Lecture 2 : Story proofs, Axioms of Probability

Example

10 people, split ~~into~~ team of 6, team of 4. How many ways to do that?

Soln: $^{10}C_4 = {}^{10}C_6$ (pick a team of 6, whoever left is the team of 4)

$^{10}C_4 \to$ pick a team of 4, whoever left is the team of 6.

Example

10 people, split into 2 teams of 5. How many ways to do that?

Soln: $^{10}C_5/2$ [there is a clear difference between a team of 4 and a team of 6, but that's not the case here. Both teams are equivalent.]

from previous lecture

pick K times from a set of n objects, where order does not matter with replacement $\longrightarrow \binom{n+K-1}{K}$ ways.

Extreme cases

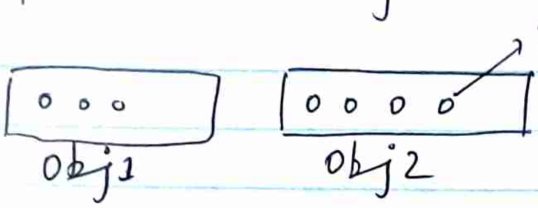
$K=0 \Rightarrow \binom{n-1}{0} = 1$ (~~If you it~~ If we have a group of people and we choose none of them there is 1 way to do that, i.e., don't choose them.)

$K=1 \Rightarrow \binom{n}{1} = n$ (it does not matter if it's ordered or unordered, with replacement or without replacement.)

$n=2 \longrightarrow$ simplest non-trivial example (special case)

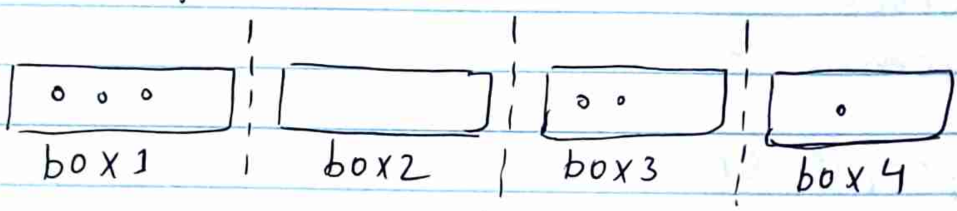
$$\binom{K+1}{K} = \binom{K+1}{1} = K+1$$

If we have 2 objects and we are picking k times.



$\#$dots is in $\underbrace{\{0, 1, \dots k\}}_{\text{set}}$

Proof :- Equivalently, how many ways are there to put k indistinguishable particles into n distinguishable boxes?



$n = 4$
$k = 6$

$$\underbrace{\circ\circ\circ \,|\,|\, \circ\circ \,|\, \circ}_{k \text{ dots}, (n-1) \text{ separators}} \longrightarrow \binom{9}{6} \text{ or } \binom{9}{3}$$

[Choose positions for dot($\cdot$), the remaining positions are position of '|' and vice versa]

$$\binom{n+k-1}{k} \text{ or } \binom{n+k-1}{n-1}$$

Bose-Einstein Condensate

Story proof - proof by interpretation, not by algebra or calculus

① $\binom{n}{k} = \binom{n}{n-k}$

② $n\binom{n-1}{k-1} = k\binom{n}{k}$ → pick k people out of n with 1 of them designated as President

Two different approaches:—

(i) we select who is in the club (i.e., k people) and then one of them elected as president. so, multiplied by k as we can select president from any of the k people.

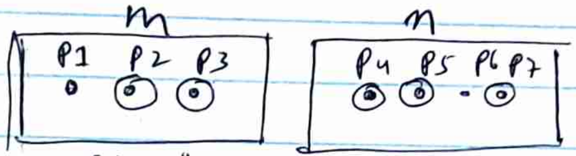
$$k\binom{n}{k} \text{ or } \binom{n}{k}\binom{k}{1}$$

③ $\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$

(Vandermonde Identity)

→ picking k people, out of $(m+n)$

m $\quad$ n



Pick j $\qquad$ Pick $k-j$

$j = 2$ (in this case) $\qquad$ $k-j = 3$

(ii) We select the president, then choose the $(k-1)$ people from the remaining people.

$n\binom{n-1}{k-1}$ or $\binom{n}{1}\binom{n-1}{k-1}$

for president $\qquad$ for remaing people

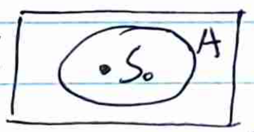
Non-naive definition of probability →

P must satisfies these axioms/rules —

Rule ① $P(\phi) = 0$, $P(S) = 1$

emptyset $\qquad$ ↳ impossible event

S



Rule ② $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$

(Countable infinitely many) $\qquad$ (if $A_1, A_2, \ldots A_n$ are disjoint) (non-overlapping)

we need a notion of probability space

→ A probability space consists of two ingredients S and P, where S is ~~the S~~ a sample space and P is a function that takes an event $A \subseteq S$ as an input, $P(A) \in [0,1]$ as output.

→ Here, S does not need to be finite as we have in naive definition of probability.