COMPLEX NUMBERS.

$$a + ib$$

$\in$ Real Number.

$i = \sqrt{-1}$ or $i^2 = -1$

iota

$$\sqrt{9} = 3 \qquad x^2 = 9$$

$$x = \pm 3$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$\sqrt{-9} = \sqrt{-1}\sqrt{9}$$

$$= \sqrt{-1}\,(3)$$

$$= 3i$$

$$2 + 3i \qquad 2 + \sqrt{-9}$$

$$x^2 + x + 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{-7}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$$

$$Z = a + ib$$

$Re(Z)$

real part of complex number Z.

$Im(Z)$

Imaginary part of complex number Z

$$Z = Re(Z) + i\, Im(Z)$$

Any number can be represented as a complex number.

$2+3i$

$3+0i$ } purely real $\quad R \subset C$

$0+2i \Rightarrow 2i$ } purely imaginary ✓
$0-3i \Rightarrow -3i$

$i=\sqrt{-1}$
$i^2=-1$
$i^3=-i$
$i^4=1$

$i^{4n} = 1$
$i^{4n+1} = i^{4n}\cdot i = 1\cdot i = i$
$i^{4n+2} = i^{4n}\cdot i^2 = 1\cdot i^2 = -1$
$i^{4n+3} = i^{4n}\cdot i^3 = 1\cdot i^3 = -i$

$i^5=i$
$i^6=-1$
$i^7=-i$
$i^8=1$

Divide power by 4
remainder (0) = 1
remainder (1) = i
remainder (2) = -1
remainder (3) = $-i$

Sum of any four consecutive powers of i is 0
i.e $\quad i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0 \qquad n \geq 0$

Find a) $i^{25} \to i$ b) $i^{127} \to -i$ c) $i^{2009} \to i$ d) $i^{-2521} \to \frac{1}{i^{2521}} = \frac{1}{i} = -i$

e) $i^4+i^5+i^6+i^7 \longrightarrow 0$

f) $\dfrac{i^{85}+i^{157}+i^{271}+i^{370}}{i^{75}+i^{147}+i^{261}+i^{360}} = \underbrace{i^{10}}_{-1}\dfrac{\cancel{(i^{75}+i^{147}+i^{261}+i^{360})}}{\cancel{(i^{75}+i^{147}+i^{261}+i^{360})}}$

if $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$

$$z_1 = z_2 \Rightarrow \begin{matrix} Re(z_1) = Re(z_2) \\ Im(z_1) = Im(z_2) \end{matrix}$$

$$\Rightarrow a_1 + ib_1 = a_2 + ib_2$$

$$\underline{(2+3i)^2 = a-2 + bi}$$

a & b = ? if $a, b \in \mathbb{R}$

$$4 + 9i^2 + 2(2)(3i) = a-2 + bi$$

$$4 - 9 + 12i = a-2 + bi$$

$$-5 + 12i = (a-2) + bi$$

$$a - 2 = -5 \Rightarrow a = -3$$

$$12 = b$$

Conjugate of a complex number.

if $z = a + ib$ is a complex number

conjugate of z is denoted by $\bar{z} = a - ib$

eg. $z = 2 + 3i$ $\quad$ $z = 2$ $\quad$ $z = 3i$

$\bar{z} = 2 - 3i$ $\quad$ $\bar{z} = 2$ $\quad$ $\bar{z} = -3i$

$$z\bar{z} = (a+ib)(a-ib)$$

$$= a^2 - (ib)^2 = a^2 - i^2 b^2 = \underline{\underline{a^2 + b^2}} \text{ real.}$$

$$\underline{z\bar{z} = a^2 + b^2}$$

properties of conjugate. ($z = a+ib$) ($\bar{z} = a-ib$)

i) If $z = \bar{z}$ z is purely real.

ii) if $z = -\bar{z}$ z is purely imaginary

iii) $Re(z) = Re(\bar{z}) = \dfrac{z+\bar{z}}{2}$

iv) $Im(z) = \dfrac{z-\bar{z}}{2i}$

v) $\overline{(\bar{z})} = z$

vi) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$z_1+z_2 = (a_1+ib_1)+(a_2+ib_2)$

$z_1+z_2 = a_1+a_2 + i(b_1+b_2)$

$\overline{z_1+z_2} = a_1+a_2 - i(b_1+b_2)$

$\bar{z}_1 = a_1 - ib_1$

$\bar{z}_2 = a_2 - ib_2$

$\bar{z}_1+\bar{z}_2 = a_1+a_2 - i(b_1+b_2)$

vii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

viii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

Algebraic Operations with Complex Numbers.

