FUNCTIONS TUTORIAL.

Pg 59

4, 6, 8, 9, 10

Pg 60

1, 2, 3, 5, 6, 8, 10, 12

Pg 61

15, 16, 20, 22, 23, 25

Pg 62

27, 33, 35, 38

Pg 63

39, 43, 45

Pg 64

8, 9, 12

Pg 65 - 67

Comp 2, 3

Pg 68

3

Pg 59.

$\log_e$

(4) $f(x) = \ln\left(\dfrac{1-x}{1+x}\right)$

$$f(x) - f(y) = \ln\left(\frac{1-x+y-xy}{1+x-y-xy}\right)$$

$$\ln\left(\frac{1-x}{1+x}\right) - \ln\left(\frac{1-y}{1+y}\right) = \ln\left(\frac{\frac{1-x}{1+x}}{\frac{1-y}{1+y}}\right)$$

$$= \ln\left(\frac{(1-x)(1+y)}{(1+x)(1-y)}\right)$$

$$= \ln\left(\frac{1-x+y-xy}{1+x-y-xy}\right)$$

(6) $f(x) = \dfrac{\sin^4 x + \cos^2 x}{\sin^2 x + \cos^4 x} = \dfrac{\sin^2 x(1-\cos^2 x) + \cos^2 x}{\sin^2 x + \cos^2 x(1-\sin^2 x)}$

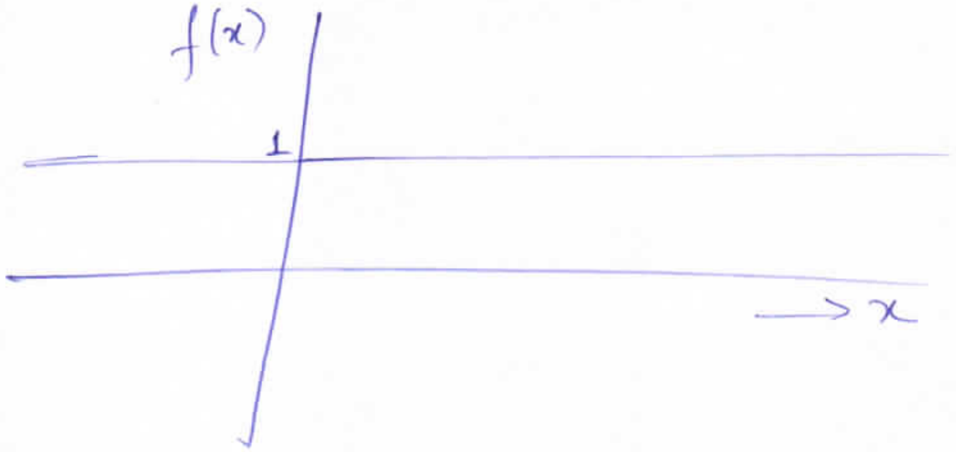
$f(2002)$

$$= \frac{\sin^2 x - \sin^2 x\cos^2 x + \cos^2 x}{\sin^2 x + \cos^2 x - \sin^2 x\cos^2 x}$$

$$= \frac{1 - \sin^2 x\cos^2 x}{1 - \sin^2 x\cos^2 x} = 1$$

$f(x) = 1$

$f(2002) = 1.$



8

$$e^{f(x)} = \frac{10+x}{10-x} \qquad x \in (-10, 10)$$

$$f(x) = k f\left(\frac{200x}{100+x^2}\right) \qquad \text{find } k$$

$$\log_e e^{f(x)} = \log_e\left(\frac{10+x}{10-x}\right)$$

$$f(x) \log_e e = \ln\left(\frac{10+x}{10-x}\right)$$

$$f(x) = \ln\left(\frac{10+x}{10-x}\right)$$

$$\ln\left(\frac{10+x}{10-x}\right) = k \ln\left(\frac{10 + \frac{200x}{100+x^2}}{10 - \frac{200x}{100+x^2}}\right)$$

$$\ln\left(\frac{10+x}{10-x}\right) = k \ln\left(\frac{1000+10x^2+200x}{1000+10x^2-200x}\right)$$

$$= k \ln\left(\frac{\cancel{10}(x^2+20x+100)}{\cancel{10}(x^2-20x+100)}\right)$$

$$= k \ln\left(\frac{(x+10)^2}{(10-x)^2}\right)$$

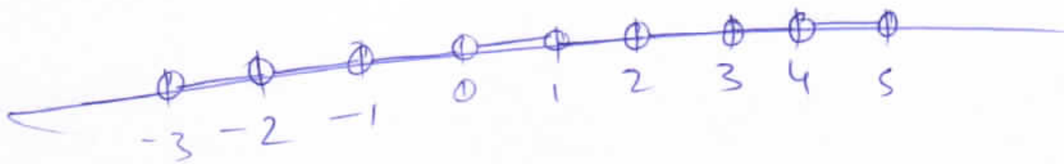
$$= k \ln\left(\frac{10+x}{10-x}\right)^2$$

$$\ln\left(\frac{10+x}{10-x}\right) = 2k \ln\left(\frac{10+x}{10-x}\right) \Rightarrow k = \frac{1}{2}$$

