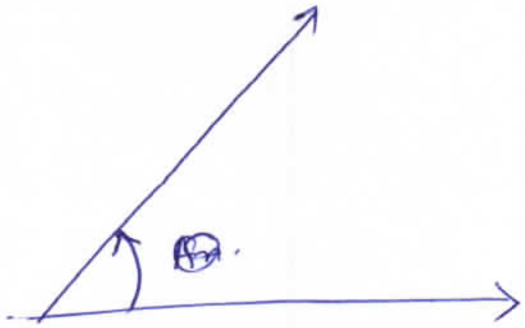
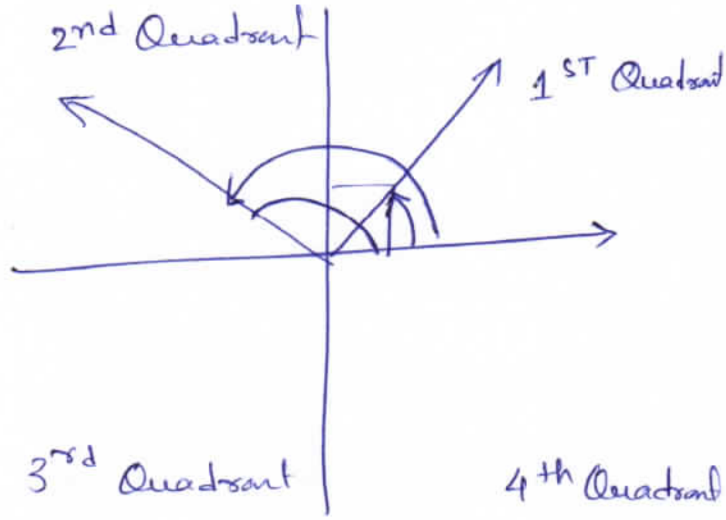
TRIGONOMETRY

Angles

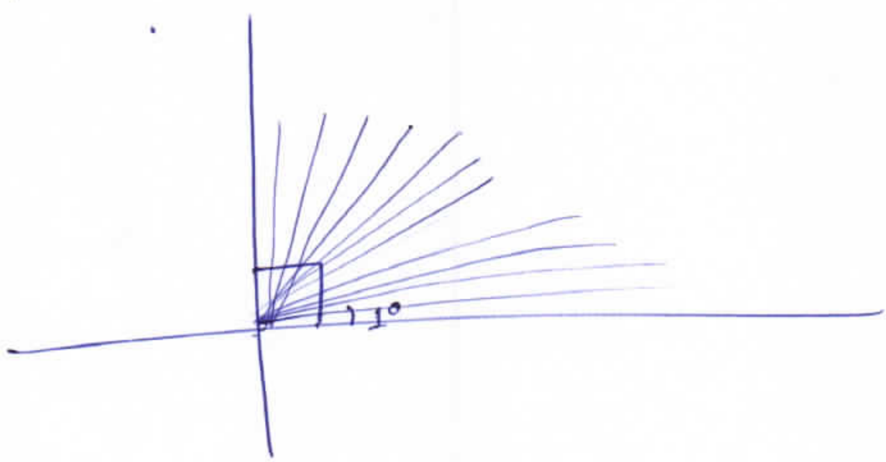




Anticlockwise +ve

Clockwise −ve

Sexagesimal System

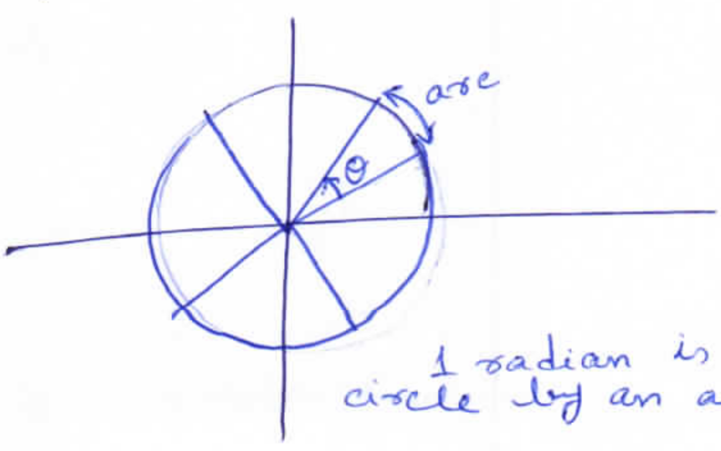


1 rt angle = $90°$

1 turn = 4 rt angle

= $360°$

Circular System



$$\text{Radian}(\theta) = \frac{\text{arc length}}{\text{radius}}$$

1 turn

$$\theta = \frac{2\pi \cancel{r}}{\cancel{r}} = 2\pi \text{ rad}$$

1 radian is angle subtended at center of circle by an arc that is equal to the radius.