$z_1 = a+ib \qquad z_2 = c+id.$

$$z_1+z_2 = a+ib+c+id$$
$$= (a+c)+i(b+d)$$

$$z_1-z_2 = (a+ib)-(c+id)$$
$$= a-c+i(b-d)$$

$$z_1z_2 = (a+ib)(c+id) = ac+iad+ibc+i^2bd$$
$$= (ac-bd)+i(ad+bc)$$

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$
$$= \frac{ac+bd}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

Find a & b if $a,b \in R$.

i) $\cancel{2+3i+bi = a}$

$2+3i-bi = a-2-2i+bi$

ii) $(a+2i)(2+2i) = b+6i$

iii) $\dfrac{a+i}{2-i} = 1+bi$

i) $2+2+3i+2i = a + bi + bi$

$4+5i = a+2bi$

$a = 4$

$2b = 5$

$b = 5/2$

ii) $(a+2i)(2+2i) = b+6i$

$2a-4+4i+2ai = b+6i$

$-4-2i = b-2a-2ai$

$b = -2$

$b-2a = -4$

$-2a = -2 \Rightarrow a = 1$

iii) $\dfrac{a+i}{2-i} = 1+bi$

$a+i = (1+bi)(2-i)$

$a+i = 2+b-i+2bi$

$-2+2i = b-a+2bi$

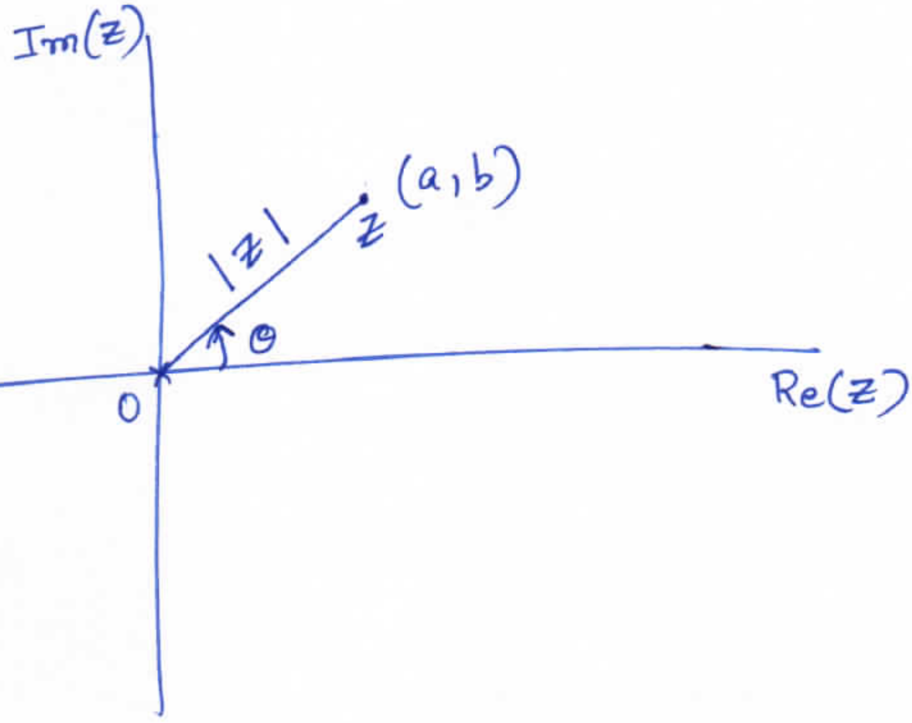
$a = 3$

$b-a = -2$

$2b = 2 \longrightarrow b = 1$

Graphical representation of a complex number:

$z = a + ib.$
$a, b \in R$
Can be uniquely located as a point on a 2-D plane known as the argand plane.



$$\text{modulus of } z = |z| = \sqrt{a^2+b^2} = \sqrt{\{Re(z)\}^2 + \{Im(z)\}^2}$$

