CONTINUITY & DIFFERENTIABILITY TUTORIAL

Pg 131-132	3, 7, 9, 11, 18, 19, 20
Pg 133	3, 5, 7, 8
Pg 134-135	10, 12, 15, 19, 21, 25, 30
Pg 136	8, 9
Pg 138	Comp 2

Section A

③ $f(x) = \begin{cases} \frac{\sin 3x}{x} & , x \neq 0 \\ 2 & , x = 0 \end{cases}$

L·H·L

$\lim_{x \to 0^-} f(x) = \frac{\sin 3x}{x}$

=

L·H·L = R·H·L = 3

$\lim_{x \to 0} f(x) = 2$

As L·H·L = R·H·L and $\neq f(0)$

Not continuous at $x = 0$

⑦ $f(t) = \begin{cases} \frac{\cos t}{(\pi/2) - t} & , t \neq \pi/2 \\ 1 & , t = \pi/2 \end{cases}$

at $f(\pi/2) = 1$

$\lim_{t \to \frac{\pi}{2}} \frac{\cos t}{(\pi/2) - t} \Rightarrow \lim_{t \to \frac{\pi}{2}} \frac{-\sin t}{0 - 1} = \sin t$

$\sin(\frac{\pi}{2}) = 1$

$LHL = RHL = f(\frac{\pi}{2})$ $\therefore$ continuous

9) i) $f(x) = \begin{cases} \dfrac{\sqrt{5x+2} - \sqrt{4x+4}}{x-2}, & x \neq 2 \\ k, & x = 2 \end{cases}$

$$\lim_{x\to 2} f(x) = \cancel{LHL}: \quad \frac{\sqrt{5x+2}-\sqrt{4x+4}}{x-2} \times \frac{\sqrt{5x+2}+\sqrt{4x+4}}{\sqrt{5x+2}+\sqrt{4x+4}}$$

$$= \frac{5x+2-4x-4}{(x-2)\left(\sqrt{5x+2}+\sqrt{4x+4}\right)}$$

$$= \frac{\cancel{(x-2)}}{\cancel{(x-2)}\left(\sqrt{5x+2}+\sqrt{4x+4}\right)}$$

$$= \frac{1}{\sqrt{12}+\sqrt{12}}$$

$k = \dfrac{1}{4\sqrt{3}} = f(2)$ for $f(x)$ to be continuos at $x=2$

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(11) $f(x) = \begin{cases} 2+\sqrt{1-x^2} & |x| \leq 1 \\ 2e^{(1-x)^2} & |x| > 1 \end{cases}$

at $x \to 1$

LHL : $2+\sqrt{1-1} = 2$

$f(1) = 2+\sqrt{1-1} = 2$

RHL $= 2e^{(1-1)^2} = 2e^0 = 2$

since LHL = RHL = $f(1)$, it is continuous at $x=1$

at $x \to -1$

LHL : $2+\sqrt{1-1} = 2$

$f(-1) = 2+\sqrt{1-1} = 2$

RHL $= 2e^{(1+1)^2} = 2e^4$

LHL $= f(-1) \neq$ RHL $\therefore$ it isn't continuous at -1

9) (ii) $f(x) = \begin{cases} \dfrac{\ln(1+ax) - \ln(1-bx)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$

$$\lim_{x \to 0} \frac{\ln(1+ax)}{x} - \frac{\ln(1-bx)}{x}$$

$$= \lim_{x \to 0} \quad a - (-b)$$

$$= \underline{\underline{a+b}}$$

$$k = \underline{\underline{a+b}}$$

Pg 132

(18) $f(x) = \begin{cases} x\left(\dfrac{e^{1/x} - 1}{e^{1/x} + 1}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$

To prove $f(x)$ is not derivable at $x = 0$

L.H.L.