2π rad $= 360^\circ$

π rad $= 180^\circ$

$\frac{\pi}{2}$ rad $= 90^\circ$

$\frac{\pi}{4}^c$

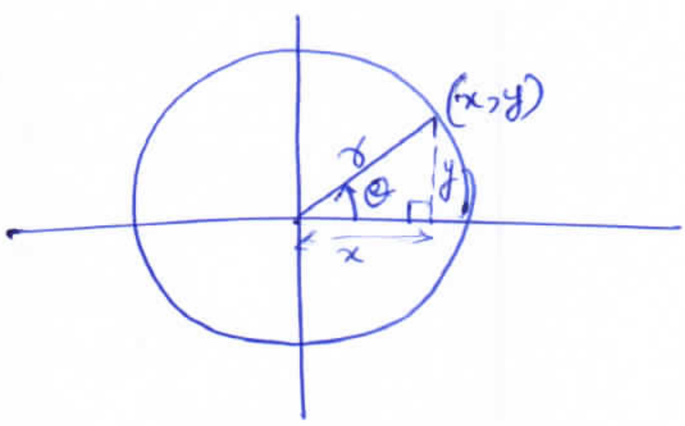
$\frac{\pi}{4}$ rad $= 45^\circ$

$\frac{\pi}{6}$ rad $= 30^\circ$

$\downarrow$ radian

$$\frac{30^\circ}{180^\circ} \times \pi = \frac{\pi}{6} \text{ rad.}$$

Basic Identities



$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$

$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{r}{y}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$

$x^2+y^2 = r^2$

$\sin^2\theta+\cos^2\theta = \frac{y^2}{r^2}+\frac{x^2}{r^2}$

$\sin^2\theta+\cos^2\theta = 1$ ✓

$$1 + \cot^2\theta = 1 + \frac{x^2}{y^2} = \frac{x^2+y^2}{y^2} = \frac{r^2}{y^2} = \operatorname{cosec}^2\theta.$$

$$\underline{1 + \cot^2\theta = \operatorname{cosec}^2\theta.}$$

$$1 + \tan^2\theta = 1 + \frac{y^2}{x^2} = \frac{x^2+y^2}{x^2} = \frac{r^2}{x^2} = \sec^2\theta.$$

$$\underline{1 + \tan^2\theta = \sec^2\theta.}$$

$$\Downarrow$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1.$$

$$\boxed{\sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta}}$$

$$\boxed{\operatorname{cosec}\theta - \cot\theta = \frac{1}{\operatorname{cosec}\theta + \cot\theta}}$$

Prove that

1. $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$
2. If $\operatorname{cosec}\theta - \sin\theta = a^3$ & $\sec\theta - \cos\theta = b^3$ find $a^2b^2(a^2+b^2)$ given $ab \neq 0$
3. If $\sin\theta + \sin^2\theta = 1$ find $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta$

Ans 1 $(\sin^4\theta)^2 - (\cos^4\theta)^2 = (\sin^4\theta + \cos^4\theta)(\sin^4\theta - \cos^4\theta)$

$a^2+b^2 = (a+b)^2 - 2ab$

$$\{(\sin^2\theta)^2 + (\cos^2\theta)^2\} \qquad (\sin^2\theta)^2 - (\cos^2\theta)^2$$

$$(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta \qquad (\sin^2\theta - \cos^2\theta)\underbrace{(\sin^2\theta + \cos^2\theta)}_{1}$$

$$(1 - 2\sin^2\theta\cos^2\theta) \qquad (\sin^2\theta - \cos^2\theta)$$

Ans 2

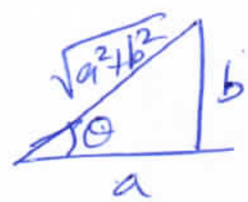
$$\frac{1}{\sin\theta} - \sin\theta = a^3 \qquad\qquad \frac{1}{\cos\theta} - \cos\theta = b^3.$$

