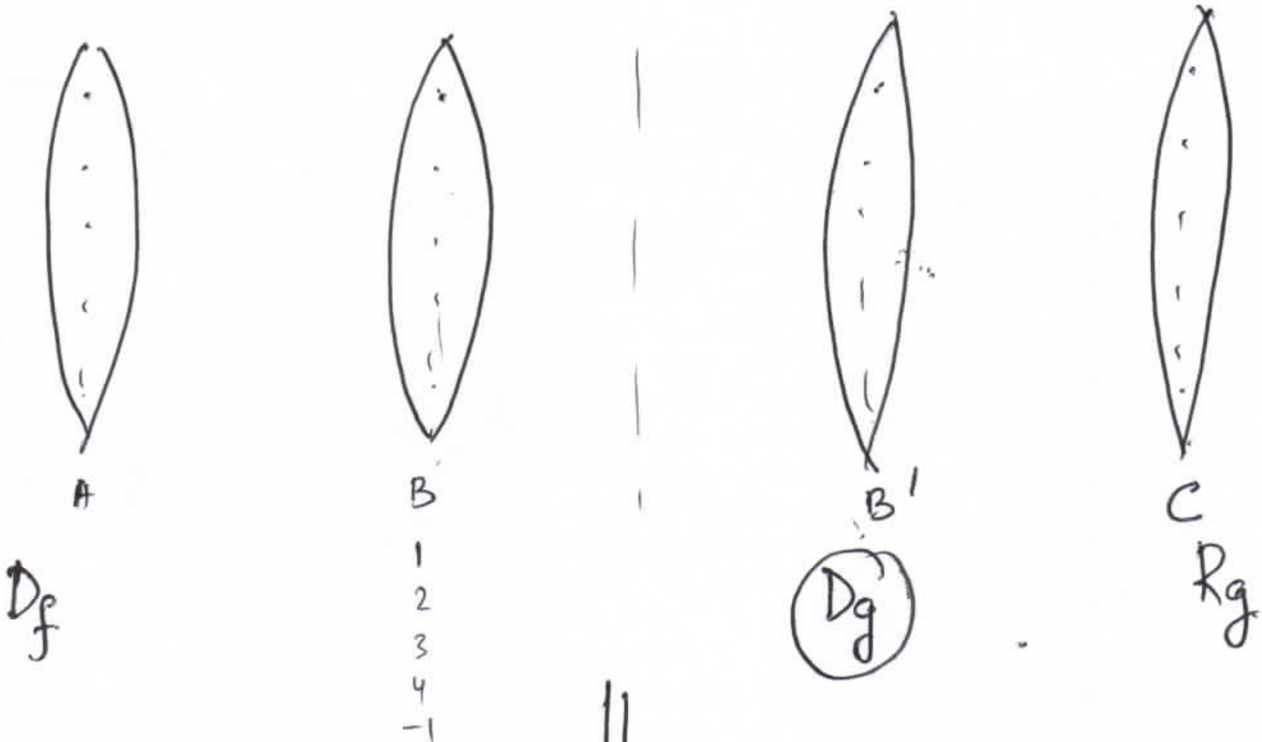
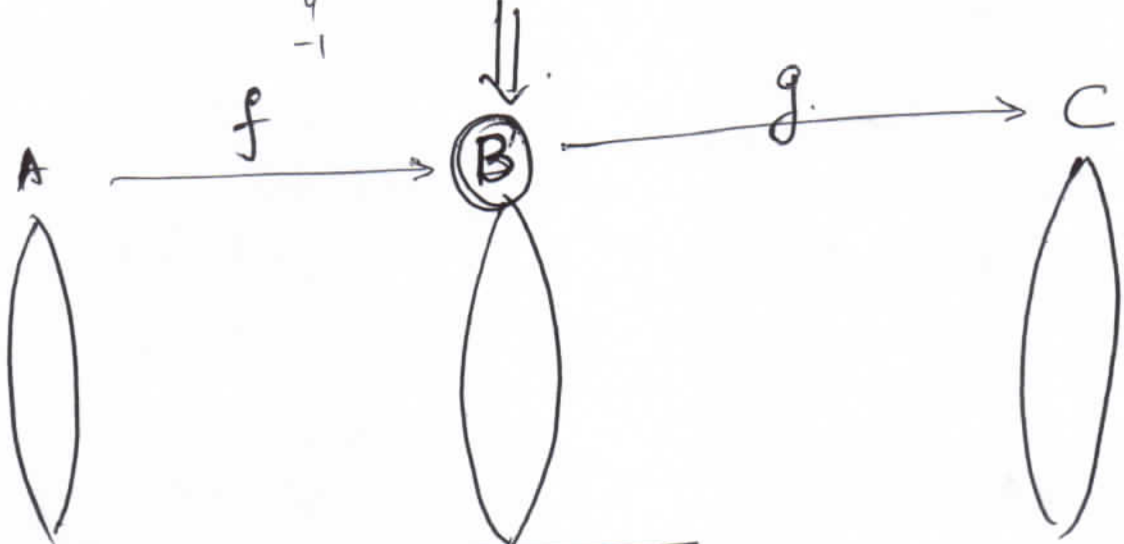
COMPOSITION OF FUNCTIONS

let $f : A \longrightarrow B$ | $g : B' \longrightarrow C$





$$\boxed{R_f \subseteq D_g}$$

$h : A \longrightarrow C$

$h(x) = g(f(x))$

outer function. inner function

$R_i \subseteq D_o$

$R_h \subseteq R_o$

$h(x) = g(f(x))$ $\qquad (g \circ f)(x) = g(f(x))$

→ Domain of $g \circ f$ = Domain of f

→ Range of $g \circ f$ = Range of g or subset of R_g

$$R_f \subseteq D_g.$$

eg. $f : \mathbb{R} \longrightarrow \mathbb{R}$ $\qquad f(x) = x + 1$

$g : \mathbb{R} \longrightarrow \mathbb{R}$ $\qquad g(x) = x^2$

find $g \circ f$ & $f \circ g$

$(g \circ f)(x) = (x+1)^2$

$g(f(x)) = g(x+1) = \underline{(x+1)^2}$

$(f \circ g)(x) = f(g(x))$

$= f(x^2) = \underline{x^2 + 1}$

$R_g \subseteq D_f$

$\mathbb{R} \subseteq \mathbb{R}$ ✓

In general $(g \circ f)(x) \neq (f \circ g)(x)$

eg. $f(x) = \operatorname{sgn}(x)$ $\quad D_f \to R_f$, $\quad R \longrightarrow \{-1, 0, 1\}$.

$g(x) = 1 + \{x\}$ $\quad R \longrightarrow [1, 2)$, $\quad D_g \longrightarrow R_g$

Find $g \circ f$ & $f \circ g$

$g \circ f$ $\qquad R_f \subseteq D_g$ ✓

$$g(f(x)) = g(\operatorname{sgn}(x)) = 1 + 0 = 1.$$

$(g \circ f)(x)$ $\qquad R \longrightarrow \{1\}$

$f \circ g$. $\qquad R_g \subseteq D_f$ ✓

$[1,2) \subset R$

$$f(g(x)) = f(1 + \{x\}) = \operatorname{sgn}(1 + \{x\})$$
$$= 1.$$

$(f \circ g)(x) = 1.$ $\qquad R \longrightarrow \{1\}.$

In general

$g \circ f \neq f \circ g$ (Not commutative)

$(g \circ f) \circ h = g \circ (f \circ h)$ (associative)

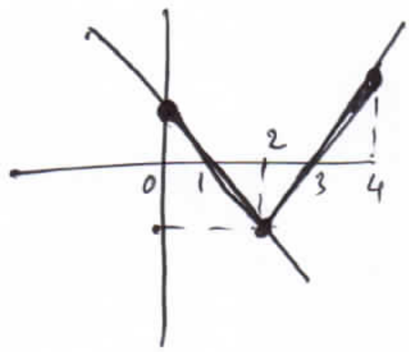
Composition of non-uniformly defined function.