$$\lim_{x\to 0^-} x\left(\frac{e^{1/x} - 1}{e^{1/x} + 1}\right) \Rightarrow 0\left(\frac{0-1}{0+1}\right) = 0$$

as $x \to 0^-$

$\frac{1}{x} \to -\infty$

$e^{1/x} \to 0$

$e^{-\infty} = \frac{1}{e^{\infty}} = 0$

R.H.L

$$\lim_{x\to 0^+} x\left(\frac{e^{1/x} - 1}{e^{1/x} + 1}\right)$$

$x \to 0^+$

$$\lim_{x\to 0^+} x\left(\frac{1 - \frac{1}{e^{1/x}}}{1 + \frac{1}{e^{1/x}}}\right) = 0\left(\frac{1-0}{1+0}\right) = 0$$

$1/x \to \infty$

$e^{1/x} \to \infty$

$$L.H.L = R.H.L = f(0) = 0$$

continuos at $x = 0$

R.H.D

$$f'(x^+) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)\left(\frac{e^{\frac{1}{x+h}} - 1}{e^{\frac{1}{x+h}} + 1}\right) - x\left(\frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{x\left(\left(\frac{e^{\frac{1}{x+h}}-1}{e^{\frac{1}{x+h}}+1}\right)-\left(\frac{e^{\frac{1}{x}}-1}{e^{1/x}+1}\right)\right)+h\left(\frac{e^{\frac{1}{x+h}}-1}{e^{\frac{1}{x+h}}+1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{x\left(\frac{\left(e^{\frac{1}{x+h}}-1\right)\left(e^{\frac{1}{x}}+1\right)-\left(e^{\frac{1}{x}}-1\right)\left(e^{\frac{1}{x+h}}+1\right)}{\left(e^{1/x+h}+1\right)\left(e^{1/x}+1\right)}\right)}{h} + \frac{e^{\frac{1}{x+h}}-1}{e^{\frac{1}{x+h}}+1}$$

$$= \lim_{h\to 0} \frac{x\left\{\frac{\cancel{e^{\frac{1}{x+h}}e^{\frac{1}{x}}}+e^{\frac{1}{x+h}}-e^{\frac{1}{x}}\cancel{-1}-\cancel{e^{\frac{1}{x}}e^{\frac{1}{x+h}}}-e^{\frac{1}{x}}+e^{\frac{1}{x+h}}\cancel{+1}}{\left(e^{\frac{1}{x+h}}+1\right)\left(e^{\frac{1}{x}}+1\right)}\right\}}{h}$$

$$+ \frac{e^{\frac{1}{x+h}}-1}{e^{\frac{1}{x+h}}-1}$$

$$= \lim_{h\to 0} \frac{2x\left\{\frac{\left(e^{\frac{1}{x+h}}\right)-\left(e^{1/x}\right)}{\left(e^{1/x+h}+1\right)\left(e^{1/x}+1\right)}\right\}}{h} + \frac{e^{\frac{1}{x+h}}-1}{e^{\frac{1}{x+h}}-1}$$

$$=$$

$$e^{\frac{1}{x+h}} - e^{1/x}$$

$$e^{\frac{1}{x}}\left\{\frac{e^{\frac{1}{x+h}}}{e^{1/x}}-1\right\}$$

$$e^{\frac{1}{x}}\left\{e^{\frac{1}{x+h}-\frac{1}{x}}-1\right\}$$

$$= \lim_{h\to 0} 2x \left\{ \frac{e^{\frac{1}{x}}\left(e^{\frac{-h}{x(x+h)}}-1\right)}{h\left(e^{\frac{1}{x+h}}+1\right)\left(e^{\frac{1}{x}}+1\right)} \right\}$$

$$e^{\frac{1}{x}}\left\{e^{\frac{-h}{x(x+h)}}-1\right\}.$$

$$\lim_{h\to 0} 2x \left\{ \frac{e^{\frac{1}{x}} \times -\frac{1}{x(x+h)}}{\left(e^{\frac{1}{x}}+1\right)\left(e^{\frac{1}{x}}+1\right)} \right\} + \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}.$$