D, Range

(9) a) $f(x) = \dfrac{1}{\sqrt{x-[x]}}$

$f(x) = \dfrac{1}{\sqrt{\{x\}}}$ Domain $\in \mathbb{R} - \mathbb{Z}$

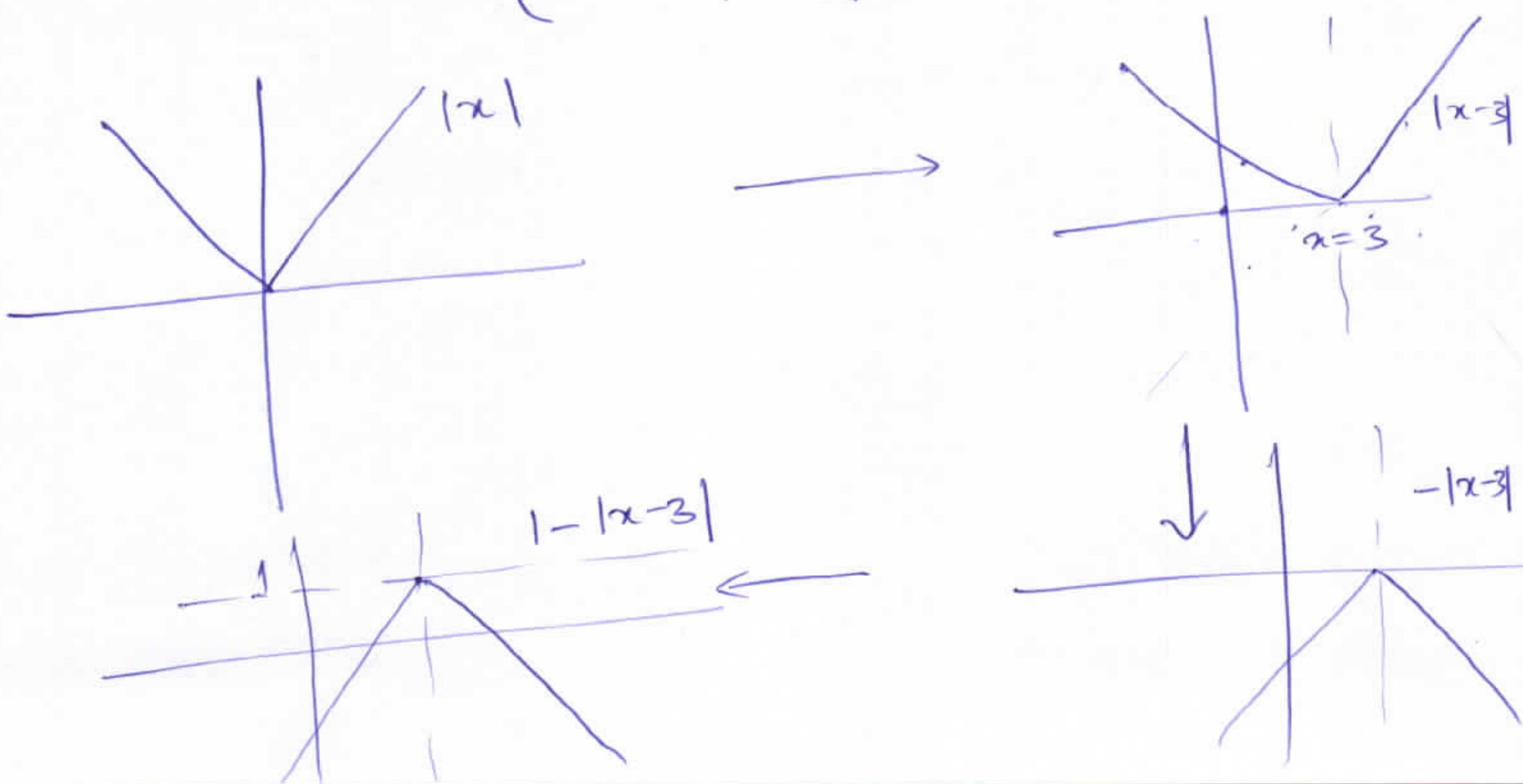


$\{x\} \longrightarrow (0, 1)$

$\dfrac{1}{\sqrt{\{x\}}} \longrightarrow (1, \infty)$ Range.

(10) ii) $f(x) = 1 - |x-3|$ Domain $\in \mathbb{R}$.

