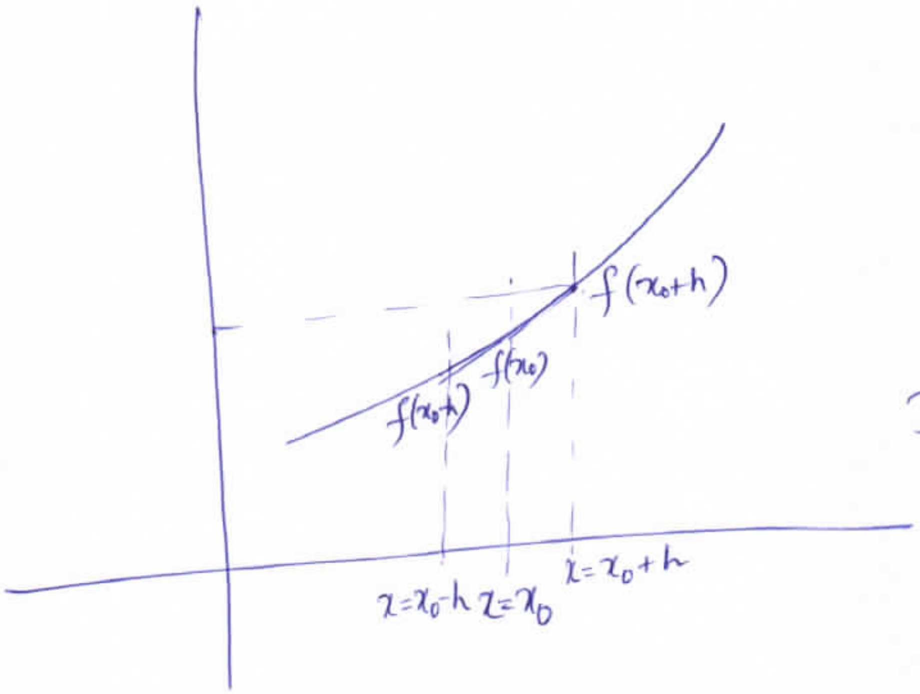
$kf(x)$ is continuous at $x = a$.

$f \cdot g$ is continuous at $x = a$

$\frac{f}{g}$ is continuous at $x = a$

provided $g(a) \neq 0$

DIFFERENTIABILITY



Right Part Slope

$$f'(x_0^+) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{(x_0+h) - (x_0)}$$

$$= RHD \text{ (Right hand Derivative)}$$

Left Part slope.