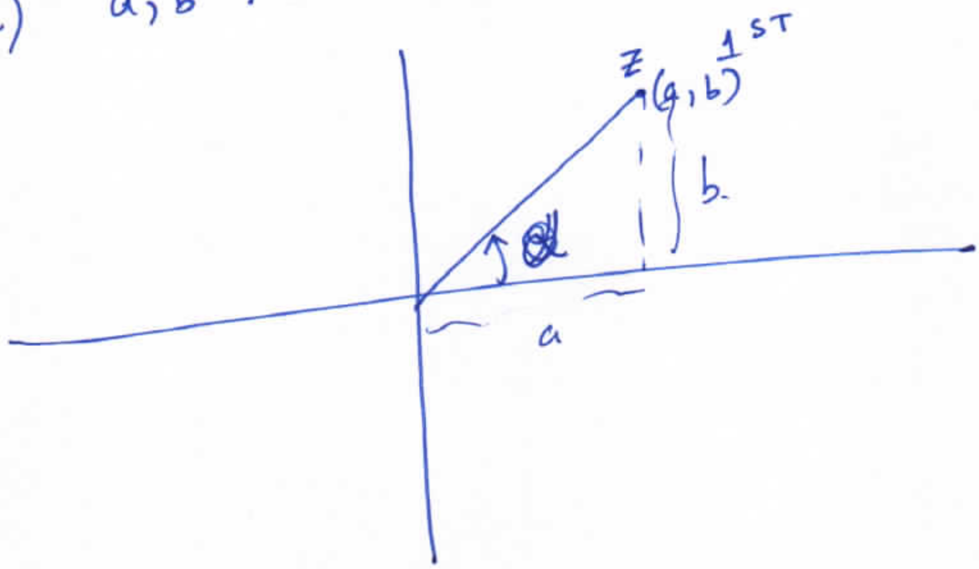
$$z\bar{z} = |z|^2$$

$$\theta = \text{argument of } z = \arg(z) = 2n\pi + \theta$$

if $-\pi < \theta \le \pi$ then θ is the principal argument of z (amplitude)

Calculation of principal argument / Amplitude of complex number $z = a + ib$

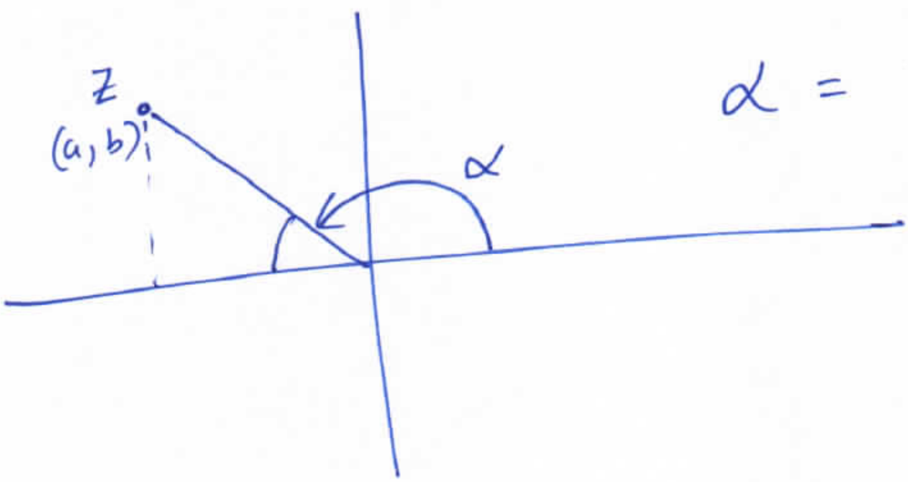
i) $a, b > 0$ a +ve b +ve



$\tan \alpha = \frac{b}{a}$.

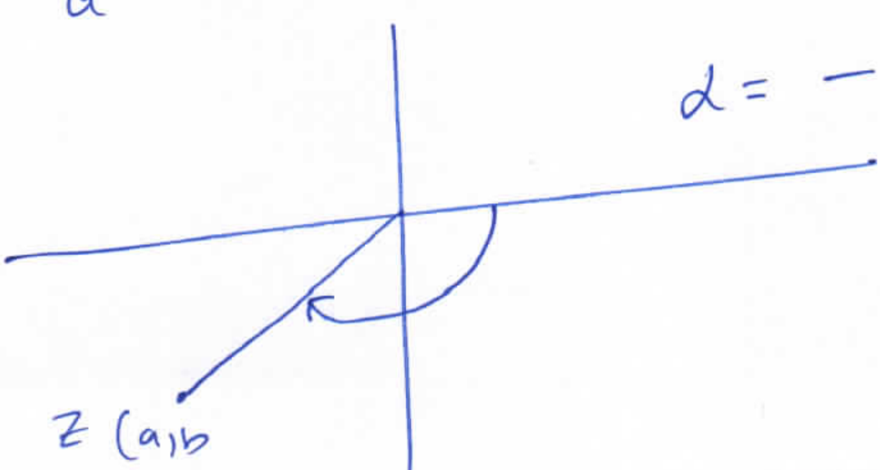
$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

