SEQUENCES & SERIES.

SEQUENCE is a set of numbers with a defined pattern.

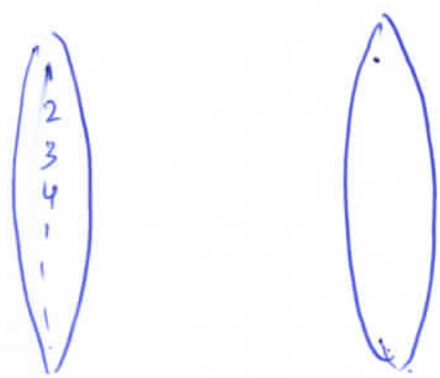
1	2	3	4	5	- - -
1^{st} term	2^{nd} term	3^{rd} term	4^{th} term		- - -

t_1 $\quad t_2$ $\quad t_3$ $\qquad\qquad t_n = n$

1	4	9	16	25	- - -
t_1	t_2	t_3	t_4	t_5	- - -

$t_n = n^2$

$S : N \longrightarrow R$



$t_n = f(n)$

$S_1 : t_n = n$

$S_2 : t_n = n^2$

$S_3 : t_n = n^2 + 1 \qquad 2, 5, 10 - - - -$

$S_4 : t_n = n^2 - \frac{1}{2} \qquad \frac{1}{2}, \frac{7}{2}, \frac{17}{2} - - - -$

SERIES

$$S' = 2+5+10+ \cdots$$

$$S' = 2+5-10+ \cdots$$

we get a series by adding or subtracting the terms in a sequence.

A sequence of numbers $t_1, t_2, t_3 \cdots$ satisfies the relation $t_{n+1} = t_n + t_{n-1}$ for $n \geq 2$

given $t_1 = t_2 = 1$

find t_4

$$t_{3+1} = t_3 + t_{3-1}$$

$$t_4 = t_3 + t_2 \qquad t_4 = 2+1 = 3$$

$$t_{2+1} = t_2 + t_{2-1}$$

$$t_3 = t_2 + t_1$$

$$= 1+1 = 2$$

New terms depends on values of old terms. This type of sequence is a recursive sequence.

Fibonacci Series.

0 1 1 2 3 5 8 13

t_1 t_2 $\qquad t_{n+1} = t_n + t_{n-1} \qquad n \geq 2$

$t_1 = 0$ $t_2 = 1$

Find t_2, if $t_0, t_1, t_2, t_3 \dots$ satisfies the relation $t_{n+1} = 3t_n - 2t_{n-1}$ $\quad n \geq 1$

$t_0 = 2$ & $t_1 = 3$

$$t_{1+1} = 3t_1 - 2t_0$$

$$t_2 = 3 \times 3 - 2 \times 2$$
$$= 5$$

① Arithmetic Sequence or Arithmetic Progression (AP)

This is a sequence in which the difference between any term and its preceeding term is a constant. This constant is called the common difference of this AP.

$$t_2 - t_1 = \text{const} = d$$
$$t_3 - t_2 = d$$
$$t_4 - t_3 = d$$
$$\vdots$$
$$t_{n-1} - t_{n-2} = d$$
$$t_n - t_{n-1} = d.$$

$$t_n - t_1 = (n-1)d \qquad a_1 = a$$

$$t_n = t_1 + (n-1)d$$

$$a_n = a + (n-1)d.$$

S 1 2 3 4 5 6 - -

$d = 1$ $a = 1$

P 2 3 4 5 6 - - - -

$d = 1$ $a = 2$

Q 2 4 6 8 10 12 - -

$d = 2$ $a = 2$

R 1 3 5 7 9 11 -

$d = 2$ $a = 1$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$S_2 = \frac{2}{2}\left(2(1) + (2-1)1\right)$$

$$= 3$$

$$S_4 = \frac{4}{2}\left(2(2) + 3(1)\right)$$

$$= 14$$

$$a_n = a + (n-1)d$$

S_n be a finite series for this AP till n terms

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

$$S_n = a + (a+d) + (a+2d) + \cdots (a + (n-1)d)$$

$$\cancel{= n(a) + (d + 2d + 3d + \cdots (n-1)d)}$$

$$S_n = (a + (n-1)d) + (a + (n-2)d) + \cdots + a$$

$$2S_n = (2a + (n-1)d) + (2a + (n-1)d) + \cdots (2a + (n-1)d)$$

$$2S_n = n(2a + (n-1)d)$$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$S_n = \frac{n}{2}(a + a + (n-1)d)$$

$$= \frac{n}{2}(a_1 + a_n)$$

$S_2(S)$ $S_4(P)$ $S_5(Q)$ $S_7(R)$

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

$$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$$

$$S_n - S_{n-1} = a_n$$

Q1) Show that if $t_1, t_2, t_3, \ldots t_n$ are in A.P

a) $t_1 \pm k, t_2 \pm k, t_3 \pm k \ldots t_n \pm k$ are also in A.P

b) $t_1 k, t_2 k, t_3 k \ldots t_n k$ are in A.P

c) $t_1/k, t_2/k, t_3/k \ldots t_n/k$ are in A.P.