$f(x) = -1 + |x-2| \qquad 0 \le x \le 4$

$g(x) = 2 - |x| \qquad -1 \le x \le 3.$

$g \circ f$

$$f(x) = \begin{cases} -1-(x-2) \\ = 1-x & 0 \le x < 2 \\ -1+x-2 \\ = x-3 & 2 \le x \le 4 \end{cases}$$



$$g(x) = \begin{cases} 2-(-x) \\ = 2+x & -1 \le x < 0 \\ = 2-x & 0 \le x \le 3 \end{cases}$$

$g \circ f = g(f(x))$

$$g(f(x)) = \begin{cases} 2+f(x) & -1 \le f(x) < 0 \\ 2-f(x) & 0 \le f(x) \le 3 \end{cases}$$

$2+f(x)$

$-1 \le f(x) < 0$

$f(x) \quad 1-x \cdot \ (0 \le x < 2$

$-1 \le 1-x < 0$

$0 < x-1 \le 1$

$1 < x \le 2 \quad f(x) = (1-x)$

$\to 2+(1-x)$

$\boxed{3-x \qquad 1 < x \le 2}$

$f(x) \quad x-3. \quad (2 \le x \le 4)$

$-1 \le x-3 < 0$

$2 \le x < 3 \quad f(x) = x-3$

$\to 2+x-3$

$\boxed{x-1 \qquad 2 \le x < 3}$

$2-f(x)$

$0 \le f(x) \le 3$

$f(x) \quad (1-x)$

$0 \le 1-x \le 3$

$-3 \le x-1 \le 0$

$-2 \le x \le 1$

$0 \le x \le 1 \quad f(x) = 1-x$

$\to 2-(1-x)$

$\boxed{x+1 \qquad 0 \le x \le 1}$

$f(x) \quad x-3$

$0 \le x-3 \le 3$

$3 \le x \le 6$

$3 \le x \le 4 \quad f(x) = x-3$

$\to 2-(x-3)$

$\boxed{5-x \qquad 3 \le x \le 4}$

$$(g\circ f)(x) = \begin{cases} x+1 & 0 \le x \le 1 \\ 3-x & 1 < x < 2 \\ x-1 & 2 \le x < 3 \\ 5-x & 3 \le x \le 4 \end{cases}$$

Q) $$f(x) = \begin{cases} x+1 & -\infty < x \le 1 \\ 2x+1 & 1 \le x \le 2 \end{cases}$$

$$g(x) = \begin{cases} x^2 & -1 \le x < 2 \\ x+2 & 2 \le x \le 3 \end{cases}$$

Find $g\circ f$

Sol^n $$g(f(x)) = \begin{cases} \{f(x)\}^2 & -1 \le f(x) < 2 \quad \text{— (a)} \\ f(x)+2 & 2 \le f(x) \le 3 \quad \text{— (b)} \end{cases}$$

(a) $g(f(x)) = \{f(x)\}^2$

$-1 \le f(x) < 2$

$f(x) \quad x+1$	$f(x) \quad 2x+1$
$-1 \le x+1 < 2$	$-1 \le 2x+1 < 2$
$-2 \le x < 1$ $\quad f(x) = x+1$	$-2 \le 2x < 1$
$\boxed{(x+1)^2 \quad -2 \le x < 1}$	$-1 \le x < 1/2$ $\quad \times$

(b) $g(f(x)) = f(x)+2$

$2 \le f(x) \le 3$

$f(x) \quad x+1$	$f(x) \quad 2x+1$
$2 \le x+1 \le 3$	$2 \le 2x+1 \le 3$
$1 \le x \le 2$ $\quad f(x)=2x+1$	$1 \le 2x \le 2$
	$\frac{1}{2} \le x \le 1$ $\quad x+1$

$(2x+1)+2 = 2x+3$

$1 \le x \le 2$

$f(x)+2 = x+3 \quad \frac{1}{2} \le x \le 1$

$x=1 \quad g(f(x)) = x+3$

$$g \circ f(x) = \begin{cases} (x+1)^2 & -2 \le x < 1 \\ x+3 & x=1 \\ 2x+3 & 1 < x \le 2 \end{cases}$$

$$g \circ f(x) = \begin{cases} (x+1)^2 & -2 \le x < \tfrac{1}{2} \\ x+3 & \tfrac{1}{2} \le x \le 1 \\ 2x+3 & 1 < x \le 2 \end{cases}$$

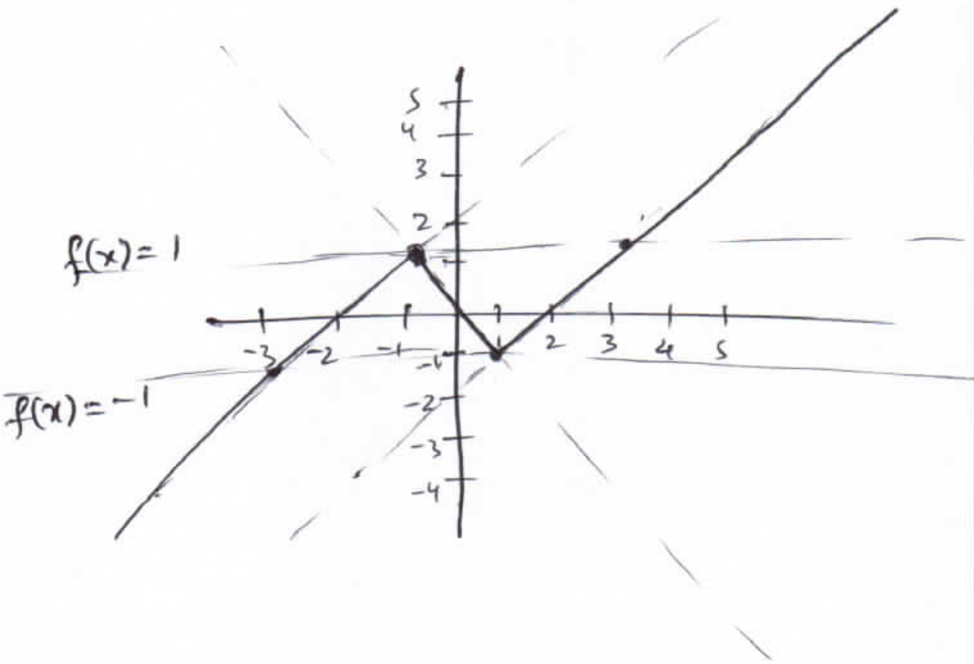
×

$$f(x) = \begin{cases} x+2 & x < -1 \\ -x & -1 \le x \le 1 \\ x-2 & x > 1 \end{cases}$$

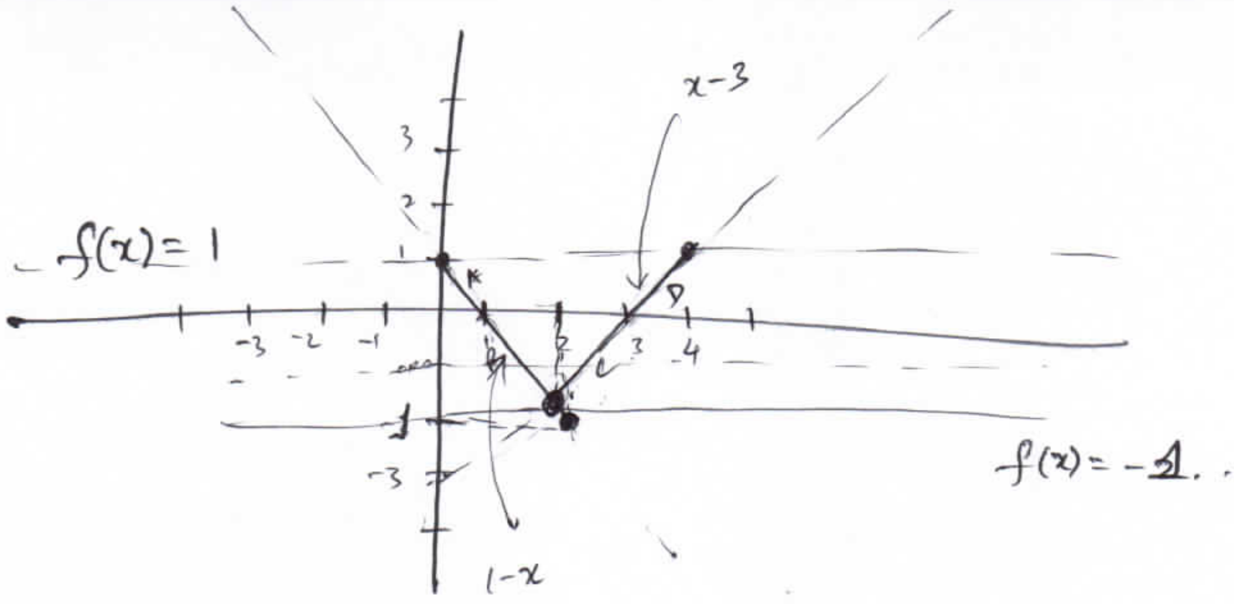
$$g(x) = \begin{cases} x-2 & x < -1 \\ -x & -1 \le x \le 1 \\ x+2 & x > 1 \end{cases}$$