$(-\infty, 1]$



(10) $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$ $\qquad \dfrac{\sqrt{1-|x|}}{\sqrt{2-|x|}}$

(1) $\qquad$ (2)

$1-|x| \geq 0$

$|x| \leq 1$

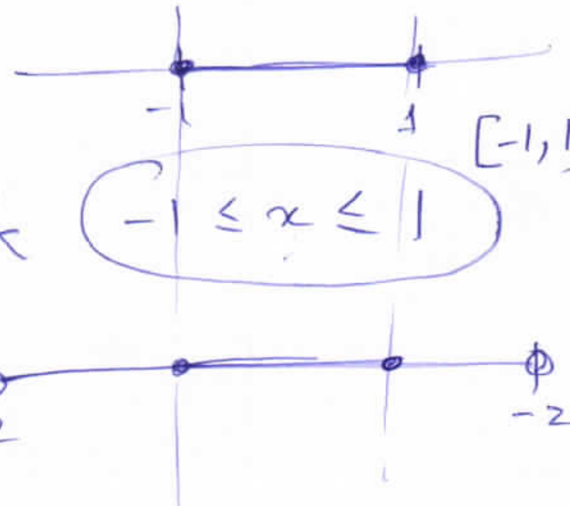
$-1 \leq x \leq 1$

$2-|x| > 0$

$|x| < 2$

$-2 < x < 2$

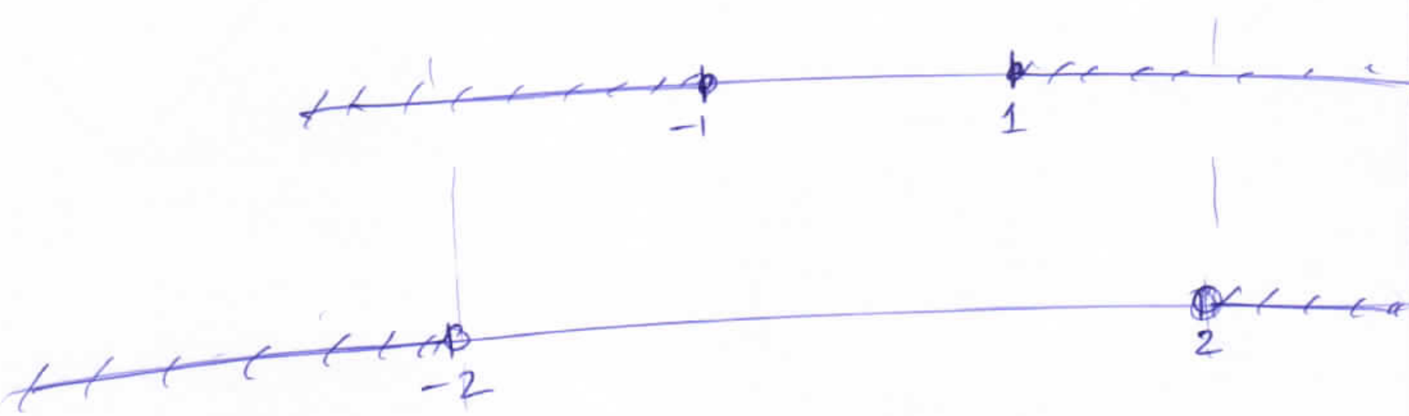
& $-1 \leq x \leq 1$ $\quad [-1, 1]$



$1-|x| \leq 0$ & $2-|x| < 0$

$|x| \geq 1$ $\qquad |x| > 2$

$x \geq 1$ or $x \leq -1$ $\qquad x > 2$ or $x < -2$



$(-\infty, -2) \cup (2, \infty)$.

(1) $f(x) = \cos(\ln x)$ then $f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$

$$\cos(\ln x)\cos(\ln y) - \frac{1}{2}\left[\cos\left(\ln \frac{x}{y}\right) + \cos(\ln(xy))\right]$$

$$\cos(\ln x)\cos(\ln y) - \frac{1}{2}\left[\cos(\ln x - \ln y) + \cos(\ln x + \ln y)\right]$$

$$\cancel{\cos(\ln x)\cos(\ln y)} - \frac{1}{\cancel{2}}\left[\cancel{2}\cancel{\cos(\ln x)\cos(\ln y)}\right] = 0$$

(2) $f(x) = \frac{1-x}{1+x}$ $\qquad f(\cos 2\theta) = \frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta$

$$f(f(\cos 2\theta)) = f(\tan^2\theta) = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \cos 2\theta$$

(3)