ii) a −ve b +ve



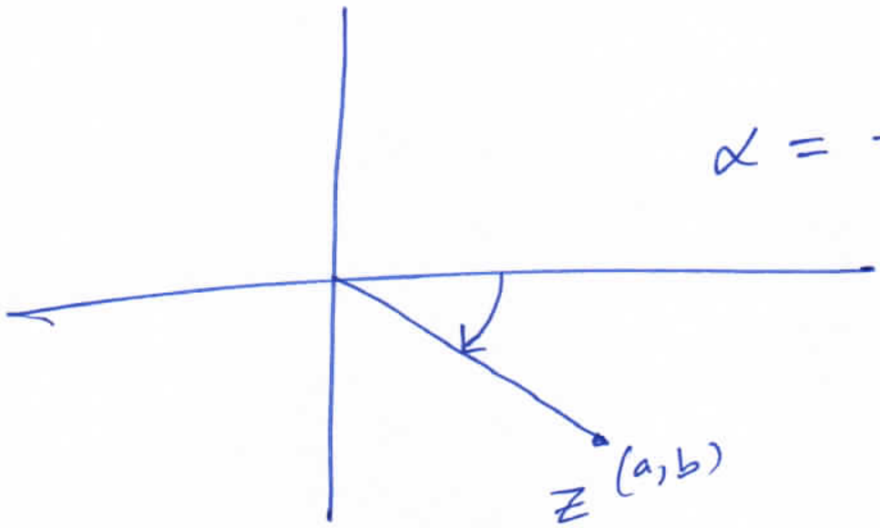
$\alpha = \pi - \left|\tan^{-1}\left(\frac{b}{a}\right)\right|$

iii) a −ve b −ve



$\alpha = -\pi + \tan^{-1}\left|\frac{b}{a}\right|$

iv) a +ve b -ve



$$\alpha = -\tan^{-1}\left|\frac{b}{a}\right|$$

Q) Find argument, amplitude & modulus of

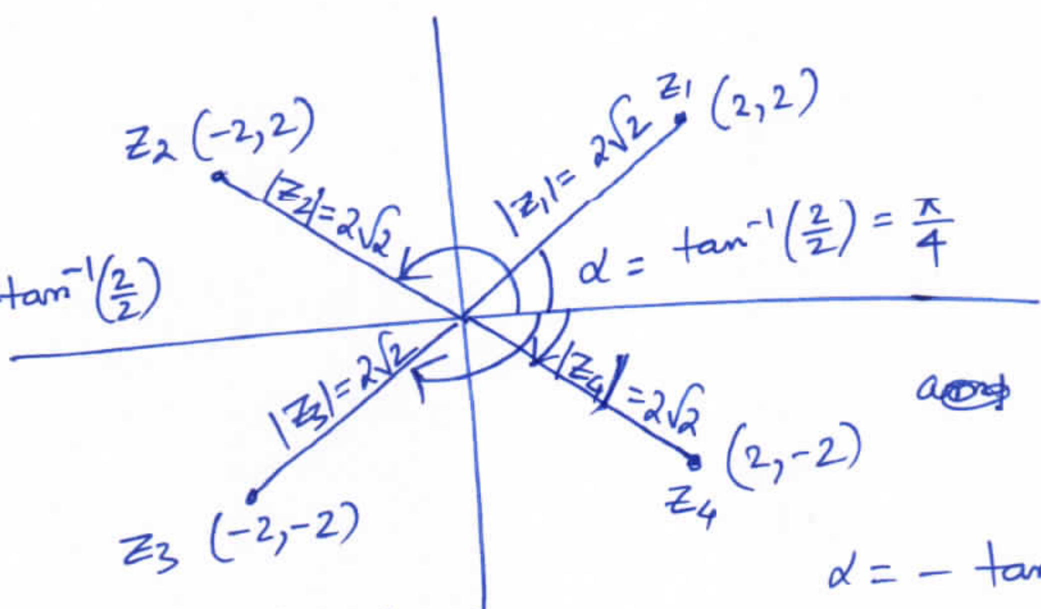
i) $z_1 = 2+2i$

$z_2 = -2+2i$

$z_3 = -2-2i$

$z_4 = 2-2i$

$\arg(z) = 2n\pi + \text{amp}(z)$



$\text{amp}(z_2)$

$= \pi - \tan^{-1}\left(\frac{2}{2}\right)$

$= 3\pi/4$

$\arg = 2n\pi + \frac{\pi}{4}$

arg

$$\alpha = -\pi + \tan^{-1}\left|\left(\frac{-2}{-2}\right)\right|$$

$$= -\pi + \pi/4 = -\frac{3\pi}{4}$$

$\arg(z_3) = 2n\pi + (-3\pi/4)$

$$\alpha = -\tan^{-1}\left|\frac{2}{-2}\right|$$

$$= -\tan^{-1}|1| = \frac{-\pi}{4}$$

$\arg(z_4) = 2n\pi - \frac{\pi}{4}$

Properties of argument & modulus of a complex number.