~~Q2 QI f Q1, Q2 - Q QQ~~

d) Prove that $\dfrac{1}{\sqrt{t_1}+\sqrt{t_2}} + \dfrac{1}{\sqrt{t_2}+\sqrt{t_3}} + \cdots + \dfrac{1}{\sqrt{t_{n-1}}+\sqrt{t_n}} = \dfrac{n-1}{\sqrt{t_1}+\sqrt{t_n}}$

Q2) Four different integers form an increasing A.P.
One of these numbers is equal to sum of squares of the other three numbers.
Find the numbers.

1) a) $t_2 - t_1 = d \qquad t_n - t_{n-1} = d.$

$$(t_2 \pm k) - (t_1 \pm k) = t_2 - t_1 = d$$

b) $t_2 k - t_1 k = k(t_2 - t_1) = kd$

1	2	3	4	5	. -	$d = 1$
3	6	9	12	15		$d = 3.$

c) $\frac{t_2}{k} - \frac{t_1}{k} = \frac{1}{k}(t_2 - t_1) = \frac{d}{k}$

Take LHS =

d) $\frac{1}{\sqrt{t_1}+\sqrt{t_2}} \frac{(\sqrt{t_2}-\sqrt{t_1})}{(\sqrt{t_2}-\sqrt{t_1})} + \frac{1}{\sqrt{t_2}+\sqrt{t_3}} \frac{(\sqrt{t_3}-\sqrt{t_2})}{(\sqrt{t_3}-\sqrt{t_2})} \ldots \frac{1}{\sqrt{t_n}+\sqrt{t_{n-1}}} \frac{(\sqrt{t_n}-\sqrt{t_{n-1}})}{(\sqrt{t_n}-\sqrt{t_{n-1}})}$

$\frac{\sqrt{t_2}-\sqrt{t_1}}{\sqrt{t_2}^2-\sqrt{t_1}^2} + \frac{\sqrt{t_3}-\sqrt{t_2}}{\sqrt{t_3}^2-\sqrt{t_2}^2} \cdots + \frac{\sqrt{t_n}-\sqrt{t_{n-1}}}{(\sqrt{t_n})^2-(\sqrt{t_{n-1}})^2}$

$\frac{\sqrt{t_n}-\sqrt{t_1}}{d} =$

$= \frac{\sqrt{t_n}-\sqrt{t_1}\,(\sqrt{t_n}+\sqrt{t_1})}{d(\sqrt{t_n}+\sqrt{t_1})}$

$= \frac{t_n - t_1}{d(\sqrt{t_n}+\sqrt{t_1})}$ $\qquad t_n = t_1 + (n-1)d$

$= \frac{(n-1)\cancel{d}}{\cancel{d}(\sqrt{t_n}+\sqrt{t_1})}$

$=$ RHS

$$\left.\begin{aligned} a_1 &= a \\ a_2 &= a+d \\ a_3 &= a+2d \end{aligned}\right\}$$

$$a_4 = a+3d$$

$$\vdots$$

$a-d \quad a \quad a+d$

let d be common difference.

$$a_4 = a_1^2 + a_2^2 + a_3^2$$

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ a-d & a & a+d & a+2d \end{array}$$

$$a+2d = (a-d)^2 + a^2 + (a+d)^2$$

$$a+2d = 3a^2 + 2d^2$$

$d > 0$

$a, d \in I$

$$2d^2 - 2d + (3a^2 - a) = 0$$

$$d = \frac{2 \pm \sqrt{4 - 4(2)(3a^2-a)}}{2(2)}$$

$$= \frac{\cancel{2} \pm \cancel{2}\sqrt{1 + 2a - 6a^2}}{\cancel{2}(2)}$$

$$d = \frac{1 \pm \sqrt{1+2a-6a^2}}{2}$$

$$1 + 2a - 6a^2 > 0$$

$$6a^2 - 2a - 1 < 0$$

$$(a-\alpha)(a-\beta) < 0$$

$$\alpha < a < \beta$$

$+ \quad - \quad +$

$\alpha \quad \beta$

$$a = \frac{+2 \pm \sqrt{4+24}}{12}$$

$$\alpha = \frac{2-\sqrt{28}}{12} = \frac{2-2\sqrt{7}}{12} = \frac{1-\sqrt{7}}{6}$$