If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

$f(x) = \cos 9x + \cos 10x$

2 X

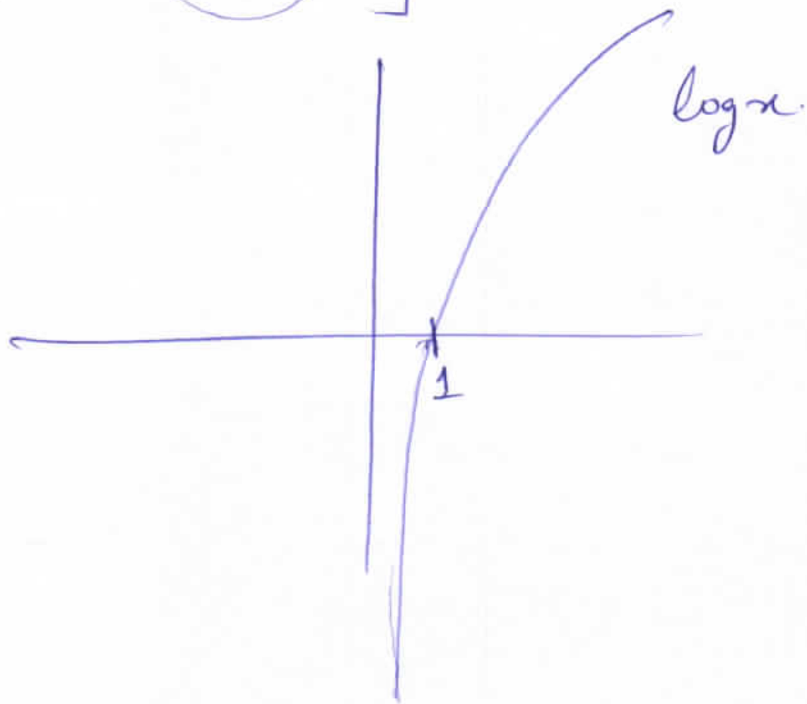
$f\left(\frac{\pi}{4}\right) = 2$ X

$f(-\pi) = 2$ X

$f(\pi) = 1$

$f\left(\frac{\pi}{2}\right) = -1$

(5) Domain of $f(x) = \left[\log_{10}\left(\frac{5x-x^2}{4}\right)\right]^{1/2}$.



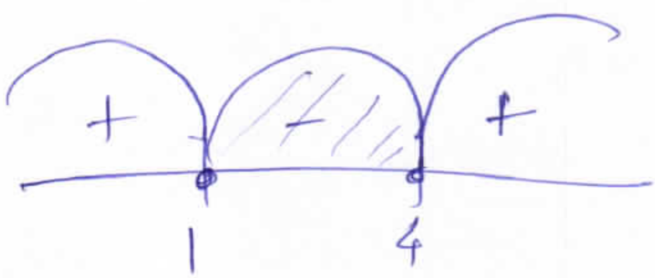
$$\frac{5x-x^2}{4} \geq 1$$

$$5x - x^2 \geq 4$$

$$5x - x^2 - 4 \geq 0$$

$$x^2 - 5x + 4 \leq 0$$

$$(x-1)(x-4) \leq 0$$



$$x \in [1,4]$$ (B)

(6) Range of $f(x) = \frac{x^2+x+2}{x^2+x+1}$ Domain $\in R$

$$y = \frac{x^2+x+2}{x^2+x+1}$$

$$x^2y + xy + y = x^2 + x + 2.$$

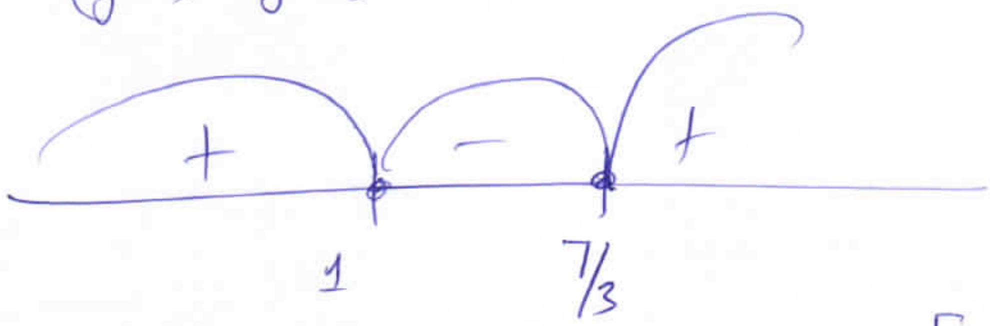
$$x^2(y-1) + x(y-1) + y - 2 = 0 \qquad y \neq 1 \ ?$$

$$D \quad (y-1)^2 - 4(y-1)(y-2) \geq 0$$

$$(y-1)[y-1-4y+8] \geq 0$$

$(y-1)(-3y+7) \geq 0$

$(y-1)(3y-7) \leq 0$



$y \in [1, 7/3]$

$\because y \neq 1$

$y \in (1, 7/3]$

(8) Range of $f(x) = \sin^2(x^4) + \cos^2(x^4)$

$= 1$

$f(x) = 1.$ $\{1\}$

(10) $f(x) = \sqrt[3]{\dfrac{2x+1}{x^2-10x-11}}$ domain ?

