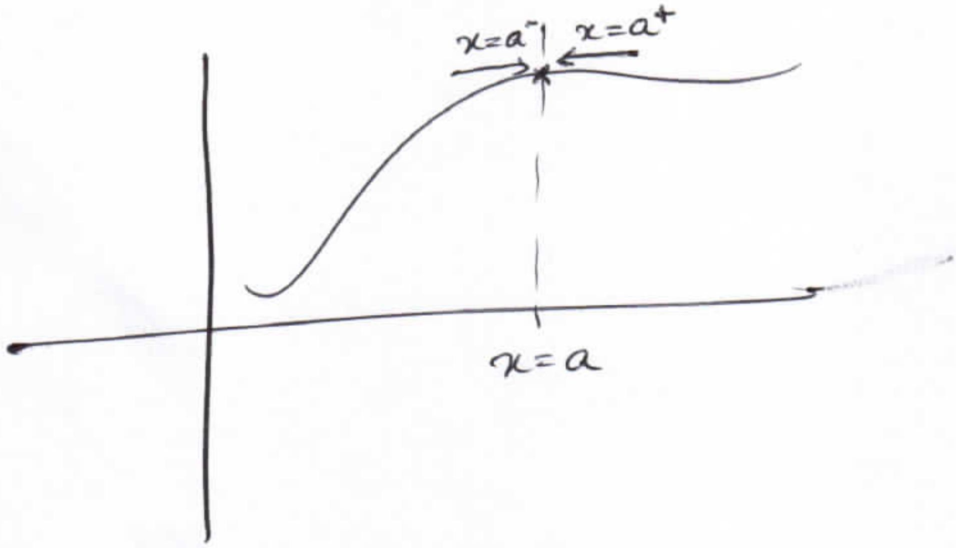
LIMITS

$$f(x) = \frac{x^2 - 5x + 6}{x - 2} \qquad \text{Domain} = \mathbb{R} - \{2\}$$



$\lim_{x \to 2} f(x)$ $\qquad f(2)$

$\lim_{x \to 2^-} f(x) \qquad = f(1.9999\ldots)$

$\lim_{x \to 2^+} f(x) \qquad = f(2.0000\ldots1)$

If $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = l$

then we say limit of $f(x)$ exists as $x \to 2$

& $\lim_{x \to 2} f(x) = l$

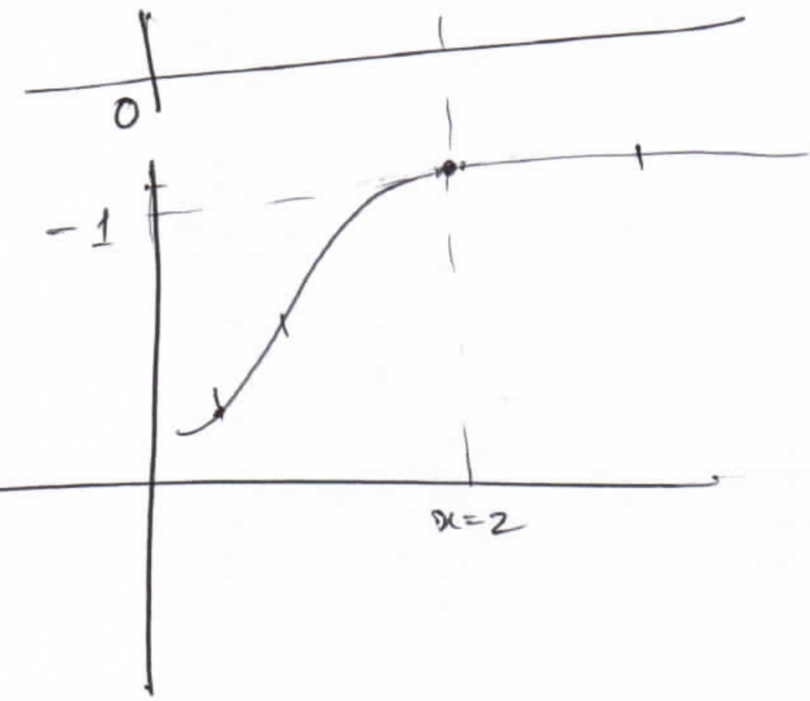
$$f(x) = \frac{(x-2)(x-3)}{(x-2)}$$

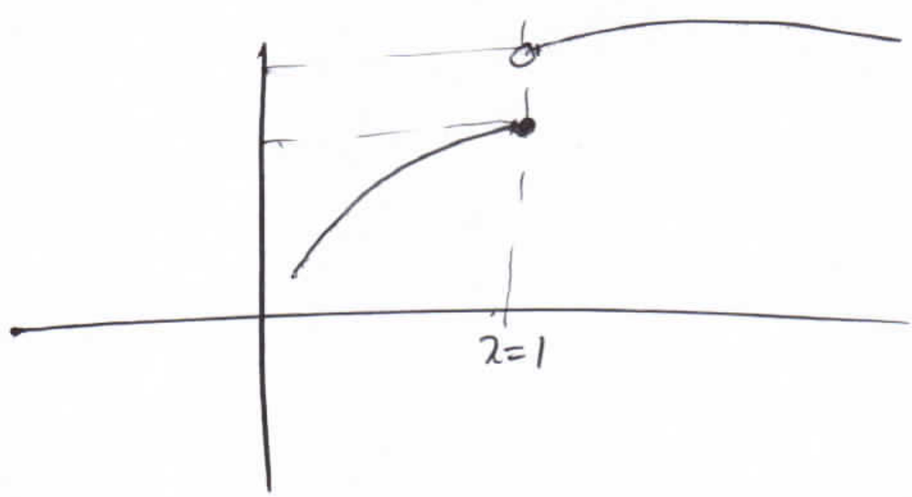
$$\lim_{x \to 2^+} f(x) = f(2.0001) \doteq \frac{0.\cancel{0001}(-1)}{\cancel{0.0001}} = -1.$$