$$\beta = \frac{1+\sqrt{7}}{6}.$$

$$\frac{1-\sqrt{7}}{6} < a < \frac{1+\sqrt{7}}{6} \qquad a \in I.$$

$$-0.\_ < a < 0.\_$$

$$a = 0$$

$$d = \frac{1 \pm \sqrt{1}}{2} = 1$$

$$-1 \quad 0 \quad 1 \quad 2 .$$

---

(2) Geometric Sequence or Geometric Progression (GP)

A Geometric progression is a sequence whose first term is non-zero & each term is obtained by multiplying preceding term by a constant. This constant is called common ratio (r) for GP.

$t_1 \quad t_2 \quad t_3 \quad \cdots \quad t_n.$

$t_2 = r t_1 \qquad t_3 = r t_2. \qquad t_n = r t_{n-1}$

$\frac{t_2}{t_1} = r \qquad \frac{t_3}{t_2} = r \qquad \frac{t_n}{t_{n-1}} = r$

$t_1 \neq 0$

If $t_1 = a$

$t_2 = ar$

$t_3 = ar^2$

$\vdots$

$t_n = ar^{n-1}$

S_n be a finite series for this G.P till n terms

$S_n = t_1 + t_2 + t_3 + \cdots \; t_n$

$= a + ar + ar^2 + \cdots \; ar^{n-1}$

$rS_n = \quad + ar + ar^2 + \cdots \; ar^{n-1} + ar^n$

$S_n - rS_n = a + 0 + 0 + \cdots + 0 - ar^n.$

$S_n(1-r) = a - ar^n$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}.$$

1 2 4 8 16 . . .

$a = 1 \quad r = 2$

$$S_4 = \frac{1(1-2^4)}{1-2} = \frac{-15}{-1} = 15.$$

$$\boxed{t_n = a r^{n-1}}$$

$t_5 = a r^4$

$t_3 = a r^2$

Q) If $t_1, t_2, t_3 \dots t_n$ are in GP. with common ratio 'r'

Show i) $t_1 k, t_2 k, t_3 k \dots$ are in GP 'r'

Also find common ratio

ii) $t_1/k, t_2/k, t_3/k \dots$ are in GP 'r'

iii) $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3} \dots$ are in GP '$\frac{1}{r}$'

iv) $t_1^r, t_2^r, t_3^r \dots$ are in GP 'r^r'

Q2) If continued product of 3 numbers in GP is 216 and sum of the product of them in pairs is 156. Find the numbers.

(2) $\frac{a}{r} \quad a \quad ar$

$\frac{a}{\cancel{r}} \times a \times a\cancel{r} = 216$

$a^3 = 216 \qquad a = 6$

$\frac{a}{r} \times a + a \times ar + \frac{a}{r} \times ar = 156$

$\frac{a^2}{r} + a^2 r + a^2 = 156$

$a^2 \left(\frac{1}{r} + r + 1 \right) = 156$

$\cancel{6}\,\overset{3}{\cancel{36}} \left(\frac{r^2 + r + 1}{r} \right) = \cancel{156}\ \cancel{26}\ 13.$

$3r^2 + 3r + 3 = 13r.$

$3r^2 - 10r + 3 = 0$

$3r^2 - 9r - r + 3 = 0$

$3r(r-3) - 1(r-3) = 0$

$r = 1/3 \qquad r = 3.$

$18 \quad 6 \quad 2 \qquad\qquad 2 \quad 6 \quad 18$

Q3) In a set of 4 numbers, the first three are in G.P and the last three are in A.P with common difference 6. If first number is same as fourth, find the four numbers.

$$a+12,\ a,\ a+6,\ a+12$$

$$\frac{a}{a+12} = \frac{a+6}{a} \Rightarrow a^2 = a^2 + 18a + 72$$

$$a = -4$$

$$8, -4, 2, 8$$

Q4) H.W

If $S_1, S_2, S_3 --- S_q$ are ~~correspo~~ sum of first n terms of q A.P's ~~respectively~~. ~~and~~ whose first terms are $1, 2, 3 --- q$ respectively and common differences as $1, 3, 5, --- (2q-1)$ respectively. Show that $S_1 + S_2 + S_3 + --- + S_q = \frac{1}{2} nq(nq+1)$

