A Differential Equation Model and its Numerical Method for Hysteresis Characteristics in Magneto-Rheological Dampers

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Complete List of Authors:	Wang, Fan; Nanjing University of Science and Technology,
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A Differential Equation Model and its Numerical Method for Hysteresis Characteristics in Magneto-Rheological Dampers

Fan Wang[1]

Abstract

Magneto-Rheological (MR) dampers have hysteresis characteristics which are crucial to their use and therefore a reasonable model is required to capture them. In the current paper, a differential equation model for MR dampers is proposed to describe the hysteresis characteristics and loading-rate dependence of MR dampers. The proposed model consists of differential and polynomial terms, and simulation results show that the model can describe the characteristics of MR dampers well. The model is further extended by higher-order differential and polynomials, both of which also obtain good results. The model has also been used to model ferromagnetic materials, showing its generality for hysteresis characteristics. For the proposed differential equation model, a numerical method incorporating Newmark method and Newton's method is also proposed, which has high efficiency and accuracy.

Keywords

Differential Equation, Numerical Method, Hysteresis Characteristics, Magneto-Rheological Damper

Introduction

Magneto-Rheological (MR) fluids are colloidal suspensions formed by dispersing small magnetic particles into an insulating carrier fluid. Magnetic particles can be magnetised to form chain-like structures under an applied magnetic field, and the properties of the suspension can be controlled by adjusting the magnetic field, giving the MR fluid controlled rheological properties [1]. This change in the MR fluid is reversible and takes only a few milliseconds and consumes very little energy [2]. The unique properties of MR fluids have attracted a great deal of interest from researchers and MR dampers with different configurations have been designed and investigated [3, 4, 5, 6]. For the simulation and control of equipment using MR dampers, a suitable MR damper model is very important [7, 8, 9].

It has been shown that MR dampers have hysteresis characteristics in their dynamics [10], since MR fluids exhibit hysteresis characteristics in the dynamic shear stress versus strain rate relationship, unlike other non-Newtonian fluids. As shown in Figure 1, the force versus velocity relationship of MR dampers has a hysteresis characteristic, the force versus velocity relationship is different during the loading and unloading process of the MR damper.

The loading process is defined as the change in velocity from maximum positive velocity to 0 and then to maximum negative velocity, which is the red line in Figure 1. The unloading process is defined as the change in velocity from maximum negative velocity to 0 and then to maximum positive velocity, which is the blue line in Figure 1. Furthermore, this hysteresis characteristics is related to the loading-rate [11, 12], the force versus velocity relationship of the same MR damper varies when the loading-rate varies, as shown in Figure 2, which is often referred to as “loading-rate dependence”. These characteristics of MR dampers make the construction of the model more complex.

[1]School of Intelligent Manufacturing, Nanjing University of Science and Technology, Jiangyin 214443, People’s Republic of China

Corresponding author:
Fan Wang, Nanjing University of Science and Technology, People’s Republic of China
Email: wangfan42@njust.edu.cn

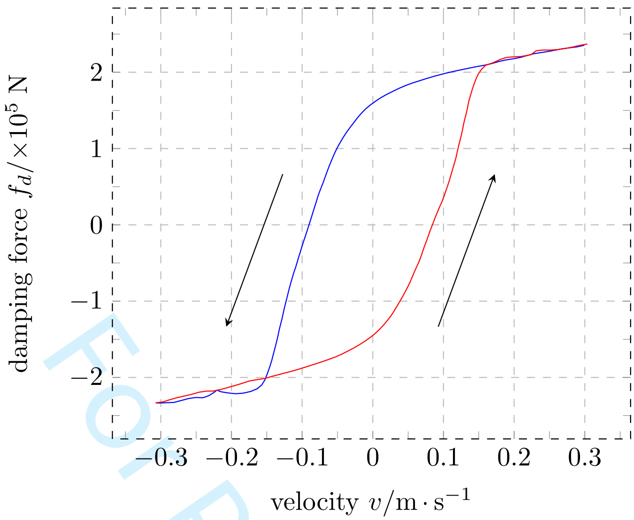


Figure 1. Hysteresis characteristics of magneto-rheological dampers (2 Hz).

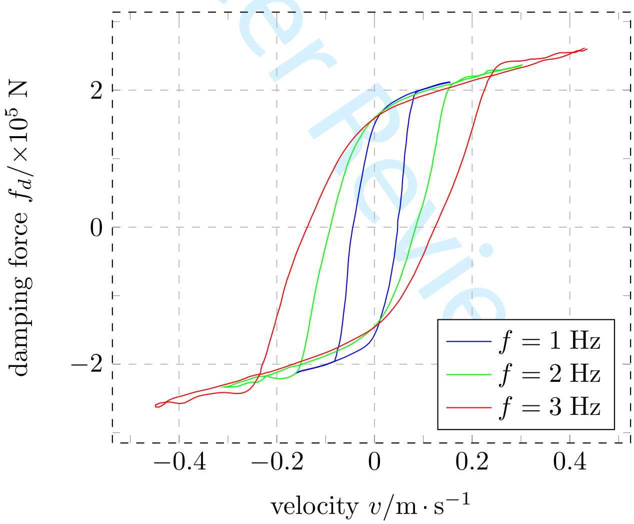


Figure 2. Loading-rate dependence of magneto-rheological damping.

For a single loading or unloading curve in Figure 1, it is natural to think of fitting it with some function of similar shape, or even with a polynomial. This is because Taylor's formula tells us that any function can be approximated by a polynomial. For example, we can fit a single curve with the following polynomial of the seventh-order polynomial:

$$f_d = c_7 v^7 + c_6 v^6 + c_5 v^5 + c_4 v^4 + 3c_v^3 + c_2 v^2 + c_1 v + c_0 \quad (1)$$

where f_d is the damping force of the MR damper, v is the velocity of the MR damper, and c_0, c_1, c_2, c_3,c_4, c_5, c_6, and c_7 are model parameters that need to be fitted with experimental data to determine. The results of the fit are shown in Figure 3, where the solid line represents the experimental results and the dashed line represents the fitting results. It is worth noting that a set of

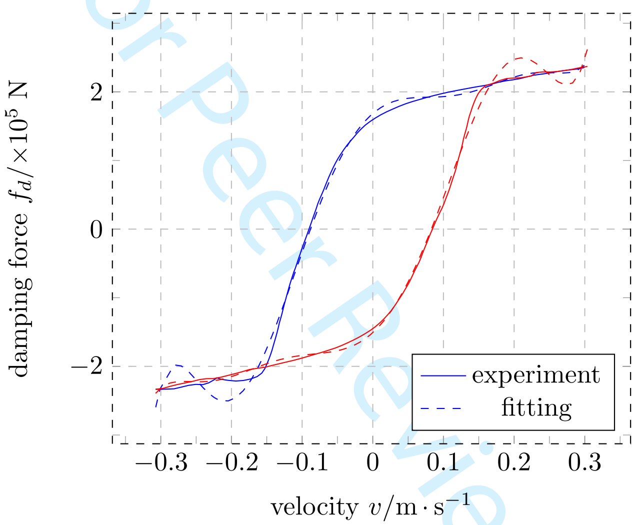


Figure 3. Polynomial fitting method for the hysteresis characteristics (2 Hz).

parameters can only characterise the loading or unloading process at a certain frequency. This means that two sets of parameters are required for the hysteresis loop and more when the loading-rate dependence needs to be characterised.