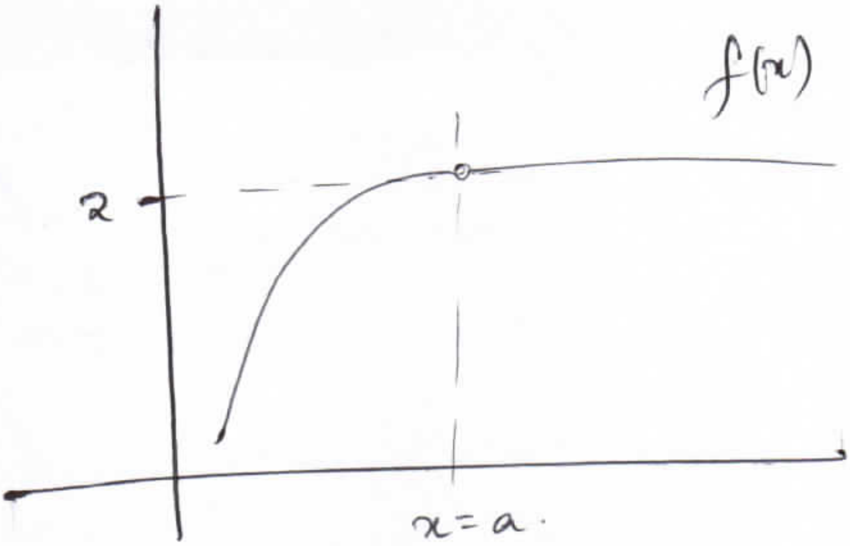
$$\lim_{x \to 2^-} f(x) = f(1.9999) = \frac{(-0.00001)(-1)}{(-0.00001)} = -1$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = -1.$$

