BINOMIAL THEOREM

$\downarrow$

Expression having
2 parameters.

$(x+y)^1 \quad (x+y)^2 \quad (x+y)^3 \quad (x+y)^4 \quad \cdots \quad (x+y)^n$

$(x+y)^1 \rightarrow x+y$

$(x+y)^2 \rightarrow x^2+2xy+y^2$

$(x+y)^3 \rightarrow x^3+3x^2y+3xy^2+y^3$

FACTORIAL of number n.

$\longrightarrow$ $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

$6! = 6 \times \underbrace{5 \times 4 \times 3 \times 2 \times 1}$
$= 6 \times 5!$

$\longrightarrow$ $\underline{n! = n \times (n-1)!}$

$\rightarrow$ $0! = 1$

$\rightarrow$ $n!$ is always even except when $n=0$ or $n=1$

$\rightarrow$ only $0!$ & $1!$ is odd.

$\rightarrow$ $n!$ ends with atleast one zero for $n \geq 5$

$\rightarrow$ factorial for a -ve number is not defined.

Q1 What is the minimum value of r if $r!$ is div by 13 $\longrightarrow$ 13

Q2. Find min value of r if $r!$ is div by 16 $\longrightarrow$ 6

Q3. Solve for $x, y \in N$ where
$1! + 2! + 3! + \cdots x! = y^2$

Q4 find $x \in N$ if it satisfies
$1! + 2! + 3! \cdots x! = 9y \; ; \; y \in I.$

Q5) True/False?
let $x = \sum_{1}^{2n} r!$ $\quad y = \sum_{1}^{n} (2r)!$

a) x is div by y.
b) y is div by x

Q6) Find max value of r s.t. $\frac{10!}{2^r}$ is integer. $\longrightarrow$ $r = 8$.

Q7) Find max value of r s.t $\frac{100!}{3^r}$ is integer

Q8) Find r_{max} if $120!$ is div by 6^r

Q9) Find number of trailing zeros in $150!$

3) $1! + 2! + 3! + 4! = 33$

$5!\ \ 6!\ \ 7!\ \ \ldots\ n!$ all ends with 0.

So numerator of $n \geq 4$

$1! + 2! + 3! + 4! + \cdots n!$

$= 3$

for the number to be perfect square it should not end with 3

$n \geq 4$ cannot be the solution.

Solution is $(1,1)$ & $(3,3)$

4) $1! + 2! + 3! \ldots x! = 9y$

$1! + 2! + 3! + 4! + 5! = 153$ div by 9.

$6! = 720$ div by 9

$7! = 6! \times 7$ div by 9.

$$\frac{(1! + 2! + 3! + 4! + 5!)}{9} + \frac{6!}{9} + \frac{7!}{9} + \frac{8!}{9} \cdots$$

for $n \geq 5$ it is div by 9.
also for $n = 3$ it is div.

5) $x = 1! + 2! + \cdots\ n! + (n+1)! + \cdots (2n)!$

$y = 2! + 4! + 6! \cdots\cdots (2n)!$

y is a subset of x.

b) is false.

$\cancel{x = (1! + 3! + 5! + 7! \cdots (2n-1)!) + y}$.

$\cancel{x = (1! + 3\times2! + 5\times4! + 7\times6! + \cdots (2n-1)(2n-2)!) + y}$.

a) is false. x is odd y is even.

6) $\dfrac{10!}{2^r} = 10\times9\times8\times7\times6\times5\times4\times3\times2\times1$

$= \dfrac{2\times5\times9\times2^3\times7\times2\times3\times5\times2^2\times3\times2\times1}{2^r}$

7) Exponent of prime 'p' in k factorial.

is $= \left[\dfrac{k}{p}\right] + \left[\dfrac{k}{p^2}\right] + \left[\dfrac{k}{p^3}\right] + \left[\dfrac{k}{p^4}\right] + \cdots$

if $k = 100$ $p = 3$.