$$\frac{1 - \sin^2\theta}{\sin\theta} = a^3 \qquad\qquad \frac{1 - \cos^2\theta}{\cos\theta} = b^3$$

$$\frac{\cos^2\theta}{\sin\theta} = a^3 \text{ — } (1) \qquad\qquad \frac{\sin^2\theta}{\cos\theta} = b^3 \text{ — } (2)$$

$$\frac{(2)}{(1)} \Rightarrow \tan^3\theta = \frac{b^3}{a^3}$$

$$\tan\theta = \frac{b}{a}$$



$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}.$$

$$\frac{\frac{\cancel{a^2}}{a^2+b^2}}{\frac{b}{\sqrt{a^2+b^2}}} = \cancel{a^3}$$

$$\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{\sqrt{a^2+b^2}}{a^2+b^2} = ab.$$

$$\Rightarrow \frac{1}{\sqrt{a^2+b^2}} = ab \Rightarrow 1 = a^2b^2(a^2+b^2)$$

Ans 3

$\sin\theta = 1 - \sin^2\theta = \cos^2\theta$.

$\sin\theta + \sin^2\theta = 1 \qquad \sin\theta(1+\sin\theta) = 1$

$\sin^6\theta + 3\sin^5\theta + 3\sin^4\theta + \sin^3\theta$

$3\sin^4\theta(1+\sin\theta)$

$3\sin^4\theta(\operatorname{cosec}\theta)$

$3\sin^3\theta$.

$\sin^6\theta + \sin^3\theta = \sin^3\theta(1+\sin^3\theta)$

$= \sin^3\theta(1+\sin\theta)(\sin^2\theta+1-\sin\theta)$

$(\sin^2\theta + \sin\theta)^3 = 1^3$.

$\sin^6\theta + \sin^3\theta + 3\sin^3\theta(\sin^2\theta+\sin\theta)$

$\sin^6\theta + \sin^3\theta + 3\sin^5\theta + 3\sin^4\theta = 1$

	0°	30°	45°	60°	90°
Sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
Cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan.	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

SIN COSEC	ALL
TAN COT	COS. SEC

If $0^\circ \leq \theta \leq 90^\circ$

$90-\theta$

1st Quadrant.

$\sin(90-\theta) = +\cos\theta$

$\cos(90-\theta) = +\sin\theta$

$\tan(90-\theta) = +\cot\theta$

$\cot(90-\theta) = +\tan\theta$

$\sec(90-\theta) = +\operatorname{cosec}\theta$

$\operatorname{cosec}(90-\theta) = +\sec\theta$

$90+\theta$

2nd Quadrant

$\sin(90+\theta) = +\cos\theta.$

$\cos(90+\theta) = -\sin\theta$

$\tan(90+\theta) = -\cot\theta$

$\cot(90+\theta) = -\tan\theta$

$\sec(90+\theta) = -\operatorname{cosec}\theta$

$\operatorname{cosec}(90+\theta) = +\sec\theta$

$180-\theta.$

2nd Quadrant.

$\sin(180-\theta) = +\sin\theta$

$\cos(180-\theta) = -\cos\theta$

$\tan(180-\theta) = -\tan\theta$

$\cot(180-\theta) = -\cot\theta$

$\sec(180-\theta) = -\sec\theta$

$\operatorname{cosec}(180-\theta) = +\operatorname{cosec}\theta.$

$180+\theta$

3rd Quadrant.

$\sin(180+\theta) = -\sin\theta$

$\cos(180+\theta) = -\cos\theta$

$\tan(180+\theta) = +\tan\theta$

$\cot(180+\theta) = +\cot\theta$

$\operatorname{cosec}(180+\theta) = -\operatorname{cosec}\theta$

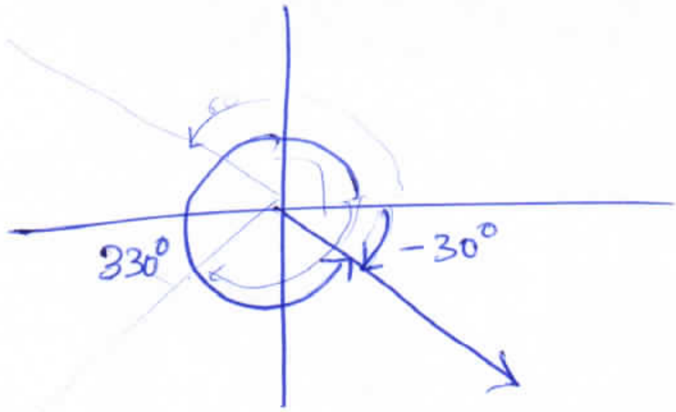
$\sec(180+\theta) = -\sec\theta$

$\forall\ \theta$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$



① $\sin 120^\circ = \sin(180-60^\circ)$
$= +\sin 60^\circ = \sqrt{3}/2$

② $\cos \frac{5\pi}{6}^c = \cos 150^\circ = \cos(180-30^\circ)$
$= -\cos 30^\circ$
$= -\sqrt{3}/2$