In the current paper, in response to the shortcomings of the polynomial model, differential terms are introduced into the model, resulting in a differential equation model. This model is used to describe the hysteresis characteristics and loading-rate dependence. Higher-order forms of the model have been proposed and compared. This model has also been used to describe the hysteresis characteristics of other materials, such as ferromagnetic materials. Further, a method for solving differential equations with high efficiency and accuracy is also proposed.

Differential equation model

To solve the problem that the polynomial approach cannot characterise the hysteresis characteristics and loading-rate dependence, in the current paper, differential terms are introduced into the polynomial to form a differential equation model. The proposed model is as follows:

$$\tau \dot{f}_d + k_1 f_d + k_3 f_d^3 + k_5 f_d^5 = v\left(t\right) \tag{2}$$

where f_d is the damping force of the MR damper, v is the velocity of the MR damper, and τ, k_1, k_3, and k_5 are the model parameters that need to be fitted with experimental data to determine. In the differential equation model, the damping force of the MR damper can be obtained by solving the velocity-dependent ordinary differential equation.

For the hysteresis characteristics, the derivative of damping force with respect to time $\dot{f}_d = \frac{\mathrm{d}f_d}{\mathrm{d}t}$ is introduced into the model. The hysteresis characteristic in Figure 1 can also be described in this way, where the velocity of the loading process is greater than that of the unloading process for the same damping force. This can be explained when the differential term is introduced. For the loading process, $\dot{f}_d$ is positive, so the velocity is greater; for the unloading process, $\dot{f}_d$ is negative, so the velocity is smaller. And for the polynomial term, only the odd-order terms are used, as the loop is roughly symmetric about the origin, so the even-order terms are ignored.

In the current paper, three sets of experimental data [13] are used to verify the capabilities of the model. The experiments were conducted on a large-scale MR damper filled with approximately 19 L of MRF-132DG-type MR fluid manufactured by Lord Corporation. A current of 2.5 A was applied to the electromagnetic coil to make MR fluid work. A hydraulic actuator was used to drive the damper with an amplitude of 25.4 mm and frequencies of 1 Hz, 2 Hz and 3 Hz. The experimental data were given in terms of the damping force versus velocity relationship, while the offset of the damping force is not taken into account.

First, experimental data with driving frequency 2 Hz are used to estimate the model parameters. Here a numerical algorithm based on least square estimation is used for parameter estimation, and the parameters are estimated as follows:

$$\tau = 0.000800039390738, \quad k_1 = 0.012142527318357,$$
$$k_3 = -0.017832162262598, \quad k_5 = 0.007840910242498$$

A comparison of the experimental and simulation results is shown in Figure 4. It can be seen that the model captures the hysteresis

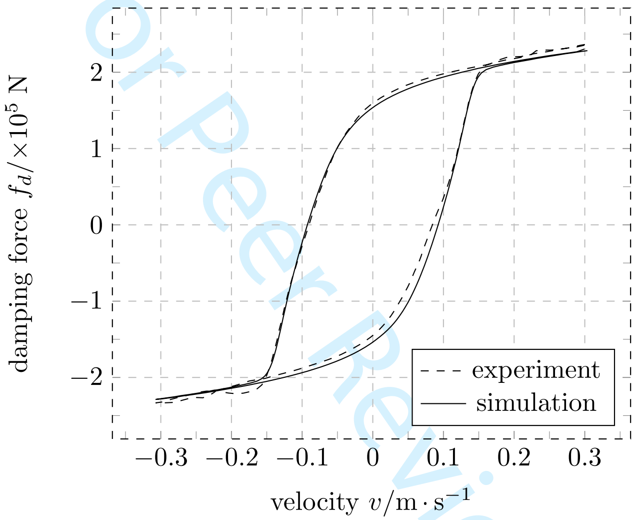


Figure 4. Comparison of experiment and simulation results (2 Hz).

characteristics very well.

The estimated model parameters are then used to simulate the hysteresis characteristics of the MR damper with driving frequencies of 1 Hz and 3 Hz. The simulated hysteresis loops are compared with their experimental counterparts, as shown in Figure 5. It can be seen that the model also captures the loading-rate dependence very well.

The above comparisons illustrate that the hysteresis characteristics of the same MR damper at three different loading rates are modelled by a single differential equation with the same set of model parameters. It is worth noting that the model parameters are estimated by fitting the model to the experimental data at the

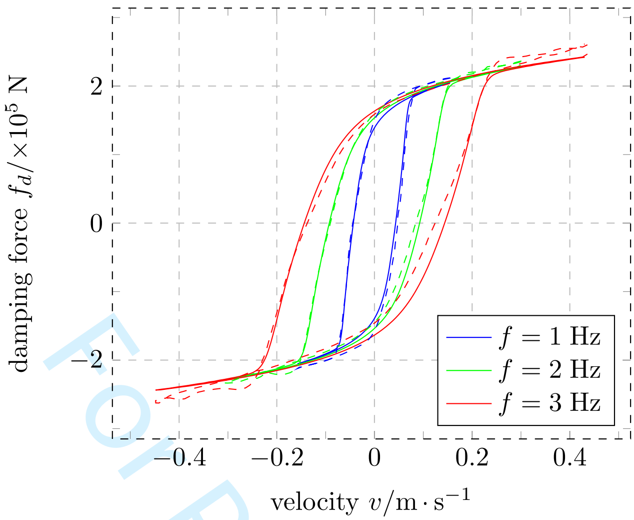


Figure 5. Comparison of the experiment (dashed lines) and simulation (solid lines) results for different loading rates.

frequency of 2 Hz, while the hysteresis loops at the frequencies of 1 Hz and, 3 Hz are predicted by the model. Based on this comparison, one can conclude that the differential equation model given by Equation (2) is capable of modelling the loading-rate dependence of the hysteresis characteristics of MR dampers.

Although only harmonic loadings are tested in the current analysis, the model has no restrictions on the other loading profiles. As the governing equations for the dynamics are formulated as differential equations, which are time-domain responses in nature, it is reasonable to believe that the model is also capable of capturing the non-harmonic loading-rate dependent characteristics of the MR dampers.

Extensions of the model

For the model of Equation (2), if the differential term is removed, leaving only the polynomial, the model degenerates to the polynomial fitting model in Figure 3. As mentioned earlier, polynomial fitting requires a large number of parameters. However, for the model proposed in the current paper, only four parameters are required, which are sufficient to account for the hysteresis characteristics and loading-rate dependence.