$= \left[\dfrac{100}{3}\right] + \left[\dfrac{100}{3^2}\right] + \left[\dfrac{100}{3^3}\right] + \left[\dfrac{100}{3^4}\right] + \left[\dfrac{100}{3^5}\right] + \cdots$

$48 = 33 + 11 + 3 + 1 \quad 0$

8) $\dfrac{120!}{6^r} = \dfrac{120!}{3^r 2^r}$

exponent of 3

$$\left[\frac{120}{3}\right] + \left[\frac{120}{3^2}\right] + \left[\frac{120}{3^3}\right] + \left[\frac{120}{3^4}\right] + \cdots$$

$= 40 + 13 + 4 + 1$

$= 58.$

exponent of 2

$$= \left[\frac{120}{2}\right] + \left[\frac{120}{2^2}\right] + \left[\frac{120}{2^3}\right] + \cdots$$

$= 60 + 30 + 15 + 7 + 3 + 1$

$= 116$

$$\cancel{\frac{120!}{2^{116} 3^{58}}} \cdots = \frac{2^{116} \cdot 3^{58} \cdots}{2^r 3^r}$$

$= \underline{58}.$

9) $\dfrac{150!}{10^r} = \dfrac{150!}{5^r 2^r}.$

$$\left[\frac{150}{5}\right] + \left[\frac{150}{5^2}\right] + \left[\frac{150}{5^3}\right] + \cdots$$

$= 30 + 6 + 1$

$= 37$ zeroes.

BINOMIAL COEFFICIANTS

^nC_r $\quad n \geq r$

$n \geq 1$

$= \dfrac{n!}{r!(n-r)!}$ $\quad r \geq 0$ $\quad {}^5C_3 = \dfrac{5!}{3!\,2!} = \dfrac{\cancel{5 \times 4 \times 3 \times 2 \times 1}}{\cancel{3!\,2!}}$

$= 10$

1) $^nC_r = {}^nC_{n-r}$ $\quad {}^nC_{n-r} = \dfrac{n!}{(n-r)!(r!)}$

2) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ $\leftarrow$ prove this.

3) $^nC_0 = {}^nC_n = 1$ | $^nC_1 = {}^nC_{n-1} = n$ | $^nC_2 = {}^nC_{n-2} = \dfrac{n(n-1)}{2}$ ....

$\cancel{^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots\cdots {}^nC_n}$

BINOMIAL THM

$n \in N.$

$(x+y)^n$

$= {}^nC_0 x^n + {}^nC_1 x^{n-1}y^1 + {}^nC_2 x^{n-2}y^2 + \cdots + {}^nC_r x^{n-r}y^r + \cdots + {}^nC_n y^n$

eg $(x+y)^3 = {}^3C_0 x^3 + {}^3C_1 x^2y^1 + {}^3C_2 xy^2 + {}^3C_3 y^3$

$= x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^5 = {}^5C_0x^5 + {}^5C_1x^4y + {}^5C_2x^3y^2 + {}^5C_3x^2y^3 + {}^5C_4xy^4 + {}^5C_5y^5$

$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

In expansion of $(x+y)^n$ we have $(n+1)$ terms.

$T_r = {}^nC_{r-1}\, x^{n-r+1}\, y^{r-1}$

Find 7th term in expansion of $(2x-3y)^{11}$

$T_7 = {}^{11}C_6 (2x)^{11-6} (-3y)^6$

$= {}^{11}C_6 \times 2^5 x^5 (-3)^6 y^6$

$= {}^{11}C_6\, 2^5 3^6\, x^5 y^6$

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \cdots {}^nC_nx^n$

$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + \cdots (-1)^n\,{}^nC_nx^n$

${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots {}^nC_n = 2^n$ (put $x=1$ in 1st of above 2 eq)

${}^nC_0 - {}^nC_1 + {}^nC_2 + \cdots (-1)^n\,{}^nC_n = 0$

$\underbrace{{}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots}_{2^{n-1}} = \underbrace{{}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots}_{2^{n-1}}$

Q) Find constant term in expansion of

$$\left(x-\frac{1}{x}\right)^6$$

let T_{r+1} be the constant term $= {}^6C_r x^{6-r}\left(-\frac{1}{x}\right)^r$

$$= {}^6C_r(-1)^r \frac{x^{6-r}}{x^r} = {}^6C_r(-1)^r x^{6-2r}$$

$$6-2r=0$$

$r=3$ $\quad T_4$ is const.