③ $\tan \frac{7\pi}{6}$

④ $\sin \frac{11\pi}{6}$

⑤ $\tan 330^\circ$

$\sin(330^\circ)$
$= \sin(360-30^\circ)$
$= -\sin 30^\circ$

$\sin(-30^\circ) = \sin(330^\circ)$
$= -\sin 30^\circ$

$\cos\left(\pi - \frac{\pi}{6}\right) = -\cos\frac{\pi}{6}$
$= -\sqrt{3}/2$

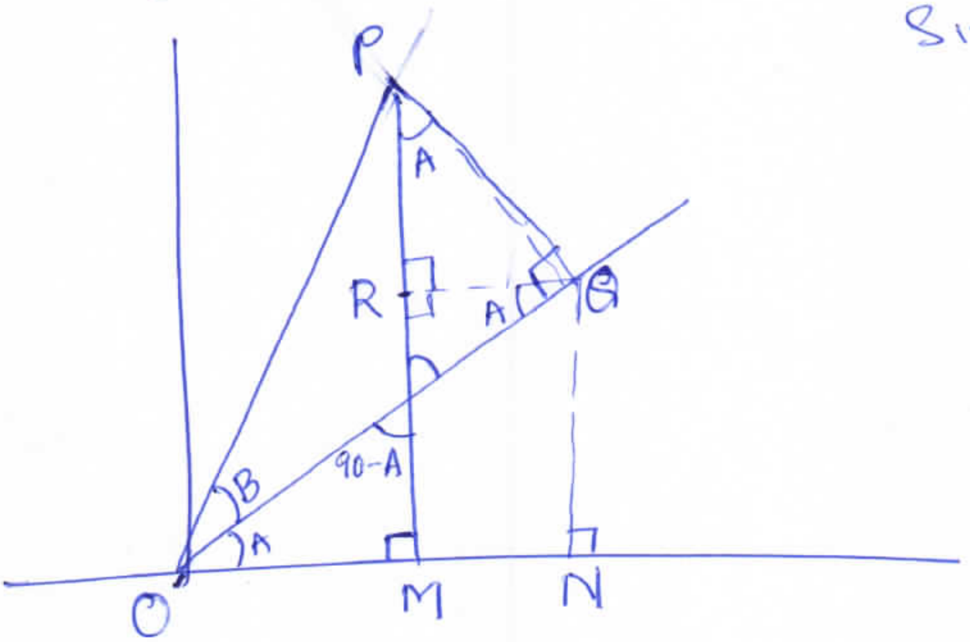
$\tan\left(\pi + \frac{\pi}{6}\right)$
$+\tan\left(\frac{\pi}{6}\right) = +\frac{1}{\sqrt{3}}$

$\sin\left(\frac{11\pi}{6}\right) \quad \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$
$= -1/2$

$\tan(360-30) = -\tan 30 = -\frac{1}{\sqrt{3}}$

$\sin 75^\circ \longrightarrow \sin(30^\circ + 45^\circ)$

$\sin(A+B)$



$$\sin(A+B) = \frac{PM}{OP}$$

$$= \frac{PR+RM}{OP}$$

$$= \frac{PR}{OP} + \frac{RM}{OP}$$

$$= \frac{PR}{OP} + \frac{QN}{OP}$$

$$= \frac{PR}{PQ} \times \frac{PQ}{OP} + \frac{QN}{OQ} \times \frac{OQ}{OP}$$

$$= \cos A \sin B + \sin A \cos B.$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \sin(90-(A+B)) = \sin((90-A)-B)$$

$$= \sin(90-A)\cos B - \cos(90-A)\sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

$$\sin 75^\circ = \sin(30^\circ+45^\circ) = \sin 30^\circ\cos 45^\circ + \cos 30^\circ\sin 45^\circ$$

$$= \frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\tan 15^\circ = \tan(45^\circ-30^\circ)$$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ\tan 30^\circ} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Double Angles & Triple Angles

