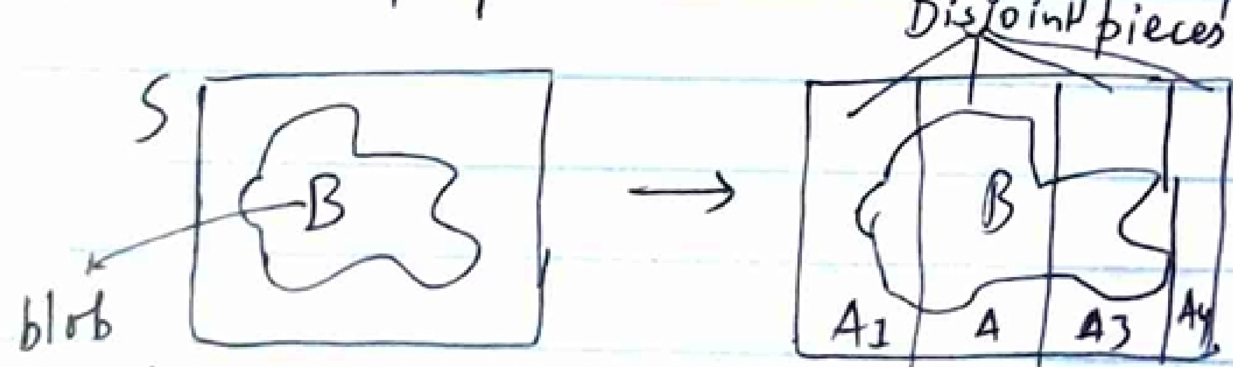
Lecture 5 : Conditioning Continued, Law of Total Probability

Thinking conditionally is a condition for thinking

How to solve a problem?

(1) try simple and extreme cases.

(2) break up problem into simpler pieces.



Let $A_1, A_2, \ldots, A_n$ be partition of S.

Then $P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n)$

$= P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2) + \cdots + P(B|A_n) \cdot P(A_n)$

Law of total probability

Example Get random 2-card hand from standard deck.

Find $P(\text{both aces} \mid \text{have ace})$,

$P(\text{both aces} \mid \text{have Ace of spades})$

at least 1 $= P(1\text{ Ace}) + P(2\text{ Ace}) + P(3\text{ Ace}) + P(4\text{ Ace}) = 1 - P(\text{no Ace})$

Soln:— $P(\text{both aces} \mid \text{have ace}) = \dfrac{P(\text{both aces} \cap \text{have ace})}{P(\text{have ace})}$

$\text{both aces} \cap \text{have ace} = \text{both aces}$

$$= \frac{P(\text{both ace})}{1 - {}^{48}C_2/{}^{52}C_2} = \frac{{}^{4}C_2/{}^{52}C_2}{1 - {}^{48}C_2/{}^{52}C_2} = 1/33$$

$P(\text{both aces} \mid \text{have Ace of spades})$

$$= \frac{{}^{3}C_1}{{}^{51}C_1} = 3/51 = 1/17$$

$\boxed{AS}$ $\boxed{?}$

Ace of spade — Any card other than Ace of spade

Examples Patient gets tested for disease afflicts 1% of population, test positive. Suppose test is advertised as "95% accurate", ~~spp~~ suppose this means

D : patient has disease

T : patient tests positive

$$P(T \mid D) = 0.95 = P(T^c \mid D^c)$$

$$P(D \mid T) = ?$$

$$P(D \mid T) = \frac{P(T \mid D) \cdot P(D)}{P(T)}$$

$$= \frac{P(T \mid D) * P(D)}{\underbrace{P(T \mid D) \cdot P(D) + P(T \mid D^c) \cdot P(D^c)}_{}}$$

Law of total probability

$$= \frac{0.95 \times 1/100}{0.95 \times 1/100 + 0.05 \times 99/100}$$

$$= 0.95/5.90 \approx 0.16$$

Common Mistakes with Conditional Probability

① Confusing $P(A|B)$, $P(B|A)$ ["Prosecutor's fallacy"]

Ex - Sally clark case (SIDS)

Assume that she is innocent, probability of a baby spontaneously dying for no apparent reason $= \underset{\text{(1st baby)}}{\frac{1}{8500}} \cdot \underset{\text{(2nd baby)}}{\frac{1}{8500}}$ (expert said) → this assume independence

$\approx \frac{1}{73 \times 10^6}$

p(evidence | innocence)

Relevant thing we want to compute P(innocence | evidence)

≈ 1 (prior prob. of innocence)

② Confusing $P(A)$ "Prior" → before we have evidence

with

$P(A|B)$ "Posterior" → After we have evidence

③ Confusing independence with conditional independence

Conditional Independence

Events A, B are conditionally independent given some other event C, if

$$P(A \cap B | C) = P(A|C) \cdot P(B|C)$$

Note :- Does conditional independence given C imply independence? ⟶ NO

Example

Chess opponent of unknown strength. May be that game outcomes are conditionally independent given strength of opponent but not independent unconditionally.

Note :- Does independence imply conditional independence given C? ⟶ NO

Example

A : Fire alarm goes off.

Caused by -

F : Fire

C : popcorn

Suppose F, C are independent,

But $P(F \mid \underbrace{A, C^c}_{A \cap C^c}) = 1$

Probability that there is a fire given that the alarm goes off, and no one's making popcorn

⟶ They are not conditionally independence given that the alarm goes off.