$$\lim_{x \to 2} f(x) = -1$$

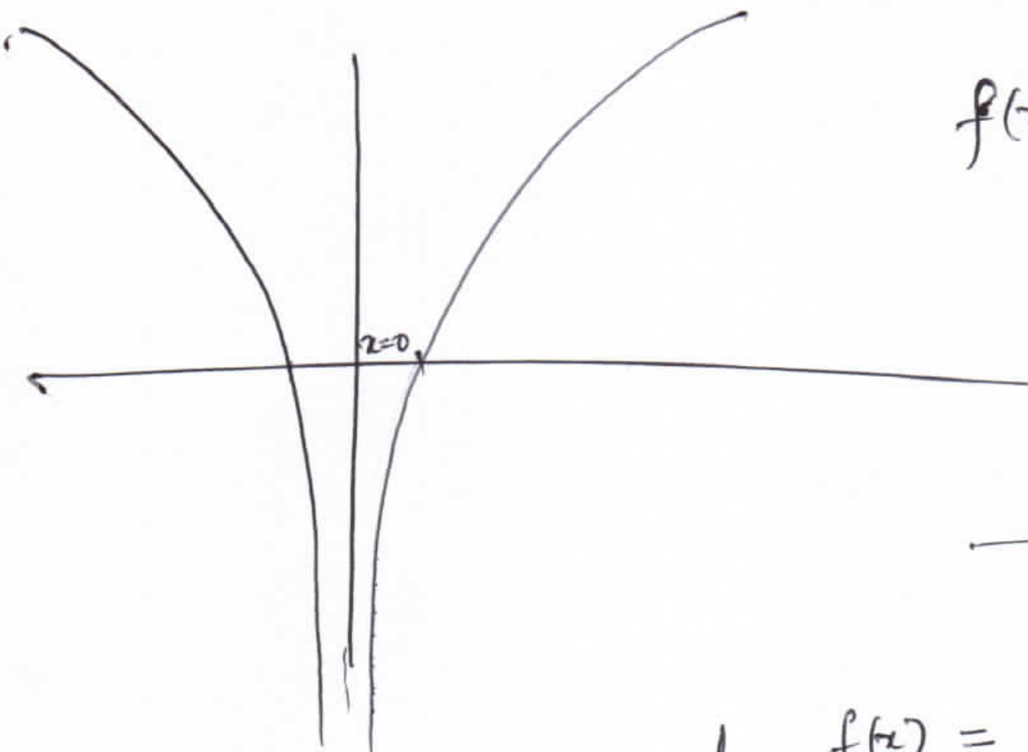




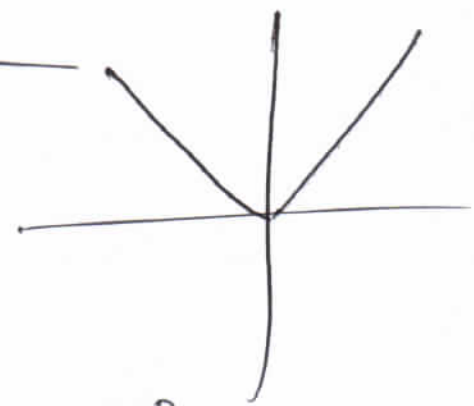


$$\lim_{x \to a} f(x) = 2.$$

Limit exists at $x=2$



$f(x) = \log|x|$



$$\lim_{x \to 0^-} f(x) = -\infty$$

$$\lim_{x \to 0^+} f(x) = -\infty$$

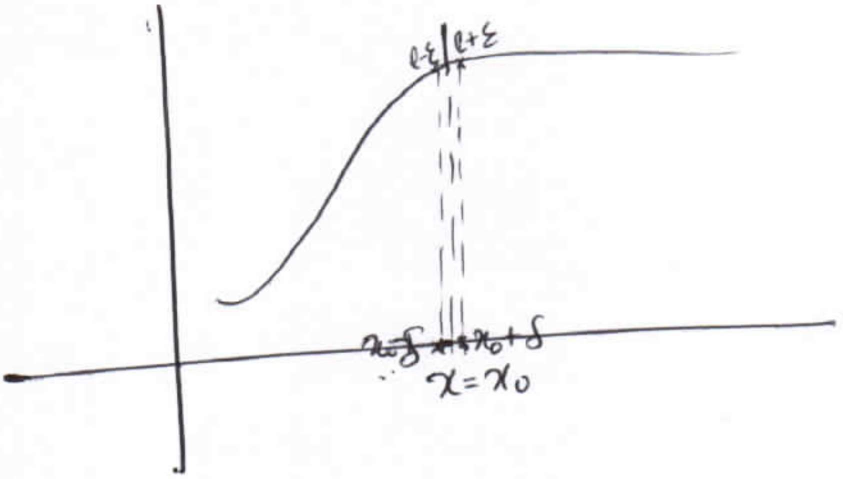
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = -\infty$$

Limit does not exist

if $|x - x_0| < \delta$ (δ is very small)

$|f(x) - l| < \varepsilon$ (ε is very small as compared to $f(x)$)

then we say $\lim_{x \to x_0} f(x) = l$



L·H·L (left hand limit)

$\lim_{x \to x_0^-} f(x) = l - \varepsilon$

$\lim_{x \to x_0^+} f(x) = l + \varepsilon$

R·H·L (Right hand limit)

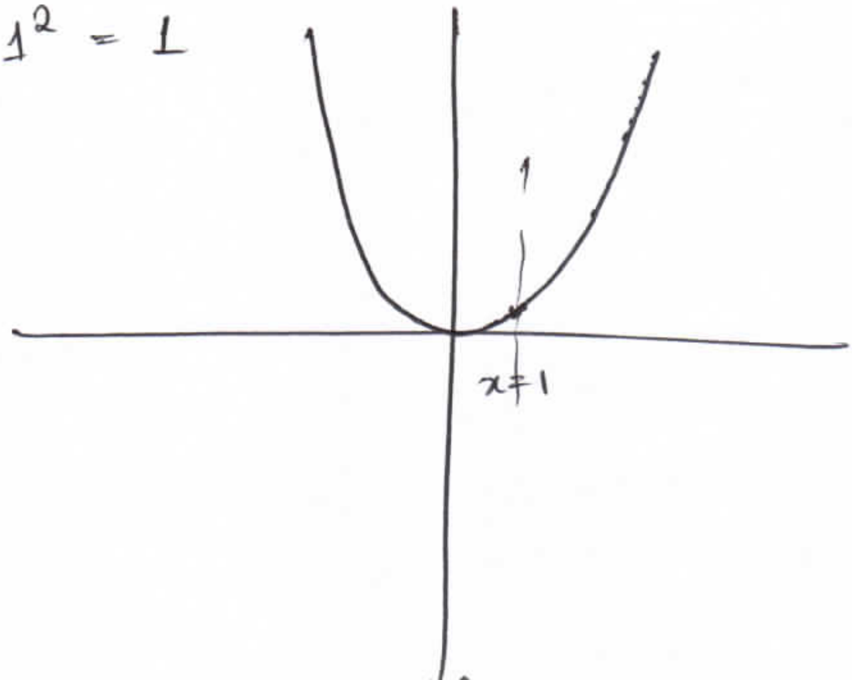
$\lim_{x \to x_0} f(x) = l$

if L·H·L = R·H·L = finite quantity (l)

then limit at $x = x_0$ is said to exist.

We can find limiting value of $f(x)$ at any point.

$$\lim_{x \to 1} x^2 = 1^2 = 1$$



Generally we find limits of $f(x)$ at $x \to x_0$. when function value is indeterminate at $x = x_0$.

