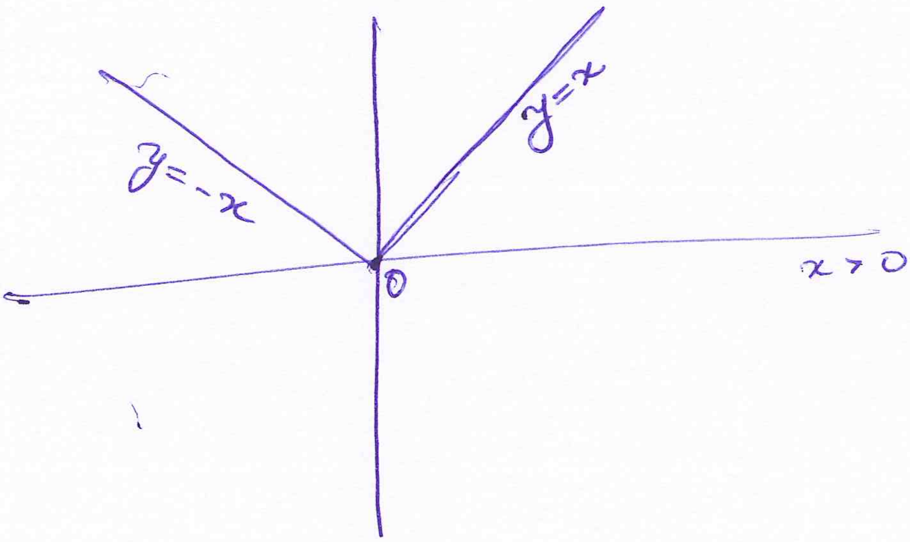
Some Special Functions.

1) Modulus function.

$$y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Domain $\in \mathbb{R}$.

Range $\in [0, \infty)$



$|x| = 2 \Rightarrow x = \pm 2$

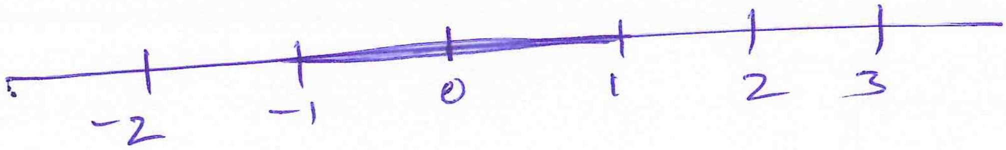
$|x| - 1 \leq 0$

$\Rightarrow |x| \leq 1$

$-1 \leq x \leq 1$

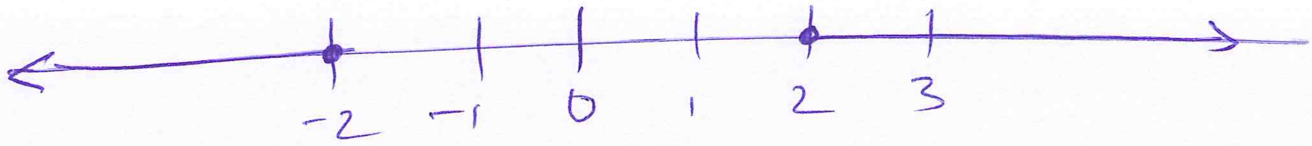
$|x| \leq a$

$-a \leq x \leq a$



$|x| \geq 2.$

$(-\infty, -2] \cup [2, \infty)$



Solve for x.

i) $|x-3| + |4-x| = 1.$

ii) $|x| + |x+4| = 4$

$x-3 \geq 0$

$y = |x|$ $\quad x \quad x > 0$

$\quad -x \quad x < 0$

$x-3$

$3 \leq x \leq 4$ $\qquad x \geq 4.$

$4-x.$ $\qquad -(4-x)$

$x-3 < 0$

$-(x-3) + (4-x) = 1$

$-2x + 7 = 1$

$2x = 6$

$x = 3$

X.

$(x-3) - (4-x) = 1$

$2x - 7 = 1$

$x = 4$

$x - 3 + 4 - \cancel{x} = 1$

X.