$$f'(x_0^-) = \lim_{h\to 0} \frac{f(x_0-h) - f(x_0)}{(x_0-h) - (x_0)}$$

$$= LHD \text{ (Left hand Derivate)}$$

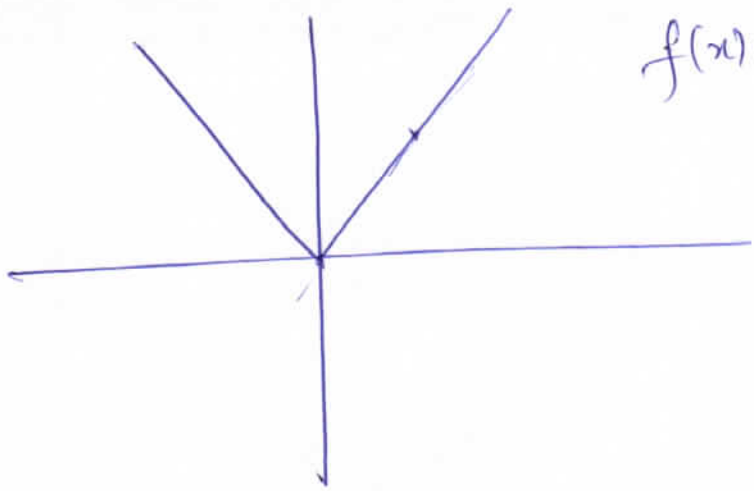
h is very small

$h \longrightarrow 0$

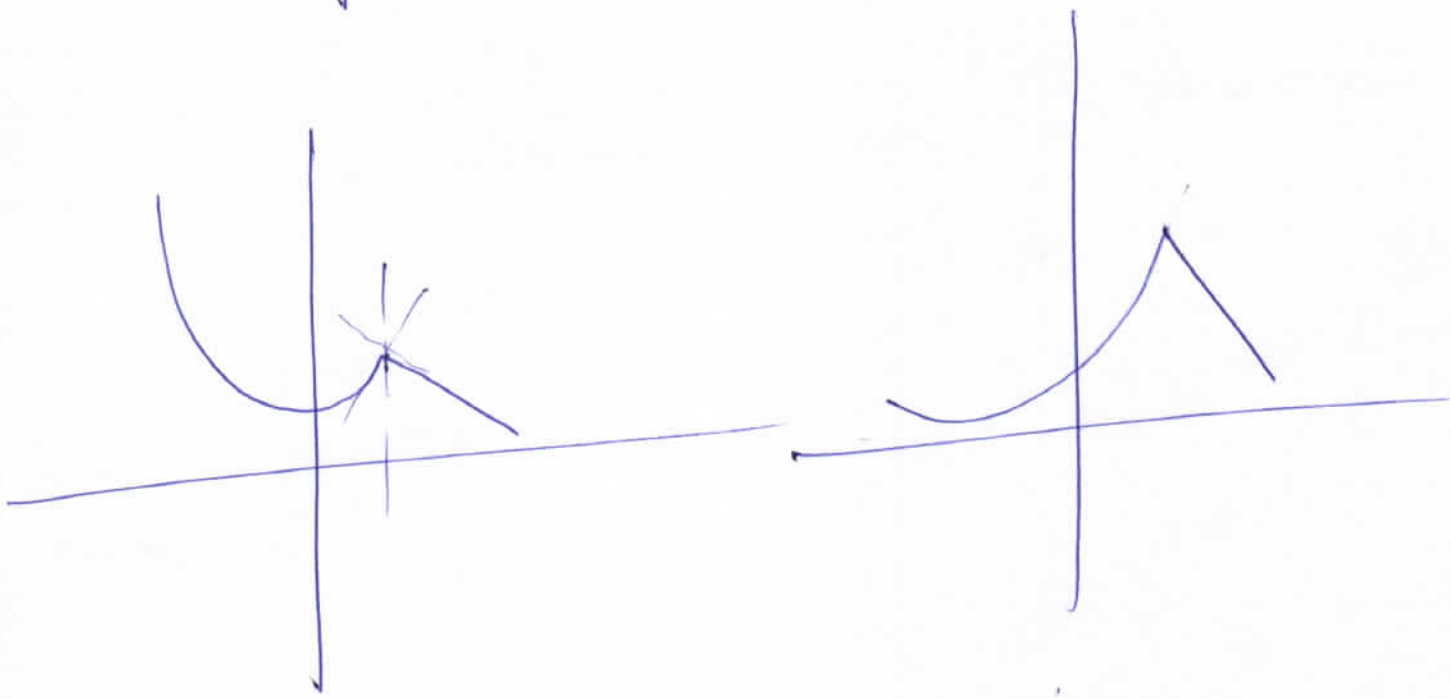
for the function to be differentiable at $x = x_0$

$$L \cdot H \cdot D = R \cdot H \cdot D = \text{finite}.$$

$$\lim_{h\to 0} \frac{f(x_0-h) - f(x_0)}{-h} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \text{finite}$$



$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



Graphically if there is a kink (sharp corner) at any point on the curve then the function is not differentiable at that point.

If function is NOT continuous at $x=a$
$\Longrightarrow$ NOT DIFFERENTIABLE at $x=a$.

A function is DIFFERENTIABLE at $x=a$ if we can draw a unique tangent to the function at $x=a$.

Check differentiability of $f(x)$

① $f(x) = \begin{cases} x & x < 1 \\ 2-x & 1 \le x \le 2 \\ -(x^2-3x+2) & x > 2 \end{cases}$

② $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

③ $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

④ $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Sol^n ② $x = 0$

L.H.L

$f(0^-) = 0$

R.H.L

$f(0^+) = 0$ $f(0) = 0$

continuos.

L.H.D

$\lim\limits_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim\limits_{h \to 0} \frac{f(-h) - 0}{-h} = \lim\limits_{h \to 0} \frac{-(-h)}{-h} = \lim\limits_{h \to 0} -1$

$= -1$

R·H·D

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \Rightarrow \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

L·H·D ≠ R·H·D ⟹ Not Differentiable.