i) if $z_1 = z_2 \Rightarrow |z_1| = |z_2|$ $\quad arg(z_1) = arg(z_2)$

ii) if $arg(z)$ is $(2n+1)\frac{\pi}{2} \Rightarrow z$ is purely Imaginary

iii) if $arg(z)$ is $n\pi \Rightarrow z$ is purely real

iv) $|z| \geq 0$

if $|z| = 0 \Rightarrow z = 0$

v) $z\bar{z} = |z|^2 = |\bar{z}|^2$

vi) $|z| = |\bar{z}|$

vii) $|z_1 z_2| = |z_1||z_2|$

viii) $|z|$ is a real number

ix) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

x) $|z^n| = |z|^n$

xi) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

xii) $amp(\bar{z}) = -amp(z)$

xiii) $amp(z) = 0 \Rightarrow z$ is positive real.

xiv) ~~$arg(z_1 z_2) = arg(z_1) + arg(z_2)$~~ $\quad + 2k\pi \quad k \in (0, 1, -1)$

$amp(z_1 z_2) = amp(z_1) + amp(z_2)$

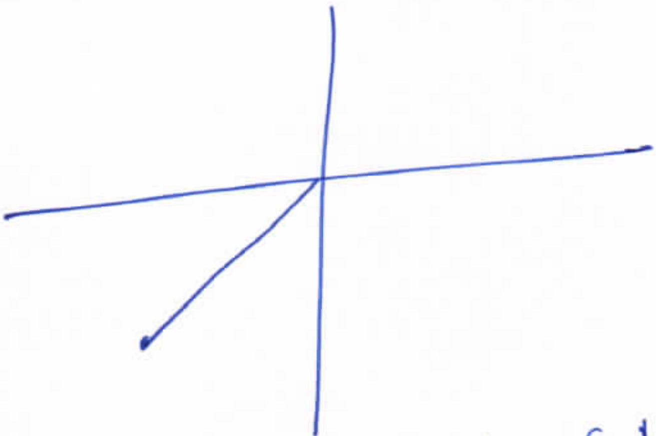
xv) $amp\left(\frac{z_1}{z_2}\right) = amp(z_1) - amp(z_2) + 2k\pi \quad k \in (0, 1, -1)$

eg. $z_1 = 1+\sqrt{3}i$ $\quad$ $\operatorname{amp}(z_1) = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = 60^\circ$

$z_2 = -2+2i$ $\quad$ $\operatorname{amp}(z_2) = \pi - \tan^{-1}|1| = 135$

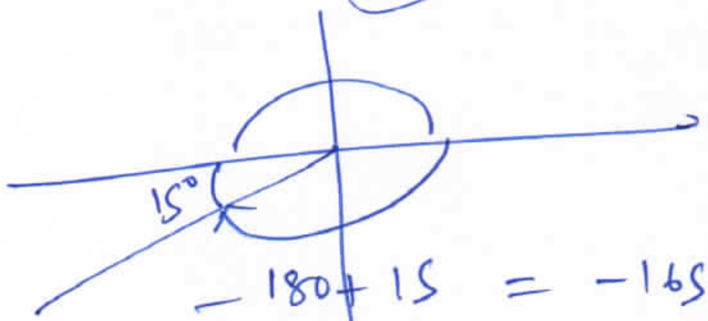
find $\operatorname{amp}(z_1 z_2)$

$$z_1 z_2 = (1+\sqrt{3}i)(-2+2i)$$
$$= -2+2i-2\sqrt{3}i+2\sqrt{3}i^2$$
$$= (-2-2\sqrt{3}) + (2-2\sqrt{3})i$$



$$\alpha = -\pi + \tan^{-1}\left|\frac{2-2\sqrt{3}}{-2-2\sqrt{3}}\right|$$
$$= -180^\circ + \tan^{-1}\left|\frac{-1.464}{-5.464}\right|$$
$$= -180^\circ + 15^\circ = \underline{\underline{-165^\circ}}$$

$$\operatorname{amp}(z_1 z_2) = \operatorname{amp}(z_1) + \operatorname{amp}(z_2) + 2k\pi.$$
$$= 195^\circ + 2k\pi.$$