$x \in [3, 4]$

ii) $|x| + |x+4| = 4$

$x \geq 0$

$+x + (x+4) = 4$

$2x = 0$

$x = 0$ ✓

$x < 0$

$-4 \leq x < 0$ ✓

$-x + (x+4) = 4$

$x < -4$

$-x - (x+4) = 4$

$2x = 0$

$x = 0$

✗

$[-4, 0]$

In general $|x| + |y| = |x+y|$ iff $xy > 0$

Solve

i) $|x^2 - 1 + \sin x| = |x^2 - 1| + |\sin x|$

(ii) $\left|\frac{x}{x-1}\right| + |x| = \frac{x^2}{|x-1|}$

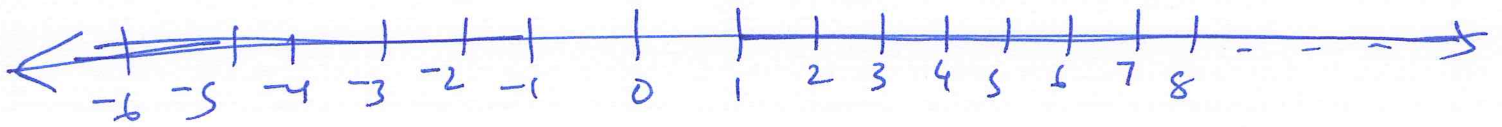
$x^2 - 1 > 0$ & $\sin x > 0$

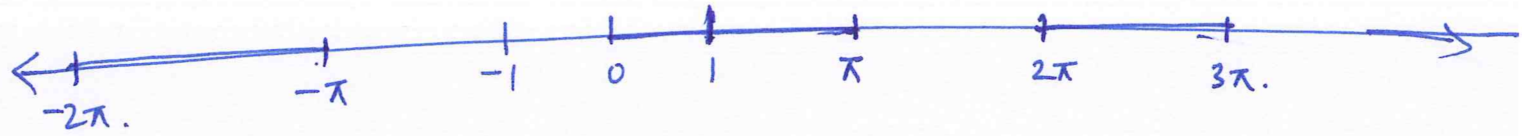
$(x-1)(x+1) > 0$

$+ \quad -1 \quad - \quad 1 \quad +$

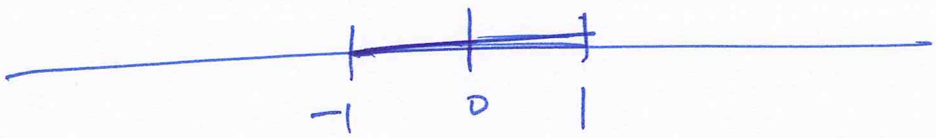
$\{(-\infty, -1) \cup (1, \infty)\} \cap (0, \pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi) \cup (-2\pi, -\pi) \cup (-4\pi, -3\pi)$

$x^2 - 1 < 0$ & $\sin x < 0$

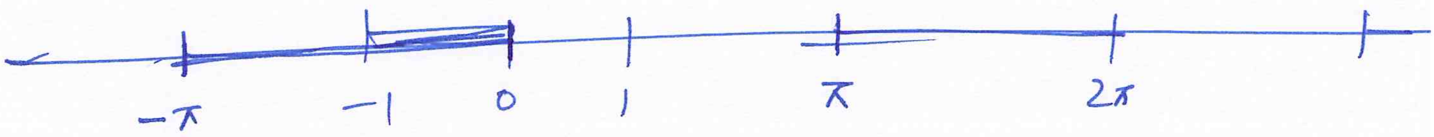




$x^2-1<0$



$\sin x<0$



$(-1,0)$

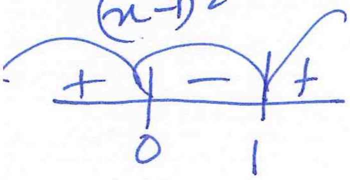
$$\left|\frac{x}{x-1}\right|+|x|=\left|\frac{x}{x-1}+x\right|$$