For the model of Equation (2), only first-order differential is used. If the second-order differential is also used, the model becomes:

$$\rho \ddot{f}_d + \tau \dot{f}_d + k_1 f_d + k_3 f_d^3 + k_5 f_d^5 = v\,(t) \tag{3}$$

The first-order differential and second-order differential models are compared using 2 Hz as an example, as shown in Figure 6. It can be

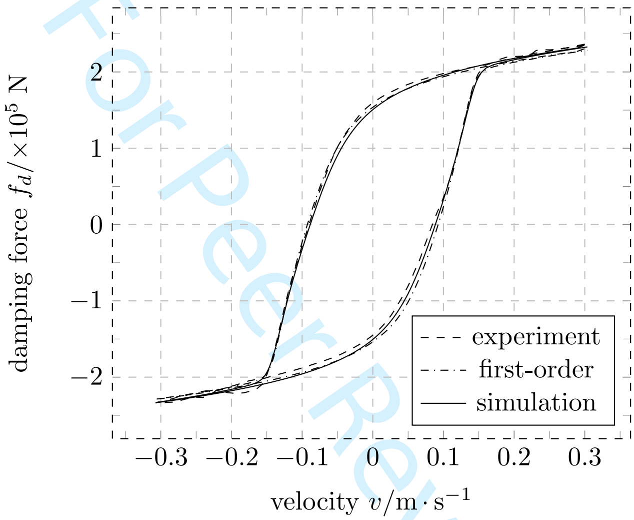


Figure 6. Comparison of experimental and simulation results with first-order differential and second-order differential (2 Hz).

seen that the simulation gives better results, but this comes at the cost of increased computational effort. The parameters are estimated as follows:

$$\rho = 0.000000179537831, \quad \tau = 0.000830121316692,$$
$$k_1 = 0.021578750337282, \quad k_3 = -0.017788493061992$$
$$k_5 = 0.006968520533849$$

The comparison of the parameters also shows that the coefficient before the second-order differential ρ is very small and the other parameters are similar to when only the first-order differential is used. This suggests that the characteristics of the MR damper are well described using only first-order differential and that higher-order differential would increase the computational effort without giving a more significant improvement in the results.

Similarly, the polynomial term can also be improved the by using a seventh-order polynomial:

$$\tau \dot{f}_d + k_1 f_d + k_3 f_d^3 + k_5 f_d^5 + k_7 f_d^7 = v(t) \quad (4)$$

Figure 7 shows the experimental and simulation results with fifth-order polynomial and seventh-order polynomial. Similar to the use

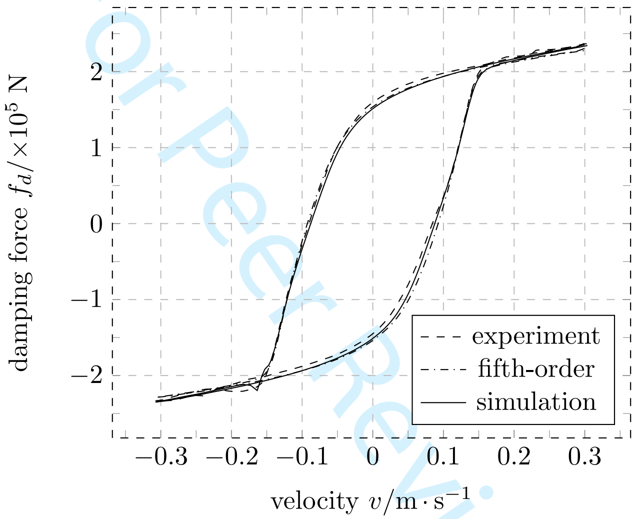


Figure 7. Comparison of experimental and simulation results with fifth-order polynomial and seventh-order polynomial (2 Hz).

of second-order differential, the simulation is improved, but the computational effort increases. The parameters are estimated as follows:

$$\tau = 0.000831927839105, \quad k_1 = 0.037931179389992,$$
$$k_3 = -0.042026263353530, \quad k_5 = 0.016508959848687,$$
$$k_7 = -0.001064492444477$$

The comparison of the parameters also shows that the coefficient before the seventh-order polynomial k_7 is very small, while the other parameters are similar to those when only the fifth-order polynomial is used. This suggests that the MR damper is already well characterised using only the fifth-order polynomial and that higher-order polynomials increase the computational effort without providing a more significant improvement in results.

For the proposed model, it is also used for the hysteresis characteristics of ferromagnetic materials. Experiment data [14] are used to verify the capability of the model. ZDKH electrical steel with a thickness of 0.23 mm and a width of 4 mm was measured in a Single Sheet Tester. The material was measured at the different frequencies of 50 Hz, 500 Hz, and 1000 Hz. The simulation results are again compared with its experiment counterpart in Figure 8. It can be

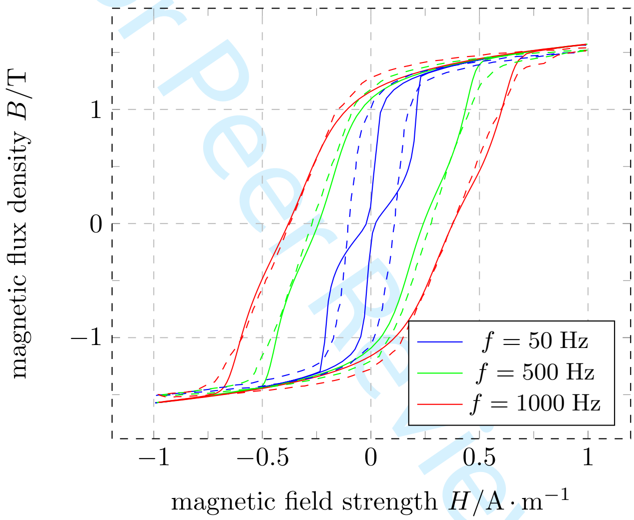


Figure 8. Application of models in ferromagnetic materials.

seen that the simulation results agree very well with the experiment results. This shows that the model can also capture hysteresis loops of ferromagnetic materials and also illustrates the generality of the model for modelling hysteresis characteristics in general.

Numerical method of the model

For the model proposed in the current paper, it is essentially an ordinary differential equation, while its numerical method is a key influence on the application of the model, requiring a balance between efficiency and accuracy. A numerical method for solving this ordinary differential equation model is proposed, which combines Newmark method and Newton's method, and allows the solution to be computationally efficient with high accuracy.

Newmark method assumes that $f_d(t+\Delta t)$ and $\dot{f}_d(t+\Delta t)$ at the moment $t+\Delta t$ can be expressed by $f_d(t)$, $\dot{f}_d(t)$ and $\ddot{f}_d(t)$ at the moment t and $\ddot{f}_d(t+\Delta t)$ at the moment $t+\Delta t$:

$$\dot{f}_d(t+\Delta t) = \dot{f}_d(t) + \left[(1-\beta)\,\ddot{f}_d(t) + \beta\ddot{f}_d(t+\Delta t)\right]\Delta t \tag{5a}$$

$$f_d(t+\Delta t) = f_d(t) + \dot{f}_d(t)\,\Delta t + \left[\left(\frac{1}{2}-\gamma\right)\ddot{f}_d(t) + \gamma\ddot{f}_d(t+\Delta t)\right]\Delta t^2 \tag{5b}$$

where β and γ are parameters that are adjusted according to the accuracy and stability requirements. Through the derivation, $\dot{f}_d(t+\Delta t)$ and $\ddot{f}_d(t+\Delta t)$ at the moment $t+\Delta t$ can be expressed using $f_d(t+\Delta t)$ at moment $t+\Delta t$, and $f_d(t)$, $\dot{f}_d(t)$ and $\ddot{f}_d(t)$ at moment t:

$$\ddot{f}_d(t+\Delta t) = \frac{1}{\gamma\Delta t^2}\left[f_d(t+\Delta t) - f_d(t)\right] - \frac{1}{\gamma\Delta t}\dot{f}_d(t) - \left(\frac{1}{2\gamma}-1\right)\ddot{f}_d(t) \tag{6a}$$

$$\dot{f}_d(t+\Delta t) = \frac{\beta}{\gamma\Delta t}\left[f_d(t+\Delta t) - f_d(t)\right] - \left(\frac{\beta}{\gamma}-1\right)\dot{f}_d(t) - \Delta t\left(\frac{\beta}{2\gamma}-1\right)\ddot{f}_d(t) \tag{6b}$$

