① $S = 1^2+2^2+3^2+4^2+ \cdots n^2 = \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

② $S = 1^3+2^3+3^3+4^3+ \cdots + n^3 = \sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

③ $S = 1+2+3+4+5+ \cdots n = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (proved already)

①

$$(r+1)^3 - r^3 = 3r^2 + 3r + 1$$

L.H.S

$$(\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (4^3 - \cancel{3^3}) + \cdots ((n+1)^3 - \cancel{n^3})$$

$$(n+1)^3 - 1^3$$

R.H.S

$$= 3(1^2+2^2+3^2+ \cdots n^2) + 3(1+2+3+4+ \cdots n) + n$$

$$\sum_1 r^2$$

$$(n+1)^3 - 1^3 = 3\sum_1 r^2 + 3\frac{n(n+1)}{2} + n$$

$$\sum_1 r^2 = \frac{n(n+1)(2n+1)}{6}$$

② $(r+1)^4 - r^4 = ((r+1)^2 + r^2)((r+1)^2 - r^2)$

$$= (2r^2+2r+1)(2r+1)$$

$$(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$$

$r = 1$ to n.

L.H.S.

$$(n+1)^4 - 1 = 4\sum_1 r^3 + \frac{6\,n(n+1)(2n+1)}{6} + \frac{4\,n(n+1)}{2} + n.$$

$$\sum_1 r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Q1) Find the sum of series.

$$S = \frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \cdots \text{ n terms.}$$

Q2) Find $S = 1^3 + 3\cdot 2^2 + 3^3 + 3\cdot 4^2 + 5^3 + 3\cdot 6^2 \cdots$ n terms

n is even

eg. $S = 1+2+3+4 \cdots n.$

$t_r = r.$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Ans Q1

$$t_r = \frac{1^3+2^3+3^3+\cdots r^3}{1+3+5+\cdots(2r-1)} = \frac{\left\{\frac{r(r+1)}{2}\right\}^2}{\frac{r}{2}\left(1+(2r-1)\right)} = \frac{\frac{\cancel{r^2}(r+1)^2}{4}}{\frac{\cancel{r}}{\cancel{2}}\times \cancel{2r}}$$

$$t_r = \frac{(r+1)^2}{4} = \frac{r^2+2r+1}{4} = \frac{r^2}{4} + \frac{r}{2} + \frac{1}{4}$$

$$S = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n} \frac{r^2}{4} + \frac{r}{2} + \frac{1}{4} = \frac{1}{4}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{r=1}^{n} r + \frac{1}{4}\sum_{r=1}^{n} 1$$

$$= \frac{1}{4}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{2}\frac{n(n+1)}{2} + \frac{1n}{4}$$

$$= \frac{n}{4}\left\{\frac{(n+1)(2n+1)}{6} + (n+1) + 1\right\}$$

$$= \frac{n}{24}\left\{2n^2 + 9n + 13\right\}$$

let $n = 2m$

$$S = \{1^3 + 3^3 + 5^3 + \cdots (2m-1)^3\} + \{3\cdot 2^2 + 3\cdot 4^2 + \cdots 3\cdot(2m)^2\}$$

$$t_r = (2r-1)^3 \qquad t_r = 3\cdot(2r)^2$$

$$\sum_{r=1}^{m} t_r + \sum_{r=1}^{m} t_r$$

$$\sum_{r=1}^{m} (2r-1)^3 + \sum_{r=1}^{m} 3\cdot(2r)^2$$

$$\sum_{r=1}^{m} 8r^3 - 1 - 3(2r)^2 + 3(2r)$$

$$8\sum_{r=1}^{m} r^3 \cancel{- 12\sum_{r=1}^{m} r^2} + 6\sum_{r=1}^{m} r - \sum_{r=1}^{m} 1 \cancel{+ 12\sum_{r=1}^{m} r^2}$$

$$8\left(\frac{m(m+1)}{2}\right)^2 + 6\left(\frac{m(m+1)}{2}\right) - m$$

$$2\left(m\, m(m+1)^2\right) + 3m(m+1) - m$$

$$m\{2m(m^2+2m+1) + 3m + 3 - 1\}$$

$$m\{2m^3 + 4m^2 + 2m + 3m + 2\}$$

$$m\{2m^3 + 4m^2 + 5m + 2\}$$

$$\frac{n}{2}\left\{2\frac{n^3}{8} + \frac{4(n^2)}{4} + \frac{5n}{2} + 2\right\} = \frac{n}{8}\{n^3 + 4n^2 + 10n + 8\}$$