$$=\left|\frac{\cancel{x}+x^2-\cancel{x}}{x-1}\right|$$

$$=\frac{|x^2|}{(x-1)}=\frac{x^2}{|x-1|}$$

$\frac{x}{x-1}>0$ & $x>0$

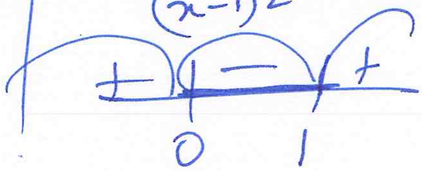
$\frac{x(x-1)}{(x-1)^2}>0$



$(1,\infty)$ ✓

$\frac{x}{x-1}<0$ & $x<0$

$\frac{x(x-1)}{(x-1)^2}<0$

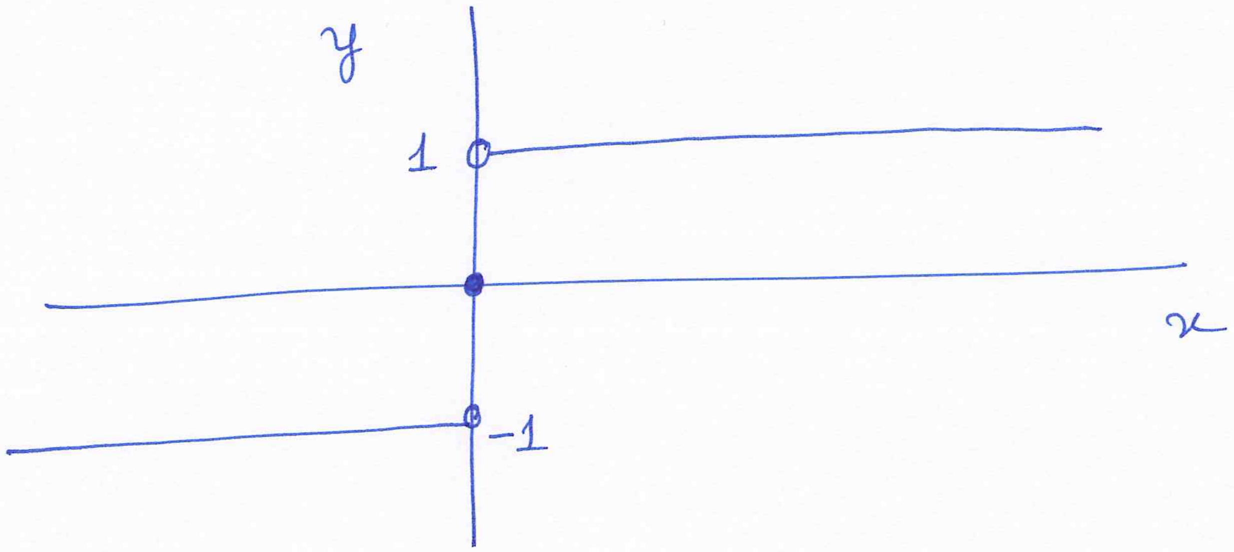


(2) Signum function.

$$f(x) = \operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Domain $\in \mathbb{R}$.