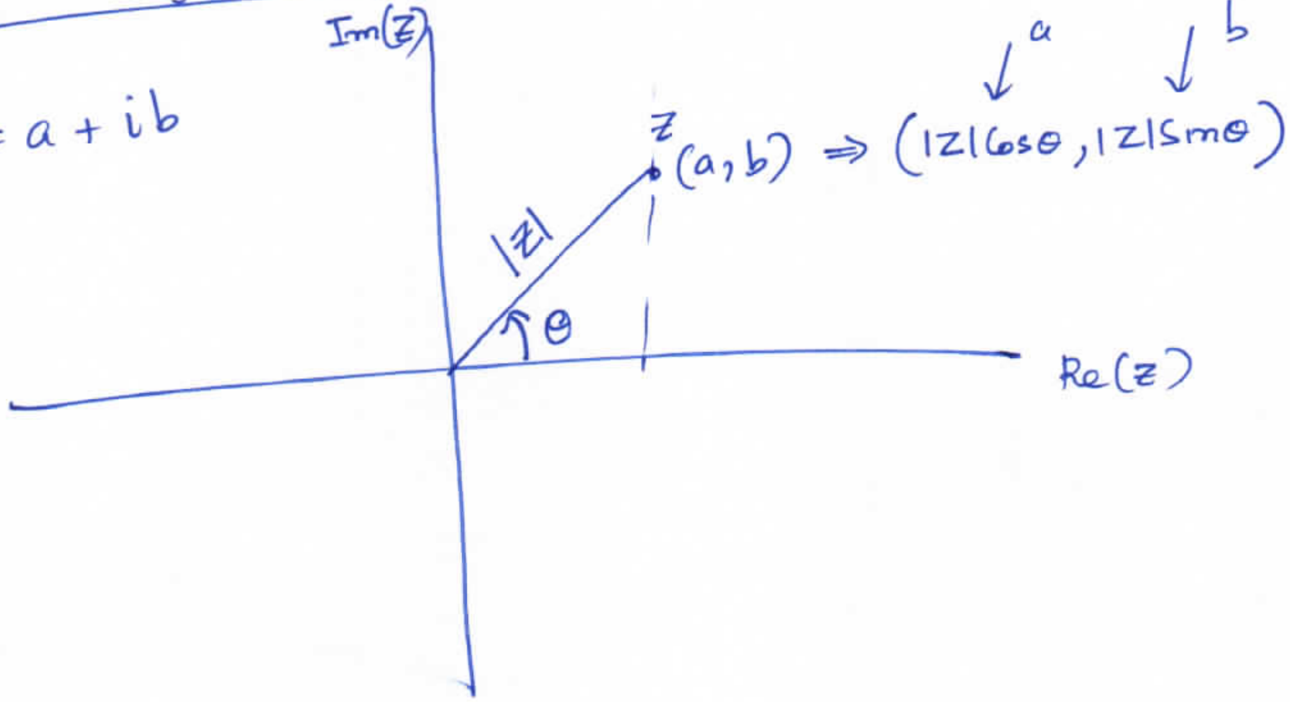


$$-180 + 15 = \underline{\underline{-165^\circ}}$$

$$\operatorname{amp}\left(\frac{z}{\bar{z}}\right) = \operatorname{amp}(z) - \operatorname{amp}(\bar{z}) + 2k\pi)$$
$$= \operatorname{amp}(z) + \operatorname{amp}(z) + 2k\pi$$
$$= 2\operatorname{amp}(z) + 2k\pi$$

Polar form of Complex Number

$z = a + ib$



$$
\begin{aligned}
z &= a + ib \\
&= |z|\cos\theta + i|z|\sin\theta \\
&= |z|(\cos\theta + i\sin\theta) \\
&= |z|\{\cos(\arg(z)) + i\sin(\arg(z))\}
\end{aligned}
$$

$$\boxed{\text{Euler's formula} \Rightarrow e^{i\theta} = \cos\theta + i\sin\theta}$$

$z = |z|\, e^{i\theta}$

$z = |z|\, e^{i\arg(z)}$

modulus of z.

$|z| = r$.

$\arg(z) = \theta$.

argument of z.

$z = re^{i\theta}$

$$z = r e^{i\theta}$$

$$z^{-1} = \left(r e^{i\theta}\right)^{-1} = r^{-1} e^{-i\theta}$$

$$= \frac{1}{r} e^{-i\theta}$$

$$z^{-1} = \frac{1}{|z|} e^{-i\theta}.$$

$$z^{-1} = \frac{1}{|z|} \left\{ \cos(-\theta) + i \sin(-\theta) \right\}.$$

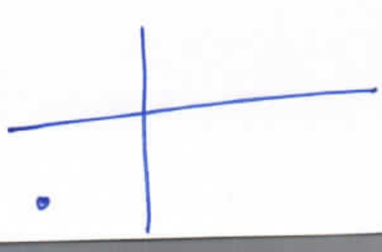
$$= \frac{1}{|z|} \left\{ \cos\theta \cancel{+} i \sin\theta \right\}$$

$$z^{-1} = \frac{\cos\theta}{|z|} - i \frac{\sin\theta}{|z|}$$

Q) Write polar form of $z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

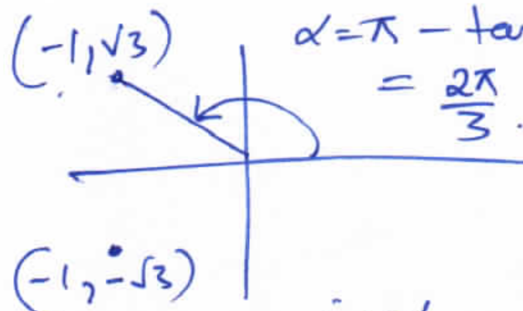
Q) Find $\dfrac{-1 - i\sqrt{3}}{-1 + i\sqrt{3}}$

$$|z| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$



$$\alpha = -\pi + \tan^{-1}\left|\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right| = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$$