TYPES of indeterminate form.

i) $\frac{0}{0}$ $\quad f(x) = \frac{\sin x}{x} \quad \lim_{x \to 0} \frac{\sin x}{x}$

ii) $\frac{\infty}{\infty}$ $\quad \lim_{x \to \infty} \frac{x^2+2x}{x^2-4x+1}$

iii) $0 \cdot \infty$ $\quad \lim_{x \to 2} (x-2) \tan \frac{\pi}{x}$

iv) $\infty - \infty$ $\quad \lim_{x \to \infty} (x^2+2x) - (2x^2+4x)$

v) ∞^0 $\quad \lim_{x \to \infty} x^{1/x}$

vi) 1^∞ $\quad \lim_{x \to 0} (1+x)^{1/x}$

vii) 0^0 $\quad \lim_{x \to 0} x^x$

$$f(x) = \cancel{e^{x}} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Some expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \infty$$

$$a^x = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{3!} + \frac{(x \ln a)^4}{4!} + \cdots \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \leq 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \qquad -1 \leq x < 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots \qquad |x| < 1$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \cdots \qquad |x| < 1$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \cdots \qquad |x| < 1$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \cdots \qquad |x| < 1$$

Some General Properties of limits.

If $\lim_{x \to a} f(x) = l_1$ & $\lim_{x \to a} g(x) = l_2$

i) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

$= l_1 \pm l_2$

(ii) $\lim_{x \to a} k f(x) = k \lim_{x \to a} f(x) = k l_1$

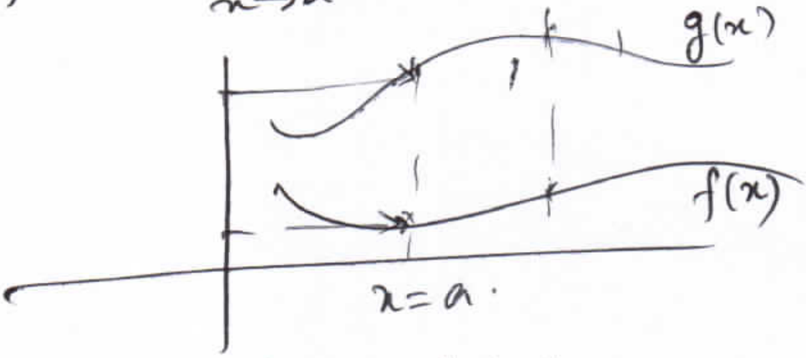
iii) $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x) = l_1 \times l_2$

iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{l_1}{l_2}$ provided. $l_2 \neq 0$

v) $\lim_{x \to a} f(x)^{g(x)} = \left\{\lim_{x \to a} f(x)\right\}^{\lim_{x \to a} g(x)} = (l_1)^{l_2}$

(vi) if $f(x) \le g(x) \ \forall x$

then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x) \Rightarrow l_1 \le l_2$.



vii)

$\lim_{x \to a} |f(x)| = |\lim_{x \to a} f(x)| = |l_1|$

Method of Calculating limit.

i) Direct Substitution.

$$\lim_{x\to 2}(x^2-2x+4) = 2^2-2(2)+4 = 4$$

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x\to 2}(x+2) = 4$$

$$\lim_{x\to 2}\frac{x^2+4}{x-2}$$

$$\lim_{x\to 2^-}\frac{x^2+4}{x-2}\to -\infty \qquad \lim_{x\to 2^+}\frac{x^2+4}{x-2}\longrightarrow \infty$$

L.H.L $\neq$ R.H.L

limit does not exist.

General factorizations:

$$a^2-b^2 = (a-b)(a+b)$$

$$a-b = (\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

$$a^4-b^4 = (a^2+b^2)(a-b)(a+b)$$

$$x^n-a^n = (x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-1})$$

$$= x^n-\cancel{ax^{n-1}}+\cancel{ax^{n-1}}-\cancel{a^2x^{n-2}}+\cancel{a^2x^{n-2}}-\cancel{a^3x^{n-3}}\cdots+a^{n-1}$$

Some Standard formulae

① $$\boxed{\lim_{x\to a} \frac{x^n - a^n}{x-a} = n a^{n-1}}$$

Proof

$$\lim_{x\to a} \frac{\cancel{(x-a)}(x^{n-1} + a x^{n-2} + a^2 x^{n-3} + \cdots a^{n-1})}{\cancel{(x-a)}}$$

$$(a^{n-1} + a^{n-1} + \cdots a^{n-1})$$

$$= n a^{n-1}$$

② $$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

Proof

$$\lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}{x}$$

$$\lim_{x\to 0} \frac{\cancel{x}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)}{\cancel{x}}$$

$$= 1 - \frac{0^2}{3!} + \frac{0^4}{5!} - 0 \cdots$$

$$= 1$$

$$\boxed{\lim_{x\to 0} \frac{\sin kx}{x} = K}$$

Proof $$\lim_{x\to 0} \frac{kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} \cdots}{x}$$

$$\lim_{x\to 0} \frac{\cancel{x}\left(k - \frac{k^3x^2}{3!} + \frac{k^5x^4}{5!}\right)}{\cancel{x}}$$

$$= K$$

(3) $$\boxed{\lim_{x \to 0} \frac{\tan x}{x} = 1}$$

Proof.