$x^2-10x-11 \neq 0$

$x^2-11x+x-11 \neq 0$

$(x+1)(x-11) \neq 0$

$x \neq -1 \quad x \neq 11$

$\mathbb{R} - \{-1, 11\}$

(12) Range of $f(x) = \frac{\sin(\pi[x^2+1])}{(x^4+1)} = \frac{\sin n\pi}{x^4+1}$

$= 0$

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(15) $f(x+y) = f(x)f(y) \quad \forall\ x,y \in R$

$f(0) \neq 0$

$F(x) = \frac{f(x)}{1+(f(x))^2}$

$f(x) = a^x$

$f(x+y) = a^{x+y} = a^x \cdot a^y = f(x)f(y)$

$f(x) = a^x$

$F(x) = \frac{a^x}{1+(a^x)^2} = \frac{a^x}{1+a^{2x}}$

$F(-x) = \frac{a^{-x}}{1+a^{-2x}} = \frac{\frac{1}{a^x}}{1+\frac{1}{a^{2x}}} = \frac{\frac{1}{a^x}}{\frac{a^{2x}+1}{a^{2x}}}$

$= \frac{a^{2x}/a^x}{a^{2x}+1}$

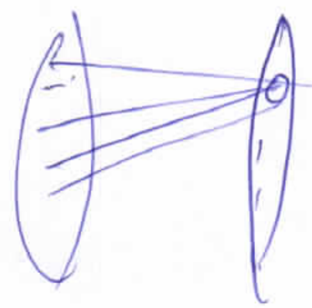
even function.

$= \frac{a^x}{a^{2x}+1} = F(x)$

(16) $f : R \to R$ $\quad f(x) = \dfrac{e^{|x|} - e^{-x}}{e^{x} + e^{-x}}$

$$f(x) = \begin{cases} \dfrac{e^{-x} - e^{-x}}{e^{x} + e^{-x}} = 0 & x < 0 \; (\quad) \\ \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}} & x \geq 0 \end{cases}$$

<u>not one-one</u>.



$$f(x) = \frac{e^{x} \cancel{+} e^{-x}}{e^{x} + e^{-x}}$$

$$= \frac{e^{x} - \frac{1}{e^{x}}}{e^{x} + \frac{1}{e^{x}}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{> 0}{> 2}.$$

$x \geq 0$

$[0, \infty)$

$\downarrow$

Range $= [0, 1) \subset R$ Into function.

(20) $[x]$ $\{x\}$.

$$[x] + \sum_{r=1}^{1000} \frac{\{x+r\}}{1000}$$

$$[x] + \frac{\{x+1\} + \{x+2\} + \{x+3\} + \cdots \{x+1000\}}{1000}$$

$$[x] + \frac{\{x\} \times \cancel{1000}}{\cancel{1000}}$$

$x = 2.2$

$\{x\} = 0.2$

$$[x) + \{x\} = \underline{\underline{x}}$$ (c)

$x+1 = 3.2$

$\{x+1\} = 0.2$

(22) $f(x) = \sin\left(\ln\left(x + \sqrt{x^2+1}\right)\right)$

$$f(-x) = \sin\left(\ln\left(-x + \sqrt{(-x)^2+1}\right)\right)$$

$$= \sin\left(\ln\left(\sqrt{x^2+1} - x\right)\right)$$

$$= \sin\left(\ln\left[\frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x}\right]\right)$$

$$= \sin\left(\ln\left[\frac{1}{x + \sqrt{x^2+1}}\right]\right)$$

$$= \sin\left(\ln 1 - \ln\left(x + \sqrt{x^2+1}\right)\right)$$

$$= \sin\left(-\ln\left(x + \sqrt{x^2+1}\right)\right)$$

$$= -\sin\left(\ln\left(x + \sqrt{x^2+1}\right)\right) = -f(x)$$ odd function

(23) $f(x) = \sin 3\pi\{x\} + \tan \pi[x] \longrightarrow$

Period of $f(x)$

$f(x) = \sin 3\pi\{x\} + 0$

$= \sin 3\pi\{x\}$

$f(x)$ is periodic with period 1.

$f(x+1) = \sin 3\pi\{x+1\}$
$= \sin 3\pi\{x\} = f(x)$

$f(x+2) = \sin 3\pi\{x+2\}$
$= \sin 3\pi\{x\} = f(x)$

T is the fundamental period
nT is a period

$f(x+T) = f(x)$
$f(x+nT) = f(x)$

(25) On $[0,1]$

$$f(x) = x \quad \text{if } x \text{ is rational}$$
$$= 1-x \quad \text{if } x \text{ is irrational.}$$

$$f(f(x)) = x$$

$f(x) = x$ $\quad f(f(x)) = f(x)$ $\quad$ x is rational.

$$= x$$

$f(x) = 1-x$ $\quad f(f(x)) = 1-f(x)$ $\quad$ x is irrational.