$$= {}^6C_3(-1)^3x^0 = -20$$

Q) Find coeff of x^{13} in $\left(2x+\frac{5}{x^2}\right)^{20}$

let T_{r+1} have coeff of $x^{13} = {}^{20}C_r(2x)^{20-r}\left(\frac{5}{x^2}\right)^r$

$$= {}^{20}C_r(2)^{20-r}(5)^r\frac{x^{20-r}}{x^{2r}}$$

$$= {}^{20}C_r\, 2^{20-r}5^r\, x^{20-3r}$$

$20-3r=13$

$r=7/3$

$r \neq$ doesn't exist

Q) a) Find no. of irrational terms in $\left(4^{1/5}+7^{1/10}\right)^{45}$

b) Find no. of rational terms in $\left(\sqrt[3]{2}+7\cdot\sqrt[5]{8}\right)^{41}$

c) If 21^{st} & 22^{nd} terms in expansion of $(1-x)^{44}$ are equal find x

a) let T_{r+1} be irration $= {}^{45}C_r \, 4^{1/5(45-r)} \, 7^{1/10 \, r}$

~~(45-r) X 5 r ... 40~~

if rational $45-r \,/\, 5$ $\quad r \,/\, 10$

$r = 0, 5, 10, 15, 20, \cdots 45$ $\quad$ $r = 0, 10, 20, 30, 40$

0, 10, 20, 30, 40

except. $\underbrace{T_1, T_{11}, T_{21}, T_{31}, T_{41}}_{5 \text{ ration}}$

46 − 5 = 41 irrational.

b) T_{r+1} be rational $= {}^{41}C_r \, 2^{1/3(41-r)} \, 7^r \, 8^{1/5 \, r}$

$41-r \,/\, 3$ $\quad r \,/\, 5.$

2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41 $\quad$ 0, 5, 10, 15, 20, 25, 30, 35, 40

T_6, T_{21}, T_{36}

c) $T_{21} = {}^{44}C_{20}(-x)^{20}$

$=$

$T_{22} = {}^{44}C_{21}(-x)^{21}$

$${}^{44}C_{20}\cancel{(-x)^{20}} = {}^{44}C_{21}(-x)^{\cancel{20}+1}$$

$$x = -\frac{{}^{44}C_{20}}{{}^{44}C_{21}} = -\frac{\frac{\cancel{44!}}{20!\,24!}}{\frac{\cancel{44!}}{21!\,23!}} = \frac{-\overset{21}{\cancel{21!}}\,\overset{1}{\cancel{23!}}}{\cancel{20!}\,\underset{24}{\cancel{24!}}} = \boxed{-\frac{7}{8}}$$

8) Find coeff of x^8 in expansion of

$$(\underbrace{2+3x}_{a} + \underbrace{4x^3}_{b})^{20}$$

$$\underbrace{{}^{20}C_0(2+3x)^{20} + {}^{20}C_1(2+3x)^{19}4x^3 + {}^{20}C_2(2+3x)^{18}(4x^3)^2}_{} + {}^{20}C_3(2+3x)^{17}(4x^3)^3 + \cdots$$

8   5   2

$${}^{20}C_0\left({}^{20}C_8\,2^{20-8}(3x)^8\right),\ {}^{20}C_1\left({}^{19}C_5\,2^{19-5}(3x)^5\right)\times 4x^3$$

$${}^{20}C_2\left({}^{18}C_2\,2^{18-2}(3x)^2\right)(4x^3)^2$$

$$\left({}^{20}C_0\,{}^{20}C_8\,2^{12}3^8 + {}^{20}C_1\,{}^{19}C_5\,2^{14}3^5 4 + {}^{20}C_2\,{}^{18}C_2\,2^{16}3^2.4^2\right)x^8$$

For $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$

i) General term $T_{r+1} = {}^nC_r x^r$

ii) rth term from beginning $= T_r$

is equal to $(n-r+2)$ term from end.

for eg. T_2 2nd term from beginning

is $(n-2+2) = n$ term from end.