$$\lim_{x \to 0} \frac{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots}{x}$$

$$\lim_{x \to 0} \frac{\cancel{x}\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \cdots\right)}{\cancel{x}}$$

$$= 1 + \frac{0^2}{3} + \frac{2}{15} \times 0^4 + (\,)0 \cdots$$

$$= 1$$

$$\boxed{\lim_{x \to 0} \frac{\tan kx}{x} = k}$$

$$\boxed{\lim_{x \to 0} \frac{\sin mx}{\tan nx} = \frac{m}{n}} \qquad \boxed{\lim_{x \to 0} \frac{\tan kx}{\sin px} = \frac{k}{p}}$$

$$\lim_{x \to 0} \frac{\frac{\sin mx}{x}}{\frac{\tan nx}{x}} = \frac{\lim_{x \to 0} \frac{\sin mx}{x}}{\lim_{x \to 0} \frac{\tan nx}{x}} = \frac{m}{n}.$$

④ $$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}}$$

Proof

$$\lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x^2} \quad // \quad 2\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}\right)^2$$

$$= 2\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x} \times \lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}$$

$$= 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$$\boxed{\lim_{x\to 0} \frac{1-\cos kx}{x^2} = \frac{k^2}{2}}$$

⑤ $$\boxed{\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1} \qquad \boxed{\lim_{x\to 0} \frac{\ln(1+kx)}{x} = k}$$

proof. $$\lim_{x\to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots}{x}$$

$$\lim_{x\to 0} \frac{\cancel{x}\left(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right)}{\cancel{x}}$$

$$= 1 - \frac{0}{2} + \frac{0^2}{3} + -0 - \cdots$$

$$= 1$$

$$\boxed{\lim_{x\to 0} \frac{\ln(1-x)}{x} = -1} \qquad \boxed{\lim_{x\to 0} \frac{\ln(1-kx)}{x} = -k}$$

eg. $\lim_{x\to 2} \frac{\tan x}{x} = \frac{\tan 2}{2}$

$\lim_{x\to 0} \frac{\tan x}{x} = 1$

$\lim_{x\to 2} \frac{\tan(x-2)}{(x-2)}$

$\lim_{h\to 0} \frac{\tan h}{h} = 1$

$x \to 2$

$x-2 \to 0$ $\qquad x-2 = h$

$\lim_{h\to 0} \frac{\tan h}{h} = 1.$

eg. $\lim_{x\to 0} \frac{\ln(1+x^2)}{x^2} = 1$ $\quad \boxed{\lim_{\{\,\}\to 0} \frac{\ln(1+\{\,\})}{\{\,\}} = 1}$

$\lim_{x\to 0} \left\{ \frac{\ln(1+x^2)}{x\times x} \right\} x = \lim_{x\to 0} \frac{\ln(1+x^2)}{x^2} \times \lim_{x\to 0} x$

$\downarrow$

$1 \times 0 = 0$

(6) $\boxed{\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a.}$

Proof $\lim_{x\to 0} \frac{\left(1 + x\ln a + \frac{(x\ln a)^2}{2!} + \cdots\right) - 1}{x}$

$$\lim_{x\to 0} \frac{x\ln a + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \cdots}{x}$$

$$\lim_{x\to 0} \frac{\cancel{x}\left(\ln a + \frac{x\ln^2 a}{2!} + \frac{x^2\ln^3 a}{3!} + \cdots\right)}{\cancel{x}}$$

$$= \ln a + 0 + 0 \cdots$$

$$= \ln a.$$

$$\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = \ln e = 1}$$

Q1) $\lim_{x\to 3} \frac{x^5 - 3^5}{x-3} = 5\cdot 3^{5-1} = 5\cdot 3^4 = 5\times 81 = 405$

Q2) $\lim_{x\to 0} \frac{\sin 3x}{x} = 3$

Q3) $\lim_{x\to 0} \frac{\sin 8x}{\sin 4x} = \frac{\lim_{x\to 0}\frac{\sin 8x}{x}}{\lim_{x\to 0}\frac{\sin 4x}{x}} = \frac{8}{4} = 2$

Q4) $\lim_{x\to 0} \frac{\sqrt{2+x^2} - \sqrt{2-x^2}}{x^2} = \lim_{x\to 0} \frac{\left(\sqrt{2+x^2} - \sqrt{2-x^2}\right)\left(\sqrt{2+x^2} + \sqrt{2-x^2}\right)}{x^2\left(\sqrt{2+x^2} + \sqrt{2-x^2}\right)}$

$= \lim_{x\to 0} \frac{2\cancel{x^2}}{\cancel{x^2}\left(\sqrt{2+x^2} + \sqrt{2-x^2}\right)}$

$= \frac{2}{\sqrt{2+0} + \sqrt{2-0}}$

$= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$

Q5) $\lim_{x\to 0} \frac{\sin 5x - \sin 2x}{\sin 7x + \sin 4x} = \lim_{x\to 0} \frac{\frac{\sin 5x}{x} - \frac{\sin 2x}{x}}{\frac{\sin 7x}{x} + \frac{\sin 4x}{x}}$