find $g \circ f$

$$g(f(x)) = \begin{cases} f(x) - 2 & f(x) < -1 \\ -f(x) & -1 \le f(x) \le 1 \\ f(x)+2 & f(x) > 1 \end{cases}$$



$$g(f(x)) = \begin{cases} x & x < -3. \\ -(x+2) & -3 \le x < -1 \\ -(-x) & -1 \le x \le 1 \\ -(x-2) & 1 < x \le 3 \\ x-2 & x > 3 \end{cases}$$



$2 + 1 - x$ $\qquad$ $1 \le x < 2$

$2 + x - 3$ $\qquad$ $2 \le x \le 3$

$2 - (1 - x)$ $\qquad$ $0 \le x < 1$

$2 - (x - 3)$ $\qquad$ $3 < x \le 4$

$1 + x$ $\qquad$ $0 \le x < 1$

$3 - x$ $\qquad$ $1 \le x < 2$

$x - 1$ $\qquad$ $2 \le x \le 3$

$5 - x$ $\qquad$ $3 < x \le 4$

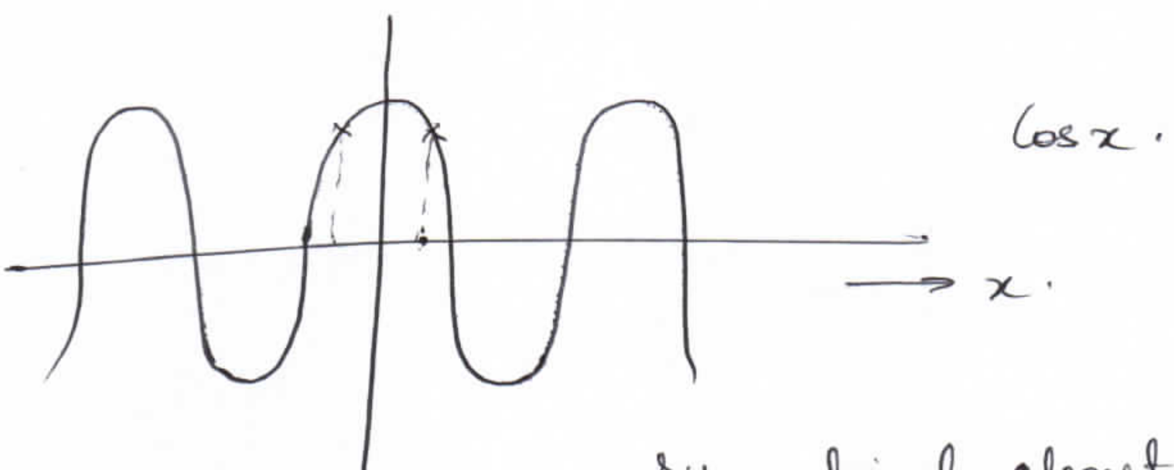
~~PERIODIC FUNCTION~~

EVEN & ODD FUNCTION.

A function f is said to be even iff $\cancel{f(x)}$ $f(-x) = f(x)$ $\forall x \in$ Domain

$$f(x) = \cos x$$

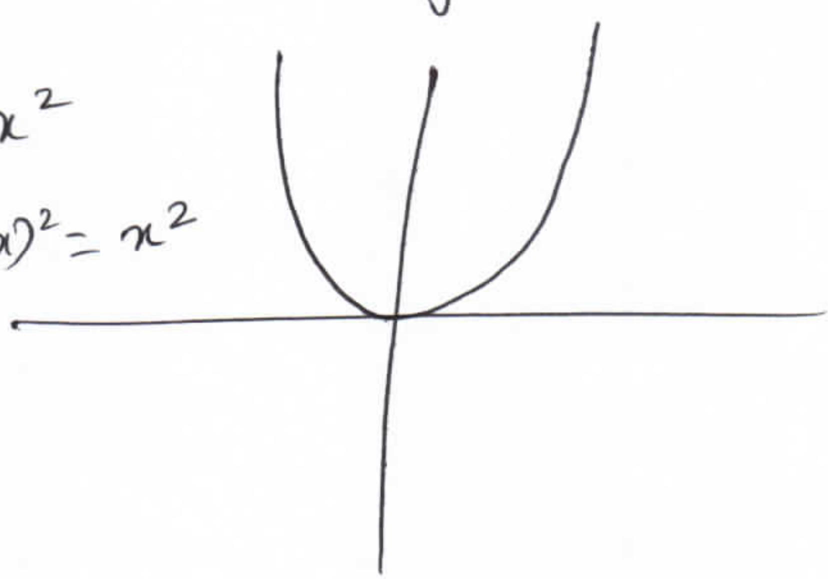
$$\cos(-x) = \cos x.$$



Symmetrical about the y axis.

$$f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2$$

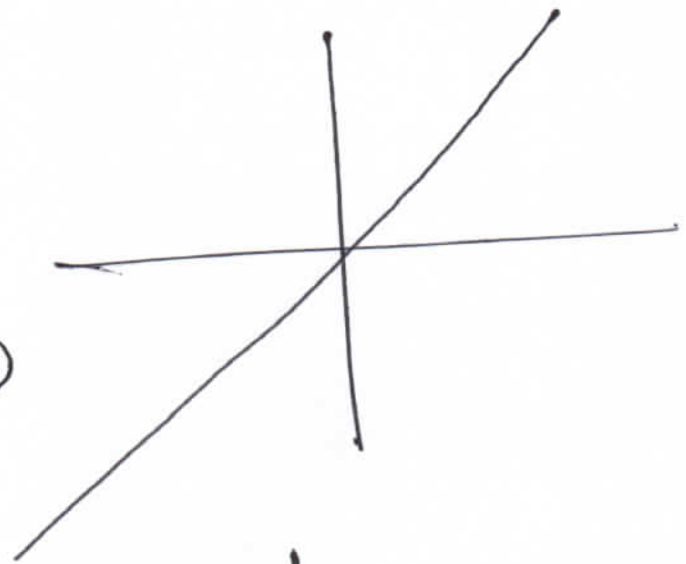


A function f is said to be odd function iff $f(-x) = -f(x)$ $\forall\, x \in$ Domain.

$f(x) = x$

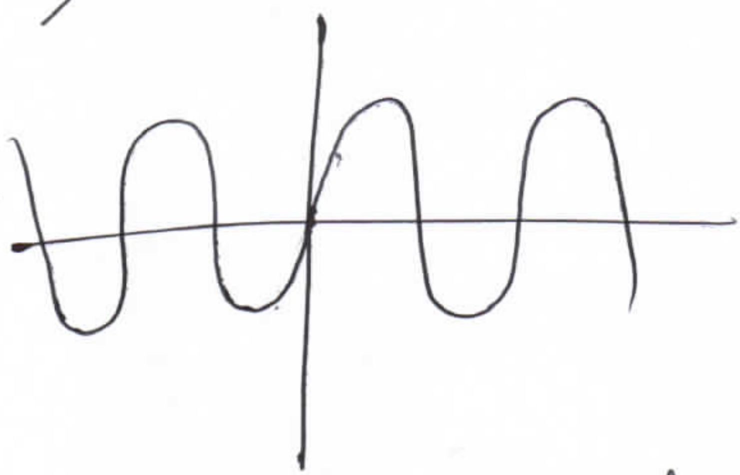
$f(-x) = -x$.