Consider the differential equation at the moment $t+\Delta t$:

$$\rho\ddot{f}_d(t+\Delta t) + \tau\dot{f}_d(t+\Delta t) + k_1 f_d(t+\Delta t) + k_3 f_d^3(t+\Delta t) + k_5 f_d^5(t+\Delta t) = v(t+\Delta t) \tag{7}$$

Substitute Equation (6a) and (6b) into Equation (7),

$$k f_d(t+\Delta t) + k_3 f_d^3(t+\Delta t) + k_5 f_d^5(t+\Delta t) = p \tag{8}$$

where

$$k = \frac{1}{\gamma \Delta t^2}\rho + \frac{\beta}{\gamma \Delta t}\tau + k_1$$
$$p = \frac{1}{\gamma \Delta t}\left(\frac{1}{\Delta t}\rho + \beta\tau\right) f_d(t) + \left[\frac{1}{\gamma \Delta t}\rho + \left(\frac{\beta}{\gamma} - 1\right)\tau\right]\dot{f}_d(t)$$
$$+ \left[\left(\frac{1}{2\gamma} - 1\right)\rho + \Delta t\left(\frac{\beta}{2\gamma} - 1\right)\tau\right]\ddot{f}_d(t) + v(t + \Delta t)$$

Equation (8) is a nonlinear equation,

$$g(f_d) = k f_d + k_3 f_d^3 + k_5 f_d^5 - p \tag{9}$$

which can be solved by Newton's method

$$f_d^{(k+1)} = f_d^{(k)} - \frac{g\left[f_d^{(k)}\right]}{g'\left[f_d^{(k)}\right]} \tag{10}$$

where $k = 0, 1, 2, \cdots$ is the number of iterations.

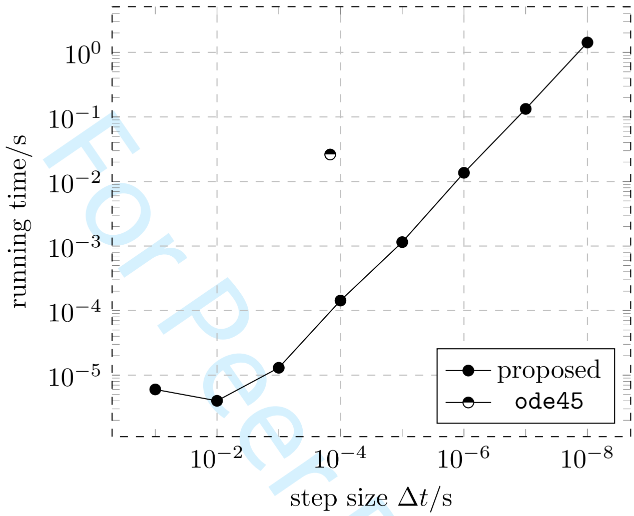
For the solution at the moment $t + \Delta t$, it is natural to think that the solution at time t can be used as the initial value of Newton's method. In addition, when Δt is suitable, since $t + \Delta t$ and t are very close, an accurate solution can even be obtained by only one calculation, that is, it does not need to be solved with multiple iterations:

$$f_d(t + \Delta t) = f_d(t) - \frac{g[f_d(t)]}{g'[f_d(t)]} \tag{11}$$

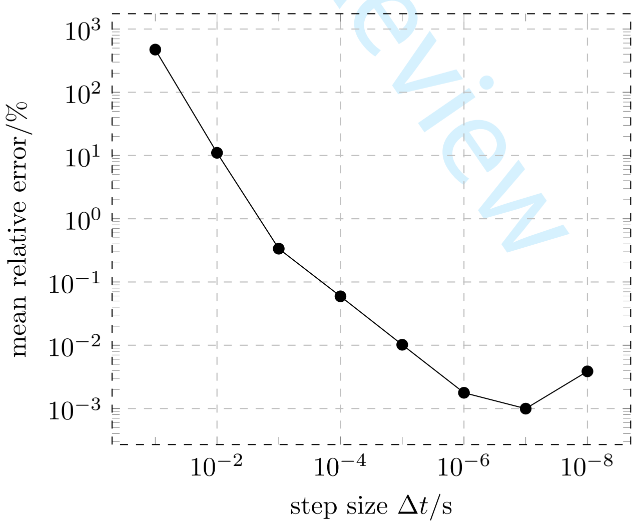
Thus $f_d(t + \Delta t)$ can be obtained, and then $\ddot{f}_d(t + \Delta t)$ and $\dot{f}_d(t + \Delta t)$ can be obtained using Equation (6a) and (5a).

Especially, when $\beta = \frac{1}{2}$ and $\gamma = \frac{1}{4}$, it is the constant average acceleration method, that is, assuming that the acceleration from time t to time $t + \Delta t$ is constant, so the velocity changes linearly. And when $\beta = \frac{1}{2}$ and $\gamma = \frac{1}{6}$, it is the linear acceleration method, that is, assuming that the acceleration changes linearly from time t to time $t + \Delta t$.

The situation when $f = 2$ Hz is taken as an example to discuss the stability, accuracy, and running time of the proposed numerical method. In the current paper, the result obtained by the `ode45` function in MATLAB is compared with the proposed method. `ode45` function is a numerical method of ordinary differential equations with adaptive step length (variable step length), and its overall truncation error is Δx^5.



(a) running time



(b) accuracy

Figure 9. Efficiency and accuracy of the proposed method.

Figure 9a shows the running time of the proposed method under different step size Δt, where the running time is operating under the same condition. The running time of the `ode45` function used for comparison is also marked in the figure. It can be seen that the running time is proportional to the number of calculation cycles, and the proposed method has a shorter running time compared with the Runge–Kutta method. Figure 9b shows the mean relative error of the results calculated by the proposed method under different step size Δt. It can be seen that the proposed method has better stability and can still work normally at a smaller step size, while the Runge–Kutta method does not converge under a very small step size, because it is an explicit method.

Explicit methods do not require iteration in each time step, so the amount of computation in each time step is smaller than that of implicit methods, but the step size should not be too large, otherwise it will easily lead to scattered results, that is, the stability of the explicit methods is not good. Implicit methods require iteration within each time step, which is relatively more computationally intensive, but its convergence speed and stability are better, and in general a larger step size can be chosen than explicit methods, and the computation results will not be scattered. The proposed method combines the advantages of the two methods mentioned above, it is essentially an implicit method and is therefore stable and gives a good solution even with large step sizes, however unlike other implicit methods, it does not require iteration and is therefore more computationally efficient. Taken together, the proposed method has the advantage of being used for the solution of the proposed model with high efficiency and accuracy.

Conclusions

In the current paper, a differential equation model for MR dampers is proposed to describe the hysteresis characteristics and loading-rate dependence of MR dampers. The proposed model consists of differential and polynomial terms, and simulation results show that the model can describe the characteristics of MR dampers well. The model is further extended by higher-order differential and polynomials, both of which also obtain good results. The model has also been used to model ferromagnetic materials, showing its generality for hysteresis characteristics. A method for solving differential equations is also proposed, which has the advantages of both the explicit and implicit methods, that is, less computational effort and better stability. The proposed differential equation model has a simpler form and higher accuracy and can be used for modelling MR dampers or other materials with hysteresis characteristics.