$= \frac{5-2}{7+4} = \frac{3}{11}$

$$\lim_{x\to 0} \frac{\cancel{2} \cos\frac{7x}{2} \sin\frac{3x}{2}}{\cancel{2} \sin\frac{11x}{2} \cos\frac{3x}{2}} = \lim_{x\to 0} \frac{\cos\frac{7x}{2}}{\cos\frac{3x}{2}} \times \frac{\sin\frac{3x}{2}/x}{\sin\frac{11x}{2}/x}$$

$$= \frac{1}{1} \times \frac{3/2}{11/2} = \frac{3}{11}$$

$$\lim_{x\to\infty} \frac{x^2 - x + 1}{2x^2 + x}$$

$$\lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{x}{x^2} + \frac{1}{x^2}}{2\frac{x^2}{x^2} + \frac{x}{x^2}}$$

$$\lim_{x\to\infty} \frac{1 - \frac{1}{x} + \frac{1}{x^2}}{2 + \frac{1}{x}}$$

$$= \frac{1}{2}.$$

$$\lim_{x\to\infty} \frac{x}{x^2+1}$$

$$\lim_{x\to\infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1 + \frac{1}{x^2}}$$

$$= \frac{0}{1+0} = 0$$

$$\lim_{x\to\infty} \frac{x^2+2x}{x+1}$$

$$= \infty$$

L - Hospital's Rule.

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

$= \lim_{x\to a} \frac{f'(x)}{g'(x)} = \cdots\cdots = \lim_{x\to 0} \frac{f^n(x)}{g^n(x)} = \text{finite}$

continue till.

Q $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{3\cos 3x}{5\cos 5x} = \frac{3}{5}$ ✓

Q $\lim_{x\to\infty} \frac{x^2-4x+1}{2x^2+12x} = \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2x-4}{4x+12} \quad \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2}{4} = \frac{2}{4} = \frac{1}{2}$

Q) $\lim_{x\to 0} \frac{x^2}{\sin x} \quad \frac{0}{0}$

$= \lim_{x\to 0} \frac{2x}{\cos x} = \frac{0}{1} = 0$

$\lim_{x\to 0} \frac{x^2/x}{\sin x/x} = \frac{\lim_{x\to 0} x}{\lim_{x\to 0} \frac{\sin x}{x}}$

$= \frac{0}{1}$

Q) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \frac{0}{0}$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$

$= \lim_{x\to 0} \frac{\sec^2 x - \cos x}{3x^2} = \frac{0}{0}$

$\frac{d(\sec^2 x \tan x)}{dx}$

$= \sec^2 x\,(\sec^2 x) + \tan x \frac{d(\sec^2 x)}{dx}$

$= \lim_{x\to 0} \frac{2\sec x(\sec x \tan x) + \sin x}{6x} \quad \frac{0}{0}$

$= \lim_{x\to 0} \frac{2\sec^2 x \sec^2 x + \tan x(4\sec^2 x \tan x) + \cos x}{6}$

$= \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin x(1-\cos x)}{\cos x \; x^3}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{(1-\cos x)}{x^2} \times \frac{1}{\cos x}$$

$$= 1 \times \frac{1}{2} \times 1 = \frac{1}{2}$$

Q) $\lim_{n\to\infty} \frac{1+2+3+\cdots n}{n^2} = \lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n^2} = \lim_{n\to\infty} \frac{n^2+n}{2n^2}$

$= \frac{1}{2}$

Q) $\lim_{x\to\infty} \frac{3\sin 5x}{7x} = 0$

1^∞ form.

eg. $$\boxed{\lim_{f(x)\to 0} (1+f(x))^{1/f(x)} = e}$$

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

$$\lim_{x\to 2} (x-1)^{1/x-2} = \lim_{x-2\to 0} (1+(x-2))^{1/(x-2)} = e$$

let $\lim_{f(x)\to 0} (1+f(x))^{1/f(x)} = y$

taking log both sides

$$\ln\left(\lim_{f(x)\to 0} (1+f(x))^{1/f(x)}\right) = \ln y$$

$$\frac{1}{f(x)} \ln\left(\lim_{f(x)\to 0} (1+f(x))\right) = \ln y$$

$$\lim_{f(x)\to 0} \frac{\ln(1+f(x))}{f(x)} = \ln y$$

$$\Rightarrow 1 = \ln y \qquad y = e$$

$$\boxed{\lim_{f(x)\to 0} (1+kf(x))^{1/f(x)} = e^k}$$

$$\boxed{\lim_{f(x)\to \infty} \left(1+\frac{1}{f(x)}\right)^{f(x)} = e}$$

$$\boxed{\lim_{f(x)\to \infty} \left(1+\frac{k}{f(x)}\right)^{f(x)} = e^k}$$

In general.