$f(-x) = -f(x)$



$f(x) = \sin x$

$f(-x) = -\sin x$.

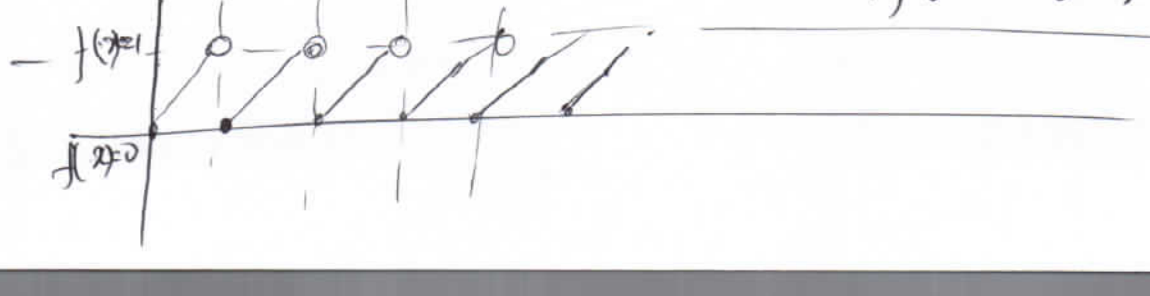


Symmetrical about the origin.

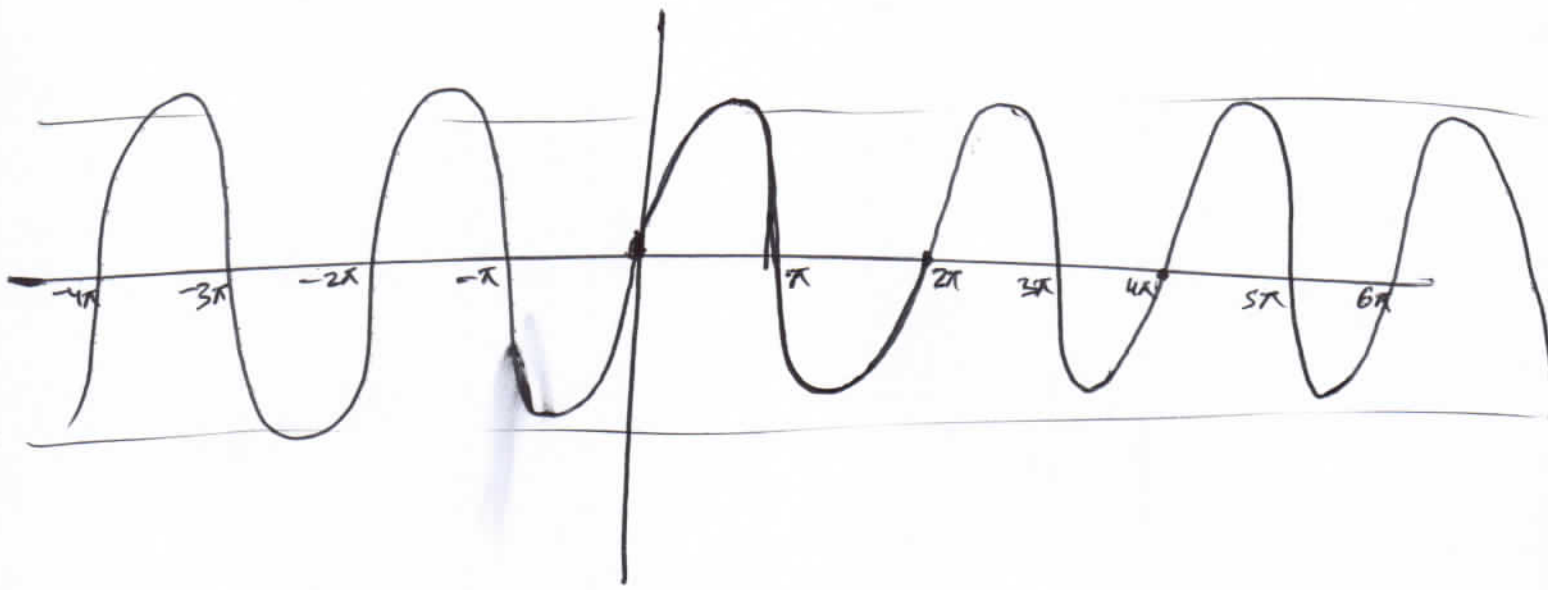
PERIODIC FUNCTIONS:

A function is called periodic if it repeats at regular intervals.

$f(x) = \{x\}$



If there exists 'T' least +ve real number such that $f(x+T) = f(x) \;\forall\; x \in$ Domain then T is called the fundamental period of $f(x)$



$\sin x \longrightarrow 2\pi.$

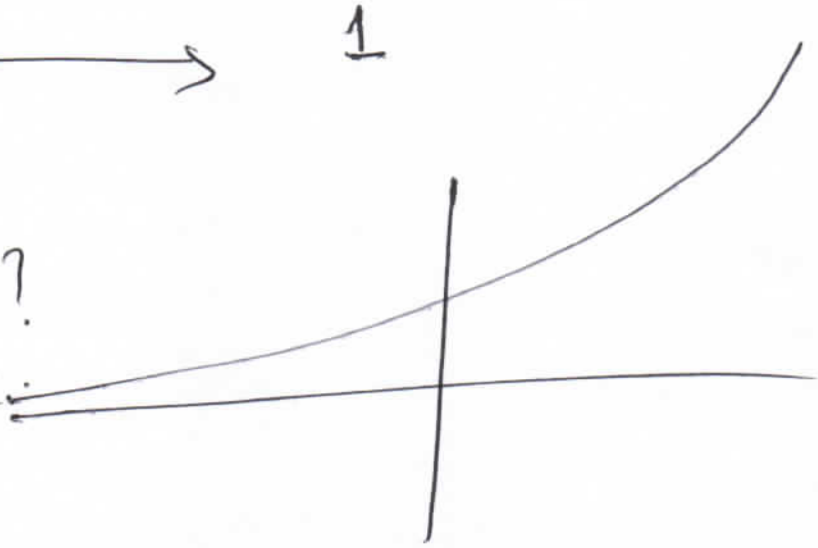
$\cos x \longrightarrow 2\pi.$

$\tan x \longrightarrow \pi$

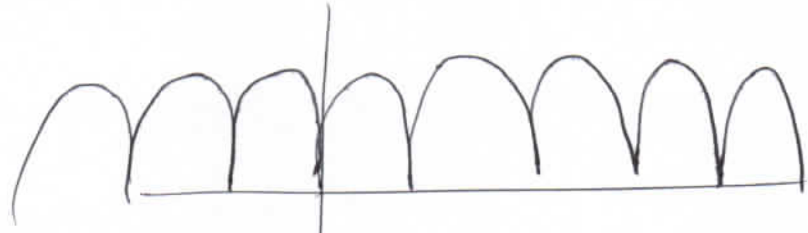
$\{x\} \longrightarrow 1$

Is e^x periodic.?

NOT PERIODIC.