Declaration of conflicting interests

The authors declare no conflict of interest in preparing this article.

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Differential Equation, Numerical Method, Hysteresis Characteristics, Magneto-Rheological Damper

Introduction

Magneto-Rheological (MR) fluids are colloidal suspensions formed by dispersing small magnetic particles into an insulating carrier fluid. Magnetic particles can be magnetised to form chain-like structures under an applied magnetic field, and the properties of the suspension can be controlled by adjusting the magnetic field, giving the MR fluid controlled rheological properties [1]. This change in the MR fluid is reversible and takes only a few milliseconds and consumes very little energy [2]. The unique properties of MR fluids have attracted a great deal of interest from researchers and MR dampers with different configurations have been designed and investigated [3, 4, 5, 6]. For the simulation and control of equipment using MR dampers, a suitable MR damper model is very important [7, 8, 9].

It has been shown that MR dampers have hysteresis characteristics in their dynamics [10], since MR fluids exhibit hysteresis characteristics in the dynamic shear stress versus strain rate relationship, unlike other non-Newtonian fluids. As shown in Figure 1, the force versus velocity relationship of MR dampers has a hysteresis characteristic, the force versus velocity relationship is different during the loading and unloading process of the MR damper.

The loading process is defined as the change in velocity from maximum positive velocity to 0 and then to maximum negative velocity, which is the red line in Figure 1. The unloading process is defined as the change in velocity from maximum negative velocity to 0 and then to maximum positive velocity, which is the blue line in Figure 1. Furthermore, this hysteresis characteristics is related to the loading-rate [11, 12], the force versus velocity relationship of the same MR damper varies when the loading-rate varies, as shown in Figure 2, which is often referred to as “loading-rate dependence”. These characteristics of MR dampers make the construction of the model more complex.

[1]School of Intelligent Manufacturing, Nanjing University of Science and Technology, Jiangyin 214443, People’s Republic of China

Corresponding author:
Fan Wang, Nanjing University of Science and Technology, People’s Republic of China
Email: wangfan42@njust.edu.cn

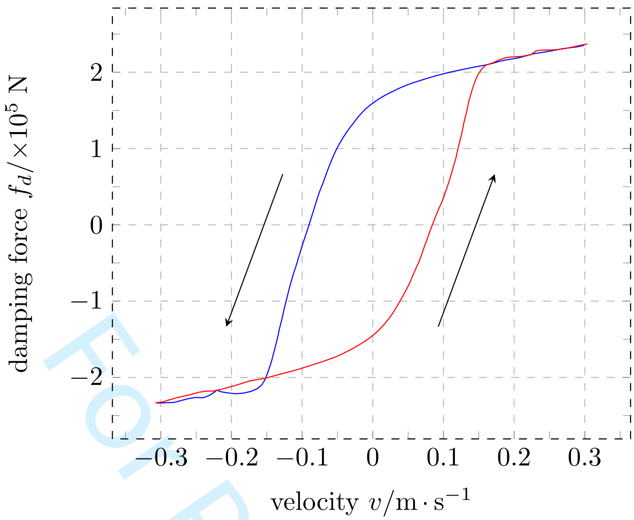


Figure 1. Hysteresis characteristics of magneto-rheological dampers (2 Hz).

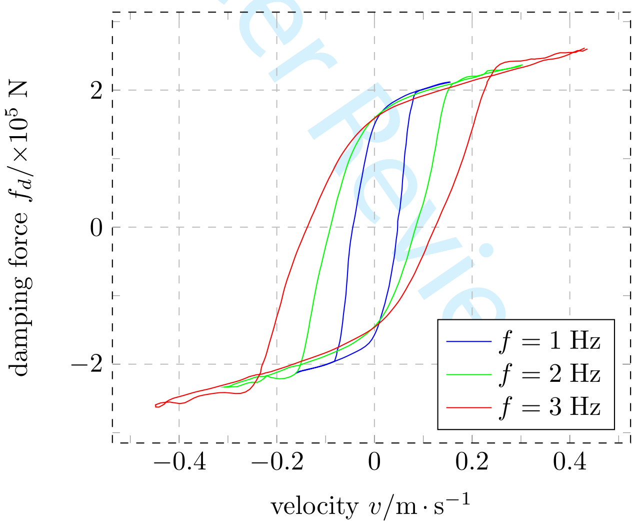


Figure 2. Loading-rate dependence of magneto-rheological damping.

For a single loading or unloading curve in Figure 1, it is natural to think of fitting it with some function of similar shape, or even with a polynomial. This is because Taylor's formula tells us that any function can be approximated by a polynomial. For example, we can fit a single curve with the following polynomial of the seventh-order polynomial:

$$f_d = c_7 v^7 + c_6 v^6 + c_5 v^5 + c_4 v^4 + 3c_v^3 + c_2 v^2 + c_1 v + c_0 \qquad (1)$$

where f_d is the damping force of the MR damper, v is the velocity of the MR damper, and c_0, c_1, c_2, c_3,c_4, c_5, c_6, and c_7 are model parameters that need to be fitted with experimental data to determine. The results of the fit are shown in Figure 3, where the solid line represents the experimental results and the dashed line represents the fitting results. It is worth noting that a set of

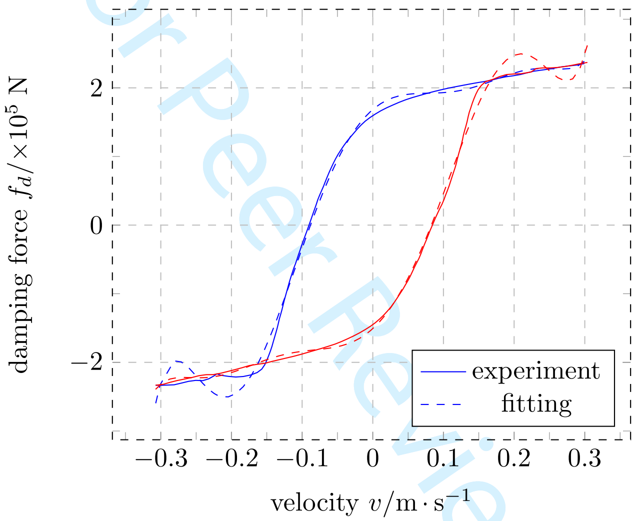


Figure 3. Polynomial fitting method for the hysteresis characteristics (2 Hz).

parameters can only characterise the loading or unloading process at a certain frequency. This means that two sets of parameters are required for the hysteresis loop and more when the loading-rate dependence needs to be characterised.

In the current paper, in response to the shortcomings of the polynomial model, differential terms are introduced into the model, resulting in a differential equation model. This model is used to describe the hysteresis characteristics and loading-rate dependence. Higher-order forms of the model have been proposed and compared. This model has also been used to describe the hysteresis characteristics of other materials, such as ferromagnetic materials. Further, a method for solving differential equations with high efficiency and accuracy is also proposed.

