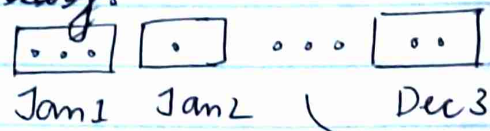
Lecture 3 : Birthday Problem, Properties of Probability

Birthday Problem

K people and we want to find probability that 2 have same birthday.
Exclude Feb 29, assume other 365 days are equally likely, assume independence of birthday.

Solu: If $K > 365$ (people), prob. is 1



People are labelled

(pigeonhole principle) [If we have more dots than boxes, then atleast one box will have more than one dots.]

Let $K \leq 365$ people.

$$P(\text{no match}) = \frac{365 \cdot 364 \cdot 363 \cdots (365-K+1)}{365^K}$$

of same birthday

If $K=1$, $P(\text{no match}) = \frac{365}{365^1} = 1$

$K=2$, $P(\text{no match}) = \frac{\cancel{365} \cdot 364}{365^{\cancel{2}}} = \frac{364}{365}$

$$P(\text{match}) \approx \begin{cases} 50.7\% & \text{, if } K=23 \\ 97\% & \text{, if } K=50 \\ > 99.999\% & \text{, if } K=100 \end{cases}$$

that two people have same birthday

that two people likely to have same birthday

Intuition: The basic intuition is that looking at K (number of people) is actually not a relevant quantity here. The most relevant quantity is not K but $\binom{K}{2}$, i.e., K pair of people.

$$\binom{K}{2} = \frac{K(K-1)}{2}$$

if we do $\binom{23}{2} = 253$ pair of people — anyone of these pair of people may have the same birthday

Non-naive definition of probability (Axioms): $0 \leq P(A) \leq 1$

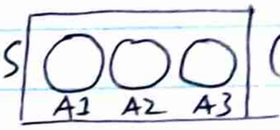
(I) $P(\phi) = 0$ (An empty set is an event, but it's an event that never occurs, so $P(\phi)=0$)

$P(S) = 1$ → full sample space

(II) $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$, if $A_1, A_2, \dots, A_n$ are disjoint events, i.e., they don't overlap

↓

can also applies to finite ~~events~~ union case



(finite case)

Consequences of above axioms properties:

① $P(A^c) = 1 - P(A)$

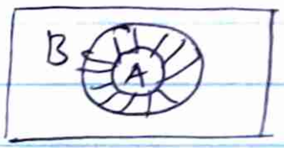
proof: $1 = P(S) = P(A \cup A^c)$

$= P(A) + P(A^c)$, since $A \cap A^c = \phi$ (Using axioms)

or, $\boxed{P(A^c) = 1 - P(A)}$

② If $A \subseteq B$ (if A occurs then B occurs but not vice versa), then $P(A) \leq P(B)$

proof: $B = A \cup (B \cap A^c)$

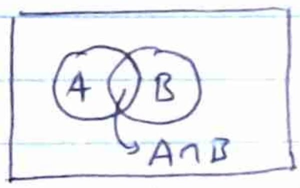


$P(B) = P(A) + P(B \cap A^c) \geq P(A)$

(from axiom 2) A and $(B \cap A^c)$ are disjoint

or, $\boxed{P(A) \leq P(B)}$

③ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Double counted the intersection

proof: $P(A \cup B) = P(A \cup (B \cap A^c))$

A & $(B \cap A^c)$ are disjoint, so we can apply axiom 2

or, $P(A \cup B) = P(A) + P(B \cap A^c)$

$\overset{?}{=} P(A) + P(B) - P(A \cap B)$ → We wish this is true but we don't know as of now whether this is true or not

↓ compare this with

$P(A) + P(B \cap A^c)$

It's true iff $P(B \cap A^c) = P(B) - P(A \cap B)$

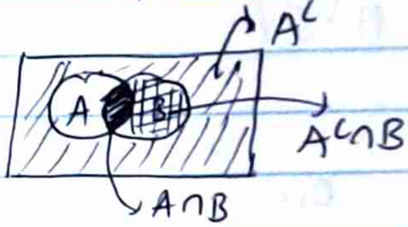
or, $P(A \cap B) + P(B \cap A^c) = P(B)$

This is true since

$(A \cap B)$ & $(A^c \cap B)$ are disjoint.

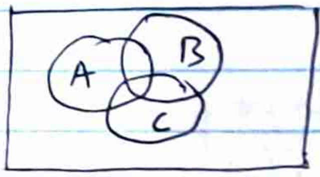
So, from second axiom of non-naive definition of probability, the result is $P(B)$.

Hence, proved.



Inclusion-exclusion

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



General form of Inclusion & Exclusion

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{j=1}^{n} P(A_j) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Alternate '+' & '−'

Example

deMontmort's Problem (1713), matching problem

originated in a gambling game

N cards, labelled $1, 2, \ldots, n$

find the probability that the position of a card in a deck is the number written on the card.

inclusion & exclusion is the easiest way to solve the problem

Solu: Let A_j be the event, i.e., jth card matches.
In other words, the jth card in the deck is the number j.

$P(A_1 \cup A_2 \cup \ldots \cup A_n) = ?$

↳ Probability that atleast 1 card matches

$P(A_j) = 1/n$ (Since all positions are equally likely for card labelled j)

$P(A_1 \cap A_2) = \frac{(n-2)!}{n!}$

1st card is labelled 1, second labelled 2, other (n-2) cards can be in any order whatsoever.

n! possible permutation of a deck of card, naive defn of probability

1st card has 1 labelled on it & 2nd card has 2 labelled on it

$= 1/n(n-1)$

$P(A_1 \cap A_2 \cap \ldots \cap A_k) = \frac{(n-k)!}{n!}$

first k cards are exactly 1 upto k ~~(ordered)~~ labelled correctly), remaining (n-k) cards can be in any order whatsoever

$\therefore P(A_1 \cup A_2 \cup \ldots \cup A_n) =$

$n \cdot 1/n - \binom{n}{2} \cdot 1/n(n-1) + \binom{n}{3} \, 1/n(n-1)(n-2) - \ldots$

possible 2 pairs

possible 3 pairs

$= n \cdot 1/n - \frac{n(n-1)}{2!} \cdot 1/n(n-1) + \frac{n(n-1)(n-2)}{3!} \cdot 1/n(n-1)(n-2)$

$= 1/1! - 1/2! + 1/3! - 1/4! + \ldots + (-1)^{n+1} \, 1/n!$

$\simeq (1 - 1/e)$

taylor series of e^x

Case when all of them match. There is only 1 way that could happen that the cards are 1 through n.