$\sin 2A$

$\downarrow$

$\sin(A+A)$

$\sin A\cos A + \cos A\sin A$

$= 2\sin A\cos A$

$\cos 2A$

$\downarrow$

$\cos(A+A)$

$= \cos A\cos A - \sin A\sin A$

$= \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

$= 2\cos^2 A - 1$

$\tan 2A \longrightarrow \tan(A+A)$

$= \dfrac{\tan A + \tan A}{1 - \tan A \cdot \tan A}$

$= \dfrac{2\tan A}{1 - \tan^2 A}$

Find $\sin 3A$ $\quad \cos 3A$ $\quad \tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$= 3\sin A - 4\sin^3 A$

$= 4\cos^3 A - 3\cos A$

$\downarrow$

$\sin(A+2A)$

$= \sin A\cos 2A + \cos A\sin 2A$

$= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A\cos A)$

$= \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A)$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A$

Half Angles

$\sin\left(\frac{A}{2}\right)$

$$\cos 2A = 1 - 2\sin^2 A$$

$$\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right)$$

$$2\sin^2\frac{A}{2} = 1 - \cos A$$

$$\sin^2\frac{A}{2} = \frac{1-\cos A}{2}$$

$$\boxed{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}}$$

$+, -$ depends

$0 \le \frac{A}{2} \le 180^\circ$ $+$

$180^\circ \le \frac{A}{2} \le 360^\circ$ $-$

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos A = 2\cos^2\frac{A}{2} - 1.$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\pm\sqrt{\frac{1-\cos A}{2}}}{\pm\sqrt{\frac{1+\cos A}{2}}} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$= \pm\frac{\sin A}{1+\cos A}$$

$$= \pm\frac{1-\cos A}{\sin A}$$

Product $\longrightarrow$ Sum transformation.

$$2\sin A\cos B = \underline{\sin(A+B)} + \underline{\sin(A-B)}$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

$$2\cos A\cos B = \cos(A-B) + \cos(A+B)$$

Sum $\longrightarrow$ Product transformation.

$$\sin A + \sin B = \sin(x+y) + \sin(x-y)$$

$$= 2\sin x\cos y$$

$$= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$A = x+y$
$B = x-y$

$x = \frac{A+B}{2}$ $\quad y = \frac{A-B}{2}$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$= 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Find

$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - 1/\sqrt{3}}{1 + 1/\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$

$\tan 22.5^\circ = \tan\left(\frac{45^\circ}{2}\right) = \frac{1 - \cos 45^\circ}{\sin 45^\circ} = \frac{1 - 1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2} - 1$

$\tan 67.5^\circ = \tan\left(\frac{135^\circ}{2}\right) = \frac{1 - \cos 135^\circ}{\sin 135^\circ} = \frac{1 + 1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2} + 1$

$\tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1/\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$

$\sin 18^\circ = \frac{\sqrt{5}-1}{4}$

$\sin 54^\circ = \frac{\sqrt{5}+1}{4}$

$\sin 2A = 2 \sin A \cos A$

$\sin 18^\circ = ?$

$\sin 72^\circ = 2 \sin 36^\circ \cos 36^\circ$

$\cancel{\cos} 18^\circ = 2(2 \sin 18^\circ \cancel{\cos 18^\circ})(1 - 2\sin^2 18^\circ)$

$1 = 4 \sin 18^\circ - 8 \sin^3 18^\circ$

$8 \sin^3 18^\circ - 4 \sin 18 + 1 = 0$

$(2 \sin 18 - 1)(4 \sin^2 18 + 2 \sin 18 - 1) = 0$

$\neq 0$

$\sin 18 = \frac{-2 \pm \sqrt{4 + 16}}{2 \times 4}$

$= \frac{-2 \pm 2\sqrt{5}}{2 \times 4}$

$= \boxed{\frac{\sqrt{5}-1}{4}}$

Homework.