$\sin^2 x$



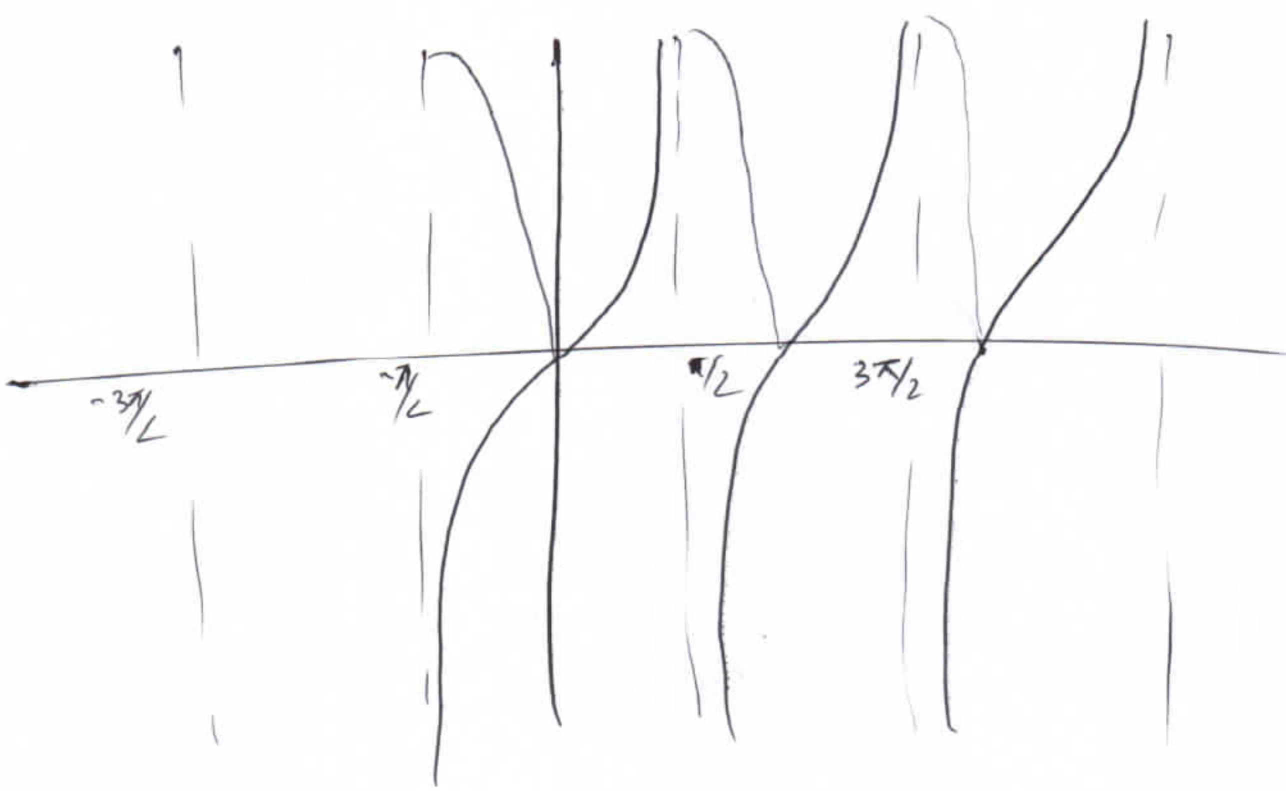
$\sin^n x \longrightarrow \pi$ if n is even.

$\qquad\qquad\quad 2\pi$ if n is odd.

$\cos^n x \longrightarrow \pi$ if n is even

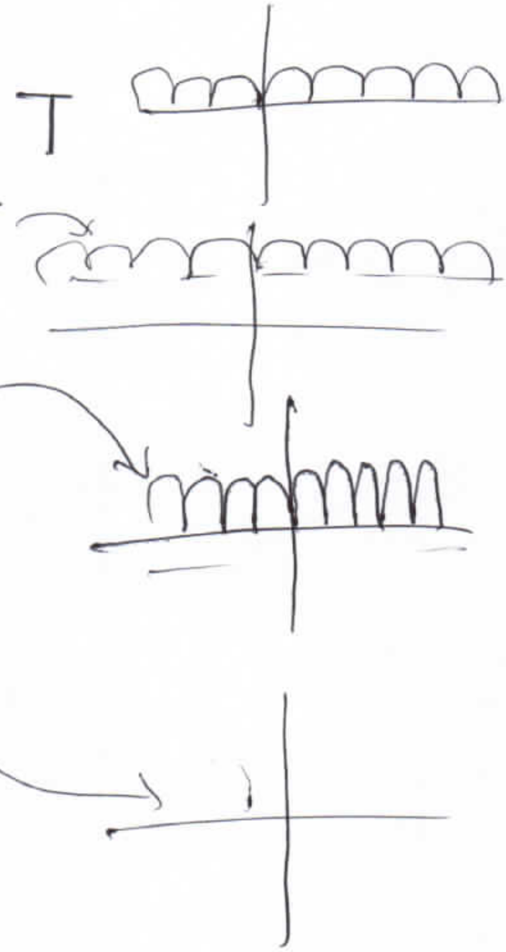
$\qquad\qquad\quad 2\pi$ if n is odd

$\tan^n x \longrightarrow \pi \quad \forall\ n.$



properties of periodic function.

if $f(x) \longrightarrow T$



$f(x)+c \longrightarrow T$

shift graph vertically by c units.

$c>0$ up

$c<0$ down

$cf(x) \longrightarrow T$

$f(x+c) \longrightarrow T$

Shift graph horizontally by c units

$c>0$ left

$c<0$ right.

$$f(ax) \longrightarrow \frac{T}{|a|}$$

$a>0$ compresses

$a<0$ expands

$$f(ax+b) \longrightarrow \frac{T}{|a|}$$

$\sin(x+30^\circ) \longrightarrow 2\pi.$

$\sin 2x \longrightarrow 2\pi/2 = \pi$

$2\tan x \longrightarrow \pi.$

$2\tan^2 2x \longrightarrow \pi/2$

$\sin^2 2x \longrightarrow \pi/2$

$\cos^3 2x \longrightarrow \frac{2\pi}{2} = \pi.$

If $f(x) \longrightarrow T_1$

$g(x) \longrightarrow T_2$

then the period of $h(x) = f(x) \pm g(x)$

$\mathrm{LCM}(T_1, T_2)$

except when $f(x)$ & $g(x)$ are <u>co function</u> or even function.

$\downarrow$

$f(A) = g(B)$

$\sin(x) = \cos(\frac{\pi}{2} - x)$

$f(x) = \{x\} + \sin \pi x \qquad \text{LCM}(1,2) = 2$

$f(x) = \{x\} + \cos x. \qquad \text{LCM}(1, 2\pi) = \text{NOT POSSIBLE}$

$f(x) = |\sin x| + |\cos x| \qquad \text{LCM}(\pi,\pi) = \pi = nT \to \pi/2$

$f(x) = \sin^2 x + \cos^4 x \qquad \text{LCM}(\pi,\pi) = \pi$

$f(x) = \sin \frac{3x}{11} + \tan \frac{3x}{13}$

$f(x) = \sin^2 x + \cos^4 x.$

$= \frac{1-\cos 2x}{2} + \left(\frac{1+\cos 2x}{2}\right)^2$

$= \frac{1}{2} - \frac{1}{2}\cos 2x$

$\sin^2 x + \cos^2 x(1-\sin^2 x)$

$= \sin^2 x + \cos^2 x - \sin^2 x \cos^2 x$

$= 1 - \frac{1}{4} 4\sin^2 x \cos^2 x$

$= 1 - \frac{1}{4}(2\sin x \cos x)^2$

$= 1 - \frac{1}{4}\sin^2 2x \qquad \boxed{\frac{\pi}{2}}$

$f(x) = \sin \frac{3x}{11} + \tan \frac{3x}{13}$

$\text{LCM}\left(\frac{22\pi}{3}, \frac{13\pi}{3}\right) = \frac{286\pi}{3}$

NOTES

1) $\sin^n x$, $\cos^n x$, $\operatorname{cosec}^n x$, $\sec^n x$

period 2π — n is odd

period π — n is even.

2) $\tan^n x$, $\cot^n x$

period π — $\forall n$

3) An algebraic function is always non periodic

4) Constant function is periodic with undefined period.

5) If f is periodic ~~fun~~ function.
then gof is also a periodic function.

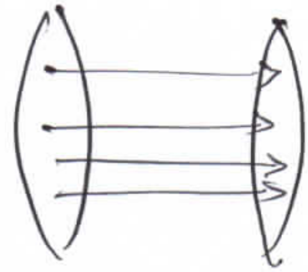
period of gof may or may not be equal to period of f.