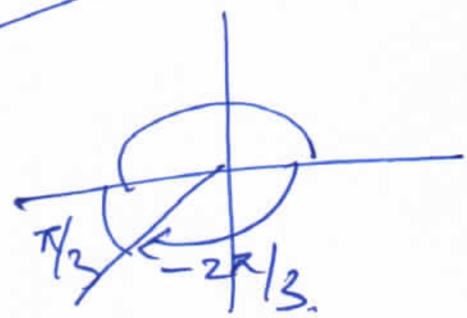
$$Z = |Z| e^{i\theta}$$
$$= 1 e^{i\left(-\frac{2\pi}{3}\right)}$$
$$= e^{i\left(-\frac{2\pi}{3}\right)}$$



$$\alpha = \pi - \tan^{-1}\sqrt{3} = \frac{2\pi}{3}.$$

Q2. $$\frac{-1 - i\sqrt{3}}{-1 + i\sqrt{3}} = \frac{e^{-i\frac{2\pi}{3}}}{e^{i2\pi/3}} = e^{-i4\pi/3} = e^{-i\left(-\frac{2\pi}{3}\right)} = e^{i2\pi/3}.$$

$$\frac{a^x}{a^y} = a^{x-y}$$



$$re^{i\theta} = r\left(\cos\theta + i\sin\theta\right)$$
$$1e^{i2\pi/3} = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$
$$= 1\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

De Moivre's Thm.

a) $$\{\cos\theta + i\sin\theta\}^n = \cos n\theta + i\sin n\theta$$

$$(e^{i\theta})^n = e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta)$$

b) $$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdots(\cos\theta_n + i\sin\theta_n)$$

$$= \cos(\theta_1+\theta_2+\theta_3\cdots\theta_n) + i\sin(\theta_1+\theta_2+\theta_3\cdots\theta_n)$$

$$e^{i\theta_1}\cdot e^{i\theta_2}\cdot e^{i\theta_3}\cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots\theta_n)}$$

$$\downarrow$$

$$\cos(\theta_1+\theta_2\cdots\theta_n) + i\sin(\theta_1+\theta_2+\cdots\theta_n)$$

Suppose i have to find n^{th} root of a complex number.

$$Z = re^{i\theta} = re^{i(\theta+2\pi)} = re^{i(\theta+4\pi)} = \cdots$$

$$Z = re^{i(\theta+2k\pi)}$$

$$Z^{1/n} = \left\{re^{i(\theta+2k\pi)}\right\}^{1/n}$$

$$= r^{1/n}e^{i\frac{(\theta+2k\pi)}{n}} \qquad k \in 0,1,2\cdots n-1$$

$$= r^{1/n}\left\{\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right\}.$$

find square root of $z = 2+2\sqrt{3}i$

$$|z| = \sqrt{(2)^2+(2\sqrt{3})^2} = \sqrt{4+12} = 4$$

$$\alpha = \tan^{-1}\frac{2\sqrt{3}}{2} = \frac{\pi}{3}.$$

$$z = 4e^{i\frac{\pi}{3}}$$

$$= 4e^{i\left(2k\pi+\frac{\pi}{3}\right)}$$

$$z^{1/2} = 4^{1/2}e^{i\frac{\left(2k\pi+\frac{\pi}{3}\right)}{2}}$$

$$= 2e^{i\left(k\pi+\frac{\pi}{6}\right)}$$

$$= 2\left\{\cos\left(k\pi+\frac{\pi}{6}\right)+i\sin\left(k\pi+\frac{\pi}{6}\right)\right\}$$

$k=0$

$$z_1 = 2.\left\{\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right\} = 2\left\{\frac{\sqrt{3}}{2}+i\left(\frac{1}{2}\right)\right\}$$

$$= \underline{\sqrt{3}+i}$$

$k=1$

$$z_2 = 2\left\{\cos\left(\pi+\frac{\pi}{6}\right)+i\sin\left(\pi+\frac{\pi}{6}\right)\right\} = 2\left\{-\frac{\sqrt{3}}{2}+i\left(-\frac{1}{2}\right)\right\}$$

$$= \underline{-\sqrt{3}-i}$$

For Square root Calculation.

