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# Lagrange Interpolation on Chebyshev Points of Two Variables

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We study interpolation polynomials based on the points in $[-1, 1] \times [-1, 1]$ that are common zeros of quasi-orthogonal Chebyshev polynomials and nodes of near minimal degree cubature formula. With the help of the cubature formula we establish the mean convergence of the interpolation polynomials. © 1996 Academic Press, Inc.

## 1. INTRODUCTION

It is well known that the zeros of Chebyshev polynomials in one variable are in many ways optimal nodes for the purpose of polynomial interpolation; see, for example, [10]. For $n \geqslant 1$, the Chebyshev polynomial of degree $n$ is defined by

$$T_n^*(x) = \cos n\theta, \qquad x = \cos\theta;$$

its zeros are given explicitly as

$$x_k = \cos\frac{(2k-1)\,\pi}{2n}, \qquad k = 1, ..., n.$$

It is well known that $T_n^*$ are orthogonal with respect to the Chebyshev weight function $w(x) = (1-x^2)^{-1/2}$ on $[-1, 1]$. The zeros of $T_n^*$ are also the nodes of Gaussian quadrature formula with respect to $w$ and they yield compact interpolation formulae.

There are many difficulties with polynomial interpolation in several variables. First of all, to ensure that interpolation problems are well-posed, one needs to find, for each given set of points, a proper subspace of polynomials in which the interpolation polynomial can be uniquely chosen.

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Another difficulty is how to choose sequences of points so that the interpolation process converges to the function being interpolated under mild conditions imposed on the function. Other difficulties include the lack of compact formulae for the interpolation polynomials and the lack of analytic means to describe the interpolation points in general. In this respect, one may ask the question what are good interpolation points, by which we mean the set of points that resolve these difficulties. In one variable, the zeros of Chebyshev polynomials, or zeros of other orthogonal polynomials, are good interpolation points; they yield compact formulae and elegant convergence results. In several variables, however, the question is largely unanswered; beyond the points of tensor product type, which is in essence one variable, there are few examples of interpolation points which are explicitly known and yield good results.

In this paper we will work with several sequences of points in two variables which are analog of Chebyshev points in several respects. These points come from nodes of minimal or near minimal numerical cubature formula. A minimal cubature formula is precise for polynomials of certain degree with minimal number of nodes; it is the analog of Gaussian quadrature of one variable. Minimal or near minimal cubature formulae are, in general, difficult to find; their nodes are common zeros of quasi-orthogonal polynomials in several variables. For the product Chebyshev weight on square region, however, several examples are known. It is shown only recently in [14] that if such a cubature formula exists, then a Lagrange interpolation polynomial can be uniquely defined; no example has been examined yet. The purpose of this paper is to study two examples with respect to the product Chebyshev weight. We describe the construction of Lagrange interpolation polynomials in detail in Section 2 and prove the mean convergence of the Lagrange interpolation in Section 3.

## 2. CONSTRUCTION OF INTERPOLATION POLYNOMIALS

Although most results on cubature formula and common zeros of polynomials are developed in a general framework for all weight functions, we shall restrict our exposition in this section to the case of product Chebyshev weight functions of two variables. For an account of the results in general we refer to [3, 5, 7, 14] and the references given there.

### 2.1. *Product Chebyshev Polynomials*

We denote the classical Chebyshev weight of the first kind by $w_0$, which is defined by

$$w_0(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1,$$

and zero outside $[-1,1]$. The orthonormal polynomials with respect to $w_0$ are

$$T_0(x)=1, \qquad T_k(x)=\sqrt{2}\cos k\theta, \qquad k\geqslant 1, \qquad x=\cos\theta.$$

We warn the reader that our definition of Chebyshev polynomials differ from the usual one by a factor of $\sqrt{2}$, since our $T_n$ are orthonormal with respect to $w_0$. Moreover, in the definition of $w_0$ we already incorporate the factor $\pi^{-1}$, so that the integral of $w_0$ on $[-1,1]$ is 1.

The product Chebyshev weight function on $[-1,1]^2$ is defined by

$$W_0(x,y)=w_0(x)\,w_0(y)=\frac{1}{\pi^2}\frac{1}{\sqrt{1-x^2}\,\sqrt{1-y^2}}, \qquad (x,y)\in[-1,1]^2.$$

It is easy to verify that the polynomials defined by

$$P_k^n(x,y)=T_{n-k}(x)\,T_k(y), \qquad 0\leqslant k\leqslant n, \quad n\in\mathbb{N}_0,$$

where each $P_k^n$ is of degree exactly $n$, are orthonormal with respect to $W_0$. One convenient notation in dealing with orthogonal polynomials is the vector

$$\mathbb{P}_n=(P_0^n,\ldots,P_n^n)^T, \qquad n\in\mathbb{N}_0,$$

whose components are orthonormal polynomials of degree exactly $n$. Introducing the matrices

$$A_{n,1}=\frac{1}{2}\begin{bmatrix} 1 & & \bigcirc & 0\\ & \ddots & & \vdots\\ & & 1 & 0\\ \bigcirc & & \sqrt{2} & 0\end{bmatrix} \quad\text{and}\quad A_{n,2}=\frac{1}{2}\begin{bmatrix} 0 & \sqrt{2} & & \bigcirc\\ 0 & & 1 & \\ \vdots & & & \ddots\\ 0 & \bigcirc & & 1\end{bmatrix},$$

it can be readily verified that the product Chebyshev polynomials satisfy the three-term relation

$$x_i\mathbb{P}_n(\mathbf{x})=A_{n,i}\mathbb{P}_{n+1}(\mathbf{x})+A^{\mathrm{T}}_{n-1,i}\,\mathbb{P}_{n-1}(\mathbf{x}), \qquad i=1,2, \quad \mathbf{x}=(x_1,x_2),$$

where we write $\mathbf{x}=(x_1,x_2)$ for convenience; in all other places in the paper we write $\mathbf{x}=(x,y)$. Every system of orthogonal polynomials in several variables satisfies a three-term relation in such a vector–matrix form; the relation plays an essential role in the study of orthogonal polynomials in several variables and cubature formulae; see [12, 14] and the references given there.

