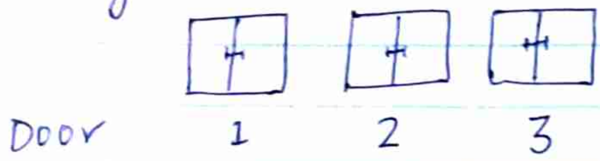
Lecture 6: Monty Hall, Simpson's Paradox

Monty Hall Problem



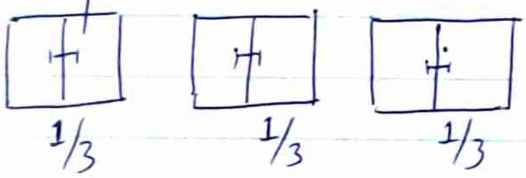
Keep your original choice or do you want to switch?

- Under these assumptions, we should switch. If ~~you~~ we switch, ~~your~~ probability of success is $2/3$ and if we stick with our original choice, our probability of success is $1/3$.

Assumption :-

- 1 door has car behind it, 2 doors have goats.
- Monty knows which door has car behind it.
- Monty always opens a goat door.
- If he has a choice of which door to open (in case when the player picks the right/car door initially, 2 remaining door has goat behind them), he picks with equal probabilities

Initial prob. →

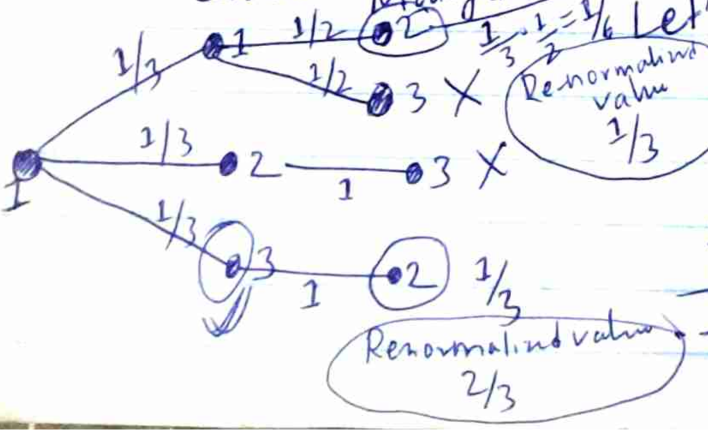


But that doesnot mean that conditionally after we observe what happens, it's still equally likely, because we have information.

Note : If Monty opened door 2, we know door 2 has a goat, and Monty opened door 2.

Tree diagram solution

Choose door 1 (contestant) → Conditioned on Monty opening door 2



Let's suppose Monty Hall opens door 2, then door 3 cases are irrelevant. Now, we renormalize.

$$\frac{1/6}{\frac{1}{3}+1/6}, \frac{1/3}{1/3+1/6} = \frac{1/6}{1/2}, \frac{1/3}{1/2}$$
$$= 1/3, 2/3$$

As per above tree ⇒ So, what this says — "Conditioning on Monty Hall opening Door 2, there is $2/3$ chance now that the car is behind Door 3 and there is a $1/3$ chance that it's behind Door 1".

$\therefore P(\text{success if switch} \mid \text{Monty opens door 2}) = 2/3$.

Conditional Probability Argument & Law of Total Probability (LOTP)

wish we knew where the car is —

S : Event that we succeed (assuming switch)

D_j : Door j has the car ($j \in \{1,2,3\}$)

$$P(S) = P(S|D_1) \cdot P(D_1) + P(S|D_2) \cdot P(D_2) + P(S|D_3) \cdot P(D_3)$$

$$\left[P(D_1) = P(D_2) = P(D_3) = 1/3 \right]$$

$$= P(S|D_1) \cdot 1/3 + P(S|D_2) \cdot 1/3 + P(S|D_3) \cdot 1/3$$

(Assuming we picked door 1 initially)

$$= 0 \cdot 1/3 + 1 \cdot 1/3 + 1 \cdot 1/3$$

we picked door 1 initially, & car is behind door 1 $\therefore P(\text{Switch}|D_1) = 0$ (bad case)

we picked door 1, & car is behind door 2, so it's a good idea to switch

same logic but this time car is behind door 3

$$= 2/3$$

$P(S) \rightarrow$ uncondition probability that our strategy will be successful.

By symmetry, $P(S \mid \text{Monty opens door 2}) = 2/3$

→ Door 2 and door 3 are completely symmetrical until Monty opens one of them. So, that means both the conditional and unconditional probabilities of success are $2/3$.

Simpson's Paradox

Is it possible to have two doctors where the first doctor has a higher success rate at every single possible type of surgery imaginable than the second one? yet, the second doctor, overall has a higher success rate?

Dr. Hibbert

	heart	bandage
Success	70	10
failure	20	0

Dr. Nick

	heart	bandage
Success	2	81
failure	8	9

Two types of surgery
(I) heart
(II) bandage removal

Hibbert succeeded 80 % of the time, while Nick's success rate is 83 %.

Conditional → Dr. Hibbert is better ~~than~~ in both surgeries.

Unconditional → Dr. Nick has higher percentage rate.

→ condition on which type of surgery.

$$\boxed{\frac{1}{3} + \frac{2}{5} \neq \frac{3}{8}}$$ ← (fraction Addition analogy)

Simpson's paradox in terms of conditional prob.

A: Event that surgery is successful

B: treated by Dr. Nick, $\bar{B}/B^c$: treated by Dr. Hibbert

C: heart surgery

$P(A|B\cap C) = P(A|B, C) < P(A|B^c, C)$

$P(A|B, C^c) < P(A|B^c, C^c)$

But, when we aggregate (don't cond. on the type of surgery, we are just looking at the overall)

$P(A|B) > P(A|B^c)$

← C is called the "confounder" or "control".

Because of the fact that if we don't condition on the type of heart surgery, what happens is knowing that we got treated by Dr. Nick gives us information about what type of surgery we had, which ~~in turn~~ then in turn effects the prob. of success.

$$P(A|B) = \underbrace{P(A|B,C)}_{< P(A|B^c,C)} \cdot P(C|B) + \underbrace{P(A|B,C^c)}_{< P(A|B^c,C^c)} \cdot P(C^c|B)$$

weights

$P(C|B)$ is prob. of performing heart surgery for Dr. Nick which is very ~~difficult from~~ different from the prob. of heart surgery for Dr. Hibbert. So, the weights changes and that's what enables Simpson's paradox to happen.