$$= 1-(1-x)$$
$$= x$$

(27) $f(x) = \sin^4 x + \cos^4 x$ $\quad$ period $= ?$

$$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x.$$
$$= 1 - \frac{1}{2} \times (2\sin x \cos x)^2$$
$$= 1 - \frac{1}{2}\sin^2 2x$$
$$= 1 - \frac{1}{2}\left(\frac{1-\cos 4x}{2}\right)$$
$$= 1 - \frac{1}{4} + \frac{1}{4}\cos 4x$$
$$= \frac{3}{4} + \frac{1}{4}\cos 4x \Rightarrow T = \frac{2\pi}{4} = \frac{\pi}{2}$$

(33)

$$y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$$

$$y = \frac{10^x - \frac{1}{10^x}}{10^x + \frac{1}{10^x}}$$

$$y = \frac{10^{2x} - 1}{10^{2x} + 1}$$

$$y(10^{2x} + 1) = 10^{2x} - 1$$

$$10^{2x}(y - 1) = -1 - y.$$

$$10^{2x} = \frac{y+1}{1-y}$$

$$\log_{10} 10^{2x} = \log_{10} \frac{1+y}{1-y}$$

$$2x \log_{10} 10 = \log_{10} \frac{1+y}{1-y}$$

$$x = \frac{1}{2} \log_{10} \frac{1+y}{1-y}$$

$$f^{-1}(x) = \frac{1}{2} \log_{10} \left(\frac{1+x}{1-x}\right)$$ (A)

(35) $f(x) = \frac{1-x}{1+x}$. domain of $f^{-1}(x)$

Domain of $f(x) \Rightarrow \mathbb{R} - \{-1\}$.

$y = \frac{1-x}{1+x}$.

$y + yx = 1 - x$.

$x(y+1) = 1 - y$

$x = \frac{1-y}{y+1}$

Range of $f(x) =$

$\mathbb{R} - \{-1\}$.

Domain of $f^{-1}(x)$

$f : A \longrightarrow B$ one-one & onto

$f^{-1} : B \longrightarrow A$. one-one & onto

(38) $f(x) = 10(\cos^2 \pi x + \underbrace{x - [x]}_{\{x\}})$ $+ \sin^2 \pi x$.

period $= ?$ $\text{LCM}(1, 1)$

$= \text{LCM}(1, 1)$ $\frac{\pi}{\pi} = 1$.

$= 1$ (A)

(39)

$$f(x) = \begin{cases} x^2 & 0 < x < 2 \\ 2x-3 & 2 \le x < 3 \\ x+2 & x \geq 3 \end{cases}$$

$f(3/2) = \frac{9}{4}$

$f\left(\frac{9}{4}\right) = 2\left(\frac{9}{4}\right) - 3 = \frac{3}{2}$

$f\left(\frac{3}{2}\right) = $ (A)

(43)

$f(x) f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

$f(x) = 1 \pm x^n$

$f(4) = 65$ $\nearrow$ $f(x) = 1 + x^n$

$\cancel{1+x^4 = 65}$. $1 + 4^n = 65$.

$4^n = 64$

$n = 3$

$f(x) = 1 + x^3$

$f(3) = 1 + 3^3 = 1 + 27 = 28$ (B)

(45) $f : [-3,5] \longrightarrow (\quad)$

$$g(x) = |3x+4|$$

domain of $f(g(x))$

$$f(|3x+4|)$$

$$-3 \le |3x+4| \le 5$$

$$|3x+4| \le 5.$$

$$-5 \le 3x+4 \le 5$$

$$-9 \le 3x \le 1$$

$$-3 \le x \le \tfrac{1}{3}$$

$$x \in [-3, \tfrac{1}{3}]$$ (B)

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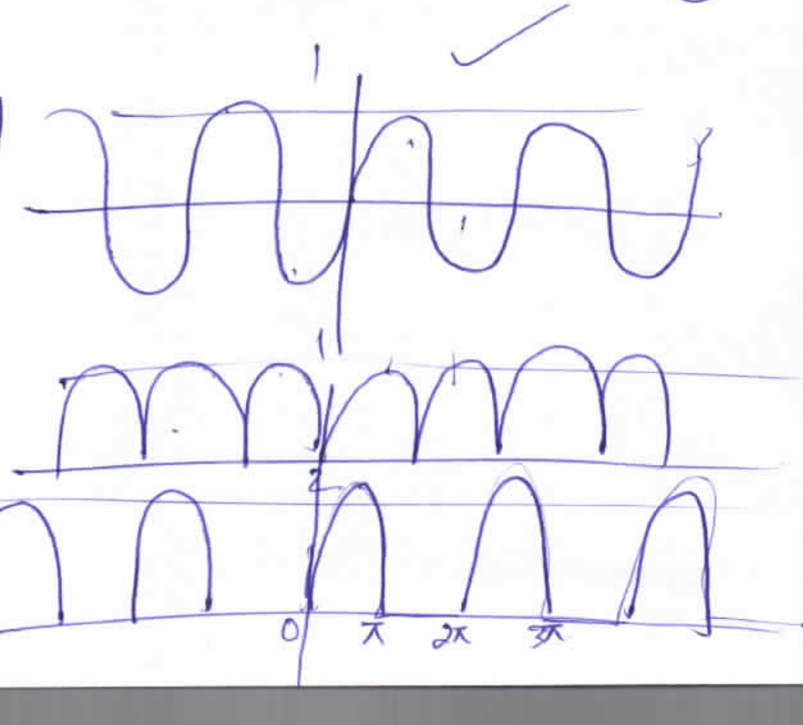
(8) $f(x) = \sin x + |\sin x|$