For $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{y} \in \mathbb{R}^2$, the reproducing kernel of the product Chebyshev polynomials is defined by

$$\mathbf{K}_n(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n-1} \sum_{j=0}^{k} P_j^k(\mathbf{x})\, P_j^k(\mathbf{y}) = \sum_{k=0}^{n-1} \mathbb{P}_k^T(\mathbf{x})\, \mathbb{P}_k(\mathbf{y}).$$

It is shown in [15] that there is a compact formula for $\mathbf{K}_n(\,\cdot\,,\,\cdot\,)$. Let

$$\mathbf{x} = (\cos\theta_1, \cos\theta_2), \qquad \mathbf{y} = (\cos\phi_1, \cos\phi_2).$$

Then the compact formula is given by

$$\begin{aligned} \mathbf{K}_n(\mathbf{x}, \mathbf{y}) = {} & D_n(\theta_1+\phi_1, \theta_2+\phi_2) + D_n(\theta_1+\phi_1, \theta_2-\phi_2) \\ & + D_n(\theta_1-\phi_1, \theta_2+\phi_2) + D_n(\theta_1-\phi_1, \theta_2-\phi_2), \end{aligned} \tag{2.1}$$

where the function $D_n$ is defined by

$$D_n(\theta_1, \theta_2) = \frac{1}{2} \frac{\cos(n-1/2)\,\theta_1 \cos(\theta_1/2) - \cos(n-1/2)\,\theta_2 \cos(\theta_2/2)}{\cos\theta_1 - \cos\theta_2}. \tag{2.2}$$

These formulae will lead to a compact formula for the Lagrange interpolation polynomials considered below.

### 2.2. *Cubature Formula*

By a cubature formula of degree $2n-1$ with respect to $W_0$ we mean a linear functional $f \mapsto \mathscr{I}_n(f)$,

$$\mathscr{I}_n f = \sum_{k=0}^{N} f(\mathbf{x}_k)\, \lambda_k, \qquad \lambda_k > 0, \quad \mathbf{x}_k \in \mathbb{R}^2,$$

where $N$ is an integer depending on $n$, such that

$$\int_{[-1,1]^2} P(x, y)\, W_0(x, y)\, dx\, dy = \mathscr{I}_n(P), \qquad \forall P \in \Pi_{2n-1}^2,$$

where $\Pi_m^2$ denote the space of polynomials of (total) degree $m$ in two variables. The points $\mathbf{x}_k$, $1 \leqslant k \leqslant N$, are called nodes and $\lambda_1, ..., \lambda_N$ weights. Such a formula is called minimal, if $N$, the number of nodes, is minimal among all cubature formulae of degree $2n-1$ with respect to $W_0$. According to a general result of Möller [5] on centrally symmetric weight functions, of which $W_0$ is an example, the number of nodes in the cubature formula satisfies

$$N \geqslant \dim \Pi_{n-1}^2 + \left[\frac{n}{2}\right] = \binom{n+1}{2} + \left[\frac{n}{2}\right]. \tag{2.3}$$

Clearly, cubature formulae that attain this lower bound are minimal ones. For general weight functions the lower bound (2.3) may not hold. In [5] Möller characterized cubature formulae that attain the lower bound (2.3). Using this characterization, Morrow and Patterson [6] gave the first minimal formulae for $W_0$. The minimal cubature formulae are not unique; there are other examples known (cf. [2]).

We are interested in those minimal or near minimal formulae for $W_0$ that can be given explicitly. We give two of them below and indicate how they are constructed later. We denote by $z_k$ the points

$$z_k = z_{k,n} = \cos \frac{k\pi}{n}, \qquad 0 \leqslant k \leqslant n.$$

The first formula is a minimal formula given in [6]:

For $n = 2m$,

$$\begin{aligned} \frac{1}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} f(x, y) \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}} &= \frac{2}{n^2} \sideset{}{''}\sum_{i=0}^{n/2} \sum_{j=0}^{n/2-1} f(z_{2i}, z_{2j+1}) \\ &+ \frac{2}{n^2} \sum_{i=0}^{n/2-1} \sideset{}{''}\sum_{j=0}^{n/2} f(z_{2i+1}, z_{2j}), \qquad f \in \Pi^2_{2n-1}, \end{aligned} \tag{2.4}$$

where $\sum''$ means that the first and last terms in the summation are halved.

It is readily verified that this formula uses $n^2/2 + n$ nodes, which is the lower bound of (2.3). The second formula is shown to exist in [14]; here it is given explicitly for the first time:

For $n = 2m - 1$,

$$\begin{aligned} \frac{1}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} f(x, y) \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}} &= \frac{2}{n^2} \sideset{}{'}\sum_{i=0}^{(n-1)/2} \sideset{}{'}\sum_{j=0}^{(n-1)/2} f(z_{2i}, z_{2j}) \\ &+ \frac{2}{n^2} \sideset{}{'}\sum_{i=0}^{(n-1)/2} \sideset{}{'}\sum_{j=0}^{(n-1)/2} f(z_{n-2i}, z_{n-2j}), \qquad f \in \Pi^2_{2n-1}, \end{aligned} \tag{2.5}$$

where $\sum'$ means that the first term in the summation is halved.

It is easy to verify that the formula uses $(n+1)^2/2$ nodes which is one more than the lower bound (2.3).

Several minimal cubature formulae are known to exist, but their nodes cannot be given by explicit formula as above. There are other formulae of the similar type which can be written down explicitly, such as the one for $n=2m$ in [6, p. 964]. We choose these two to illustrate our method; the others can be dealt with similarly.

The existence of a cubature formula of degree $2n-1$ depends on the solution of nonlinear matrix equations. For $W_0$ these equations are first considered by Möller [5]; here we follow the formulation of [14]. A cubature formula exists when the following equations in variable $V$ are solvable,

$$A_{n-1,1}(VV^{\mathrm{T}}-I)\,A^{\mathrm{T}}_{n-1,2}=A_{n-1,2}(VV^{\mathrm{T}}-I)\,A^{\mathrm{T}}_{n-1,1} \tag{2.6a}$$

and

$$V^{\mathrm{T}}A^{\mathrm{T}}_{n-1,1}A_{n-1,2}V=V^{\mathrm{T}}A^{\mathrm{T}}_{n-1,2}A_{n-1,1}V, \tag{2.6b}$$

where $V$ is a matrix of size $(n+1)\times\sigma$ and it is necessary that

$$\sigma=\left[\frac{n}{2}\right] \qquad \text{or} \qquad \sigma=\left[\frac{n}{2}\right]+1.$$

When these two equations are solvable for $V$, a cubature formula of degree $2n-1$ exists whose nodes are the common zeros of the polynomials $U^{\mathrm{T}}\mathbb{P}_n$, where $U^{\mathrm{T}}$ is the orthogonal complement of $V$; i.e., $U$ satisfies

$$U^{\mathrm{T}}V=0, \qquad U:(n+1)\times(n+1-\sigma).$$

The solution with $\sigma=[n/2]$ corresponds to the minimal cubature formula with the number of nodes equal to the lower bound (2.3); the solution with $\sigma=[n/2]+1$ corresponds to a near minimal cubature formula with one more node than the lower bound (2.3).