Arithmetico - Geometric Series

A.P $\quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \cdots$

G.P. $\quad 1 \quad x \quad x^2 \quad x^3 \quad x^4 \quad \cdots$

AGP $\quad 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 \cdots$

$$S_n = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \cdots (a+(n-1)d)r^{n-1}$$

$$rS_n = ar + (a+d)r^2 + (a+2d)r^3 + \cdots \{a+(n-1)d\}r^n$$

$$S_n(1-r) = a + dr + dr^2 + dr^3 + \cdots dr^{n-1} - \{a+(n-1)d\}r^n$$

$$S_n(1-r) = d(r + r^2 + r^3 + \cdots r^{n-1}) + a(1-r^n) + (n-1)dr^n$$

$$S_n(1-r) = d(r + r^2 + r^3 + \cdots r^{n-1}) + a(1-r^n) + (n-1)dr^n$$

$$= dr(1 + r + r^2 + \cdots r^{n-2})$$

$$S_n(1-r) = dr(1)\left(\frac{1-r^{n-1}}{1-r}\right) + a(1-r^n) + (n-1)dr^n$$

$$S_n = \frac{dr(1-r^{n-1})}{(1-r)^2} + \frac{a}{1-r} - \frac{\{a+(n-1)d\}r^n}{1-r}$$

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{\{a+(n-1)d\}r^n}{(1-r)}$$

$$S = 1 + 2x + 3x^2 + 4x^3 + --- n \text{ terms}.$$

$a = 1 \qquad r = x. \qquad d = 1$

$$S = \frac{1}{1-x} + \frac{x(1-x^{n-1})}{(1-x)^2} - \frac{\{1+(n-1)\}x^n}{(1-x)}$$

G.P $\Rightarrow S_n = \frac{a(1-r^n)}{1-r}$

$n \to \infty \qquad S_\infty = \frac{a(1-0)}{1-r} \qquad \because \lim_{n\to\infty} r^n = 0$

$-1 < r < 1$

$$S_\infty = \frac{a}{1-r}$$

$$S_n = \frac{a}{1-r} + dr\frac{(1-r^{n-1})}{(1-r)^2} - \frac{\{a+(n-1)d\}r^n}{1-r}$$

$n \to \infty \qquad -1 < r < 1$

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

Find $S_n = 1 + 4x + 7x^2 + 10x^3 + --- n$ terms.

if $S_\infty = \frac{35}{16}$ find x

$a = 1 \qquad d = 3 \qquad r = x.$

$$S_n = \frac{1}{1-x} + 3x\frac{(1-x^{n-1})}{(1-x)^2} - \frac{\{1+(n-1)3\}x^n}{1-x}$$

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

$$\frac{35}{16} = \frac{1}{1-x} + \frac{3x}{(-x)^2}$$

$$\frac{35}{16} = \frac{1-x+3x}{(1-x)^2} = \frac{1+2x}{x^2-2x+1}$$

$$35x^2 - 70x + 35 - 32x - 16 = 0$$

$$35x^2 - 102x + 19 = 0 \qquad \therefore r = \frac{1}{5} \qquad r = \frac{19}{7} \times \; (-1 < r < 1)$$

Find $S_n = 1 + 3 + 7 + 15 + 31 + \cdots$ n terms.