If $\lim_{x \to a} f(x) = 1$

& $\lim_{x \to a} g(x) = \infty$

$$\boxed{\lim_{x \to a} \{f(x)\}^{g(x)} = e^{\lim_{x \to a} g(x)\{f(x)-1\}}}$$

$$\lim_{x \to 0} (1+x)^{1/x} = e^{\lim_{x \to 0} \frac{1}{x}(1+x-1)}$$
$$= e^{\lim_{x \to 0} \frac{x}{x} 1} = e^{1} = e$$

$f(x) = 1+x$
$g(x) = \frac{1}{x}$

Q) $\lim_{x \to \infty} \left(\frac{x^2+2x-1}{x^2-2x+2}\right)^{2x+1}$

$f(x) = \frac{x^2+2x-1}{x^2-2x+2}$

$g(x) = 2x+1$

$= e^{\lim_{x \to \infty} g(x)\{f(x)-1\}}$

$= e^{\lim_{x \to \infty} (2x+1)\left\{\frac{x^2+2x-1}{x^2-2x+2} - 1\right\}}$

$= e^{\lim_{x \to \infty} \frac{(2x+1)(4x-3)}{(x^2-2x+2)}} = e^8$

Q) $\lim_{x \to 5} (x-4)^{\frac{4}{x-5}}$

$f(x) = x-4 \quad g(x) = \frac{4}{x-5}$

$= e^{\lim_{x \to 5} g(x)\{f(x)-1\}}$

$= e^{\lim_{x \to 5} \frac{4}{\cancel{(x-5)}} \cancel{(x-5)}}$

$= e^4$

Q $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

$f(x) = \sin x$
$g(x) = \tan x$.

$= e^{\lim_{x\to\pi/2} \tan x (\sin x - 1)}$

$= e^{\alpha}$

$\alpha = \lim_{x \to \frac{\pi}{2}} \tan x(\sin x - 1)$

$= \lim_{x \to \frac{\pi}{2}} \frac{\sin^2 x - \sin x}{\cos x}$

$= \lim_{x \to \pi/2} \frac{2\sin x \cos x - \cos x}{-\sin x}$

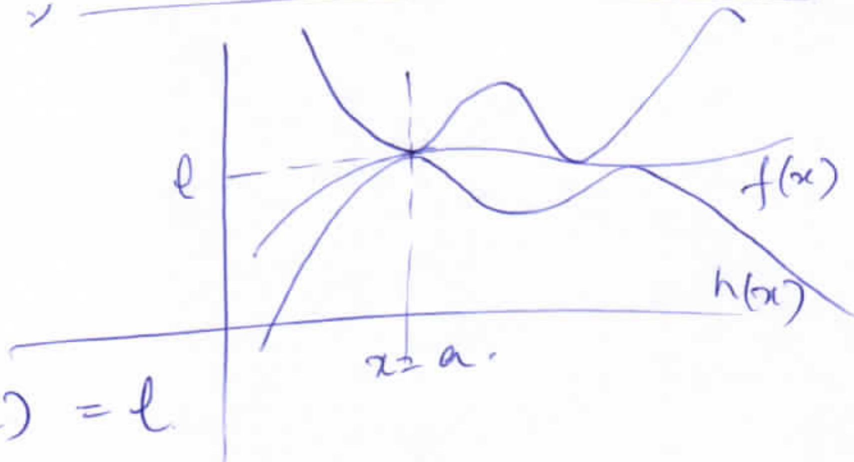
$\alpha = 0$

$= e^0 = 1.$

SANDWICH THEOREM.

If $h(x) \leq f(x) \leq g(x)$

If $\lim_{x\to a} h(x) = \lim_{x \to a} g(x) = l$



$$\lim_{x\to a} h(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

$$l \le \lim_{x\to a} f(x) \le l$$

$$\Downarrow$$

$$\lim_{x\to a} f(x) = l$$

eg. $$\lim_{n\to\infty} \frac{[x] + [2x] + [3x] + \cdots [nx]}{n^2}$$

$$x-1 < [x] \le x.$$

$$2x-1 < [2x] \le 2x$$

$$\vdots$$

$$nx-1 < [nx] \le nx$$

$$\frac{(x-1)+(2x-1)+\cdots(nx-1)}{n^2} < \frac{[x]+[2x]+[\ ]\cdots[nx]}{n^2} \le \frac{x+2x+\cdots nx}{n^2}$$

$$\frac{\frac{n(n+1)}{2}x - n}{n^2} < f(x) \le \frac{\frac{n(n+1)}{2}x}{n^2}$$

$$\lim_{n\to\infty} \frac{n\left[\frac{(n+1)x}{2} - 1\right]}{n^2} < \lim_{n\to\infty} f(x) \le \lim_{n\to\infty}\left(\frac{\frac{n^2}{2}x + \frac{nx}{2}}{n^2}\right)$$

$$\frac{x}{2} < \lim_{n\to\infty} f(x) \le \frac{x}{2}$$

$$\lim_{n\to\infty} f(x) = \boxed{\frac{x}{2}}.$$

Limits of SPECIAL FUNCTIONS

$$\lim_{x\to 6} \frac{|x-6|}{x-6}$$

$$\underbrace{\lim_{x\to 6^+} \frac{x-6}{x-6} = 1.}_{R\cdot H\cdot L} \quad \neq \quad \underbrace{\lim_{x\to 6^-} \frac{-(\cancel{x-6})}{\cancel{x-6}} = -1.}_{L\cdot H\cdot L}$$