In particular, for $n=2m$, one solution of (2.6) can be easily verified to be

$$V^{\mathrm{T}}=\begin{bmatrix} \sqrt{2} & & & \bigcirc & 0 & \bigcirc & & & -\sqrt{2} \\ & 1 & & & 0 & & & -1 & \\ & & \ddots & & \vdots & & \iddots & & \\ \bigcirc & & & 1 & 0 & -1 & & & \bigcirc \end{bmatrix}, \tag{2.7}$$

which corresponds to the cubature formula (2.4); the nodes are the common zeros of the polynomials

$$U^{\mathrm{T}}\mathbb{P}_n, \qquad \text{where } U^{\mathrm{T}}=\begin{bmatrix} 1 & & \bigcirc & 0 & \bigcirc & & -1 \\ & \ddots & & \vdots & & \iddots & \\ \bigcirc & & 1 & 0 & -1 & & \bigcirc \\ 0 & & 0 & \sqrt{2} & 0 & & 0 \end{bmatrix}, \tag{2.8}$$

or, more transparently,

$$T_{n-k+1}(x)\, T_{k-1}(y)-T_{k-1}(x)\, T_{n-k+1}(y), \qquad 1\leqslant k\leqslant n/2+1.$$

For $n=2m-1$, one solution of (2.6) can be verified to be

$$V^{\mathrm{T}}=\begin{bmatrix} \sqrt{2} & & & \bigcirc & \bigcirc & & & -\sqrt{2} \\ & 1 & & & & & 1 & \\ & & \ddots & & & \iddots & & \\ \bigcirc & & & 1 & -1 & & & \bigcirc \end{bmatrix}, \tag{2.9}$$

which corresponds to the cubature formula (2.5); the nodes are the common zeros of polynomials

$$U^{\mathrm{T}}\mathbb{P}_n, \qquad U^{\mathrm{T}}=\begin{bmatrix} 1 & & \bigcirc & \bigcirc & & -1 \\ & \ddots & & & \iddots & \\ \bigcirc & & 1 & -1 & & \bigcirc \end{bmatrix}, \tag{2.10}$$

or, more transparently,

$$T_{n-k+1}(x)\, T_{k-1}(y)-T_{k-1}(x)\, T_{n-k+1}(y), \qquad 1\leqslant k\leqslant (n+1)/2,$$

and an additional polynomial of degree $n+1$ which we will not give explicitly (see [14]).

We need to know the polynomials $U^{\mathrm{T}}\mathbb{P}_n$ in order to compute the nodes of the cubature formula; the polynomial subspace in which the interpolation polynomials are uniquely defined will be given with the help of $V$. The method we described above can be used to construct cubature formulae for $W_0$. We have indicated the construction of (2.4) and (2.5); in fact, the cubature formula (2.5) is found exactly this way. We first find a solution $V$ of (2.6a) and (2.6b), then determine the nodes by solving the common zeros of $U^{\mathrm{T}}\mathbb{P}_n$. Once the nodes are determined, the weights of the cubature formula are uniquely determined by a formula given below; see (2.13).

### 2.3. *Lagrange Interpolation Polynomials*

The connection between the cubature formula and polynomial interpolation has been established recently in [14]. If a cubature formula exists, we can consider the Lagrange interpolation problem based on the nodes of the cubature formula, which asks for a unique polynomial solution of

$$P(\mathbf{x}_k) = f(\mathbf{x}_k), \qquad 1 \leqslant k \leqslant N,$$

for any function $f$ in a subspace of $\Pi_n^2$. Since $N < \dim \Pi_n^2$, we have to specify the subspace so that the interpolation is unique. Clearly, no polynomial that vanishes on all $\{\mathbf{x}_k\}$ can belong to the subspace. In [14], it is proved that the proper subspace is given by

$$\mathscr{V}_n^2 = \Pi_{n-1}^2 \cup \operatorname{span}\{V^+ \mathbb{P}_n\}, \tag{2.11}$$

where $V^+$ is the unique Moore–Penrose generalized inverse of $V$; since $V$ has full rank, we have

$$V^+ = (V^{\mathrm{T}} V)^{-1} V^{\mathrm{T}}.$$

Moreover, the Lagrange interpolation polynomial, denoted by $L_n f$, can be given explicitly. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, recall that $\mathbf{K}_n(\,\cdot\,,\,\cdot\,)$ denotes the reproducing kernel; we set

$$\mathbf{K}_n^*(\mathbf{x}, \mathbf{y}) = \mathbf{K}_n(\mathbf{x}, \mathbf{y}) + [V^+ \mathbb{P}_n(\mathbf{x})]^{\mathrm{T}} V^+ \mathbb{P}_n(\mathbf{y}).$$

Using a modified Christoffel–Dabroux formula, we showed in [14] that $\mathbf{K}_n^*(\mathbf{x}_k, \mathbf{x}_j) = 0$ for $k \neq j$, while it follows from the definition that $\mathbf{K}_n^*(\mathbf{x}_k, \mathbf{x}_k) \neq 0$. Therefore, it follows that

$$(L_n f)(\mathbf{x}) = \sum_{k=1}^{N} f(\mathbf{x}_k) \frac{\mathbf{K}_n^*(\mathbf{x}, \mathbf{x}_k)}{\mathbf{K}_n^*(\mathbf{x}_k, \mathbf{x}_k)}. \tag{2.12}$$

Actually, the existence of the cubature formula is established in [14] with the help of (2.12); it is shown there that the cubature formula is given by the integration of $L_n f$, just as in the case of one variable:

$$\int_{[-1,1]^2} (L_n f)(\mathbf{x}) \, W_0(\mathbf{x}) \, d\mathbf{x} = \sum_{k=1}^{N} f(\mathbf{x}_k) \, \lambda_k = \mathscr{I}_n f.$$

In particular, from the orthogonality of $P_j^k$ and the definition of $\mathbf{K}_n^*(\,\cdot\,,\,\cdot\,)$ it follows immediately that the weights in the cubature formula are given by

$$\lambda_k = [\mathbf{K}_n^*(\mathbf{x}_k, \mathbf{x}_k)]^{-1}. \tag{2.13}$$

By the definition of $\mathbf{K}_n^*(\cdot,\cdot)$, (2.1), and (2.2), the weights in the cubature formula (2.4) and (2.5) can be computed using (2.13).