Differential equation model

To solve the problem that the polynomial approach cannot characterise the hysteresis characteristics and loading-rate dependence, in the current paper, differential terms are introduced into the polynomial to form a differential equation model. The proposed model is as follows:

$$\tau \dot{f}_d + k_1 f_d + k_3 f_d^3 + k_5 f_d^5 = v(t) \tag{2}$$

where f_d is the damping force of the MR damper, v is the velocity of the MR damper, and τ, k_1, k_3, and k_5 are the model parameters that need to be fitted with experimental data to determine. In the differential equation model, the damping force of the MR damper can be obtained by solving the velocity-dependent ordinary differential equation.

For the hysteresis characteristics, the derivative of damping force with respect to time $\dot{f}_d = \frac{\mathrm{d}f_d}{\mathrm{d}t}$ is introduced into the model. The hysteresis characteristic in Figure 1 can also be described in this way, where the velocity of the loading process is greater than that of the unloading process for the same damping force. This can be explained when the differential term is introduced. For the loading process, $\dot{f}_d$ is positive, so the velocity is greater; for the unloading process, $\dot{f}_d$ is negative, so the velocity is smaller. And for the polynomial term, only the odd-order terms are used, as the loop is roughly symmetric about the origin, so the even-order terms are ignored.

In the current paper, three sets of experimental data [13] are used to verify the capabilities of the model. The experiments were conducted on a large-scale MR damper filled with approximately 19 L of MRF-132DG-type MR fluid manufactured by Lord Corporation. A current of 2.5 A was applied to the electromagnetic coil to make MR fluid work. A hydraulic actuator was used to drive the damper with an amplitude of 25.4 mm and frequencies of 1 Hz, 2 Hz and 3 Hz. The experimental data were given in terms of the damping force versus velocity relationship, while the offset of the damping force is not taken into account.

First, experimental data with driving frequency 2 Hz are used to estimate the model parameters. Here a numerical algorithm based on least square estimation is used for parameter estimation, and the parameters are estimated as follows:

$$\tau = 0.000800039390738, \quad k_1 = 0.012142527318357,$$
$$k_3 = -0.017832162262598, \quad k_5 = 0.007840910242498$$

A comparison of the experimental and simulation results is shown in Figure 4. It can be seen that the model captures the hysteresis

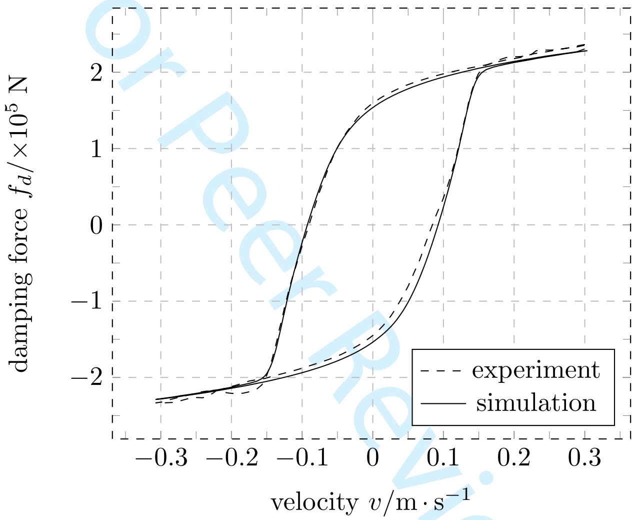


Figure 4. Comparison of experiment and simulation results (2 Hz).

characteristics very well.

The estimated model parameters are then used to simulate the hysteresis characteristics of the MR damper with driving frequencies of 1 Hz and 3 Hz. The simulated hysteresis loops are compared with their experimental counterparts, as shown in Figure 5. It can be seen that the model also captures the loading-rate dependence very well.

The above comparisons illustrate that the hysteresis characteristics of the same MR damper at three different loading rates are modelled by a single differential equation with the same set of model parameters. It is worth noting that the model parameters are estimated by fitting the model to the experimental data at the

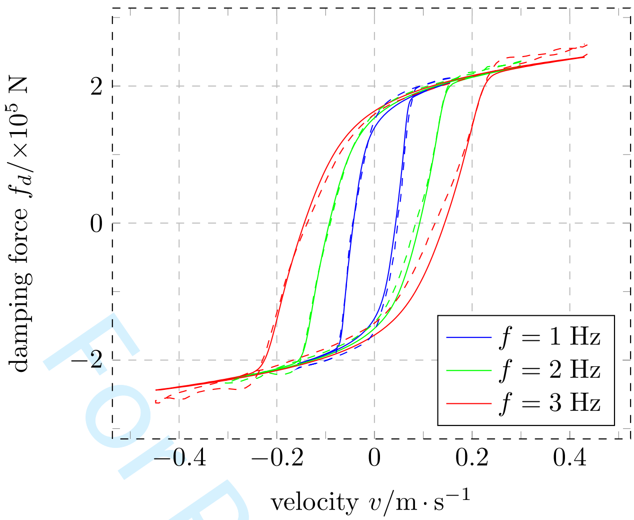


Figure 5. Comparison of the experiment (dashed lines) and simulation (solid lines) results for different loading rates.

frequency of 2 Hz, while the hysteresis loops at the frequencies of 1 Hz and, 3 Hz are predicted by the model. Based on this comparison, one can conclude that the differential equation model given by Equation (2) is capable of modelling the loading-rate dependence of the hysteresis characteristics of MR dampers.

Although only harmonic loadings are tested in the current analysis, the model has no restrictions on the other loading profiles. As the governing equations for the dynamics are formulated as differential equations, which are time-domain responses in nature, it is reasonable to believe that the model is also capable of capturing the non-harmonic loading-rate dependent characteristics of the MR dampers.

Extensions of the model

For the model of Equation (2), if the differential term is removed, leaving only the polynomial, the model degenerates to the polynomial fitting model in Figure 3. As mentioned earlier, polynomial fitting requires a large number of parameters. However, for the model proposed in the current paper, only four parameters are required, which are sufficient to account for the hysteresis characteristics and loading-rate dependence.

For the model of Equation (2), only first-order differential is used. If the second-order differential is also used, the model becomes:

$$\rho \ddot{f}_d + \tau \dot{f}_d + k_1 f_d + k_3 f_d^3 + k_5 f_d^5 = v\,(t) \tag{3}$$

The first-order differential and second-order differential models are compared using 2 Hz as an example, as shown in Figure 6. It can be

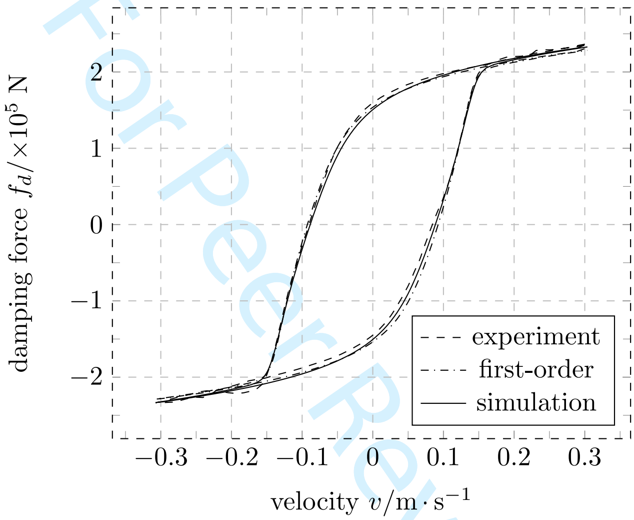


Figure 6. Comparison of experimental and simulation results with first-order differential and second-order differential (2 Hz).

seen that the simulation gives better results, but this comes at the cost of increased computational effort. The parameters are estimated as follows:

$$\rho = 0.000000179537831, \quad \tau = 0.000830121316692,$$
$$k_1 = 0.021578750337282, \quad k_3 = -0.017788493061992$$
$$k_5 = 0.006968520533849$$

The comparison of the parameters also shows that the coefficient before the second-order differential ρ is very small and the other parameters are similar to when only the first-order differential is used. This suggests that the characteristics of the MR damper are well described using only first-order differential and that higher-order differential would increase the computational effort without giving a more significant improvement in the results.