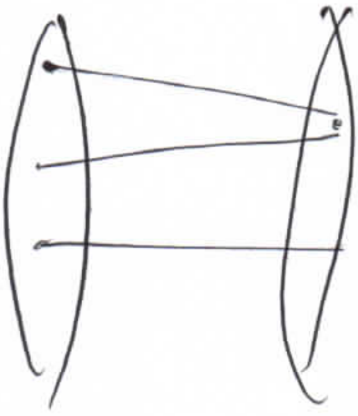
6) LCM of rational & irrational number is not possible to determine.

Classification of functions.

→ One - One

→ Many - One





How to check if function is one - one.

(i) If $y = f(x)$

if $f(x_1) = f(x_2)$ is solved.

& only solution obtained is $x_1 = x_2$ then

$f(x)$ is one - one.

eg. $f(x) = 2x + 1$

$f(x_1) = f(x_2)$

$2x_1 + 1 = 2x_2 + 1$

$x_1 = x_2$

$f(x) = x^2$

$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$x_1^2 - x_2^2 = 0$

$(x_1 - x_2)(x_1 + x_2) = 0$

$x_1 = x_2$

$x_1 = -x_2$

not one-one

$$f(x) = e^{1/x} - 1$$

$$f(x_1) = f(x_2)$$

$$e^{1/x_1} \cancel{-1} = e^{1/x_2} \cancel{-1}$$

$$e^{1/x_1} = e^{1/x_2}$$

$$\underline{x_1 = x_2}.$$ ONE . ONE.

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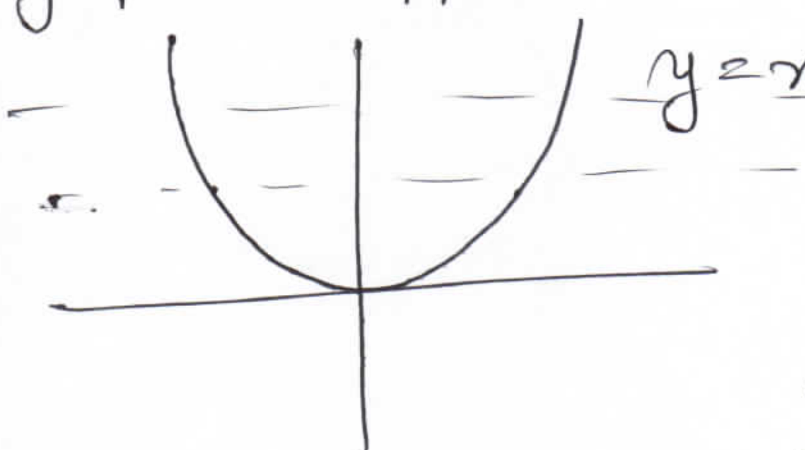
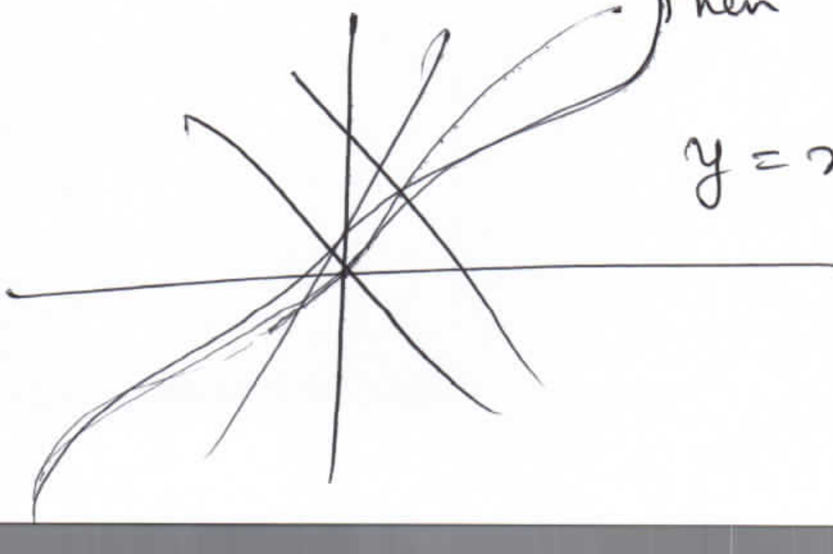
(2) Graphical Approach.



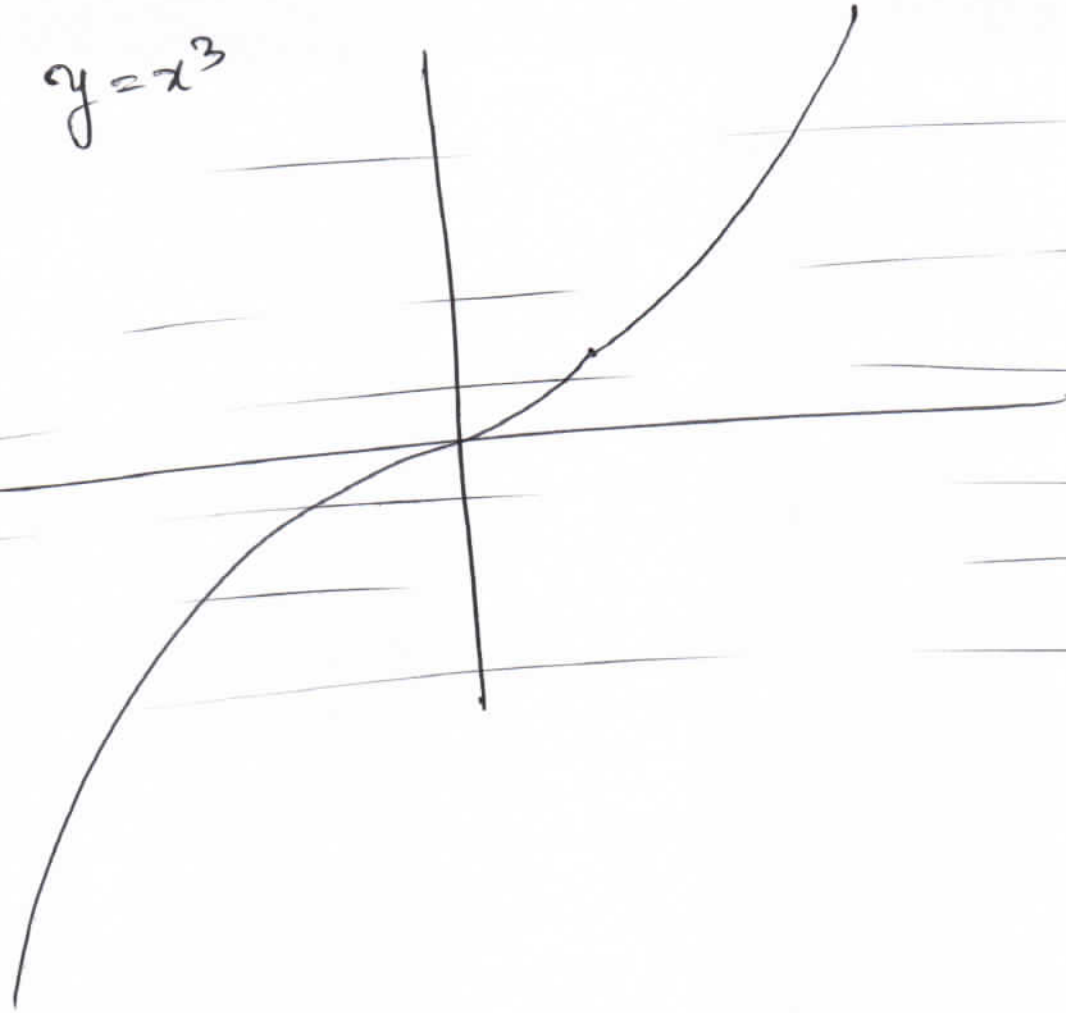
$y = x^2$.

Draw lines // to x axis if at any of these lines cuts the graph at more than one point then it is many-one function.