In the following we give explicit formulae for the Lagrange interpolation polynomials based on the nodes of (2.4) and (2.5) and indicate how the formulae are derived.

*The case* $n=2m$. The interpolation points are

$$\begin{aligned} \mathbf{x}_{2i,2j+1} &= (z_{2i}, z_{2j+1}), \qquad 0 \leqslant i \leqslant m, \quad 0 \leqslant j \leqslant m-1, \\ \mathbf{x}_{2i+1,2j} &= (z_{2i+1}, z_{2j}), \qquad 0 \leqslant i \leqslant m-1, \quad 0 \leqslant j \leqslant m. \end{aligned} \tag{2.14}$$

The Lagrange interpolation polynomials are given by (2.12) with

$$\mathbf{K}_n^*(\mathbf{x}, \mathbf{x}_{k,l}) = \tfrac{1}{2}[\mathbf{K}_n(\mathbf{x}, \mathbf{x}_{k,l}) + \mathbf{K}_{n-1}(\mathbf{x}, \mathbf{x}_{k,l})] - \tfrac{1}{2}(-1)^k \cdot [T_n(x) - T_n(y)], \tag{2.15}$$

and

$$\begin{aligned} &\mathbf{K}_n^*(\mathbf{x}_{0,2j+1}, \mathbf{x}_{0,2j+1}) = n^2, \qquad \mathbf{K}_n^*(\mathbf{x}_{2i,2j+1}, \mathbf{x}_{2i,2j+1}) = n^2/2, \qquad i>0, \\ &\mathbf{K}_n^*(\mathbf{x}_{2i+1,0}, \mathbf{x}_{2i+1,0}) = n^2, \qquad \mathbf{K}_n^*(\mathbf{x}_{2i+1,2j}, \mathbf{x}_{2i+1,2j}) = n^2/2, \qquad j>0. \end{aligned} \tag{2.16}$$

If we recall the formula (2.1) for $\mathbf{K}_n(\cdot,\cdot)$, we see that (2.15) and (2.16) give a compact formula for Lagrange interpolation polynomials; moreover, by (2.13), (2.16) gives the weights in the cubature formula. To verify these formulae, we use the notation $\mathbf{x}_{k,l}$ for one of the points whenever we do not need the explicit formula in (2.14). From (2.7) and (2.8), it is easy to verify

$$V^{\mathrm{T}} V = 2 \begin{bmatrix} 2 & & & \bigcirc \\ & 1 & & \\ & & \ddots & \\ \bigcirc & & & 1 \end{bmatrix},$$

$$V^{+} = \begin{bmatrix} \sqrt{2}/2 & & & 0 & & & -\sqrt{2}/2 \\ & 1 & & 0 & & -1 & \\ & & \ddots & \vdots & \iddots & & \\ & & 1 & 0 & -1 & & \end{bmatrix},$$

$$(V^{+})^{\mathrm{T}} V^{+} + \frac{1}{4} U U^{\mathrm{T}} = \frac{1}{2} I - \frac{1}{8} \begin{bmatrix} 1 & & -1 \\ & \bigcirc & \\ -1 & & 1 \end{bmatrix},$$

where $I$ stands for the identity matrix and the last matrix in the formula contains only four nonzero entries located at the corners. Since $U^{\mathrm{T}}\mathbb{P}_n(\mathbf{x}_k)=0$, it readily follows that

$$\mathbb{P}_n^{\mathrm{T}}(\mathbf{x})(V^+)^{\mathrm{T}}\,V^+\mathbb{P}_n(\mathbf{x}_{k,l})$$
$$=\tfrac{1}{2}\mathbb{P}_n^{\mathrm{T}}(\mathbf{x})\,\mathbb{P}(\mathbf{x}_{k,l})-\tfrac{1}{8}\,[P_0^n(\mathbf{x})-P_n^n(\mathbf{x})]\cdot[P_0^n(\mathbf{x}_{k,l})-P_n^n(\mathbf{x}_{k,l})]. \tag{2.17}$$

From the fact that $P_0^n(\mathbf{x})=\sqrt{2}\,T_n(x)$ and $P_n^n(\mathbf{x})=\sqrt{2}\,T_n(y)$, it follows that $P_0^n(\mathbf{x}_{k,l})-P_n^n(\mathbf{x}_{k,l})$ takes the value $(-1)^k\,2\sqrt{2}$. Since

$$\mathbb{P}_n^{\mathrm{T}}(\mathbf{x})\,\mathbb{P}_n(\mathbf{y})=\mathbf{K}_n(\mathbf{x},\mathbf{y})-\mathbf{K}_{n-1}(\mathbf{x},\mathbf{y}),$$

(2.15) follows from the definition of $\mathbf{K}_n^*(\,\cdot\,,\,\cdot\,)$. In particular, set $\mathbf{x}=\mathbf{y}=\mathbf{x}_{k,l}$ in (2.15) and use (2.14); we conclude that

$$\mathbf{K}_n^*(\mathbf{x}_{k,l},\mathbf{x}_{k,l})=\tfrac{1}{2}[\mathbf{K}_n(\mathbf{x}_{k,l},\mathbf{x}_{k,l})+\mathbf{K}_{n-1}(\mathbf{x}_{k,l},\mathbf{x}_{k,l})]-1. \tag{2.18}$$

Using the formula (2.18) and combining with (2.2), we can compute the cubature weights. After tedious computation, (2.16) can be verified.