Similarly, the polynomial term can also be improved the by using a seventh-order polynomial:

$$\tau \dot{f}_d + k_1 f_d + k_3 f_d^3 + k_5 f_d^5 + k_7 f_d^7 = v\,(t) \tag{4}$$

Figure 7 shows the experimental and simulation results with fifth-order polynomial and seventh-order polynomial. Similar to the use

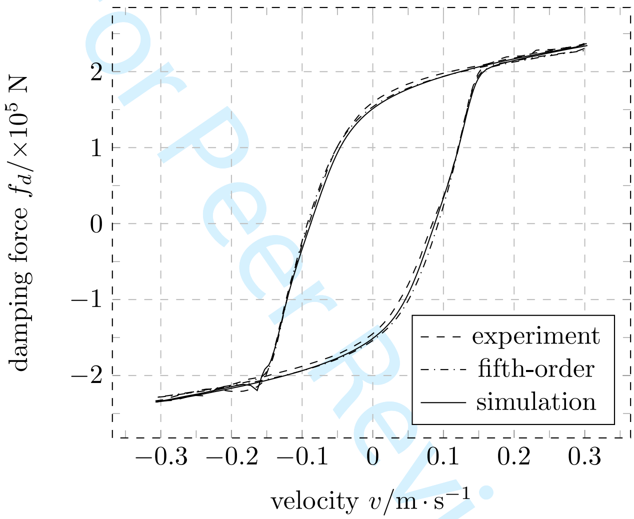


Figure 7. Comparison of experimental and simulation results with fifth-order polynomial and seventh-order polynomial (2 Hz).

of second-order differential, the simulation is improved, but the computational effort increases. The parameters are estimated as follows:

$$\tau = 0.000831927839105, \quad k_1 = 0.037931179389992,$$
$$k_3 = -0.042026263353530, \quad k_5 = 0.016508959848687,$$
$$k_7 = -0.001064492444477$$

The comparison of the parameters also shows that the coefficient before the seventh-order polynomial k_7 is very small, while the other parameters are similar to those when only the fifth-order polynomial is used. This suggests that the MR damper is already well characterised using only the fifth-order polynomial and that higher-order polynomials increase the computational effort without providing a more significant improvement in results.

For the proposed model, it is also used for the hysteresis characteristics of ferromagnetic materials. Experiment data [14] are used to verify the capability of the model. ZDKH electrical steel with a thickness of 0.23 mm and a width of 4 mm was measured in a Single Sheet Tester. The material was measured at the different frequencies of 50 Hz, 500 Hz, and 1000 Hz. The simulation results are again compared with its experiment counterpart in Figure 8. It can be

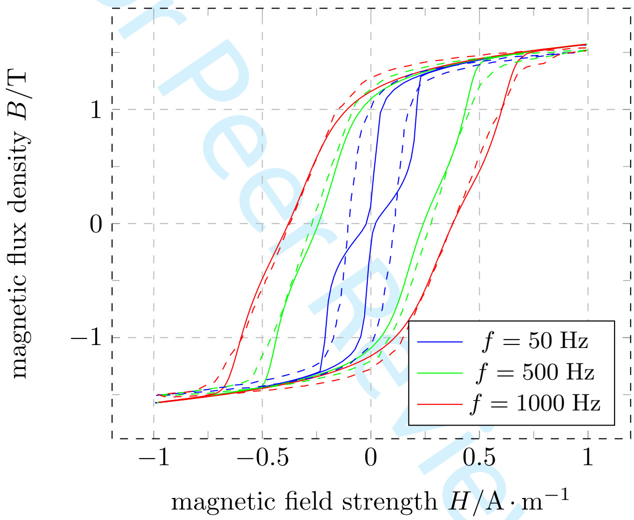


Figure 8. Application of models in ferromagnetic materials.

seen that the simulation results agree very well with the experiment results. This shows that the model can also capture hysteresis loops of ferromagnetic materials and also illustrates the generality of the model for modelling hysteresis characteristics in general.

Numerical method of the model

For the model proposed in the current paper, it is essentially an ordinary differential equation, while its numerical method is a key influence on the application of the model, requiring a balance between efficiency and accuracy. A numerical method for solving this ordinary differential equation model is proposed, which combines Newmark method and Newton's method, and allows the solution to be computationally efficient with high accuracy.

Newmark method assumes that $f_d(t+\Delta t)$ and $\dot{f}_d(t+\Delta t)$ at the moment $t+\Delta t$ can be expressed by $f_d(t)$, $\dot{f}_d(t)$ and $\ddot{f}_d(t)$ at the moment t and $\ddot{f}_d(t+\Delta t)$ at the moment $t+\Delta t$:

$$\dot{f}_d(t+\Delta t) = \dot{f}_d(t) + \left[(1-\beta)\ddot{f}_d(t) + \beta\ddot{f}_d(t+\Delta t)\right]\Delta t \tag{5a}$$

$$f_d(t+\Delta t) = f_d(t) + \dot{f}_d(t)\Delta t + \left[\left(\frac{1}{2}-\gamma\right)\ddot{f}_d(t) + \gamma\ddot{f}_d(t+\Delta t)\right]\Delta t^2 \tag{5b}$$

where β and γ are parameters that are adjusted according to the accuracy and stability requirements. Through the derivation, $\dot{f}_d(t+\Delta t)$ and $\ddot{f}_d(t+\Delta t)$ at the moment $t+\Delta t$ can be expressed using $f_d(t+\Delta t)$ at moment $t+\Delta t$, and $f_d(t)$, $\dot{f}_d(t)$ and $\ddot{f}_d(t)$ at moment t:

$$\ddot{f}_d(t+\Delta t) = \frac{1}{\gamma\Delta t^2}\left[f_d(t+\Delta t) - f_d(t)\right] - \frac{1}{\gamma\Delta t}\dot{f}_d(t) - \left(\frac{1}{2\gamma}-1\right)\ddot{f}_d(t) \tag{6a}$$

$$\begin{aligned}\dot{f}_d(t+\Delta t) = \frac{\beta}{\gamma\Delta t}\left[f_d(t+\Delta t) - f_d(t)\right] - \left(\frac{\beta}{\gamma}-1\right)\dot{f}_d(t) \\ - \Delta t\left(\frac{\beta}{2\gamma}-1\right)\ddot{f}_d(t)\end{aligned} \tag{6b}$$

Consider the differential equation at the moment $t+\Delta t$:

$$\begin{aligned}\rho\ddot{f}_d(t+\Delta t) + \tau\dot{f}_d(t+\Delta t) + k_1 f_d(t+\Delta t) \\ + k_3 f_d^3(t+\Delta t) + k_5 f_d^5(t+\Delta t) = v(t+\Delta t)\end{aligned} \tag{7}$$