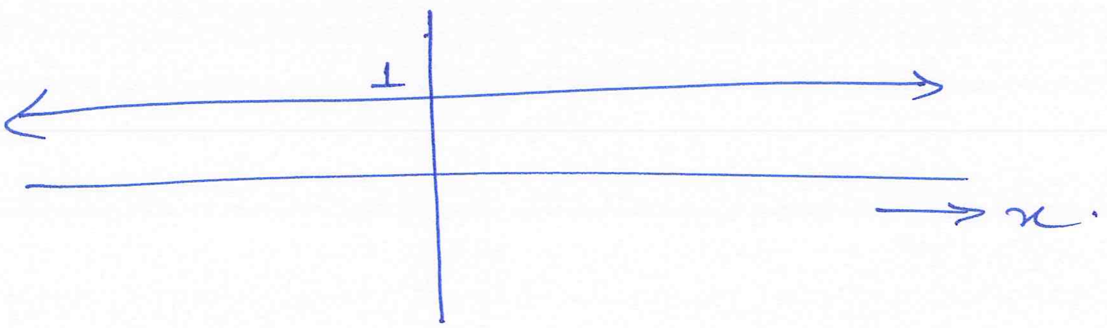
Range $= \{-1, 0, 1\}$



Sketch the graph of i) $f(x) = \operatorname{sgn}(x^2+1)$

ii) $f(x) = \operatorname{sgn}(\log_e x)$

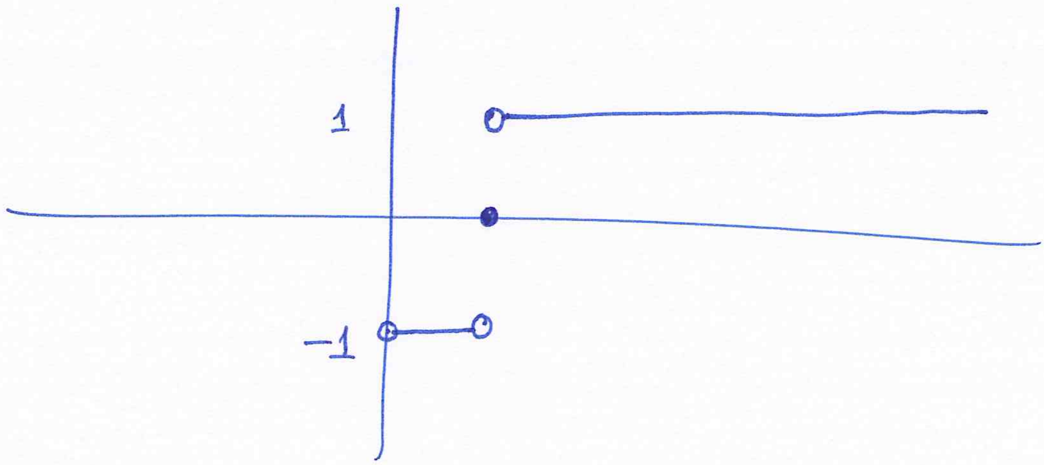
i) x^2+1 always +ve



iv)

$$\log_e x = \begin{cases} +ve & x > 1 \\ 0 & x = 1 \\ -ve & 0 < x < 1 \end{cases}$$

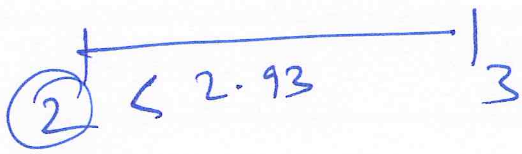
$$\text{Sgn}(\log_e x) = \begin{cases} 1 & x > 1 \\ 0 & x = 1 \\ -1 & 0 < x < 1 \end{cases}$$



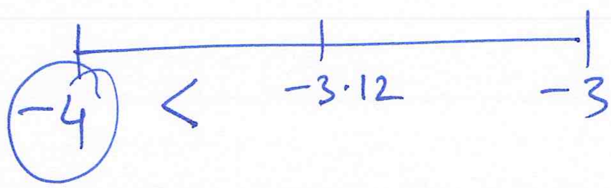
③ <u>Greatest Integer function</u> or Step up function.