Easy formula. ($z = a + ib$)

$$z^{1/2} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right] \quad \underline{b>0}$$

$$\pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right] \quad b<0$$

if $z = 2 + 2\sqrt{3}\, i$

$|z| = 4$ $\qquad a = 2$

$\therefore b > 0$

$$z^{1/2} = \pm\left[\sqrt{\frac{4+2}{2}} + i\sqrt{\frac{4-2}{2}}\right]$$

$$= \pm\left[\sqrt{3} + i\right]$$

$$= \sqrt{3} + i \ , \ -\sqrt{3} - i$$

$z = -i$ for square root.

$|z| = 1$ $\qquad a = 0$ $\qquad b < 0$

$$z^{1/2} = \pm\left[\sqrt{\frac{1+0}{2}} - i\sqrt{\frac{1-0}{2}}\right] = \pm\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

If $Z_r = \cos\left(\frac{\pi}{3^r}\right) + i\sin\left(\frac{\pi}{3^r}\right)$

find $Z_1 Z_2 Z_3 \cdots Z \cdots$ infinity

$$Z_r = e^{i\frac{\pi}{3^r}}$$

$$Z_1 Z_2 \cdots - \cdot \infty$$

$$e^{i\frac{\pi}{3}} \cdot e^{i\frac{\pi}{3^2}} \cdot e^{i\frac{\pi}{3^3}} \cdots \infty$$

$$e^{i\left(\frac{\pi}{3} + \frac{\pi}{3^2} + \frac{\pi}{3^3} \cdots \infty\right)}$$

$$e^{i\left(\frac{\frac{\pi}{3}}{1-1/3}\right)} = e^{i\pi/2} = 1\left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\}$$

$$= 1\{0 + i\} = i$$

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find cube roots of unity.

$$Z = 1.$$

$$Z = 1e^{i0} = 1e^{i(0+2k\pi)}$$

$$= 1e^{i2k\pi}$$

$$Z^{1/3} = 1^{1/3} e^{i\frac{2k\pi}{3}} = 1 e^{i\frac{2k\pi}{3}}$$

$k=0$

$$z_1 = \cos 0 + i \sin 0$$
$$= 1$$

$1.$

$k=1$

$$z_2 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$$
$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$k=2$

$$z_3 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$$
$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

ω^2

$$\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2$$
$$= \frac{1}{4} - \frac{3}{4} - 2i\frac{\sqrt{3}}{4}$$
$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$1 \quad \omega \quad \omega^2.$

$$1 + \omega + \omega^2 = 0$$

$$a^{x+y} = a^x a^y$$
$$\omega^{3n+2} = \omega^{3n}\omega^2$$
$$1$$

$$\omega^3 = 1.$$

$$\omega^{3n} = 1 \qquad \omega^{3n+1} = \omega \qquad \omega^{3n+2} = \omega^2$$

Any three consecutive powers of ω sum to 0

$$\omega^n + \omega^{n+1} + \omega^{n+2} = 0$$

Find $\sum_{r=1}^{3n+7} \omega^{r}$

$3n+7$

$3n+6$ is div 3.

$$\omega + \underbrace{\omega^2+\omega^3+\omega^4}_{0} + \underbrace{\omega^5 + - - -}_{0} \underbrace{- \ \omega^{3n+7}}_{0}$$

ω.

If α, β, γ are roots of

$x^3 - 3x^2 + 3x + 7 = 0$

Find $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$

$x^3 - 3x^2 + 3x - 1 = -8$

$(x-1)^3 = (-2)^3$

$\left(\frac{x-1}{-2}\right)^3 = 1 \Rightarrow \frac{x-1}{-2} = 1^{1/3}$

$$\frac{x-1}{-2} = 1, \omega, \omega^2$$

$$x = -1 \qquad \alpha = -1$$

$$\frac{x-1}{-2} = \omega \Rightarrow 1-2\omega \quad \beta$$

$$\frac{x-1}{-2} = \omega^2 \Rightarrow 1-2\omega^2 \quad \gamma.$$

$$\frac{-1-1}{-2w} + \frac{-\cancel{2}w}{-\cancel{2}w^2} + \frac{-2w^2}{-2}$$

$$\Rightarrow \frac{-\cancel{2}}{-\cancel{2}w} + \frac{1}{w} + w^2.$$

$$\Rightarrow \frac{2w^2 + w^2}{w \cdot w^2} = 3w^2$$

$$= 3\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$= -\frac{3}{2} - \frac{i3\sqrt{3}}{2}$$