To summarize, we have a number of

$$k \in \mathbb{N}_+ \backslash \{1\} \tag{1}$$

sailors (only one sailor is not interesting, so we remove this case from our discussions), the number of coconuts which were collected by the sailors before the night

$$n \in \mathbb{N}_+. \tag{2}$$

and the number of monkeys

$$m \in \mathbb{N}_+,\ m < k. \tag{3}$$

The first sailor who wakes up first divides the number of coconuts by the number of sailors. He gives the remaining coconuts to the monkeys. This results in k shares, which have the size (n-1)/k. The first sailor buries his (stolen) share, leaving behind the following number of coconuts:

$$n_1 = \frac{n-m}{k}(k-1) = (n-m)\frac{k-1}{k} \tag{4}$$

The second sailor, who wakes up, repeats the procedure of the first one:

$$n_2 = (n_1 - m)\frac{k-1}{k} = \big((n-m)\frac{k-1}{k} - m\big)\frac{k-1}{k} \tag{5}$$

$$= (n-1)\big(\frac{k-1}{k}\big)^2 - m\frac{k-1}{k} \tag{6}$$

Then, the third sailor wakes up and does the same again:

$$n_3 = \left(\left((n-m)\frac{k-1}{k} - m\right)\frac{k-1}{k} - m\right)\frac{k-1}{k} \tag{7}$$

$$= (n-m)\Big(\frac{k-1}{k}\Big)^3 - m\Big(\frac{k-1}{k}\Big)^2 - m\frac{k-1}{k} \tag{8}$$

So, for the i-th sailor, with $1 \leq i \leq k$, we find the general recursive rule:

$$n_i = (n_{i-1} - m)\frac{k-1}{k}. \tag{9}$$

For every sailor, we subtract off one from the previous number n_{i-1} and then multiply by $\frac{k-1}{k}$. This relationship can also be expressed as (for $i > 1$):

$$n_i = (n-m)\left(\frac{k-1}{k}\right)^i - m\left(\frac{k-1}{k}\right)^{i-1} - \ldots - m\frac{k-1}{k} \tag{10}$$

$$= (n-m)\left(\frac{k-1}{k}\right)^i - m\sum_{\substack{j=1\\i>1}}^{i-1}\left(\frac{k-1}{k}\right)^j \tag{11}$$

For our convenience, we define the initial number of coconuts as $n_0 = n$. Hence, we can write n_i for all $0 \leq i \leq k$ as:

$$n_i = \begin{cases} n, & \text{for } i = 0 \\ (n-m)\left(\frac{k-1}{k}\right)^i - m \sum\limits_{\substack{j=1 \\ i>1}}^{i-1} \left(\frac{k-1}{k}\right)^j, & \text{else} \end{cases} \tag{12}$$

In order to find an initial amount of coconuts n, we have to ensure that all n_i are dividable by k with a remainder of one (which is given to the monkey). Hence, it has to be ensured that:

$$\forall i \in \{0, \ldots, k\} : \; n_i \equiv m \mod k \tag{13}$$

Remember that in the morning, after all k sailors have woken up, the remaining coconuts are divided among all sailors. So, as stated above, also n_k has to have a remainder of one when divided by k. For $n_0 = n$ it is sufficient to ensure that n is a multiple of k plus m. For larger i, everything gets slightly more difficult. We have to find an n for Eq. (12) such that Eq. (13) is satisfied. To do so, let us first simplify the complicated expression (for now assuming $i > 1$) in Eq. (12) using the geometric series specified in the Appendix XXXXX:

$$(n-m)\left(\frac{k-1}{k}\right)^i - m \sum_{\substack{j=1 \\ i>1}}^{i-1} \left(\frac{k-1}{k}\right)^j \tag{14}$$

$$= (n-m)\left(\frac{k-1}{k}\right)^i - m \frac{\left(\frac{k-1}{k}\right)^i - \frac{k-1}{k}}{\frac{k-1}{k} - 1} \tag{15}$$

$$= (n-m)\left(\frac{k-1}{k}\right)^i - m \frac{\left(\frac{k-1}{k}\right)^i - \frac{k-1}{k}}{-\frac{1}{k}} \tag{16}$$

$$= (n-m)\left(\frac{k-1}{k}\right)^i + mk\left(\frac{k-1}{k}\right)^i - mk + m \tag{17}$$

$$= (n-m+mk)\left(\frac{k-1}{k}\right)^i - mk + m. \tag{18}$$

Now we can try to find an n that suffices Eq. (13):

$$(n - m + mk)\left(\frac{k-1}{k}\right)^i - mk + m \equiv m \mod k \tag{19}$$

$$(n - m + mk)\left(\frac{k-1}{k}\right)^i - mk \equiv 0 \mod k \tag{20}$$

$$(n - m + mk)\left(\frac{k-1}{k}\right)^i \equiv 0 \mod k \tag{21}$$

Note that we cannot get rid of the term mk, since the power of the fraction is smaller than one and not a natural number. The left-hand side of Eq. (21) should hence be a multiple of k. This can be expressed as:

$$(n - m + mk)\left(\frac{k-1}{k}\right)^i = r \cdot k, \tag{22}$$

where for now we choose $r \in \mathbb{Z}$ and then solve for n:

$$(n - m + mk)\left(\frac{k-1}{k}\right)^i = r \cdot k \tag{23}$$

$$n - m + mk = r \cdot k \left(\frac{k}{k-1}\right)^i \tag{24}$$

$$n = r \cdot k \left(\frac{k}{k-1}\right)^i - mk + m \tag{25}$$

$$n = r \cdot \frac{k^{k+1}}{(k-1)^i} - m(k-1). \tag{26}$$

Although we can now compute different values for n according to (26) which will suffice Eq. (21), we can observe a problem: For example, with $i = k = 3$ and $m = r = 1$ we receive a value of $n = 8.125$, which – although it suffices Eq. (21) – is not a natural number. This is due to the fraction in Eq. (26). However, we show in Appendix XYZABC that numerator and denominator are relatively prime to each other, hence, the fraction cannot be reduced. So, in order to obtain a natural number for n, we can drag the denominator to r and assume that r is a multiple of it by replacing $r/(k-1)^i$ with $q \in \mathbb{N}_+$:

$$n = r \cdot \frac{k^{k+1}}{(k-1)^i} - m(k-1) \tag{27}$$

$$= \frac{r}{(k-1)^i} \cdot k^{k+1} - m(k-1) \tag{28}$$

$$= q \cdot k^{k+1} - m(k-1). \tag{29}$$

Note that in Eq. (29) we have an expression which is independent of i and hence fulfills Eq. (13) for all $i \in \{2, \dots, k\}$.

You might have noticed that we actually only solved the problem for $i > 1$, since the sum in Eq. (14) is only evaluated for $i > 1$. However, Eq. (13) demands for all $i \in \{0, \dots, k\}$ – so also for $i = 0$ and $i = 1$ – that $n_i \equiv m \mod k$. We can directly see that Eq. (29) is also a valid solution for the case $i = 0$ and we can furthermore show that also the case $i = 1$ specified in Eq. (4) is only a special case of Eq. (26), with:

$$\begin{aligned}
(n-m)\frac{k-1}{k} &\equiv m \mod k && (30)\\
\Rightarrow (n-m)\frac{k-1}{k} &= s \cdot k + m && (31)\\
n &= s \cdot k \frac{k}{k-1} + m\frac{k}{k-1} + m && (32)\\
&= \frac{sk^2 + mk}{k-1} + m && (33)
\end{aligned}$$

and:

$$\begin{aligned}
\frac{sk^2 + mk}{k-1} + m &= r \cdot \frac{k^{k+1}}{(k-1)^1} - mk + m && (34)\\
sk^2 + mk &= rk^{k+1} - mk(k-1) && (35)\\
sk + m &= rk^k - mk + m && (36)\\
s &= rk^{k-1} - m \in \mathbb{N}. && (37)
\end{aligned}$$

Hence, if we insert s into Eq. (33) we obtain Eq. (26) which then leads to the general solution in Eq. (29).

Summary

Appendix

Geometric Series In this appendix we briefly mention several properties of the geometric series which are required for the derivations in other dependencies. For $q \neq 1$ and $j < N$ the geometric series can be derived as:

$$\begin{aligned}
S &= q^j + q^{j+1} + \cdots + q^{N-1}\\
qS &= q^{j+1} + q^{j+2} + \cdots + q^N\\
qS - S &= q^N - q^j\\
S(q-1) &= q^N - q^j\\
S &= \frac{q^N - q^j}{q-1} = \frac{q^j - q^N}{1-q}
\end{aligned} \qquad (38)$$

More generally, the geometric series can be written as:

$$a_0 \sum_{i=j}^{N-1} q^i = a_0 \frac{q^N - q^j}{q - 1} \tag{39}$$

Special cases

If the series starts with $i = 0$ we retrieve:

$$a_0 \sum_{i=0}^{N-1} q^i = a_0 \frac{1 - q^N}{1 - q} \tag{40}$$

For $q = 1$ we obtain:

$$a_0 \sum_{i=j}^{N-1} 1^i = a_0(N - j) \tag{41}$$

For an infinite series with $N = \infty$, convergence is achieved for values $|q| < 1$:

$$a_0 \sum_{i=j}^{\infty} q^i = a_0 \frac{q^j}{1 - q} \tag{42}$$

Powers of neighbored natural numbers are relatively prime

In this appendix we briefly show that two neighbored natural numbers k and $k+1$ are relatively prime, as well as their powers. This can be done with a proof by contradiction: Let us assume that there exists a divisor $t > 1$ for which:

$$k = q \cdot t \tag{43}$$

$$k + 1 = r \cdot t \tag{44}$$

This also implies:

$$r > q \tag{45}$$

$$r - q \geq 1 \tag{46}$$

We can also write:

$$q \cdot t + 1 = r \cdot t \tag{47}$$

$$1 = r \cdot t - q \cdot t \tag{48}$$

$$1 = (r - q) \cdot t \tag{49}$$

$$\Rightarrow r - q = 1 \wedge t = 1. \tag{50}$$

Since we required $t > 1$, we have a contradiction, which means that the two neighbored numbers k and $k+1$ are relatively prime and a fraction containing these two numbers in the numerator and denominator cannot be reduced. Furthermore, since the prime factorization of k and $k+1$ returns two disjoint sets of primes P_k and P_{k+1} ($P_k \cap P_{k+1} = \emptyset$), it can also be trivially shown that the powers k^i and $(k+1)^j$, with $i, j \in \mathbb{N}_+$ are also relatively prime.