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0} \left(\frac{e^{ax}-1}{ax}\right) \times a = a.$$

$$f'(x) = \frac{-2e^{1/x}}{x\left(e^{1/x}+1\right)^2} + \frac{e^{\frac{1}{x}}-1}{e^{1/x}+1}.$$

$$f'(0^+) = \qquad +$$

$$\lim_{x\to 0^+} \frac{-2e^{1/x}}{x\left(e^{1/x}+1\right)^2} + \frac{e^{1/x}-1}{e^{1/x}+1}$$

R.H.D.

$$+1.$$

$$\lim_{x\to 0^-} \frac{-2e^{1/x}}{x\left(e^{1/x}+1\right)^2} + \frac{e^{1/x}-1}{e^{1/x}+1}$$

L.H.D

$$-1$$

(19) $f(x) = \begin{cases} x^2+3x+a & x \le 1 \\ bx+2 & x > 1 \end{cases}$

at $x=1$

L·H·L R·H·L $f(1)$

$1^2+3(1)+a = b(1)+2$

$a+4 = b+2$

$b-a = 2$ —— (1) $a = 3$

$f'(x) = \begin{cases} 2x+3 & x \le 1 \\ b & x > 1 \end{cases}$

R·H·D $= b$

L·H·D $= 2(1)+3 = 5$

R·H·D = L·H·D $\Rightarrow b = 5$

(20) $x = a$

$\lim\limits_{x \to a} \frac{xf(a) - af(x)}{x-a} = f(a) - af'(a)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{x \to a} \frac{x f(a) - a f(x)}{x - a}$$

$$\lim_{x \to a} \frac{x f(a) - a f(a) + a f(a) - a f(x)}{x - a}$$

$$\lim_{x \to a} \frac{\cancel{(x-a)} f(a)}{\cancel{x-a}} + \frac{a f(a) - a f(x)}{x - a}$$

$$f(a) - \lim_{x \to a} a\left(\frac{f(x) - f(a)}{x - a}\right)$$

$$f(a) - \lim_{h \to 0} a\left(\frac{f(a+h) - f(a)}{h}\right)$$

let $x - a = h$.
as $x \to a$
$h \to 0$

$$f(a) - a f'(a)$$

Pg 133

(3) $f(x) = \dfrac{\log(1+ax) - \log(1-bx)}{x}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left\{\frac{a\log(1+ax)}{ax} - \frac{(-b)\log(1-bx)}{-b.x}\right\}.$$

$= a \times 1 \quad + b \times 1.$

$= a+b.$ (B)

(5) $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x & x \neq 0 \\ 2 & x = 0. \end{cases}$

$\lim_{x\to 0} f(x) = 2. \quad = f(0)$

$\Rightarrow$ continuos at $x=0$

(7) $f(x) = \begin{cases} \dfrac{x}{e^{1/x}+1} & x \neq 0 \\ 0 & x = 0. \end{cases}$

R.H.L $\lim_{x\to 0^+} \dfrac{x}{e^{1/x}+1} = 0$

L.H.L = R.H.L $= f(0)$

L.H.L $\lim_{x\to 0^-} \dfrac{x}{e^{1/x}+1} = \dfrac{0}{0+1} = 0$

Continuous at $x=0$

(8) $f(x) = \begin{cases} (1+2x)^{1/x} & x \neq 0 \\ e^2 & x = 0 \end{cases}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (1+2x)^{1/x}.$$