iii) Middle term

if $(x+y)^n \rightarrow$ no. of terms is $n+1$

If n is even middle term $= \left(\frac{n+2}{2}\right)^{th}$

if n is odd middle term $= \left(\frac{n+1}{2}\right)^{th}$ term

& $\left(\frac{n+1}{2}+1\right)^{th}$ term

for $(1+x)^n$

Middle term

n even. $\frac{n+2}{2} = \frac{n}{2}+1$

$$T_{\frac{n}{2}+1} = {}^nC_{n/2}\, x^{n/2}$$

n odd. $\frac{n+1}{2}$, $\frac{n+1}{2}+1$

$$T_{\frac{n-1}{2}+1} = {}^nC_{\frac{n-1}{2}}\, x^{\frac{n-1}{2}} \quad \& \quad T_{\frac{n+1}{2}+1} = {}^nC_{\frac{n+1}{2}}\, x^{\frac{n+1}{2}}$$

Find Middle team in expansion of

i) $(2x-3y)^{28}$ ii) $\left(3x^2+\frac{5}{x^3}\right)^{31}$

$$T_{15} = T_{14+1} = {}^{28}C_{14}(2x)^{28-14}(-3y)^{14}$$

$\downarrow$ 32

16, 17

$$T_{16} = T_{15+1} = {}^{31}C_{15}(3x^2)^{31-15}\left(\frac{5}{x^3}\right)^{15}$$

$$T_{17} = T_{16+1} = {}^{31}C_{16}(3x^2)^{31-16}\left(\frac{5}{x^3}\right)^{16}$$

find. ${}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + 4\,{}^nC_4 + \cdots\cdots + n\,{}^nC_n$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots\ {}^nC_n x^n$$

Differentiate both sides.

$$n(1+x)^{n-1} = 0 + {}^nC_1 + 2\,{}^nC_2 x + 3\,{}^nC_3 x^2 + \cdots + n\,{}^nC_n x^{n-1}$$

put $x=1$

$$n(2)^{n-1} = 0 + \underbrace{{}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \cdots + n\,{}^nC_n}$$

$n \cdot 2^{n-1}$

GREATEST BINOMIAL COEFFICIANT IN EXPANSION

of $(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + {}^nC_3 x^{n-3} y^3 + \cdots {}^nC_n y^n$

$\rightarrow$ if n is even. ${}^nC_{n/2}$

for eg. $(x+y)^{10}$
greatest coefficiant
${}^{10}C_5$

$\rightarrow$ if n is odd.

$$\underbrace{{}^nC_{\left(\frac{n-1}{2}\right)} \text{ or } {}^nC_{\left(\frac{n+1}{2}\right)}}_{\text{SAME IN VALUE.}}$$

$(x+y)^{11}$
greatest coefficiant.
${}^{11}C_{\left(\frac{11-1}{2}\right)}$ or ${}^{11}C_{\left(\frac{11+1}{2}\right)}$
${}^{11}C_5$ or ${}^{11}C_6$

GREATEST TERM IN EXPANSION

of $(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_r x^{n-r} y^r + \cdots + {}^nC_n y^n$

$T_r > T_{r+1}$	$\lvert T_r \rvert > \lvert T_{r+1} \rvert$
$T_r > T_{r-1}$	$\lvert T_r \rvert > \lvert T_{r-1} \rvert$
greatest term.	Numerically greatest. in magnitude.
$m = \dfrac{(n+1)y}{x+y}$ ✓	$m = \dfrac{(n+1)\lvert y \rvert}{\lvert x \rvert + \lvert y \rvert}$

if m is not an Integer.
$[m]^{th}$ term is the greatest.

if m is an integer.
m^{th} & $(m+1)^{th}$ term
both are the greatest term.

Q) Prove that if $|a| = \sqrt{3}|b|$ then the numerically greatest value of $(a+b)^{50}$ is the 18^{th} term.

$$m = \frac{(n+1)|y|}{|x|+|y|} = \frac{(50+1)|b|}{|a|+|b|} = \frac{51|b|}{\sqrt{3}|b|+|b|}$$

$$m = \frac{51\cancel{|b|}}{(\sqrt{3}+1)\cancel{|b|}}$$

$$= \frac{51(\sqrt{3}-1)}{2}$$

$$= 25.5(0.732)$$

$$= 18.\ldots$$

$[m] = 18.$

18^{th} term is the numerically greatest term.