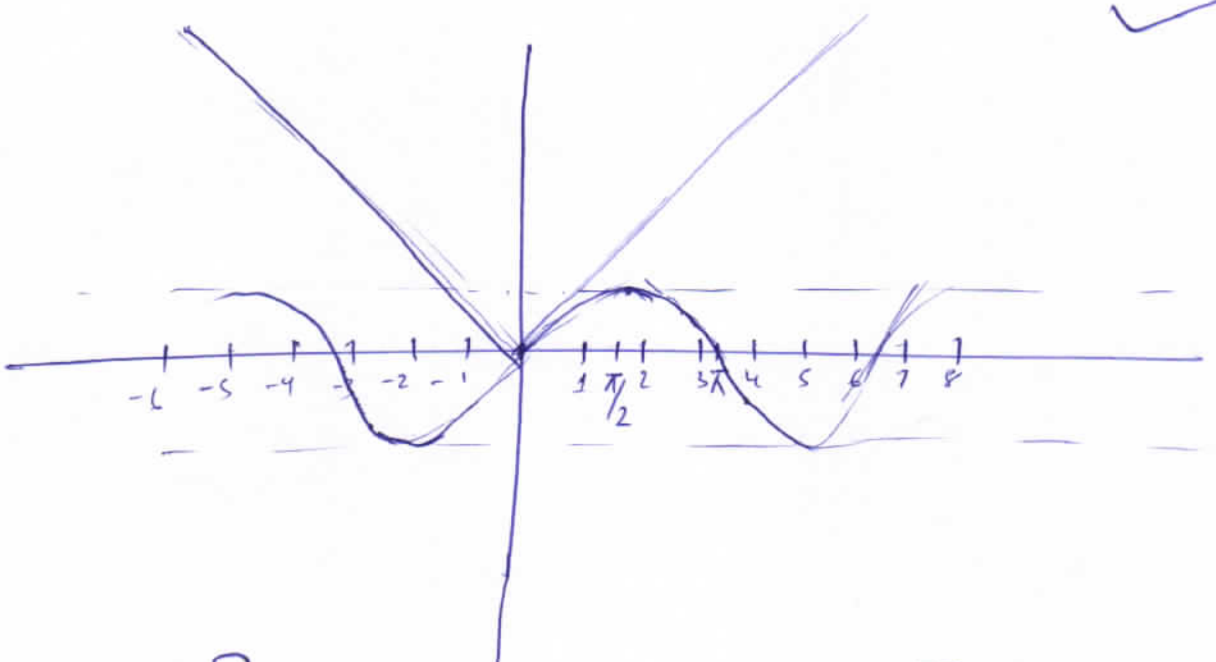


(2π)

$f(x) = \mathrm{sgn}(e^{-x})$

$f(x) = 1$ ✓ periodic

$f(x) = \min(\sin x, |x|) = \sin x$ ✓



$f(x) = \left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] + 2[-x]$

$2[x] - 1.$

$0 \leq \{x\} < 1$

$x = 2 \cdot 4$

$[x] = 2$

$0 \leq \{x\} < \frac{1}{2}$

$\left[x + \frac{1}{2}\right] = 2 = [x]$

$\left[x - \frac{1}{2}\right] = 1 = [x] - 1.$

$$f(x) = [x] + \underbrace{[x]-1 + 2(-[x]-1)}_{=-3} \quad 0 < \{x\} < \frac{1}{2}$$

$$= [x] + \underbrace{[x]-1 + 2(-[x])}_{=-1} \quad \{x\} = 0$$

$$= \underbrace{[x]+1 + [x] + 2(-[x]-1)}_{=-1} \quad \{x\} = \frac{1}{2}$$

$$= [x]+1 + [x] + 2(-[x]-1) \quad \frac{1}{2} < \{x\} < 1$$

$$= -1.$$

periodic with undefined period

ABCD

x

(9) $f(x) = x + \sin x$
NP non-periodic ✓P.

$f(x) = \cos x^2$

Non-periodic

ABC

$f(x) = \sin x + \{x\}$.
P P

LCM $(2\pi, 1)$ ✗ non-periodic

$$f(x) = \cos((x+3) - [x+3])$$

$$= \cos(\{x+3\})$$

periodic with period 1.