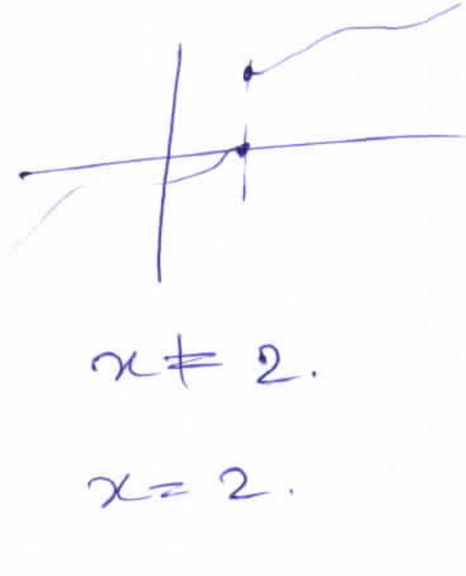
$$= e^{\alpha} \qquad \alpha = \lim_{x\to 0} \frac{1}{x}(1+2x-1)$$

$$= \lim_{x\to 0} 2$$

$$\alpha = 2$$

$$L.H.L = R.H.L = e^2 = f(0) = e^2$$

(B)



Pg 134-135

(10) $f(x) = \begin{cases} \left\{x^2 + e^{\frac{1}{2-x}}\right\}^{-1} & x \neq 2. \\ k & x = 2. \end{cases}$

R.H.L. $\lim_{x\to 2^+} \dfrac{1}{x^2 + e^{\frac{1}{2-x}}} = \dfrac{1}{4+0} = \dfrac{1}{4}$

$$R.H.L = f(2) = \frac{1}{4} = k$$

(12) $f(x) = \begin{cases} \dfrac{x^2-(a+2)x+a}{x-2} & x \neq 2. \\ 2 & x = 2 \end{cases}$

$$\lim_{x\to 2} \frac{x^2-(a+2)x+a}{x-2} = 2$$

$$x^2-(a+2)x+a = (x-2)(x-\beta)$$

$$\frac{x^2-(a+2)x+a}{x-2} = x-\beta$$

$\cancel{f(2) = 2}$.

$2-\beta = 2$. $\beta = 0$

$= (x-2)x$.

$a = 0$

$$\lim_{x\to 2.} \frac{x^2-2x}{x-2} = \lim_{x\to 2} \frac{x\cancel{(x-2)}}{\cancel{x-2}} = 2$$

15) $f(x) = \begin{cases} \sin^{-1}|x| & x \neq 0 \\ 0 & x = 0. \end{cases}$

$= \begin{cases} \sin^{-1}(-x) & x < 0 \\ 0 & x = 0 \\ \sin^{-1} x & x > 0 \end{cases}$

at $x = 0$

L.H.L $= 0$

R.H.L $= 0$ $\quad f(0) = 0$

$\Rightarrow$ continuous at $x = 0$

19) $f(x) = \begin{cases} x - 1 & -\infty < x \leq 1 \\ x^3 - 1 & 1 < x < \infty \end{cases}$

at $x = 1$

L.H.L. $= x - 1 = 1 - 1 = 0 = f(1)$

R.H.L $= x^3 - 1 = 1^3 - 1 = 0$

continuos at $x = 1$

$f'(x) = \begin{cases} 1 & -\infty < x \leq 1 \\ 3x^2 & 1 < x < \infty \end{cases}$

L.H.D $= 1 \neq$ R.H.D $= 3$ Not Differentiable

(21) $f(x) = \begin{cases} e^x + ax & x < 0 \\ b(x-1)^2 & x \geq 0 \end{cases}$

at $x = 0$

$L \cdot H \cdot L = 1.$

$R \cdot H \cdot L. = b(0-1)^2 = b.$

for continuity $\underline{\underline{b = 1}}$.

$$f'(x) = \begin{cases} e^x + a & x < 0 \\ b\,2(x-1)^1 \times 1 & x \geq 0 \end{cases}$$

$L \cdot H \cdot D. = 1 + a$

$R \cdot H \cdot D = b(2)(0-1) = -2b.$

for differentiability $L \cdot H \cdot D = R \cdot H \cdot D$

$1 + a = -2b$

$= -2$

$a = -3$

$(-3, 1)$

(25) $f(x) = \begin{cases} 1 & x < 0 \\ 1+\sin x & 0 \le x \le \pi/2 \end{cases}$