×

$(x+y)^n$

$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \ldots$ $\quad x^n\left(1+\frac{y}{x}\right)^n$

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots$

$\boxed{\text{If } n \text{ is a fraction or -ve integer}}$. $\quad (x+y)^n$
$x^n\left(1+\frac{y}{x}\right)^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \infty \qquad |x|<1$$

1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \infty$
2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \infty$
3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \infty$
4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \infty$

$$\frac{20!}{16!\,2!\,2!}\,2^{16}\,3^{2}\,4^{2}\,x^{8}$$

$$\frac{20!}{14!\,5!\,1!}\,2^{14}\,3^{5}\,4^{1}\,x^{8}$$

$$\frac{20!}{12!\,8!\,0!}\,2^{12}\,3^{8}\,4^{0}$$

Find coeffircant of x^{9}

$(2+3x+4x^3)^{20}$

$^{m+n-1}C_{n-1}$

$^{20+3-1}C_{3-1}$

$= {}^{22}C_2 = 231$

$\alpha_1+\alpha_2+\alpha_3 = 20$

$$\frac{20!}{\alpha_1!\,\alpha_2!\,\alpha_3!}\,2^{\alpha_1}(3x)^{\alpha_2}(4x^3)^{\alpha_3}$$

$$\frac{20!}{\alpha_1!\,\alpha_2!\,\alpha_3!}\,2^{\alpha_1}\,3^{\alpha_2}\,4^{\alpha_3}\,x^{\alpha_2+3\alpha_3}$$

$\alpha_2+3\alpha_3 = 9$

α_1	α_2	α_3
17	0	3
15	3	2
13	6	1
11	9	0

$$\frac{20!}{17!\,0!\,3!}\,2^{17}\,3^{0}\,4^{3}\,x^{9}$$

$$\frac{20!}{15!\,3!\,2!}\,2^{15}\,3^{3}\,4^{2}\,x^{9}$$

$$\frac{20!}{13!\,6!\,1!}\,2^{13}\,3^{6}\,4^{1}\,x^{9}$$

$$\frac{20!}{11!\,9!\,0!}\,2^{11}\,3^{9}\,4^{0}\,x^{9}$$

if $y > x$

$$(x+2y)^{-2} = (2y+x)^{-2} = (2y)^{-2}\left(1+\frac{x}{2y}\right)^{-2}$$

$$= (2y)^{-2}\left\{1 - 2\left(\frac{x}{2y}\right) + 3\left(\frac{x}{2y}\right)^2 - 4\left(\frac{x}{2y}\right)^3 + \cdots \infty\right\}$$

MULTINOMIAL THM.

$(x_1+x_2+x_3+\cdots+x_n)^m$ $\quad \Big\{ \underbrace{{}^{m+n-1}C_{n-1}}_{} \Big\}$

↑ Total Number of terms.

Coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots x_n^{\alpha_n}$

if $\underbrace{\alpha_1+\alpha_2+\alpha_3 \cdots + \alpha_n = m}$.

$$\text{Coefficient} = \frac{m!}{\alpha_1!\,\alpha_2!\,\alpha_3!\cdots\alpha_n!}$$

$(x_1+x_2)^5$.

${}^5C_3\, x_1^2 x_2^3$

$\frac{5!}{2!\,3!}$

any term $= \frac{m!}{\alpha_1!\,\alpha_2!\,\alpha_3!\cdots\alpha_n!}\left(x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\cdots x_n^{\alpha_n}\right)$

if $\alpha_1+\alpha_2+\alpha_3+\cdots\alpha_n \neq m$ term doesn't exist.

find coeff of x^8 in $(2+3x+4x^3)^{20}$

$$\frac{20!}{\alpha_1!\,\alpha_2!\,\alpha_3!}\, 2^{\alpha_1}(3x)^{\alpha_2}(4x^3)^{\alpha_3}$$

$$2^{\alpha_1}\, 3^{\alpha_2}\, 4^{\alpha_3}\, x^{\alpha_2+3\alpha_3}$$

$\alpha_1+\alpha_2+\alpha_3 = 20$

$\alpha_2 + 3\alpha_3 = 8$

$\alpha_2 = 2 \quad \alpha_3 = 2 \quad \alpha_1 = 16$

$\alpha_2 = 5 \quad \alpha_3 = 1 \quad \alpha_1 = 14$

$\alpha_2 = 8 \quad \alpha_3 = 0 \quad \alpha_1 = 12$