Lecture 28: Inequalities

Example (Conditional Expectation)

Store with a random number of customers, N.

Let X_j be the amount jth customer spends,

$N \stackrel{\text{!}}{=}$ # customers

X_j has mean μ and variance σ^2.

Assume $N, X_1, X_2, \dots$ are independent.

Find mean, variance of $X = \sum_{j=1}^{N} X_j$.

N → a Random variable

→ We're adding up a random number of random variables.

Solu: $E(X) = N\mu$ → X (a category error)

$E(X)$ → a number (Linearity)

$N\mu$ → a random variable

We wish that we knew the value of N, so that we could treat it like a constant. So, let's condition on N,

$$E(X) = \sum_{n=0}^{\infty} E(X|N=n)P(N=n) \quad \text{(Analogous to LOTP)}$$

→ This conditional expectation means we get to treat N that is known to equal n, so we know we have n customers now, in that case, we really can apply linearity.

$$= \sum_{n=0}^{\infty} \mu n P(N=n)$$

$P(N=n)$ → PMF(N)

$$= \mu \cdot E(N)$$

Adam's Law:

$$E(X) = E(E(X|N))$$
$$= E(\mu N)$$
$$= \mu E(N)$$

→ treat N as a known constant. By Linearity, $E(X|N) = \mu N$

It says the average amount of money that the store will take in is the average number of customers times the average amount that each customer spends.

$Var(X) = E(Var(X|N)) + Var(E(X|N))$ [EVE'S Law]

Note: variance of the sum of a fixed number of independent random variables is simply sum of their individual variances.
No covariance term → since they are independent.

If we treating N as a constant, the variance is N times the variance of one terms.

$$Var(X) = E(N\sigma^2) + Var(\mu N)$$
$$= \sigma^2 E(N) + \mu^2 var(N).$$

Statistical Inequalities

① Cauchy-Schwarz:

$$|E(XY)| \leq \sqrt{\underbrace{E(X^2)}_{\text{Marginal second moment of } X} E(Y^2)}$$

If X, Y uncorrelated, $E(XY) = E(X)E(Y)$.

Interpretation

If X, Y have mean 0, then $|Corr(X,Y)| = \left|\frac{E(XY) - E(X)^{0} E(Y)^{0}}{\sqrt{E(X^2)E(Y^2)}}\right| \leq 1$

so, In statistics, Cauchy-Schwarz means the correlation is between −1 and 1.

② Jensen's Inequality:

If g is convex, then $E\,g(X) \geq g(EX)$.

Example $EX^2 \geq (EX)^2$

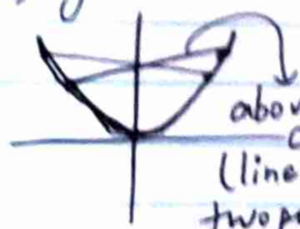
Determine if a function is convex

If the second derivative exists, $g''(x) \geq 0$

If h is concave, $Eh(X) \le h(EX)$.

$y = x^2$ Convex function

$y'' = 2$



above the curve (line through two points on the curve)

Example

Let X be positive.

$$E\left(\frac{1}{X}\right) \ge \frac{1}{E(X)}$$

$1/X$ is convex as long as X is +ve.

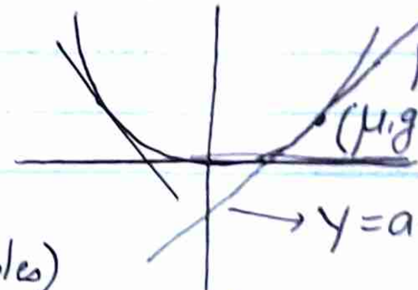
$$E(\ln X) \le \ln E(X)$$

Concave

<u>Proof of Jensen's inequality</u>

$$g(x) \ge a + bx$$

$$g(X) \ge a + bX$$ (also true as an inequality for random variables)



$$Eg(X) \ge E(a+bX) = a + bE(X)$$
$$= a + b\mu$$
$$= g(\mu)$$
$$= g(EX)$$

Any of above tangent lines we draw, it's going to stay below the curve.

③ <u>Markov Inequality</u>:

$$P(|X| \ge a) \le \frac{E|X|}{a}, \text{ for any } a > 0.$$

The strength of this inequality is not that it gives a good enough approximation, it is its simplicity and generality, that this is completely general for any random variable.

Proof: Fundamental Bridge

$P(|X| \geq a) = E I_{|X| \geq a}$ → Expected value of the indicator of the event $|X| \geq a$. If event occurs, it's 1, 0, otherwise.

$aP(|X| \geq a) = a E I_{|X| \geq a}$

~~$a E I_{|X| \geq a}$~~

$a I_{|X| \geq a} \leq |X|$ (always true)

If $I_{|X| \geq a} = 0 \Rightarrow 0 \leq |X|$ (true)

If $I_{|X| \geq a} = 1 \Rightarrow 1 \leq |X|$ (since $I_{|X| \geq a}$) (true)

So, $a E I_{|X| \geq a} \leq E|X|$

proved

Example
100 people. Is it possible that ~~95%~~ atleast 95% of the people are younger than average in group? (mean)

→ Yes, it's possible. One of these people is really, really old. That one person is going to pull up the average a lot.

Example
100 people. Is it possible that atleast 50% are older than twice the average age?

→ Impossible. Just those 50% people pulled up the average from what it was, which does not make sense.

μ → for 100 people
100μ

① Chebyshev's Inequality:

$$P(|X-\mu| \geq a) \leq \frac{Var(X)}{a^2}, \text{ for } \mu = E(X), a > 0$$

$$P(|X-\mu| \geq c\,SD(X)) \leq 1/c^2, \; c > 0$$

(crude upper bound)

This says the probability that X is more than 2 standard deviations away from ~~its~~ its mean is atmost $1/4$ (one quarter).

proof: Use Markov Inequality:

$$P(|X-\mu| \geq a) = P((X-\mu)^2 \geq a^2)$$

$$\leq \frac{E(X-\mu)^2}{a^2} = \frac{Var X}{a^2}$$