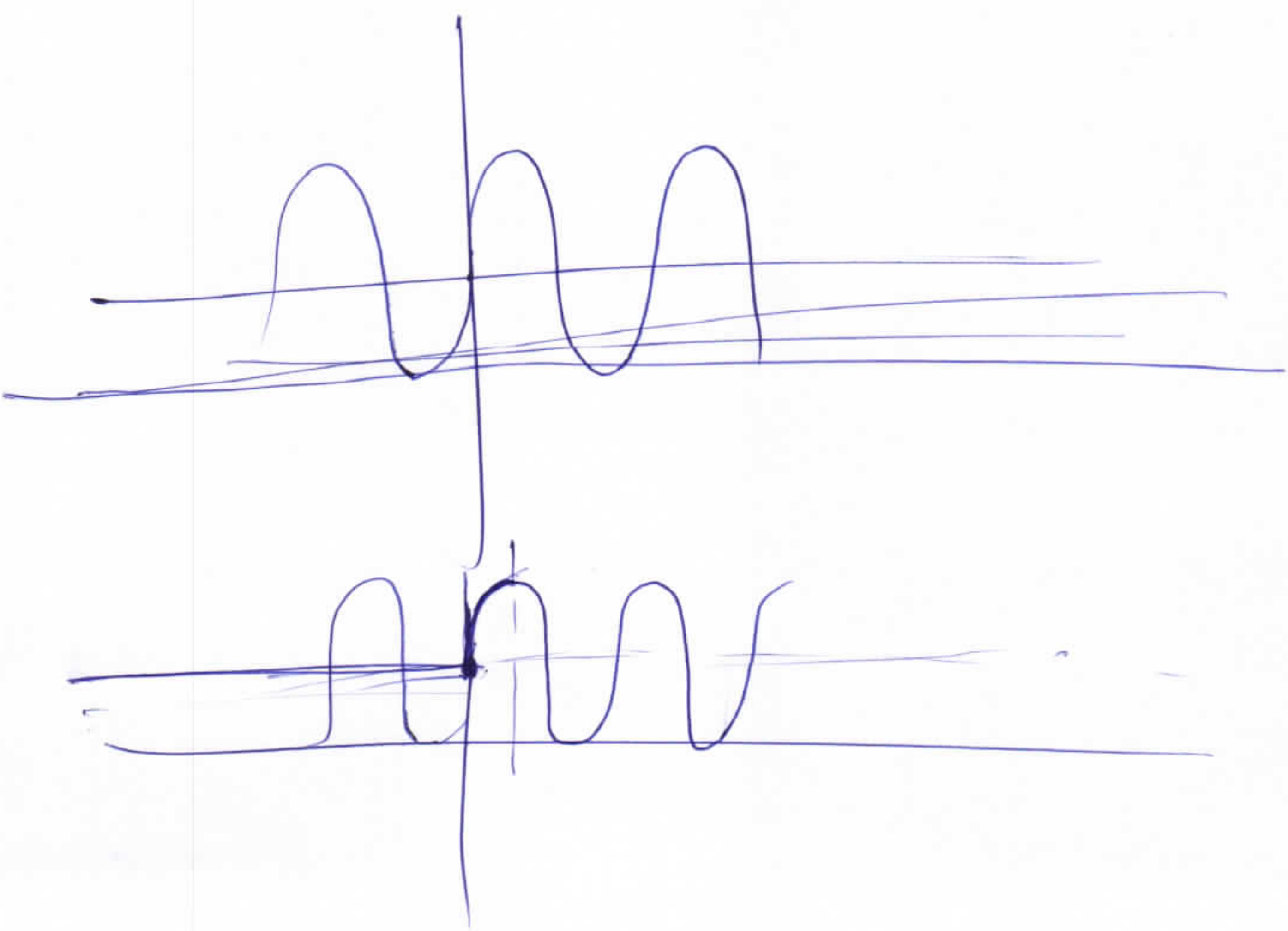
$f'(x) = ?$

$f'(x) = \begin{cases} 0 & x < 0 \\ \cancel{0+}\cos x & 0 \le x \le \pi/2 \end{cases}$

$L \cdot H \cdot D. = 0$

$\neq \; R \cdot H \cdot D = \cancel{0}\cos 0 = 1.$

Does not exist.