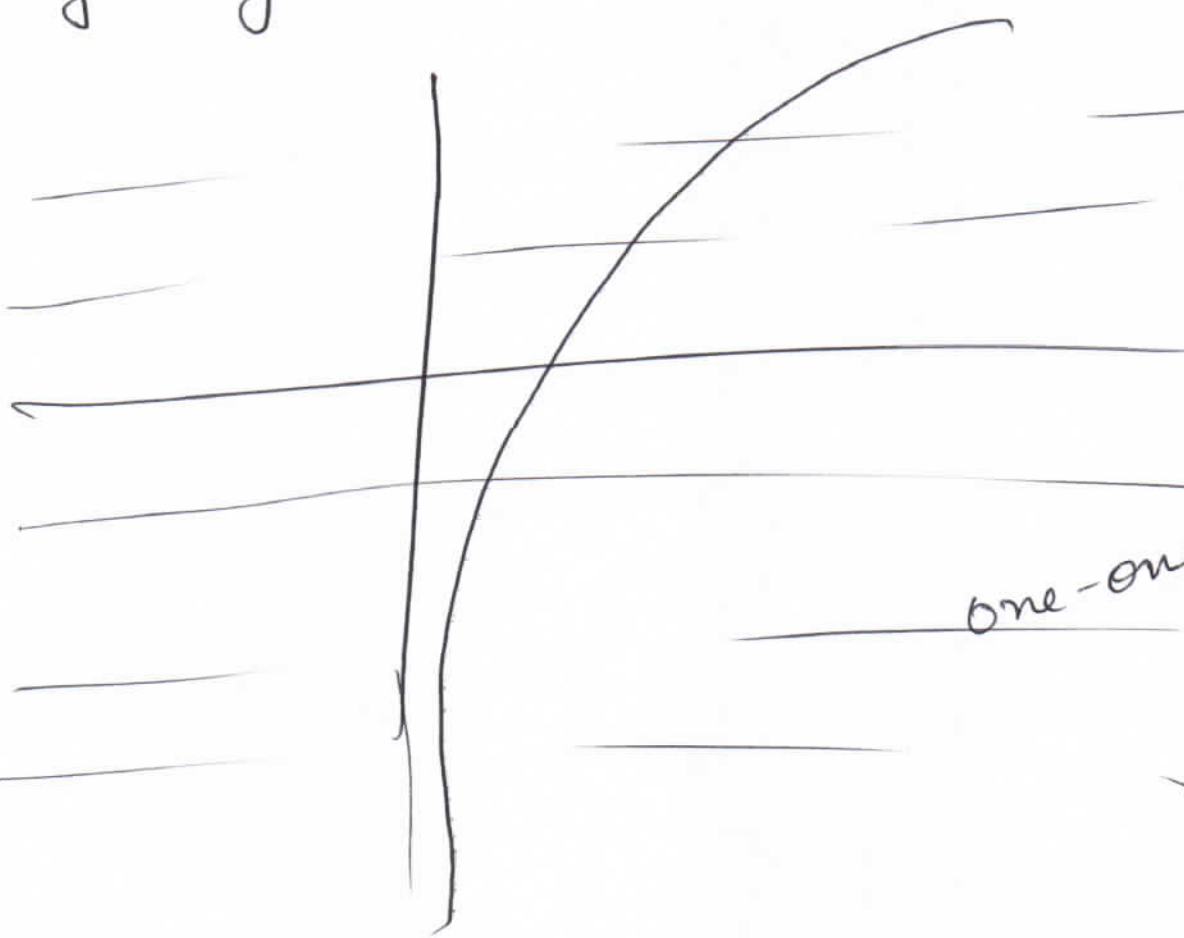


$y = x^3$

$y = x^3$

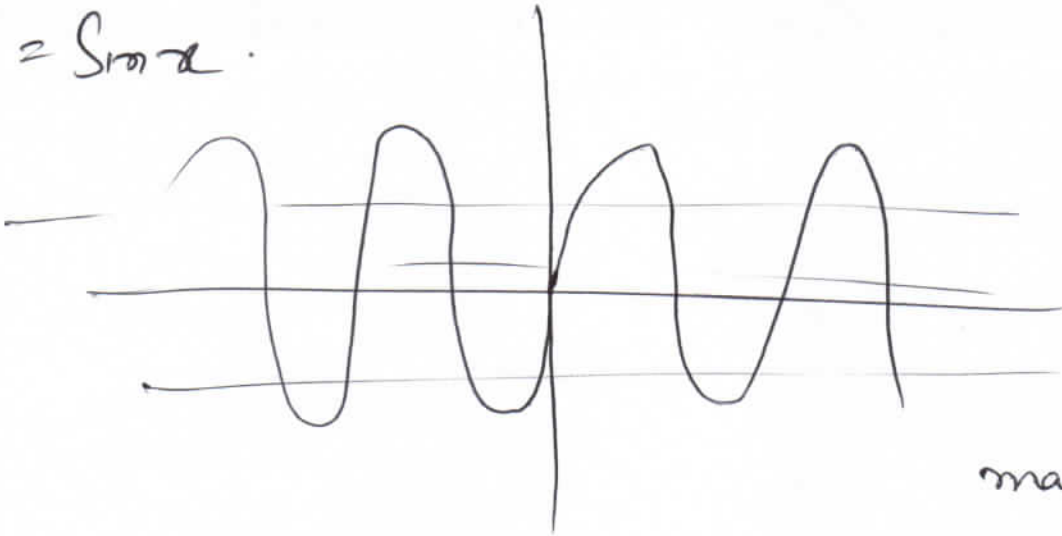


$y = \log x$



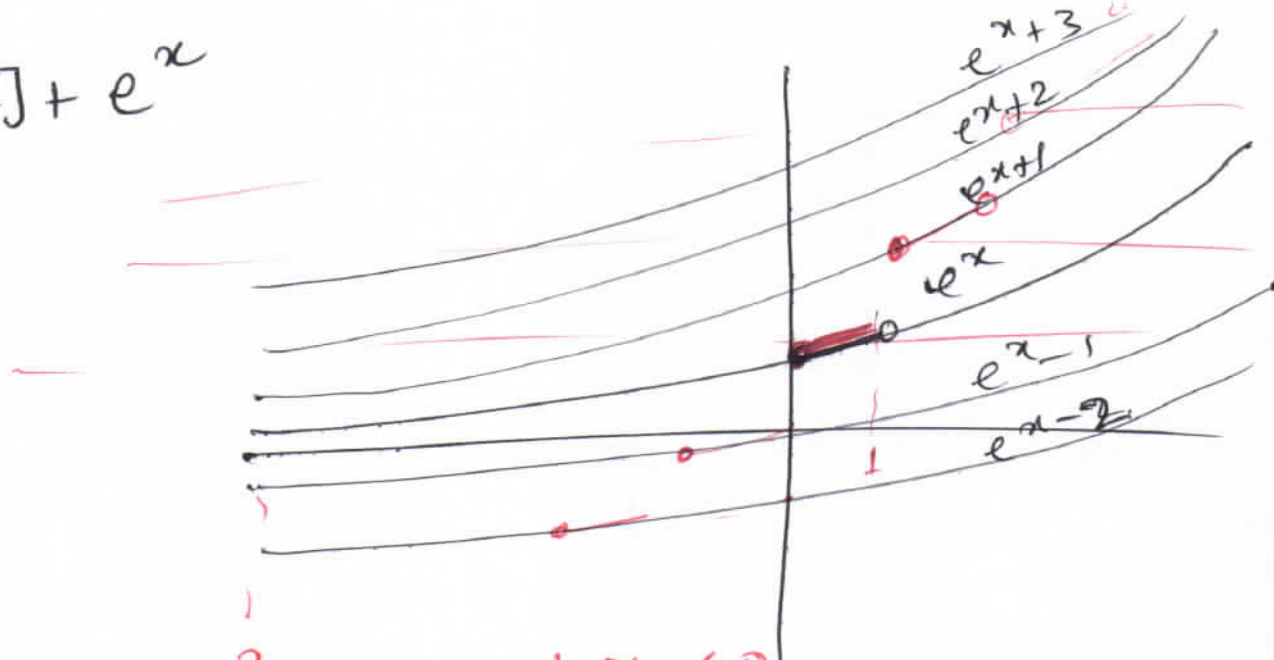
one-one.