*The case* $n=2m-1$. The interpolation points are

$$x_{2i,2j}=(z_{2i},z_{2j}),\qquad \mathbf{x}_{2i+1,2j+1}=(z_{2i+1},z_{2j+1}),\qquad 0\leqslant i,\quad j\leqslant m-1. \tag{2.19}$$

The Lagrange interpolation polynomials are given by (2.12) with

$$\mathbf{K}_n^*(\mathbf{x},\mathbf{x}_{k,l})=\tfrac{1}{2}[\mathbf{K}_n(\mathbf{x},\mathbf{x}_{k,l})+\mathbf{K}_{n-1}(\mathbf{x},\mathbf{x}_{k,l})]-\tfrac{1}{2}(-1)^k\cdot[T_n(x)+T_n(y)]. \tag{2.20}$$

and

$$\mathbf{K}_n^*(\mathbf{x}_{2i,2j},\mathbf{x}_{2i,2j})=\begin{cases} n^2/2, & \text{if}\quad 0<i,\quad j\leqslant m-1\\ n^2, & \text{if}\quad i=0\quad\text{or;}\quad j=0,\quad i+j>0\\ 2n^2, & \text{if}\quad i=j=0,\end{cases}$$

$$\mathbf{K}_n^*(\mathbf{x}_{2i+1,2j+1},\mathbf{x}_{2i+1,2j+1})$$
$$=\begin{cases} n^2/2, & \text{if}\quad 0\leqslant i,\quad j<m-1\\ n^2, & \text{if}\quad i=m-1\quad\text{or;}\quad j=m-1,\quad i+j<2m-2\\ 2n^2, & \text{if}\quad i=j=m-1.\end{cases} \tag{2.21}$$

Again, (2.20) combined with (2.1) gives the compact formulae for the Lagrange interpolation polynomials and (2.21), by (2.13), yields the weights in the cubature formula (2.5). To verify these formulae, we again use the notation $\mathbf{x}_{k,l}$. In this case, we have

$$V^{\mathrm{T}}V = 2\begin{bmatrix} 2 & & & \bigcirc \\ & 1 & & \\ & & \ddots & \\ \bigcirc & & & 1 \end{bmatrix},$$

$$V^{+} = \begin{bmatrix} \sqrt{2}/2 & & & 0 & & & -\sqrt{2}/2 \\ & 1 & & 0 & & -1 & \\ & & \ddots & \vdots & \iddots & & \\ & & 1 & 0 & -1 & & \end{bmatrix},$$

and

$$(V^{+})^{\mathrm{T}}\, V^{+} + \frac{1}{4}\, UU^{\mathrm{T}} = \frac{1}{2}\, I - \frac{1}{8}\begin{bmatrix} 1 & & 1 \\ & \bigcirc & \\ 1 & & 1 \end{bmatrix},$$

from which and the fact that $U^{\mathrm{T}}\mathbb{P}_n(\mathbf{x}_k) = 0$, it readily follows that

$$\begin{aligned} &\mathbb{P}_n^{\mathrm{T}}(\mathbf{x})(V^{+})^{\mathrm{T}}\, V^{+}\,\mathbb{P}_n(\mathbf{x}_{k,l}) \\ &\quad = \tfrac{1}{2}\mathbb{P}_n^{\mathrm{T}}(\mathbf{x})\,\mathbb{P}(\mathbf{x}_{k,l}) - \tfrac{1}{8}[P_0^n(\mathbf{x}) + P_n^n(\mathbf{x})] \cdot [P_0^n(\mathbf{x}_{k,l}) + P_n^n(\mathbf{x}_{k,l})]. \end{aligned}$$

Therefore, taking into consideration the formulae for $P_0^n$ and $P_n^n$ as in the case of $n = 2m$, we conclude that (2.20) holds. In particular, it can be verified that $\mathbf{K}_n^*(\mathbf{x}_{k,l}, \mathbf{x}_{k,l})$ is given by exactly the same formula (2.18), from which we can verify (2.21).

In fact, the existence of the cubature formula (2.5) is shown in [14], where the matrices $U$, $V$ are given but not the nodes and the weights of the formula. The formula (2.5) is given explicitly for the first time here. As mentioned before, the way to determine the nodes and the weights of the cubature formula is to compute the common zeros of (2.10) and then weights by (2.18) and (2.2). The author indeed derived the formulae (2.20) and (2.21) this way. The computation is tedious but elementary and the formulae may be verified by other means once they are found; we decide not to include the details.

## 3. MEAN CONVERGENCE OF LAGRANGE INTERPOLATION

With respect to $W_0$ we define the weighted $L^p$ space $L^p_{W_0}$ as the space of Lebesgue measurable functions $f$ on $[-1, 1]^2$ for which the norm

$$\|f\|_{W_0, p} = \left\{ \int_{[-1,1]^2} |f(x, y)|^p \, W_0(x, y) \, dx \, dy \right\}^{1/p}$$

is finite. For $0 < p < 1$, $\|\cdot\|_{W_{p,p}}$ is not a norm; nevertheless, we keep this notation for convenience. We also denote the space of continuous functions on $[-1, 1]^2$ by $C[-1, 1]^2$. Our main result is as follows.

THEOREM 3.1. *Let $L_n f$ be defined as in the previous section based on nodes* (2.14) *or* (2.19). *Let $0 < p < \infty$. Then*

$$\lim_{n \to \infty} \|f - L_n f\|_{W_{0,p}} = 0 \qquad \forall f \in C[-1, 1]^2.$$

It is worthwhile to point out that the convergence for $p = 1$ also leads to the convergence of cubature formula, namely, we have

COROLLARY 3.2. *Let $\mathscr{I}_n f$ denote the cubature formulae in* (2.4) *or* (2.5). *Then*

$$\lim_{n \to \infty} \mathscr{I}_n f = \int_{[-1,1]^2} f(\mathbf{x}) \, W_0(x) \, d\mathbf{x} \qquad \forall f \in C[-1, 1]^2.$$

Theorem 3.1 will be a consequence of the following theorem, which is of interest in itself.

THEOREM 3.3. *Let $1 < p < \infty$. Let $\mathbf{x}_k$, $1 \leqslant k \leqslant N$, be the points given in* (2.14) *or* (2.19). *Then*

$$\int_{[-1,1]^2} |P(\mathbf{x})|^p \, W_0(\mathbf{x}) \, d\mathbf{x} \leqslant A_p^p \frac{1}{N} \sum_{k=1}^{N} |P(\mathbf{x}_k)|^p \qquad \forall P \in \mathscr{V}_n^2, \tag{3.1}$$