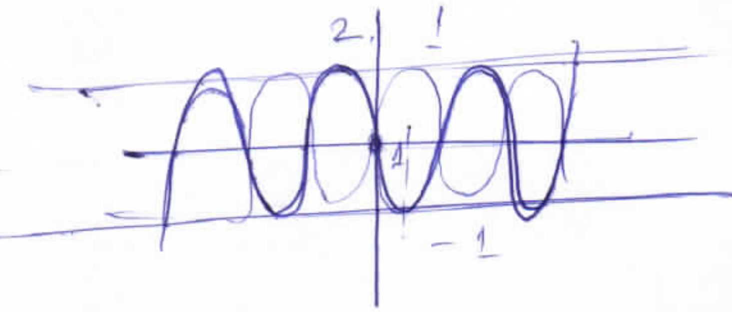
(36) $f(x) = \begin{cases} \dfrac{\sin[x]}{[x]+1} & x > 0 \\[2ex] \dfrac{\cos\left(\frac{\pi}{2}[x]\right)}{[x]} & x < 0 \\[2ex] k & x = 0 \end{cases}$

at $x = 0$

$$L\cdot H\cdot L = \frac{\cos\left(\frac{\pi}{2}(-1)\right)}{(-1)} = \frac{0}{-1} = 0$$

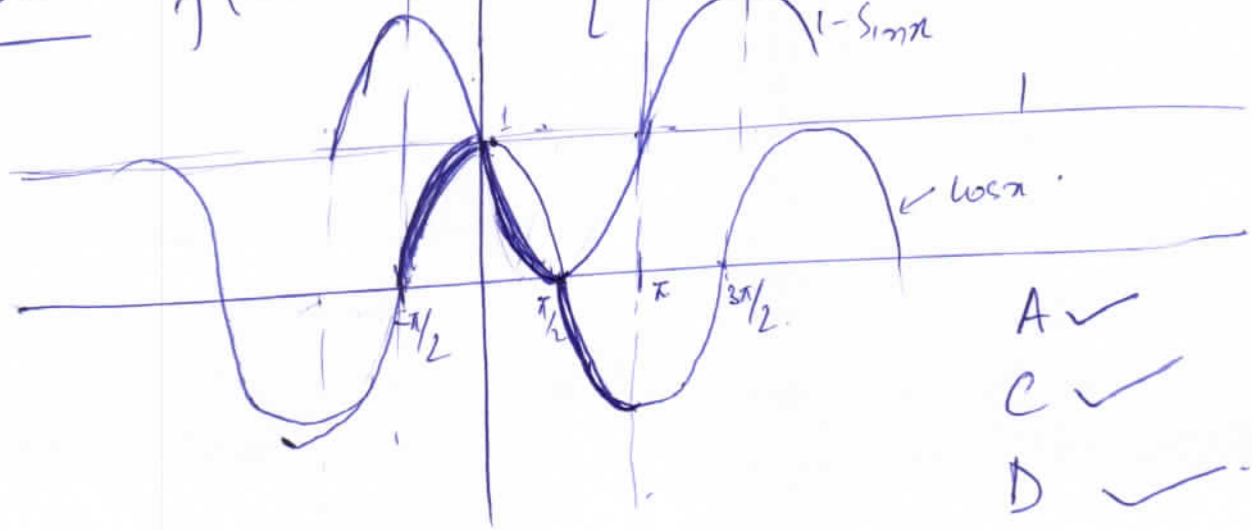
$$R\cdot H\cdot L = \frac{\sin(0)}{0+1} = 0$$

$$k = 0$$



Pg 136

(8) $f(x) = \min\{1, \cos x, 1-\sin x\} \quad -\frac{\pi}{2} \le x \le \pi.$



A ✓

C ✓

D ✓

(9) $f(x) = \lim_{n\to\infty} (\sin x)^{2n}$.