$y = \sin x$



many-one

$y = [x] + e^x$



$$y = \begin{cases} e^x - 1 & -1 \le x < 0 \\ e^x & 0 \le x < 1 \\ e^x + 1 & 1 \le x < 2 \\ e^x + 2 & 2 \le x < 3 \\ \vdots & \end{cases}$$

one-one

(3) MONOTONOCITY.

$f(x)$

$f'(x) \quad \frac{d\, f(x)}{dx}$.

If $f'(x) \geq 0$ or $f'(x) \leq 0$

increasing — One-One — decreasing

$\ln x = \log_e x$

$D = (0, \infty)$

eg. $f(x) = x^2 + \ln x$

$f'(x) = 2x + \frac{1}{x}$

$= \frac{2x^2 + 1}{x} \geq 0$

One-One.

$f(x) = 2x - \sin x$.

$f'(x) = 2 - \cos x > 0$ One-One.

$f(x) = x^3 - 4x^2 + 12x - 1$

$f'(x) = 3x^2 - 8x + 12$.

$D = 8^2 - 4(12)(3) < 0$

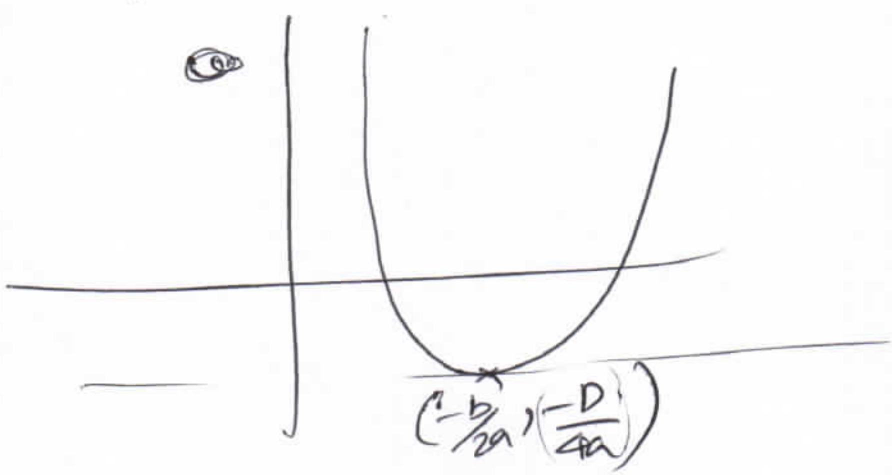
II Onto function (Surjective)

Into function.

Onto function.	Into function
Range = Co-domain	Range $\subsetneq$ Codomain

i) $f : R \longrightarrow [1, \infty)$

$f(x) = x^2 - 2x + 4$



If Codomain is not provided assume it to be $\underline{\underline{\mathbb{R}}}$

$$-\frac{(b^2 - 4ac)}{4a}$$

$$-\frac{(4-16)}{4 \times 1} = \frac{12}{4} = 3$$

Range $= [3, \infty)$

Range $\subset$ Codomain

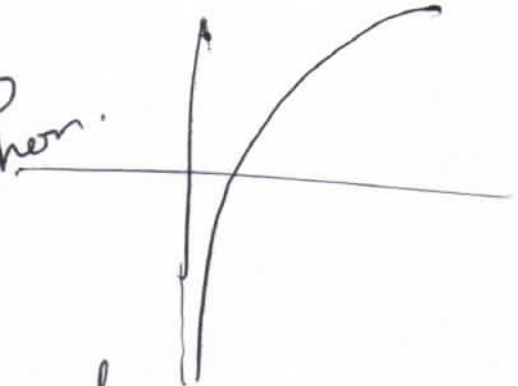
$[3, \infty) \subset [1, \infty)$ $\underline{\underline{\text{Into}}}$

$f(x) = x^2 + \ln x$. $\quad D = \mathbb{R}^+$

$f'(x) = 2x + \frac{1}{x}$ $\quad D = (0, \infty)$

$= \frac{2x^2+1}{x} > 0$

Range. $\left(f(0), f(\infty)\right)$ Onto function.
$(-\infty, \infty)$



$f(x) = x^3 - x$.

Onto function.

H.W. Check if function are Many One/One-One
Into/Onto.

i) $f(x) = x^3 - 6x^2 + 2x + 1$

ii) $f(x) = x|x| = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$

iii) $f : R \to R$.
$f(x) = 6^{|x|} + 6^{-x}$. (Use graphical method)

ONE - ONE & ONTO function is called a Bijective function.

Find Inverse of a function f

$$\Downarrow$$

$$f^{-1}$$

$f(x)$ inverse is $f^{-1}(x)$

The condition (necessary & sufficient) for existance of inverse of $f(x)$ is that $f(x)$ must be bijective.

If $f : A \longrightarrow B$ is a bijective function

$g : B \longrightarrow A$ is a bijective function

s.t $(f \circ g)(x) = x \quad \forall \; x \in B$

& $(g \circ f)(x) = x \quad \forall \; x \in A$

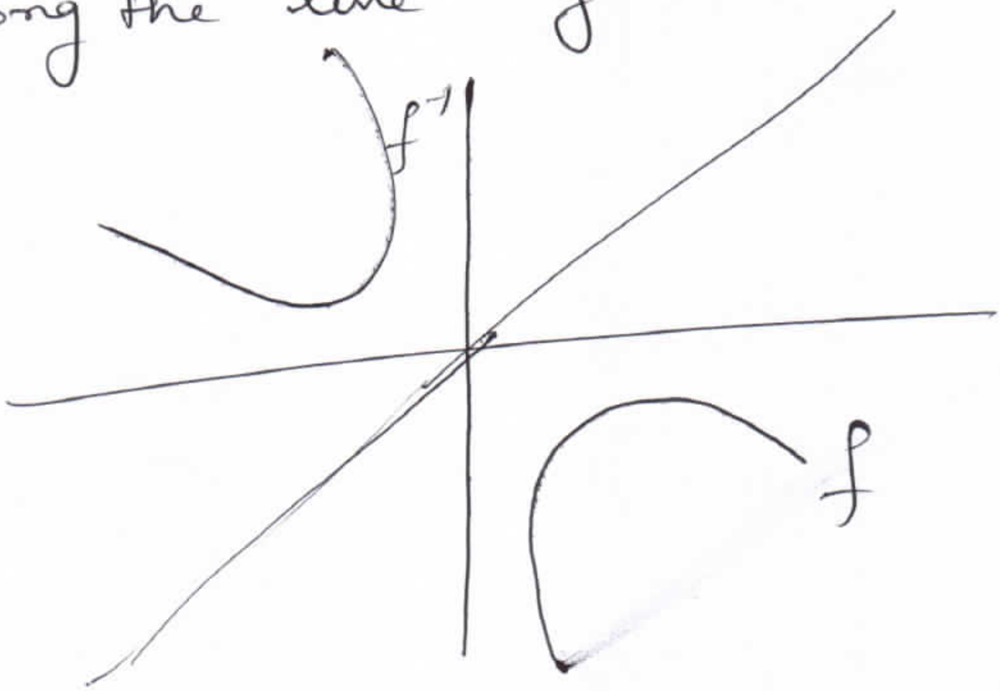
then f, g are inverse of each other.

$g = f^{-1}$ & $f = g^{-1}$

$(f \circ f^{-1})(x) = x$.

$(f^{-1} \circ f)(x) = x$.

f^{-1} is mirror image of f along the line $y = x$.



How to find the Inverse (f^{-1}) of $f(x)$

eg $f : \mathbb{R} - \{-1\} \longrightarrow R - \{1\}$

$$f(x) = \frac{x-1}{x+1}$$

① Let $y = f(x) = \dfrac{x-1}{x+1}$

② Solve for x $\quad xy + y = x - 1$

$x(y-1) = -1 - y \implies x = \dfrac{-1-y}{y-1}$

$$f^{-1}(x) = \frac{-1-x}{x-1}$$ replace y with x

$$\mathbb{R}-\{1\} \longrightarrow \mathbb{R}-\{-1\}$$