*where $A_p$ is a constant depending only on p.*

In one variable, an inequality of the type in Theorem 3.3 is called the Marcinkiewicz–Zygmund inequality; see [16, Vol. 2, p. 28] for the first such inequality established for trigonometric polynomials. Our proof will follow the approach of [13], where such inequalities are proved for the generalized Jacobi weight functions of one variable and they are used to prove the mean convergence of the interpolation polynomials based on the zeros of generalized Jacobi polynomials. For $p=2$, such a consideration has been carried out in [11] for polynomial interpolation based on the zeros of cubature formula of even degree. As we shall see, the additional term $[V^+\mathbb{P}_n(\mathbf{x})]^{\mathrm{T}}\, V^+\mathbb{P}_n(\mathbf{x}_{k,l})$ of $\mathbf{K}_n^*(\cdot,\cdot)$ which appears in the formula of Lagrange interpolation polynomials demands special consideration even when $p=2$. The proof of Theorem 3.3 uses the $L^p$ boundness of the partial sums of the orthogonal expansion with respect to the product Chebyshev polynomials. For a Lebesgue integrable function $f$, the $n$th partial sum of $f$ with respect to $P_j^k$ is defined by

$$S_n(f;\mathbf{x})=\sum_{k=0}^{n-1}\sum_{j=0}^{k} a_j^k(f)\,P_j^k(\mathbf{x}), \qquad a_j^k(f)=\int_{[-1,1]^2} f(\mathbf{x})\,P_j^k(\mathbf{x})\,W_0(\mathbf{x})\,d\mathbf{x}.$$

In terms of the reproducing kernel $\mathbf{K}_n$ we can write $S_n f$ as

$$S_n(f;\mathbf{x})=\int_{[-1,1]^2} \mathbf{K}_n(\mathbf{x},\mathbf{y})\,f(\mathbf{y})\,W_0(\mathbf{y})\,d\mathbf{y}.$$

We will need the following result, which states that $S_n$ is a bounded operator from $L^p_{W_0}$ to $L^p_{W_0}$.

LEMMA 3.4. *Let $1<p<\infty$. Then there is a constant $B_p$ which depends on $p$ only such that*

$$\|S_n f\|_{\{W_{0,p}\}}\leqslant B_p\,\|f\|_{\{W_{0,p}\}} \qquad \forall f\in L^p_{W_0}.$$

*Proof.* To prove this theorem, we relate it to its trigonometric counterpart. Let $\mathbf{x}=(\cos\theta_1,\cos\theta_2)$ and $\mathbf{y}=(\cos\phi_1,\cos\phi_2)$. Changing variables and using the formula (2.1) of $\mathbf{K}_n(\cdot,\cdot)$, we have that

$$S_n(f;\mathbf{x})=\sum_{(\varepsilon_1,\varepsilon_2)\in\{-1,1\}^2}\frac{1}{\pi^2}\int_0^\pi\int_0^\pi D_n(\theta_1+\varepsilon_1\phi_1,\theta_2+\varepsilon_2\phi_2)$$
$$\times f(\cos\phi_1,\cos\phi_2)\,d\phi_1\,d\phi_2,$$

where $D_n$ is defined in (2.2). Note that $D_n$ is even and $2\pi$-periodic in both of its variables. Set

$$\tilde{f}(\theta,\phi)=f(\cos\theta,\cos\phi),$$

which is $2\pi$-periodic in both of its variables and define

$$s_n\tilde{f}(\theta,\phi)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} D_n(\theta+\xi,\phi+\eta)\,\tilde{f}(\xi,\eta)\,d\xi\,d\eta.$$

Then it is easy to see that to prove the boundedness of $S_n$ from $L^p_{W_0}$ to $L^p_{W_0}$, it suffices to prove the boundedness of $s_n$ from $L^p$ to $L^p$, where $L^p = L^p(\mathbb{T}^2)$ is the usual $L^p$ space. More precisely, it suffices to prove that for any $f \in L^p$, $1 < p < \infty$,

$$\left(\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |s_n f(\theta, \phi)|^p \, d\theta \, d\phi\right)^{1/p}$$

$$\leqslant B'_p \left(\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(\theta, \phi)|^p \, d\theta \, d\phi\right)^{1/p}, \tag{3.2}$$

where $B'_p$ depends on $p$ only. For a function $h \in L^1(\mathbb{T}^2)$ its Fourier series is defined by

$$h(\theta, \phi) \sim \sum_{k,j} \hat{h}(k, j)\, e^{i(j\theta + j\phi)}, \qquad \hat{h}(k, j) = \frac{1}{\pi^2}\int_{[-\pi, \pi]^2} f(\theta, \phi)\, e^{-i(k\theta + j\phi)}\, d\theta \, d\phi.$$

It is easy to see, as pointed out in [1, 15], that $D_n$ is the Dirichlet kernel for the $l$-1 summation of the Fourier series and that $s_n h$ is the $l$-1 partial sum of $h$; i.e.,

$$D_{n+1}(\theta, \phi) = \sum_{|k| + |j| \leqslant n} e^{i(k\theta + j\phi)}, \qquad T_n h = h * D_n,$$

where $*$ stands for the usual convolution. Because of these formulae, the inequality (3.2), actually its $d$ variable analog, follows from results in Fourier analysis. First, by the Poisson summation formula, instead of working with Fourier series, one can deal with the Fourier integral on $\mathbb{R}^d$; to prove the inequality, it amounts to proving that the characteristic function, $\chi_{B_1}$, of the $l_1$ unit ball $B_1 = \{\mathbf{x} : |x_1| + \cdots + |x_d| \leqslant 1\}$ in $\mathbb{R}^d$ is a Fourier multiplier for $L^p(\mathbb{R}^d)$ (cf. [9, p. 260]). Second, in [8, Theorem 4 of Chap. IV] it is proved, using the $L^p$ boundedness of the Hilbert transform, that the characteristic function of a rectangular region is an $L^p$ multiplier; as pointed out in Remark 6.2.6 of [8, Chap. IV], the proof for the rectangular case can be modified to show that the characteristic function of an arbitrary polyhedron is a multiplier for $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, which includes, in particular, the $l_1$-balls. ∎

For Fourier series in one variable, the inequality (3.2) is the celebrated result of Riesz on the $L^p$ boundedness of Fourier partial sums. This indicates that the proof in [8], which we outlined above, is perhaps as simple as it can be. However, it would be interesting to find a proof of (3.2) which deals with the partial sums of Fourier series directly, without transforming the problem to $\mathbb{R}^d$; such a proof may show us a way to deal with orthogonal series with respect to other weight functions, such as the product Jacobi weight functions. In this respect, we should mention that the Dirichlet kernel of $D_n$ in $d$ variables turns out to be a divided difference with the variables acting as nodes of the divided difference; see [1, 15].

Back to the proof of Theorem 3.3, we need one more result which is the counterpart of Theorem 3.3.