(12) $f(x) = \sin[\pi^2]x + \sin[-\pi^2]x$

$f(x) = \sin 9x - \sin 10x$

$\text{LCM}\left(\frac{2\pi}{9}, \frac{2\pi}{10}\right) = 2\pi$ A ✓

$\text{LCM}\left(\frac{1}{2}, \frac{1}{3}\right) = 1.$

$f\left(\frac{\pi}{2}\right) = \sin\frac{9\pi}{2} - \sin\frac{10\pi}{2}$

$= 1 - 0 = 1$ (B) ✓

$f(-x) = \sin(-9x) - \sin(-10x)$

$= -\sin 9x + \sin 10x$

$= -(\sin 9x - \sin 10x)$

$= -f(x)$ odd. (D) ✓

ABD

Comp 2

(4) $f(x) = x^3$ $\quad f(x) = \tan x$ $\quad f(x) = \ln x$ $\quad f(x) = \sec x$

Range $\mathbb{R}$. $\quad \mathbb{R}$ $\quad \mathbb{R}$ $\quad \mathbb{R} - (-1, 1)$

$$a_0 x^n + a_1 x^{n-1} + a_2 \cdot x^{n-2} + \cdots a_n = 0$$

(5) Range $= \mathbb{R}$ when n is odd

(6) $$f(x) = \frac{x^2 + 2x + c}{x^2 + 4x + 3c}.$$

$$y = \frac{x^2 + 2x + c}{x^2 + 4x + 3c}.$$

$$x^2 y + 4xy + 3cy = x^2 + 2x + c$$

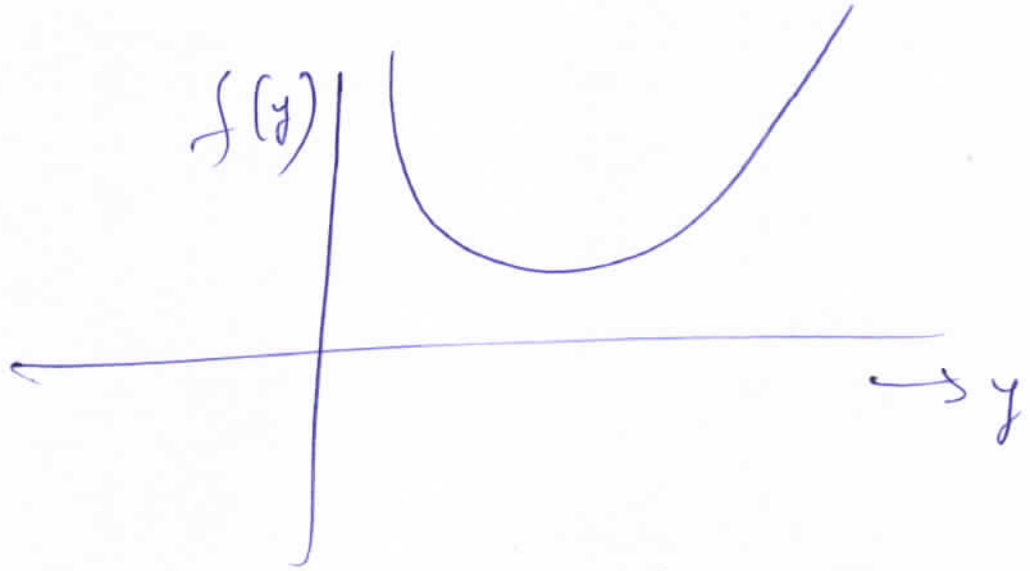
$$x^2(y-1) + x(4y-2) + 3cy - c = 0$$

$$(4y-2)^2 - 4(y-1)(3cy - c) \geq 0$$

$$16y^2 - 16y + 4 - \underline{12cy^2} + 4cy + 12cy - 4c \geq 0$$

$$y^2(16 - 12c) + y(16c - 16) - 4c + 4 \geq 0$$

$$f(y) = y^2(16-12c) + y(16c-16) - 4c+4$$



$$(16c-16)^2 - 4(16-12c)(-4c+4) < 0$$

$$16^2(c-1)^2 + 16c(16-12c) < 0$$

$$256(c^2-2c+1) + 256c - 16\times 12\, c^2 < 0$$

$$\cancel{224c^2} - 256c + 256 < 0$$

$(16\times 16 - 16\times 12)c^2$

$$64c^2 - 256c + 256 < 0$$

$$c^2 - 4c + 4 < 0$$

$$256(c^2-2c+1) - 64(4-3c)(1-c) < 0$$

$$256(c^2-2c+1) - 256 + 64\times 3c + 256c - 64\times 3c^2 < 0$$

$$64\times 4c^2 - 256\times 2c + \cancel{256} - \cancel{256} + 64\times 3c + 256c - 64\times 3c^2 < 0$$

D

$$64c^2 - 64c < 0$$

$$c^2 - c < 0$$

$$c(c-1) < 0$$

$+ \quad 0 \quad - \quad 1 \quad +$

$\underline{0 < c < 1}$