(3) $x = 0$

L·H·L = R·H·L = 0 $\quad f(0) = 0$

continuous.

$$L\cdot H\cdot D = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{f(-h)}{-h} = \lim_{h\to 0} \frac{(-h)^2 \sin\left(\frac{1}{-h}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{h^{\cancel{2}}\left(\cancel{-}\sin\frac{1}{h}\right)}{\cancel{-h}}$$

$$= \lim_{h\to 0} h \sin\frac{1}{h} = 0$$

$$R\cdot H\cdot D = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^{\cancel{2}} \sin\frac{1}{h}}{\cancel{h}} = \lim_{h\to 0} h \sin\frac{1}{h} = 0$$

L·H·D = R·H·D = 0 Differentiable.

(4) $x = 0$

$L \cdot H \cdot L = R \cdot H \cdot L = 0$ $\quad f(0) = 0$

continuous.

$$L \cdot H \cdot D = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h)}{-h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h} \sin\left(-\frac{1}{h}\right)}{-\cancel{h}}$$

$$= \lim_{h \to 0} -\sin \frac{1}{h}$$

$$R \cdot H \cdot D = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h} \sin \frac{1}{h}}{\cancel{h}}$$

$$= \lim_{h \to 0} \sin \frac{1}{h}.$$

$L \cdot H \cdot D \neq R \cdot H \cdot D$

not defferentiable.

(4) Critical points

$x = 1$ & $x = 2$

At $x = 1$

$L \cdot H \cdot L = 1$. $\quad R \cdot H \cdot L = 2 - 1 = 1$

$f(1^-) = 1$ $\quad f(1^+) = 1$ $\quad f(1) = 2 - 1 = 1$

continuos.

at $x=1$

L.H.D.

$$f'(1^-) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h) - 1}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

R.H.D.

$$f'(1^+) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{2-(1+h) - 1}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

L.H.D $\neq$ R.H.D not defferentiable at $x=1$

$x=2$

L.H.L.

$f(2^-) = 2-2 = 0$

R.H.L

$f(2^+) = -(2^2-3(2)+2)$
$= 0$

$f(2) = 2-2 = 0$

L.H.L = R.H.L $= f(2) \Rightarrow$ continuos at $x=2$.

L.H.D $\Rightarrow$ $f'(2^-)$ $\lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \frac{2-(2-h) - 0}{-h}$

$$= \lim_{h \to 0} \frac{h}{-h} = -1$$

R.H.D

$$f'(2^+) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{-\left((2+h)^2 - 3(2+h) + 2\right) - 0}{h}$$

$$= \lim_{h\to 0} - \frac{\cancel{h}(h+1)}{\cancel{h}} = \lim_{h\to 0} -(h+1)$$

$$= -1$$

$\therefore$ L.H.D = R.H.D = -1 Differentiable at $x=2$.

$$f'(x) = \begin{cases} 1 & x < 1 \\ -1 & 1 \le x \le 2 \\ -(2x-3) & x > 2 \end{cases}$$

$x=1$

L.H.D $= 1 \neq$ R.H.D $= -1$

Not Differentiable.

$x=2$

L.H.D $= -1 =$ R.H.D $= -1$

Differentiable.

(1) If $f(x) = \begin{cases} ax(x-1) + b & x < 1 \\ x - 1 & 1 \le x \le 3 \\ px^2 + qx + 2 & x > 3 \end{cases}$

Find values of a, b, p, q so that $f(x)$ is continuous everywhere but not differentiable at $x = 1$

Also $f'(x)$ is continuous at $x = 3$.

Q2) Test continuity & Differentiability of

$$f(x) = (|x| - |x-1|)^2$$

Q3) $f(x) = \begin{cases} -2 & -3 \le x < 0 \\ x - 2 & 0 \le x \le 3 \end{cases}$

If $g(x) = |f(x)| + f(|x|)$

Check continuity of $g(x)$ $\forall x \in [-3, 3]$

Q1 $x=1$ & $x=3$

at $x=1$

Continuity

$f(1^-)=f(1^+)=f(1)$

$a(1)(1-1)+b = 1-1 = 1-1$

$b=0$ ✓

at $x=3$

Continuity

$f(3^-)=f(3^+)=f(3)$

$3-1 = p(3)^2+q(3)+2 = 3-1$

$= 9p+3q+2=2$

$\Rightarrow 9p+3q=0$

$\Rightarrow 3p+q=0$ —— (1)