$f(x) = [x]$ is the integral part of x which is nearest to x but smaller than x

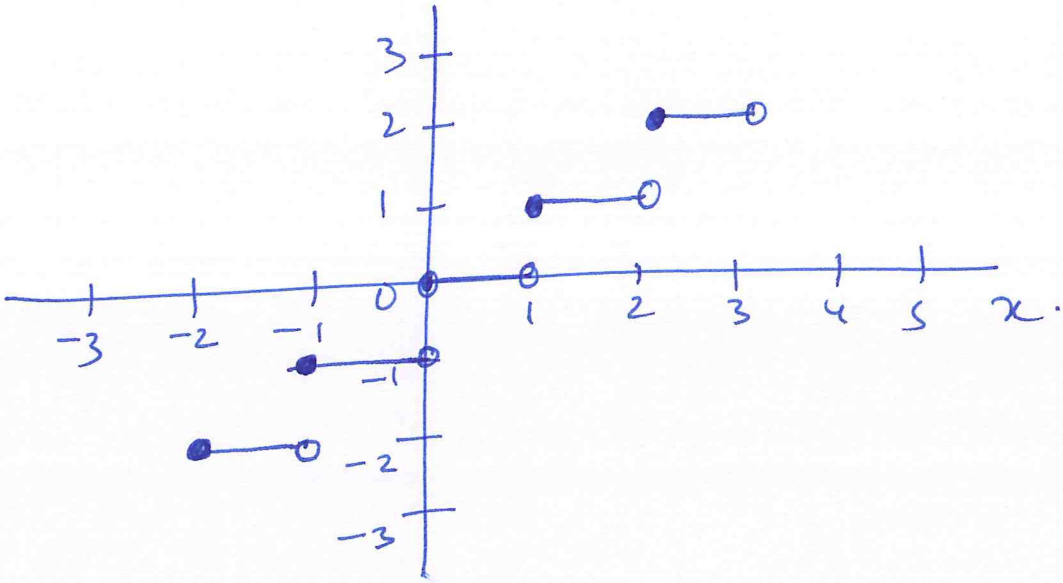
$[2.93]$



$[-3.12]$



$[x] = n \; ; \; n \le x \le n+1$



(4) Fractional Part function

$f(x) = \{x\}$ is the fractional part of x

$x = [x] + \{x\}$ $\boxed{0 \le \{x\} < 1}$

$\{2.9\} = \underset{\underset{[2.9]}{\Uparrow}}{2} + 0.9$

$\{-2.9\} = -3 + 0.1.$

$\{-x\} = 1 - \{x\}$

$\{-2.9\} = 1 - 0.9 = 0.1$

Finding Range of Function:

Step 1 : Find domain of $f(x)$

Step 2

a) If domain is a finite set.
then Range is set of corresponding $f(x)$ value.

b) If domain $\in \mathbb{R}$ or $\mathbb{R} - \{\text{finite set}\}$

i) Express x in terms of y.
ii) then find values of y for which x is defined.

c) If domain $\in$ finite interval.
then find the least & greatest values for range.

eg. i) $f(x) = \dfrac{x}{x+2}$

ii) $f(x) = \dfrac{\{x\}}{1-[x]+x}$

i) Domain = ? $\mathbb{R} - \{-2\}$

$$y = \frac{x}{x+2}$$

$$yx + 2y = x$$

$$2y = x(1-y)$$

$$\Rightarrow x = \frac{2y}{1-y}$$

$$y \neq 1.$$

Range = $\mathbb{R} - \{1\}$

ii) $f(x) = \dfrac{\{x\}}{1+\{x\}}$ Domain $\in \mathbb{R}$

$$y = \frac{\{x\}}{1+\{x\}} \qquad y + \{x\}y = \{x\}$$

$$\{x\} = \frac{y}{1-y}$$

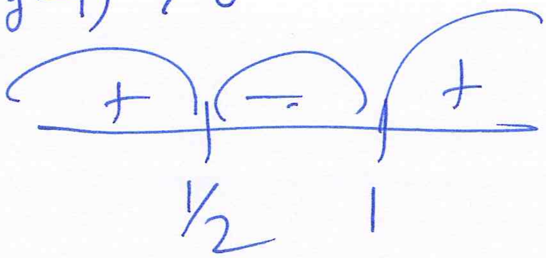
$$0 \leq \{x\} < 1 \qquad 0 \leq \frac{y}{1-y} < 1$$