Q1 Prove that $\underline{\sin 47^\circ + \sin 61^\circ} - \sin 11^\circ - \sin 25^\circ = \cos 7^\circ$

Q2 Find $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 - (\tan^2\theta + \cot^2\theta)$

Q3 Find $\sin(\alpha-\beta)$, $\cos(\alpha-\beta)$

if i) $\sin\alpha = \frac{8}{17}$ $\tan\beta = \frac{5}{12}$ α 1ST Quad β 1ST Quad

ii) $\cos\alpha = -\frac{12}{13}$ $\cot\beta = \frac{24}{7}$ α 2nd $\beta = 1^{ST}Q$

Q4 Prove $4\cos\theta\cos 2\theta\sec 3\theta\cos 4\theta = 1$

if $9\theta = \pi$

Ans1 $2\sin 54^\circ\cos 7^\circ - 2\sin 18^\circ\cos 7^\circ$

$2\cos 7^\circ(\sin 54 - \sin 18)$

$2\cos 7^\circ\left(\frac{\sqrt{5}+1}{4} - \left(\frac{\sqrt{5}-1}{4}\right)\right)$

$\cancel{2}\cos 7^\circ\left(\frac{1}{\cancel{2}}\right) = \cos 7^\circ$

Ans2 $\cancel{\left(\sin\theta + \frac{1}{\sin\theta}\right)^2 + \cos\theta}$

$\sin^2\theta + \underline{\operatorname{cosec}^2\theta + 2\sin\theta\operatorname{cosec}\theta}$
$+ \cos^2\theta + \underline{\sec^2\theta + 2\cos\theta\sec\theta}$.
$- \tan^2\theta - \cot^2\theta$.

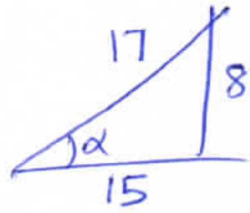
$1 + 2 + 2 + 1 + 1 = 7$

Ans 3

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

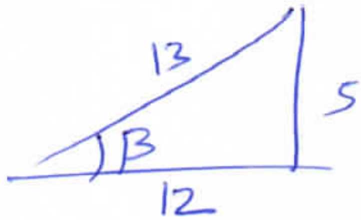
i) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\sin\alpha = \frac{8}{17}$



$\cos\alpha = \frac{15}{17}$

$\sin\beta = \frac{5}{13}$



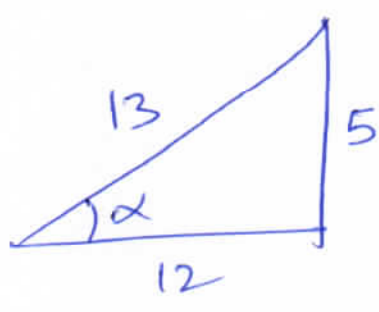
$\cos\beta = \frac{12}{13}$

$= \frac{8}{17}\times\frac{12}{13} - \frac{15}{17}\times\frac{5}{13}$.

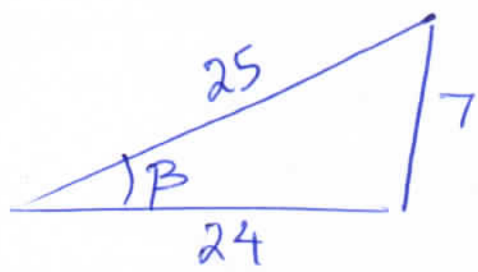
$= \frac{21}{17\times13}$.

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

ii) $\cos\alpha = -\frac{12}{13}$



$\sin\alpha = \frac{5}{13}$



$\cos\beta = \frac{24}{25}$

$\sin\beta = \frac{7}{25}$

Ans4 T.P $4\cos\theta\cos2\theta\sec3\theta\cos4\theta = 1.$

$9\theta = \pi.$ $\qquad \theta = \frac{\pi}{9}.$

$$4\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\underbrace{\sec\frac{3\pi}{9}}_{2.}\cos\frac{4\pi}{9}$$

$\cancel{(a+b)(a+b) = a^2+2ab+b^2}.$

$$\frac{2^3\sin\left(\frac{\pi}{9}\right)\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)}{\sin\left(\frac{\pi}{9}\right)}$$

$2\sin A\cos A = \sin 2A.$

$$\frac{2^2\sin\frac{2\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}}{\sin\frac{\pi}{9}}$$

$$\frac{2\sin\frac{4\pi}{9}\cos\frac{4\pi}{9}}{\sin\pi/9} = \frac{\sin\frac{8\pi}{9}}{\sin\frac{\pi}{9}} = \frac{\sin\left(\pi-\frac{\pi}{9}\right)}{\sin\frac{\pi}{9}} = \frac{\cancel{\sin\frac{\pi}{9}}}{\cancel{\sin\frac{\pi}{9}}} = 1$$