③ Harmonic Sequence or Harmonic Progression (HP)

if $t_1, t_2, t_3 \ldots t_n$ are HP.

$$\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3} \ldots \frac{1}{t_n} \quad \text{in AP.}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots \quad \text{in H.P.}$$

$$1 \quad 2 \quad 3 \quad 4 \ldots \quad \text{in A P.}$$

$$\frac{1}{t_n} - \frac{1}{t_{n-1}} = \text{const} = D.$$

$$\frac{1}{t_2} - \frac{1}{t_1} = D$$

$$\frac{1}{t_3} - \frac{1}{t_2} = D.$$

---

① If $t_1, t_2, t_3 \ldots t_n$ are in H.P prove that $t_1 t_2 + t_2 t_3 + \ldots t_{n-1} t_n = (n-1) t_1 t_n$

② The sum of three numbers in H-P is 37 and sum of their reciprocals is $\frac{1}{4}$ find the numbers.

(1) $\frac{1}{t_1}, \frac{1}{t_2} \cdots \frac{1}{t_n}$ is in AP.

$$\frac{1}{t_2} - \frac{1}{t_1} = D \Rightarrow \frac{t_1 - t_2}{t_1 t_2} = D \Rightarrow \frac{t_1 - t_2}{D} = t_1 t_2$$

$$\frac{1}{t_3} - \frac{1}{t_2} = D. \Rightarrow \frac{t_2 - t_3}{t_2 t_3} = D \Rightarrow \frac{t_2 - t_3}{D} = t_2 t_3$$

$$\vdots$$

$$\frac{1}{t_n} - \frac{1}{t_{n-1}} = D \quad \cdots \quad \frac{t_{n-1} - t_n}{D} = t_{n-1} t_n$$

$$\frac{1}{t_n} = \frac{1}{t_1} + (n-1)D \Rightarrow \frac{t_1 - t_n}{t_1 t_n} = (n-1)D$$

$$\frac{t_1 - \cancel{t_2}}{D} + \frac{\cancel{t_2} - \cancel{t_3}}{D} + \cdots \frac{\cancel{t_{n-1}} - t_n}{D} = L.H.S.$$

$$\frac{t_1 - t_n}{D} = L.H.S \qquad \frac{t_1 - t_n}{D} = \underbrace{(n-1) t_1 t_n}_{R.H.S.}$$

(2) $$\frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 37.$$

$$a-d + a + a+d = \frac{1}{4} \Rightarrow a = \frac{1}{12}$$

$$\frac{1}{\frac{1}{12} - d} + \cancel{12} + \frac{1}{\frac{1}{12} + d} = \cancel{37}\ 25.$$

$$\frac{\frac{\cancel{2}}{\cancel{12}\,6}}{\frac{1}{144} - d^2} = 25 \Rightarrow \frac{1}{150} = \frac{1}{144} - d^2$$

$$d^2 = \frac{1}{25 \times 144} \Rightarrow d = \pm \frac{1}{60} \qquad d^2 = \frac{1}{144} - \frac{1}{150} = \frac{6}{144 \times 150}$$

Q) If p^{th}, q^{th} & r^{th} terms of A.P are in G.P & both be a, b, c respectively.

Show $a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$

$$\log(AB) = \log A + \log B$$

$$\log\left(\frac{A}{B}\right) = \log A - \log B$$

$b \neq 1$

$$\log(A^{n}) = n \log A$$

$$\log_a b = c$$

$$b = a^{c}$$

$\log 1 = 0$

$$\frac{\log a}{\log b} = \log_b a$$

G.P x C.R y

first

A D

x y

$$a = A + (p-1)D = x y^{p-1}$$

$$b = A + (q-1)D = x y^{q-1}$$

$$c = A + (r-1)D = x y^{r-1}$$

$$b-c=(q-r)D$$

$$c-a=(r-p)D$$

$$a-b=(p-q)D$$

$$a^{b-c}\cdot b^{c-a}\cdot c^{a-b}=1.$$

$$\log\left(a^{b-c}\cdot b^{c-a}\cdot c^{a-b}\right)=0$$

$$\log a^{b-c}+\log b^{c-a}+\log c^{a-b}=0$$

$$(b-c)\log a+(c-a)\log b+(a-b)\log c=0$$

$$\log a=\log\left(xy^{p-1}\right)$$

$$=\log x+\log y^{p-1}$$

$$=\log x+(p-1)\log y$$

$$\log b=\log x+(q-1)\log y$$

$$\log c=\log x+(r-1)\log y.$$

$$((q-r)D)(\log x+(p-1)\log y)$$
$$+((r-p)D)(\log x+(q-1)\log y)$$
$$+((p-q)D)(\log x+(r-1)\log y)$$
$$=0$$

Arithmetic Mean :

If three terms are in A P then the middle term is called the arithmetic Mean (A M) between the other two.

a, b, c

b is AM of a & c.