$$S_n = 1 + 3 + 7 + 15 + 31 + \cdots T_{n-1} + T_n$$

$$(-)\quad S_n = \quad 1 + 3 + 7 + 15 + \cdots + T_{n-1} + T_n.$$

$$0 = 1 + 2 + 4 + 8 + 16 + \cdots + 2^{n-1} - T_n$$

$$T_n = 1 + 2 + 4 + 8 + 16 + \cdots + 2^{n-1}$$

$$T_n = 1\frac{(1-2^n)}{1-2} = 2^n - 1.$$

$$S_n = \sum T_n = \sum_{r=1}^{n} 2^r - 1$$

$$= (2 + 2^2 + \cdots 2^n) - (n)$$

$$= 2\frac{(1-2^n)}{1-2} - n = 2(2^n - 1) - n$$

$$= 2^{n+1} - (n+2)$$

Method of Difference is used when terms of series are neither in A·P or G·P but their consecutive terms difference is either in A·P or G·P.

$S_n \Rightarrow 1 + 2 + 5 + 12 + 25 + 41 + \dots$ n terms.

$\rightarrow 1 + 3 + 7 + 13 + 21 + \dots$ 1st consecutive diff

$\rightarrow 2 + 4 + 6 + 8 + \dots$ 2nd consecutive diff.

If in $(r-1)^{th}$ consecutive difference we get an A·P or a G·P.

A·P.

$$T_n = an^r + bn^{r-1} + cn^{r-2} + \dots \dots k.$$

$$T_n = an^3 + bn^2 + cn + d.$$

$$T_n = a_1(n-1)(n-2)(n-3) + b_1(n-1)(n-2) + c_1(n-1) + d_1$$

$T_1 = 1 = d_1.$

$T_2 = 2 = c_1(1) + 1 \Rightarrow c_1 = 1.$

$T_3 = 5 = b_1(2) + 2 + 1 \Rightarrow b_1 = 1.$

$T_4 = 12 = 6a_1 + 6(1) + 3(1) + 1. \Rightarrow a_1 = \frac{1}{3}.$

$$T_n = \frac{1}{3}(n-1)(n-2)(n-3) + \underbrace{(n-1)(n-2) + (n-1)}_{(n-1)(n-2+1)\ (n-1)^2} + 1.$$

$$T_n = \frac{1}{3}\left((n^2-3n+2)(n-3)\right) + n^2-2n+1+1.$$

$$= \frac{1}{3}(n^3-6n^2+11n-6) + n^2-2n+2$$

$$= \frac{1}{3}(n^3-3n^2+5n)$$

$S_n = 2 + 5 + 12 + 31 + 86 + \text{— — . } n \text{ terms.}$

1st step $\rightarrow$ 3   7   19   55   —— 1st.

4   12   36   —— 2nd.

highest power of n is no. of consecutive difference before getting the G.P.

$T_n = (an + b) + c \cdot \cancel{8}^n$

$S_n = ?$

$T_1 = 2 \quad (a+b) + c\cdot 3^1 = a + b + 3c$

$T_2 = 5. \quad (2a+b) + c\cdot 9 = 2a + b + 9c$

$T_3 = 12 \cdot (3a+b) + c\cdot 27 = 3a + b + 27c.$

$T_3 - T_2 = 7 = a + 18c$

$T_2 - T_1 = 3 = a + 6c.$

$12c = 4$

$c = \frac{1}{3}.$

$a = 1.$

$b = 0$

$T_n = (1n + 0) + \frac{1}{3} 3^n$

$\sum_1 T_n = \sum_1 n + \sum_1 3^{n-1}$

$\frac{n(n+1)}{2} + (1 + 3 + 3^2 + \cdots 3^{n-1})$

$\frac{n(n+1)}{2} \mp 1\frac{(1-3^n)}{-2} \qquad \frac{n(n+1) + 3^n - 1}{2}.$

If $a_1 a_2 a_3 \cdots a_n \cdots$ in A.P.