$\lim_{x\to \pi/2^+} f(x) = 0$ $\quad$ $\sin\left(\frac{\pi}{2}^+\right) = 1^-$

$\lim_{n\to\infty}(1^-)^{2n} = 0$

$\lim_{x\to \frac{\pi}{2}^-} f(x) = 0$ $\quad$ $\sin\left(\frac{\pi}{2}^-\right) = 1^-$

$n\to\infty \ (1^-)^{2n} = 0$

$f(1) = 1^\infty = 1$.

discontinuos at $x = \pi/2$.

discontinuos at $x = -\pi/2$.

discontinuous at infinite points

$\ldots -\frac{3\pi}{2}, -\pi/2, \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots$

B C D.

1) $f(x+y) = f(x) + f(y)$

find $f(x)$

2) $f(x+y) = f(x) \cdot f(y)$ find $f(x)$.

3) $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ find $f(x)$.

① $\underline{f(x+y) = f(x) + f(y)}$ $\longrightarrow$ $f(0+0) = f(0) + f(0)$

$= f(0) = 2f(0)$

$\Rightarrow f(0) = 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x) + f(h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h}.$$

$$f'(x) = \lim_{h\to 0} \frac{f(0+h) - 0}{h}.$$

$$= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$f'(x) = f'(0)$ $\Rightarrow \frac{df(x)}{dx} = f'(0)$

$\Rightarrow df(x) = f'(0)\,dx.$

$$f(x) = f'(0)x + c$$

$$f(0) = f'(0) \times 0 + c$$

$$0 = 0 + c \Rightarrow c = 0$$

$$\boxed{f(x) = \{f'(0)\}x}$$

$$f(x+y) = f(x) + f(y)$$

(3) $$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$$

$$f'(0) = -1 \qquad f(0) = 1$$

find $f(8)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(\frac{2x+2h}{2}\right) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(2x) + f(2h)}{2} - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(2x) + f(2h) - 2f(x)}{2h}$$

$y = 0$

$$f\left(\frac{x}{2}\right) = \frac{f(x) + f(0)}{2}$$

$$f(x) = \frac{f(2x) + f(0)}{2}$$

$$f(2x) = 2f(x) - f(0)$$

$$f(2x) = 2f(x) - 1.$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{2f(x)} - 1 + f(2h) \cancel{- 2f(x)}}{2h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(2h) - 1}{2h}$$

$$f'(x) = \lim_{h\to 0} \frac{2f(h) - 1 - 1}{2h} = \lim_{h\to 0} \frac{\cancel{2}(f(h) - 1)}{\cancel{2}h}$$