LEMMA 3.5. *Let* $1 \leqslant p \leqslant \infty$. *Let* $\mathbf{x}_k$, $1 \leqslant k \leqslant N$, *be the points given in* (2.14) *or* (2.19). *Then there is a constant* $C_p$, *depending on p only, such that*

$$\frac{1}{N} \sum_{k=1}^{N} |P(\mathbf{x}_k)|^p \leqslant C_p \int_{[-1,1]^2} |P(\mathbf{x})|^p \, W_0(\mathbf{x}) \, d\mathbf{x} \qquad \forall P \in \Pi_n^2. \tag{3.3}$$

*Proof.* The proof depends on an univariant inequality in [4], which holds for generalized Jacobi weight functions. We state it only for the Chebyshev weight function $w_0$. Let $-1 \leqslant y_s < y_{s-1} < \cdots < y_1 \leqslant 1$ be given. Write $y_j = \cos \theta_j$ and let

$$\delta = \min\{\theta_2 - \theta_1, \theta_3 - \theta_2, ..., \theta_s - \theta_{s-1}\} > 0.$$

Then for $1 \leqslant p < \infty$ and for all polynomials $Q$ of degree at most $rn$, there is a constant $c_p$ depends on $p$ only such that

$$\sum_{j=1}^{s} |Q(y_j)|^p \leqslant c_p(n + \delta^{-1}) \int_{-1}^{1} |Q(u)|^p \, w_0(u) \, du. \tag{3.4}$$

We prove the inequality (3.3) for the case $n = 2m$. Recall that $z_k = \cos k\pi/n$, we use the inequality (3.4) to conclude that

$$\begin{aligned} \sum_{k,j} |P(\mathbf{x}_{k,j})|^p &= \sum_{i=0}^{m} \sum_{j=0}^{m-1} |P(z_{2i}, z_{2j+1})|^p + \sum_{i=0}^{m-1} \sum_{j=0}^{m} |P(z_{2i+1}, z_{2j})|^p \\ &\leqslant 2c_p m \sum_{i=0}^{m} \int_{-1}^{1} |P(z_{2i}, y)|^p \, w_0(y) \, dy \\ &\quad + 2c_p m \sum_{i=0}^{m-1} \int_{-1}^{1} |P(z_{2i+1}, y)|^p \, w_0(y) \, dy, \end{aligned}$$

where for each sum on $j$ we fix $i$ and apply (3.4) on the polynomial $Q = P(z_i, \cdot)$. We then exchange the summation and the integral in the last equation and apply (3.4) on the sum of the polynomial $Q = P(\cdot, y)$ for fixed $y$ to conclude that

$$\sum_{k,j} |P(\mathbf{x}_{k,j})|^p \leqslant 8c_p^2 m^2 \int_{-1}^{1} \int_{-1}^{1} |P(x, y)|^p \, w_0(x) \, w_0(y) \, dx \, dy,$$

which proves the desired inequality with $C_p = 4c_p$ for $n = 2m$. The case $n = 2m - 1$ can be proved simlarly. ∎

We are ready to prove the Marcinkiewicz–Zygmund inequality in Theorem 3.3.

*Proof of Theorem* 3.3. For $p > 1$, we have

$$\|P\|_{W_0, p} = \sup_{\|g\|_{W_0, q}} \int_{[-1, 1]^2} P(\mathbf{x})\, g(\mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Since $P \in \Pi_n^2$, by the orthogonality we have

$$\int_{[-1, 1]^2} P(\mathbf{x})\, g(\mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x} = \int_{[-1, 1]^2} P(\mathbf{x})\, S_n(g; \mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x}. \quad (3.5)$$

We denote by $\mathbf{a}_n(g)$ the vector of the Fourier coefficients of $g$ with respect to $\mathbb{P}_n$; i.e.,

$$\mathbf{a}_n(g) = \int_{[-1, 1]^2} g(\mathbf{x})\, \mathbb{P}_n(\mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x}.$$

Using this notation, we write $S_n(g)$ as a sum,

$$S_n(g; \mathbf{x}) = S_{n-1}(g; \mathbf{x}) + \mathbf{a}_n^T(g)\, \mathbb{P}_n(\mathbf{x}), \quad (3.6)$$

and break the estimate of (3.5) into two parts. First, since $PS_{n-1}(g)$ is of degree $2n - 1$, it follows from the cubature formula and the Hölder inequality that

$$\begin{aligned} &\left| \int_{[-1, 1]^2} P(\mathbf{x})\, S_{n-1}(g; \mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x} \right| \\ &\quad = \left| \sum_{k=1}^{N} P(\mathbf{x}_k)\, S_{n-1}(g; \mathbf{x}_k)\, \lambda_k \right| \\ &\quad \leqslant \left( \sum_{k=1}^{N} |P(\mathbf{x}_k)|^p\, \lambda_k \right)^{1/p} \left( \sum_{k=1}^{N} |S_{n-1}(g; \mathbf{x}_k)|^q\, \lambda_k \right)^{1/q}. \end{aligned}$$

Applying Lemma 3.4 and Lemma 3.5 on $S_{n-1} g$ and taking into account the fact that $\|g\|_{\{W_{0,q}\}} = 1$, we conclude that

$$\left( \sum_{k=1}^{N} |S_{n-1}(g; \mathbf{x}_k)|^q\, \lambda_k \right)^{1/q} \leqslant (2C_q)^{1/q} \|S_n(g)\|_{\{W_{0,q}\}} \leqslant (2C_q)^{1/q}\, B_q, \quad (3.7)$$

where we have used (2.13) and (2.16) or (2.21) to replace $\lambda_k$ by $2/N$. Therefore, we conclude that

$$\left| \int_{[-1,1]^2} P(\mathbf{x})\, S_{n-1}(g; \mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x} \right| \leqslant (2C_q)^{1/q}\, B_q \left( \sum_{k=1}^{N} |P(\mathbf{x}_k)|^p\, \lambda_k \right)^{1/p}.$$