*· Always for modulus function check equality of L·H·L & R·H·L

$$\lim_{x\to 3} [x+8]$$

$$\lim_{x\to 3^+} [x+8] = 11 \quad R\cdot H\cdot L$$

$$\neq$$

$$\lim_{x\to 3^-} [x+8] = 10 \quad L\cdot H\cdot L$$

$$\lim_{x\to 0}\left[\frac{\sin x}{x}\right] = \lim_{x\to 0}\left[\frac{\tan x}{x}\right] = 1.$$

$$\lim_{x\to 0}\left[\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots}{x}\right]$$

$$\lim_{x\to 0}\left[\frac{\cancel{x}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}\cdots\right)}{\cancel{x}}\right]$$

$$\lim_{x\to 0}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}\cdots\right]$$

$$= 0$$

$$\lim_{x\to 0}\left[\frac{x}{\sin x}\right] = 1.$$

Inverse Trignometric functions

$$\sin^{-1}x = x+\frac{x^3}{3!}+\frac{1^2\cdot 3^2}{5!}x^5+\frac{1^2\cdot 3^2\cdot 5^2}{7!}x^7+\cdots$$

$$\cos^{-1}x = \frac{\pi}{2}-\left(x+\frac{x^3}{3!}+\frac{9}{5!}x^5+\cdots\right)$$

$$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

$$\lim_{x\to 0} \frac{\sin^{-1} 3x}{7x} = \lim_{x\to 0} \frac{3x + \frac{(3x)^3}{3!} + \frac{1^2 \cdot 3^2}{5!}(3x)^5 + \cdots}{7x}$$

$$= \lim_{x\to 0} \frac{\cancel{x}\left[3 + \frac{3^3 \cdot x^2}{3!} + \frac{1^2 \cdot 3^2 \cdot 3^5 x^4}{5!} + \cdots\right]}{7\cancel{x}}$$

$$= \frac{3}{7}$$

$$\theta = \sin^{-1} 3x \Rightarrow \sin\theta = 3x$$

$$\Rightarrow x = \frac{\sin\theta}{3}$$

$$x \to 0 \quad \frac{\sin\theta}{3} \to 0$$

$$\theta \to 0$$

$$\lim_{\theta\to 0} \frac{\theta}{7\frac{\sin\theta}{3}}$$

$$\lim_{\theta\to 0} \frac{3}{7} \times \frac{\theta}{\sin\theta} = \frac{3}{7} \times 1 = \frac{3}{7}$$

If $\lim_{x \to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3} = 1$

find a & b

$$\lim_{x \to 0} \frac{x\left(1 + a\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\right) - b\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{\cdots}\right)}{x^3}$$

$$\lim_{x \to 0} \frac{x\left(1 + a - \frac{ax^2}{2!} + \frac{ax^4}{4!} - \frac{ax^6}{6!} + \cdots\right) - b\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)}{x^3}$$

$$\lim_{x \to 0} \frac{\left(x(1+a) - \frac{ax^3}{2!} + \frac{ax^5}{4!} - \frac{ax^7}{6!} + \cdots\right) - \left(bx - \frac{bx^3}{3!} + \frac{bx^5}{5!} - \cdots\right)}{x^3}$$

$$\lim_{x \to 0} \frac{x(1+a-b) + x^3\left(\frac{b}{3!} - \frac{a}{2!}\right) + x^5\left(\frac{a}{4!} - \frac{b}{5!}\right) + x^7() + \cdots}{x^3}$$

coeff $x = 0$ $\qquad 1 + a - b = 0$ —— (1)

$$\lim_{x \to 0} \frac{\cancel{x^3}\left[\left(\frac{b}{3!} - \frac{a}{2!}\right) + x^2\left(\frac{a}{4!} - \frac{b}{5!}\right) + x^4() \cdots\right]}{\cancel{x^3}}$$

$= \dfrac{b}{3!} - \dfrac{a}{2!} = 1$ —— (2)

$b - a = 1$ —— (1)

$b - 3a = 6$ —— (2)

$(-)$ ——————

$2a = -5 \Rightarrow a = -5/2$

$b = -3/2$

$$\lim_{x\to 0} \frac{x + ax\cos x - b\sin x}{x^3} \quad \frac{0}{0}$$

$$\lim_{x\to 0} \frac{1 + a\cos x - ax\sin x - b\cos x}{3x^2}$$

$$\lim_{x\to 0} \frac{1 + (a-b)\cos x - ax\sin x}{3x^2}$$

$1 + a - b = 0 \longrightarrow$ (1)

$$\lim_{x\to 0} \frac{1 - \cos x - ax\sin x}{3x^2}$$

$$\overset{L\cdot H}{=} \lim_{x\to 0} \frac{0 + \sin x - a\sin x - ax\cos x}{6x} \quad \frac{0}{0}$$

$$\overset{L\cdot H}{=} \lim_{x\to 0} \frac{\cos x - a\cos x - a\cos x + ax\sin x}{6} \quad 1$$

$$\frac{1 - a - a + 0}{6} = 1 \qquad \frac{1-2a}{6} = 1 \quad a = -5/2$$