At $x=1$ Not differential

$f'(1^-) \neq f'(1^+)$

$2a(1)-a \neq 1.$

$a \neq 1$

$a \in \mathbb{R}-\{1\}.$

$$f'(x) = \begin{cases} 2ax-a & x<1 \\ +1 & 1\le x\le 3 \\ 2px+q. & x>3 \end{cases}$$

$f'(3^-)=f'(3^+)=f'(3)$

$1 = 2p(3)+q = 1$

$= 6p+q=1$ —— (2)

from (1) & (2) $p=1/3$ $q=-1$

Q2) $$f(x) = \begin{cases} \{-x-(-(x-1))\}^2 = 1 & x<0 \\ \{x-(-(x-1))\}^2 = (2x-1)^2 & 0\le x<1 \\ \{x-(x-1)\}^2 = 1 & x\ge 1 \end{cases}$$

$$f(x) = \begin{cases} 1 & x < 0 \\ (2x-1)^2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

for $x = 0$ & $x = 1$

At $x = 0$

$$\left.\begin{aligned} L\cdot H\cdot L &= 1 \\ R\cdot H\cdot L &= 1 \\ f(0) &= 1 \end{aligned}\right\} \text{continuous.}$$

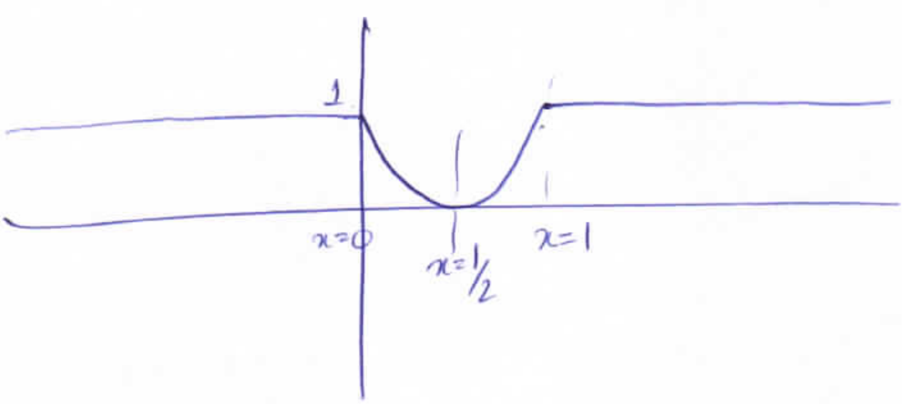
At $x = 1$

$$\left.\begin{aligned} L\cdot H\cdot L &= 1 \\ R\cdot H\cdot L &= 1 \\ f(1) &= 1 \end{aligned}\right\} \text{continuous.}$$

$$f'(x) = \begin{cases} 0 & x < 0 \\ 4(2x-1) & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}$$

$$\left.\begin{aligned} L\cdot H\cdot D &= 0 \\ R\cdot H\cdot D &= -4 \end{aligned}\right\} \text{Not Differentiable}$$

$$\left.\begin{aligned} L\cdot H\cdot D &= 4 \\ R\cdot H\cdot D. &= 0 \end{aligned}\right\} \text{Not Differentiable}$$



$g(x)$

$|f(x)|$ + $f(|x|)$

$$= \begin{cases} |-2| & -3 \le x < 0 \\ |x-2| & 0 \le x \le 3 \end{cases}$$

$$= \begin{cases} 2 & -3 \le x < 0 \\ 2-x & 0 \le x < 2 \\ x-2 & 2 \le x \le 3 \end{cases}$$

$f(|x|) = |x|-2 \quad 0 \le |x| \le 3$

$$\begin{cases} -x-2 & -3 \le x < 0 \\ x-2 & 0 < x \le 3 \end{cases}$$

$$g(x) = \begin{cases} 2 + (-x-2) = -x & -3 \le x < 0 \\ 2-x + (x-2) & 0 \le x < 2 \\ x-2 + (x-2) & 2 \le x \le 3 \end{cases}$$

$$= \begin{cases} -x & -3 \le x < 0 \\ 0 & 0 \le x < 2 \\ 2x-4 & 2 \le x \le 3 \end{cases}$$