To deal with the term corresponding to $\mathbf{a}_n^{\mathrm{T}}(g)\, \mathbb{P}_n$, we first note that since $P \in \mathscr{V}_n^2$, we have $L_n P = P$. Therefore, by the formulae (2.12), (2.13) and the definition of $\mathbf{K}_n^*(\,\cdot\,,\,\cdot\,)$,

$$\begin{aligned} &\int_{[-1,1]^2} P(\mathbf{x})\, \mathbf{a}_n^{\mathrm{T}}(g)\, \mathbb{P}_n(\mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x} \\ &\quad = \int_{[-1,1]^2} (L_n P)(\mathbf{x})\, \mathbf{a}_n^{\mathrm{T}}(g)\, \mathbb{P}_n(\mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x} \\ &\quad = \sum_{k=1}^{N} P(\mathbf{x}_k)\, \lambda_k \int_{[-1,1]^2} [\mathbb{P}_n^T(\mathbf{x}_k)(V^+)^{\mathrm{T}}\, V^+ \mathbb{P}_n(\mathbf{x})] \\ &\qquad \cdot [\mathbf{a}_n^{\mathrm{T}}(g)\, \mathbb{P}_n(\mathbf{x})]\, W_0(\mathbf{x})\, d\mathbf{x}. \end{aligned}$$

We now need the explicit formulae of $\mathbb{P}_n^{\mathrm{T}}(\mathbf{x}_k)(V_+)^{\mathrm{T}}\, V^+ \mathbb{P}_n(\mathbf{x})$ derived in the previous section. Let us assume that $n = 2m$; the case $n = 2m-1$ works similarly. By (2.15), we get

$$\begin{aligned} &\int_{[-1,1]^2} P(\mathbf{x})\, \mathbf{a}_n^{\mathrm{T}}(g)\, \mathbb{P}_n(\mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x} \\ &\quad = \tfrac{1}{2} \sum_{k=1}^{N} P(\mathbf{x}_k)\, \lambda_k \mathbf{a}_n^{\mathrm{T}}(g)\, \mathbb{P}_n(\mathbf{x}_k) \\ &\qquad - \tfrac{1}{8} [a_0^n(g) + a_n^n(g)] \sum_{k=1}^{N} P(\mathbf{x}_k)\, \lambda_k [P_0^n(\mathbf{x}_k) + P_n^n(\mathbf{x}_k)]. \end{aligned}$$

For the first sum, we use $\mathbf{a}_n^{\mathrm{T}}(g)\, \mathbb{P}_n = S_n(g) - S_{n-1}(g)$ to write it as two sums, both of which can be estimated by Hölder's inequality and the estimate in (3.7). For the second sum, we recall that

$$P_0^n(\mathbf{x}) = \sqrt{2}\, T_n(x), \qquad P_n^n(\mathbf{y}) = \sqrt{2}\, T_n(y),$$

which implies that both polynomials are uniformly bounded by $\sqrt{2}$. By Hölder's inequality, we have for $i = 0, n$,

$$\begin{aligned}|a_i^n(g)| &= \left| \int_{[-1,1]^2} P_i^n(\mathbf{x})\, g(\mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x} \right| \\ &\leqslant \left( \int_{(-1,1]^2} |P_i^n(\mathbf{x})|^p\, W_0(\mathbf{x})\, d\mathbf{x} \right)^{1/p} \left( \int_{[-1,1]^2} |g(\mathbf{x})|^q\, W_0(\mathbf{x})\, d\mathbf{x} \right)^{1/q} \\ &\leqslant \sqrt{2}\, \|g\|_{\{W_{0,q}\}} = \sqrt{2}\end{aligned}$$

and

$$\begin{aligned}&\left| \sum_{k=1}^{N} f(\mathbf{x}_k)\, \lambda_k [P_0^n(\mathbf{x}_k) + P_n^n(\mathbf{x}_k)] \right| \\ &\leqslant 2\sqrt{2} \sum_{k=1}^{N} |f(\mathbf{x}_k)|\, \lambda_k \\ &\leqslant 2\sqrt{2} \left( \sum_{k=1}^{N} |f(\mathbf{x}_k)|^p\, \lambda_k \right)^{1/p} \left( \sum_{k=1}^{N} \lambda_k \right)^{1/q} = 2\sqrt{2} \left( \sum_{k=1}^{N} |f(\mathbf{x}_k)|^p\, \lambda_k \right)^{1/p}.\end{aligned}$$

Putting these estimates together, we conclude that

$$\left| \int_{[-1,1]^2} P(x)\, \mathbf{a}_n^{\mathrm{T}}(g)\, \mathbb{P}_n(\mathbf{x})\, W_0(\mathbf{x})\, d\mathbf{x} \right| \leqslant \left( \sum_{k=1}^{N} |f(\mathbf{x}_k)|^p\, \lambda_k \right)^{1/p};$$

the constant in front of the second integral is $1 = (2\sqrt{2})^2/8$. Therefore, by (3.6), we complete the proof of the theorem. ∎

*Proof of Theorem* 3.1. Applying the inequality in Theorem 3.3 to the polynomial $L_n f \in \mathscr{V}_n^2$, we see that $\|L_n f\|_{\{W_{0,p}\}}$ is finite for $f \in C[-1,1]^2$ for $p > 1$. Moreover,

$$\|L_n f\|_{\{W_{0,p}\}} \leqslant A_p \left( \frac{1}{N} \sum_{k=1}^{N} |P(\mathbf{x}_k)|^p \right)^{1/p} \leqslant A_p \|f\|_\infty,$$

where $\|\cdot\|_\infty$ is the uniform norm of $f$ on $[-1,1]^2$. Since $L_n f$ preserves polynomials in $\mathscr{V}_n^2$, in particular, polynomials in $\Pi_{n-1}^2$, it follows that for all $P \in \mathscr{V}_n^2$,

$$\|L_n f - f\|_{\{W_{0,p}\}} \leqslant \|L_n(f-P)\|_{\{W_{0,p}\}} + \|f-P\|_{\{W_{0,p}\}} \leqslant (1+A_p)\|f-P\|_\infty,$$

for $1 < p < \infty$. Since the subspaces of polynomials $\Pi_{n-1}^2$ are dense in $C[-1,1]^2$, the desired convergence is proved for $1 < p < \infty$. For $0 < p \leqslant 1$, it follows from the Hölder inequality that

$$\int_{[-1,1]^2} |L_n f|^p\, W_0\, d\mathbf{x} \leqslant \left( \int_{[-1,1]^2} |L_n f|^2\, W_0\, d\mathbf{x} \right)^{p/2} \left( \int_{[-1,1]^2} W_0\, d\mathbf{x} \right)^{1-p/2},$$

from which it follows from the boundedness of $\|L_n f\|_{\{W_{0,p}\}}$ that $\|L_n f\|_{\{W_{0,p}\}}$ is again bounded for $f \in C[-1, 1]^2$. The convergence follows similarly as in the case $p > 1$. ∎

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