$$f'(x) = \lim_{h\to 0} \left(\frac{f(0+h) - f(0)}{h}\right)$$

$$f'(x) = f'(0)$$

$$\frac{d f(x)}{dx} = f'(0)$$

$$d f(x) = f'(0)\, dx$$

$$f(x) = f'(0)x + c$$

$$f(0) = f'(0) \times 0 + c$$

$$c = f(0) = 1.$$

$$f(x) = f'(0)x + 1.$$

$$f(x) = -x + 1.$$

$$f(8) = -8 + 1 = -7.$$

$$f(x) = ax + b.$$

Comp2

$$f\left(\frac{x+y}{3}\right) = \frac{2 + f(x) + f(y)}{3}.$$

$$f'(2) = 2.$$

$$f(x) = ax + b.$$

$$x = y = 0$$

$$f\left(\frac{0+0}{3}\right) = \frac{2 + f(0) + f(0)}{3}$$

$$f(0) = \frac{2 + 2f(0)}{3}$$

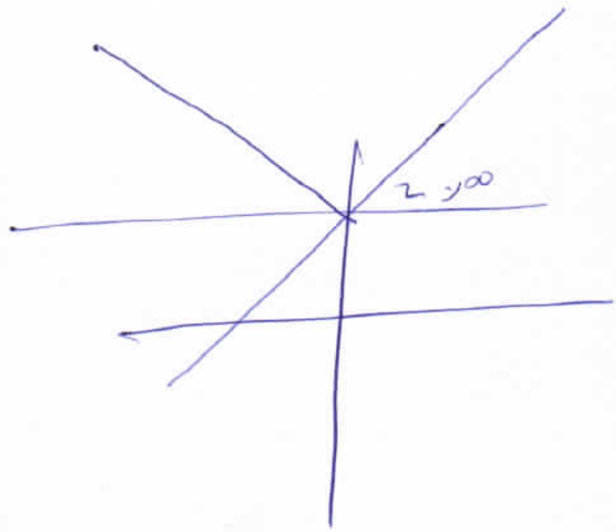
$$3f(0) = 2 + 2f(0)$$

$$f(0) = 2.$$

$f(0) = a \times 0 + b = b$

$2 = b$

$f(x) = ax + 2$



$f'(x) = a$

$f'(2) = a = 2$

$f(x) = 2x + 2$

$f(1) = 2(1) + 2 = 4$

$f(|x|) \qquad [2, \infty)$

⑥ $g(x) = |f(|x|) - 3|$

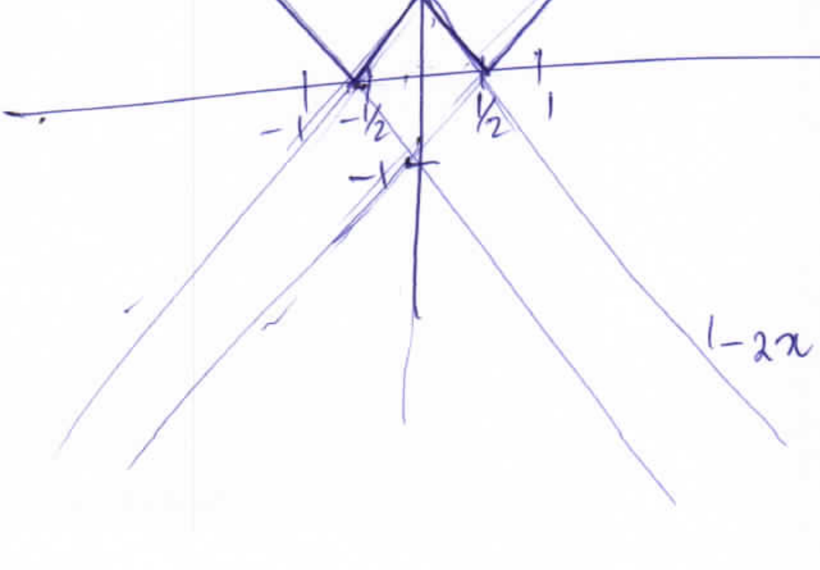
$$f(|x|) = \begin{cases} 2x + 2 & x \geq 0 \\ -2x + 2 & x < 0 \end{cases}$$

$$f(|x|) - 3 = \begin{cases} 2x - 1 & x \geq 0 \\ -2x - 1 & x < 0 \end{cases}$$

$$|f(|x|)-3| = \begin{cases} |2x-1| & x \geq 0 \\ |-2x-1| & x < 0 \end{cases}$$

$$= \begin{cases} 1-2x & 0<x<\frac{1}{2} \\ 2x-1 & x \geq \frac{1}{2} \\ -(-2x-1) & -\frac{1}{2} \leq x < 0 \\ -2x-1 & x < -\frac{1}{2} \end{cases}$$

$$= \begin{cases} -2x-1 & x < -\frac{1}{2} \\ 2x+1 & -\frac{1}{2} \leq x < 0 \\ 1-2x & 0 < x < \frac{1}{2} \\ 2x-1 & x \geq \frac{1}{2} \end{cases}$$



(c) = 3.