$a \; A_1 \; b$ in A.P

$a \; A_1 \; A_2 \; b$ in AP

$a \; A_1 \; A_2 \; A_3 \cdots A_n \; b$ in A.P

$$b - a = c - b$$
$$2b = a + c$$
$$b = \frac{a+c}{2}$$

Geometric Mean :

If three terms are in G P then the middle term is called geometric mean of other two.

a, b, c are in G.P

b is GM of a & c.

$$\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$$
$$b = \sqrt{ac}$$

Harmonic Mean :

if. a, b, c are in H.P

b is H.M of a & c.

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

$$b = \frac{2ac}{a+c}$$

---

for any two numbers a, b

A.M $= \frac{a+b}{2}$ (A)

G.M $= \sqrt{ab}$ (G)

H.M $= \frac{2ab}{a+b}$ (H)

$$G^2 = AH$$

$$G = \sqrt{AH}$$

G is the geometric mean of A & H

G lies between A & H

$\Downarrow$

Either $H \leq G \leq A$ or $A \leq G \leq H$

$$A - G = \frac{a+b}{2} - \sqrt{ab}.$$

$$= \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{a}\sqrt{b}}{2}$$

$$= \frac{(\sqrt{a} - \sqrt{b})^2}{2}$$

$$= +ve.$$

$$A \geq G. \quad \Longrightarrow \quad \boxed{A \geq G \geq H}$$

Q If a is A.M between b & c & G_1, G_2 are 2 G M between b & c

prove $G_1^3 + G_2^3 = 2abc$

$$a = \frac{b+c}{2}.$$

b, G_1, G_2, c are in G.P

$G_1 = br \quad G_2 = br^2 \quad c = br^3.$

$$G_1^3 + G_2^3 = b^3 r^3 + b^3 r^6$$
$$= b^3 r^3 (1 + r^3)$$
$$= b^3 \left(\frac{c}{b}\right)\left(1 + \frac{c}{b}\right) = \frac{b^{\cancel{3}} c(b+c)}{\cancel{b^2}} \Rightarrow 2abc$$

Series

$$S = 1+2+3+4+5 \cdots +100$$

$$\frac{n}{2}(a_1+a_n) = 100\frac{(100+1)}{2}$$

$$\cancel{S_n = \sum_{n=1}^{n} f(n)}$$

$$S_n = t_1+t_2+t_3+ \cdots t_n$$

$$= \sum_{r=1}^{n} t_r$$

$$= t_1+t_2+t_3+ \cdots +t_n$$

$$\prod_{r=1}^{n} t_r = t_1 t_2 t_3 t_4 \cdots t_n$$

$$\cancel{\sum_{n=1}^{n}(1+n+n)}$$

$$S = 1+4+9+16+25+ \cdots 100$$

$$= \sum_{r=1}^{r=10} r^2$$

~~Prog~~ Properties of $\sum$

$$\sum_{r=1}^{n} c = c + c + c + c + \cdots n \text{ term}$$
$$= n(c)$$

$$\sum_{r=1}^{n} cr = c\cdot 1 + c\cdot 2 + c\cdot 3 + \cdots c\cdot n$$
$$= c\,(1 + 2 + 3 + \cdots n)$$
$$= c \sum_{r=1}^{n} r$$

$$\sum_{r=1}^{n} r^2 - 3r = \sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r$$

$$\sum_{r=k}^{r=n} r^2 = \sum_{r=1}^{r=n} r^2 - \sum_{r=1}^{r=k-1} r^2$$

eg
$$\sum_{r=10}^{r=100} r = \sum_{r=1}^{r=100} r - \sum_{r=1}^{r=9} r.$$

Some Important Results.

$$S = \sum_{r=1}^{n} r = 1+2+3+4+ \cdots n.$$

$$S = \quad n + n-1 + n-2 + \cdots 1$$

$$2S = (n+1) + (n+1) + (n+1) \cdots (n+1)$$

$$2S = n(n+1)$$

$$S = \frac{n(n+1)}{2}$$

find $\sum_{r=1}^{5} r$

$$\frac{5(5+1)}{2} = 15.$$

find $\sum_{1}^{10} n$ $\quad \sum_{r=1}^{10} n. = 10n$

$$\frac{10\{10+1)}{2} = 55$$

$$S = \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6.}$$

$$S = \sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

H.W prove