$$\frac{y}{1-y} < 1 \Rightarrow \frac{y}{1-y} - 1 < 0$$

$$\frac{2y-1}{1-y} < 0 \longleftarrow \frac{y-1+y}{1-y} < 0$$

$$\frac{(2y-1)(1-y)}{(1-y)^2} < 0$$

$$(2y-1)(y-1) > 0$$



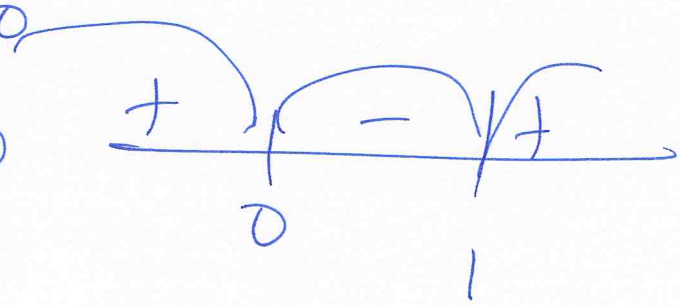
$$y < \frac{1}{2} \quad \text{or} \quad y > 1$$

&

$$\frac{y}{1-y} \geq 0$$

$$y(1-y) \geq 0$$

$$y(y-1) \leq 0$$



$$0 \leq y \leq 1$$

$$0 \leq y < \tfrac{1}{2}$$

Range $\in [0, \tfrac{1}{2})$

eg. i) $y = \log_e (3x^2 - 4x + 5)$

(ii) $f(x) = \sqrt{x-1} + \sqrt{5-x}$.

iii) $f(x) = 4 \tan x \cos x$.

iv) $f(x) = \cos(2 \sin x)$

v) $f(x) = [\{x\}] = 0$ $\quad 0 \le \{x\} < 1$.

vi) $f(x) = \dfrac{\tan([x-\pi]\pi)}{x^2-3x+2} = \dfrac{\tan n\pi}{x^2-3x+2}$ $\quad 0$

vii) $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

viii) $y = \dfrac{(x+2)(x-1)}{x(x+1)}$

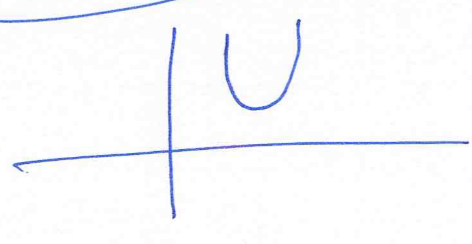
ix) $f(x) = A \sin x + B \cos x$

x) $f(x) = \log_2 \left(\dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}} \right)$

i) $\cancel{y =}$ Domain $\in \mathbb{R}$

$3x^2 - 4x + 5 > 0 \quad D < 0$

$a = 3 > 0$



$$y = \log_e (3x^2 - 4x + 5)$$

$$e^y = 3x^2 - 4x + 5$$

$$3x^2 - 4x + 5 - e^y = 0$$

$$D \geq 0$$

$$(-4)^2 - 4(3)(5 - e^y) \geq 0$$

$$\cancel{16 - 60} \; 4 - 15 + 3e^y \geq 0$$

$$3e^y - 11 \geq 0$$

$$e^y \geq 11/3$$

$$y$$

$$\log_e e^y \geq \log_e \frac{11}{3}$$

$$y \geq \log_e \frac{11}{3}$$

$$\text{Range} \in \left[\log_e \frac{11}{3}, \infty\right)$$

ii)