Substitute Equation (6a) and (6b) into Equation (7),

$$k f_d(t+\Delta t) + k_3 f_d^3(t+\Delta t) + k_5 f_d^5(t+\Delta t) = p \tag{8}$$

where

$$
\begin{aligned}
k =& \frac{1}{\gamma \Delta t^2}\rho + \frac{\beta}{\gamma \Delta t}\tau + k_1 \\
p =& \frac{1}{\gamma \Delta t}\left(\frac{1}{\Delta t}\rho + \beta\tau\right) f_d\left(t\right) + \left[\frac{1}{\gamma \Delta t}\rho + \left(\frac{\beta}{\gamma} - 1\right)\tau\right] \dot{f}_d\left(t\right) \\
&+ \left[\left(\frac{1}{2\gamma} - 1\right)\rho + \Delta t\left(\frac{\beta}{2\gamma} - 1\right)\tau\right] \ddot{f}_d\left(t\right) + v\left(t + \Delta t\right)
\end{aligned}
$$

Equation (8) is a nonlinear equation,

$$g\left(f_d\right) = k f_d + k_3 f_d^3 + k_5 f_d^5 - p \tag{9}$$

which can be solved by Newton's method

$$f_d^{(k+1)} = f_d^{(k)} - \frac{g\left[f_d^{(k)}\right]}{g'\left[f_d^{(k)}\right]} \tag{10}$$

where $k = 0, 1, 2, \cdots$ is the number of iterations.

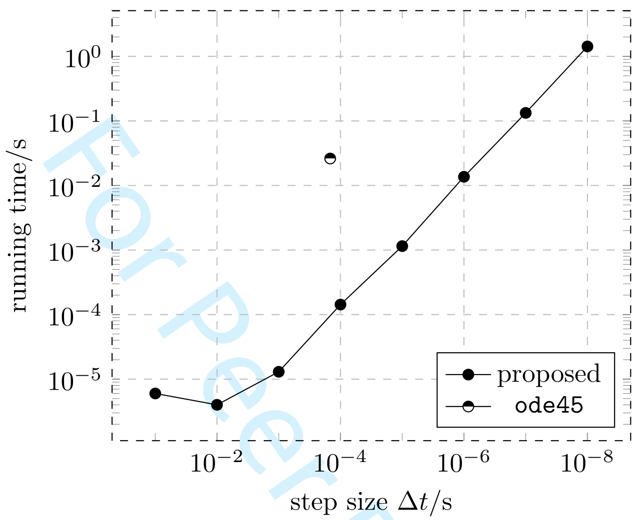
For the solution at the moment $t + \Delta t$, it is natural to think that the solution at time t can be used as the initial value of Newton's method. In addition, when Δt is suitable, since $t + \Delta t$ and t are very close, an accurate solution can even be obtained by only one calculation, that is, it does not need to be solved with multiple iterations:

$$f_d\left(t + \Delta t\right) = f_d\left(t\right) - \frac{g\left[f_d\left(t\right)\right]}{g'\left[f_d\left(t\right)\right]} \tag{11}$$

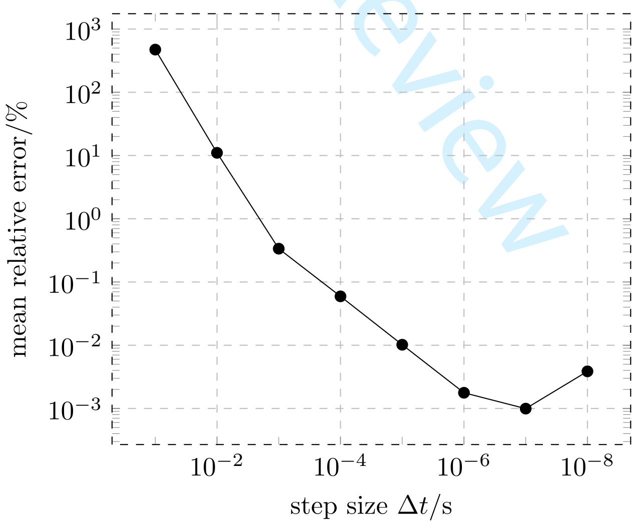
Thus $f_d\left(t + \Delta t\right)$ can be obtained, and then $\ddot{f}_d\left(t + \Delta t\right)$ and $\dot{f}_d\left(t + \Delta t\right)$ can be obtained using Equation (6a) and (5a).

Especially, when $\beta = \frac{1}{2}$ and $\gamma = \frac{1}{4}$, it is the constant average acceleration method, that is, assuming that the acceleration from time t to time $t + \Delta t$ is constant, so the velocity changes linearly. And when $\beta = \frac{1}{2}$ and $\gamma = \frac{1}{6}$, it is the linear acceleration method, that is, assuming that the acceleration changes linearly from time t to time $t + \Delta t$.

The situation when $f = 2$ Hz is taken as an example to discuss the stability, accuracy, and running time of the proposed numerical method. In the current paper, the result obtained by the `ode45` function in MATLAB is compared with the proposed method. `ode45` function is a numerical method of ordinary differential equations with adaptive step length (variable step length), and its overall truncation error is Δx^5.



(a) running time



(b) accuracy

Figure 9. Efficiency and accuracy of the proposed method.

Figure 9a shows the running time of the proposed method under different step size Δt, where the running time is operating under the same condition. The running time of the `ode45` function used for comparison is also marked in the figure. It can be seen that the running time is proportional to the number of calculation cycles, and the proposed method has a shorter running time compared with the Runge–Kutta method. Figure 9b shows the mean relative error of the results calculated by the proposed method under different step size Δt. It can be seen that the proposed method has better stability and can still work normally at a smaller step size, while the Runge–Kutta method does not converge under a very small step size, because it is an explicit method.

Explicit methods do not require iteration in each time step, so the amount of computation in each time step is smaller than that of implicit methods, but the step size should not be too large, otherwise it will easily lead to scattered results, that is, the stability of the explicit methods is not good. Implicit methods require iteration within each time step, which is relatively more computationally intensive, but its convergence speed and stability are better, and in general a larger step size can be chosen than explicit methods, and the computation results will not be scattered. The proposed method combines the advantages of the two methods mentioned above, it is essentially an implicit method and is therefore stable and gives a good solution even with large step sizes, however unlike other implicit methods, it does not require iteration and is therefore more computationally efficient. Taken together, the proposed method has the advantage of being used for the solution of the proposed model with high efficiency and accuracy.

Conclusions

In the current paper, a differential equation model for MR dampers is proposed to describe the hysteresis characteristics and loading-rate dependence of MR dampers. The proposed model consists of differential and polynomial terms, and simulation results show that the model can describe the characteristics of MR dampers well. The model is further extended by higher-order differential and polynomials, both of which also obtain good results. The model has also been used to model ferromagnetic materials, showing its generality for hysteresis characteristics. A method for solving differential equations is also proposed, which has the advantages of both the explicit and implicit methods, that is, less computational effort and better stability. The proposed differential equation model has a simpler form and higher accuracy and can be used for modelling MR dampers or other materials with hysteresis characteristics.

Declaration of conflicting interests

The authors declare no conflict of interest in preparing this article.

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