$$S_n = \frac{1}{a_1 a_2 a_3 \cdots a_r} + \frac{1}{a_2 a_3 a_4 \cdots a_{r+1}} + \frac{1}{a_3 a_4 + \cdots a_{r+2}} + \cdots + \frac{1}{a_n a_{n+1} \cdots a_{n+r-1}}$$

$$S_n = a_1 a_2 a_3 \cdots a_r + a_2 a_3 a_4 a_5 \cdots a_{r+1} + a_3 a_4 \cdots a_{r+2} + a_n a_{n+1} a_{n+2} \cdots a_{n+r-1}$$

Find $S_n = \frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{2\cdot3\cdot4\cdot5} + \frac{1}{3\cdot4\cdot5\cdot6} + \cdots \frac{1}{n(n+1)(n+2)(n+3)}$

$$3S_n = \frac{4-1}{1\cdot2\cdot3\cdot4} + \frac{5-2}{2\cdot3\cdot4\cdot5} + \frac{6-3}{3\cdot4\cdot5\cdot6} + \cdots \frac{(n+3)-n}{n(n+1)(n+2)(n+3)}$$

$$= \frac{1}{1\cdot2\cdot3} - \cancel{\frac{1}{2\cdot3\cdot4}} + \cancel{\frac{1}{2\cdot3\cdot4}} - \cancel{\frac{1}{3\cdot4\cdot5}} + \cancel{\frac{1}{3\cdot4\cdot5}} - \frac{1}{4\cdot5\cdot6} + \cdots \cancel{\frac{1}{n(n+1)(n+2)}} - \frac{1}{(n+1)(n+2)(n+3)}$$

$$3S_n = \frac{1}{1\cdot2\cdot3} - \frac{1}{(n+1)(n+2)(n+3)}$$

$$S_n = \frac{1}{3}\left(\frac{1}{1\cdot2\cdot3} - \frac{1}{(n+1)(n+2)(n+3)}\right)$$

find $S_n = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots \frac{1}{n(n+1)}$

$$1\,S_n = \frac{2-1}{1\cdot2} + \frac{3-2}{2\cdot3} + \frac{4-3}{3\cdot4} + \cdots \frac{(n+1)-n}{n(n+1)}$$

$$S_n = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

find $S_n = 1\cdot2\cdot3\cdot4 + 2\cdot3\cdot4\cdot5 + 3\cdot4\cdot5\cdot6$
$+ \cdots n(n+1)(n+2)(n+3)$

$-5S_n = 1\cdot2\cdot3\cdot4(0-5) + 2\cdot3\cdot4\cdot5(1-6) + 3\cdot4\cdot5\cdot6(2-7)$
$+ \cdots n(n+1)(n+2)(n+3)((n-1)-n+4))$

$-5S_n = \underline{0\cdot1\cdot2\cdot3\cdot4} - \cancel{1\cdot2\cdot3\cdot4\cdot5} + \cancel{1\cdot2\cdot3\cdot4\cdot5} - \cancel{2\cdot3\cdot4\cdot5\cdot6}$
$+ \cancel{2\cdot3\cdot4\cdot5\cdot6} - 3\cdot4\cdot5\cdot6\cdot7$
$+ \cdots \cancel{(n-1)n(n+1)(n+2)(n+3)}$
$\underline{- n(n+1)(n+2)(n+3)(n+4)}$

$-5S_n = 0 - n(n+1)(n+2)(n+3)(n+4)$

$$S_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$$

find $S_n = 1\cdot2 + 2\cdot3 + 3\cdot4 + \cdots + n(n+1)$

$-3S_n = 1\cdot2(0-3) + 2\cdot3(1-4) + 3\cdot4(2-5) + \cdots$
$n(n+1)((n-1)-n+2))$

$-3S_n = \underline{0\cdot1\cdot2} - \cancel{1\cdot2\cdot3}$
$+ \cancel{1\cdot2\cdot3} - \cancel{2\cdot3\cdot4} + \cancel{2\cdot3\cdot4} - 3\cdot4\cdot5$
$+ \cdots \cancel{(n-1)n(n+1)} \underline{- n(n+1)(n+2)}$

$-3S_n = 0\cdot1\cdot2 - n(n+1)(n+2)$

$$S_n = \frac{n(n+1)(n+2)}{3}$$