$x-1 \geq 0$ & $5-x \geq 0$

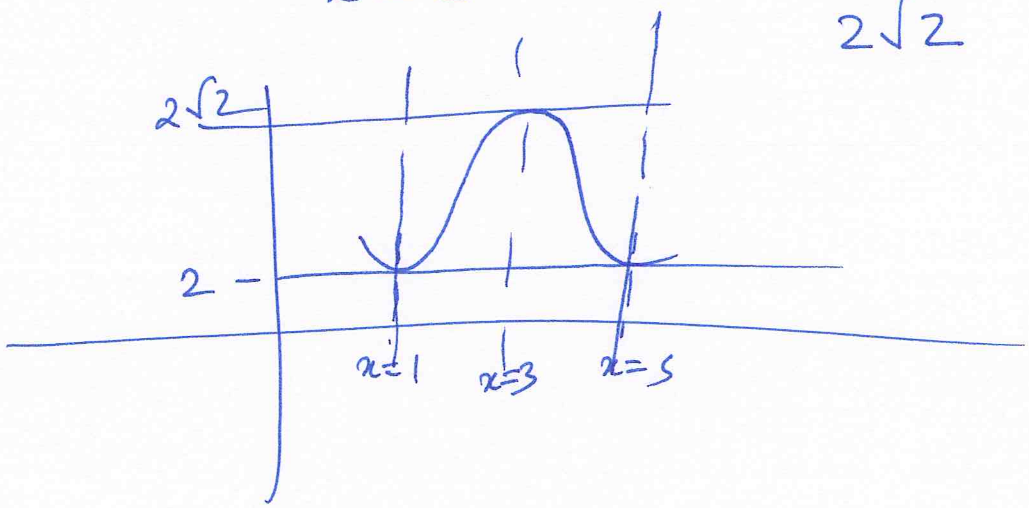
$x \geq 1$ & $x \leq 5.$

Domain $\in [1,5]$

$\sqrt{x-1} = \sqrt{5-x}$

$x = 3$

$2\sqrt{2}$



iii) $f(x) = 4\tan x \cos x.$

Domain.

$\mathbb{R} - \left\{(2n+1)\frac{\pi}{2}\right\}$

$= 4\sin x.$

$= 4(-1,1)$

$= (-4,4)$

iv) $f(x) = \cos(2\sin x)$ Domain $\in \mathbb{R}.$

$[\cos 2, 1]$

ix) $f(x) = \left[\frac{A}{\sqrt{A^2+B^2}} \sin x + \frac{B}{\sqrt{A^2+B^2}} \cos x\right] \sqrt{A^2+B^2}$

$= \left[\cos\theta \sin x + \sin\theta \cos x\right] \sqrt{A^2+B^2}$

$f(x) = \sqrt{A^2+B^2} \sin(x+\theta)$

$\left[-\sqrt{A^2+B^2}, \sqrt{A^2+B^2}\right]$

x) $-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2}$

$\frac{-\sqrt{2}+3\sqrt{2}}{\sqrt{2}} \leq \frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}} \leq \frac{\sqrt{2}+3\sqrt{2}}{\sqrt{2}}$

$\underline{\log_2 2} \leq \log_2 \mathrm{Exp} \leq \underline{\log_2 4}$

$1 \leq \log_2 \mathrm{Exp} \leq 2.$

$[1, 2]$

vii) $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$ $\qquad x^2 + x + 1 \quad D < 0$

+ve

Domain $\in \mathbb{R}$.

$$y x^2 + y x + y = x^2 - x + 1$$

$$(y-1)x^2 + (y+1)x + y - 1 = 0$$

$$D \geq 0$$

$$(y+1)^2 - 4(y-1)^2 \geq 0$$

$$y^2 + 2y + 1 - 4y^2 + 8y - 4 \geq 0$$

$$-3y^2 + 10y - 3 \geq 0$$

$$3y^2 - 10y + 3 \leq 0$$

$$(3y-1)(y-3) \leq 0$$

$+ \qquad - \qquad +$

$\frac{1}{3} \qquad\qquad 3$

$$y \in \left[\frac{1}{3}, 3\right] - \{1\}$$

$$\left[\frac{1}{3}, 1\right) \cup (1, 3]$$

viii) $y = \frac{(x+2)(x-1)}{x(x+1)}$ Domain $\in \mathbb{R} - \{0, -1\}$

$$yx^2 + yx = x^2 + x - 2$$

$$(y-1)x^2 + (y-1)x + 2 = 0$$

$$D \geq 0$$

$$(y-1)^2 - 4(y-1)(2) \geq 0$$

$y \neq 1$

$$(y-1)\{y-1-8\} \geq 0$$

$$(y-1)(y-9) \geq 0$$

$$(-\infty, 1) \cup [9, \infty)$$