Geometry and Mechanics

Transformation notation

${}^{A}A_{B}$, ${}^{A}R_{B}$, ${}^{A}g_{B}$

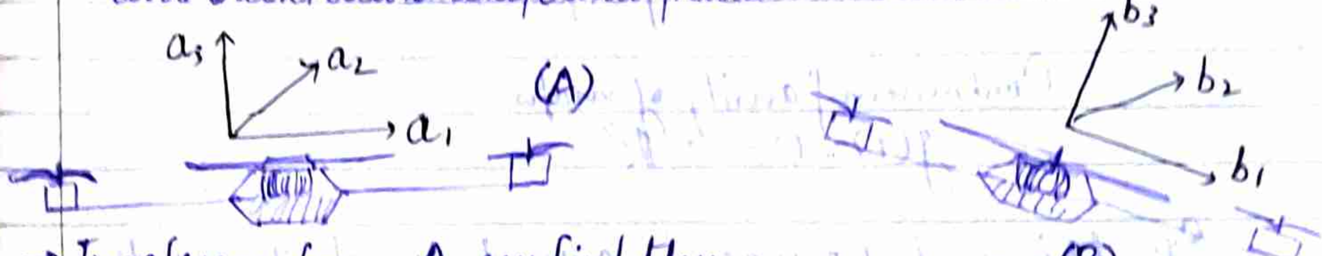
A_{ab}, R_{ab}

g_{ab}, h_{ab}

Quadrotor Kinematics

Rigid Body Transformations

Reference frame — we associate with each position and orientation a reference frame.



→ In reference frame A, we find three linearly independent basis vectors a_1, a_2 & a_3. Here, they are mutually $\perp^r$. while they don't have to be, it's convenient to choose them to be mutually ~~⊥~~ orthogonal.

→ b_1, b_2, b_3 is attached to frame B.

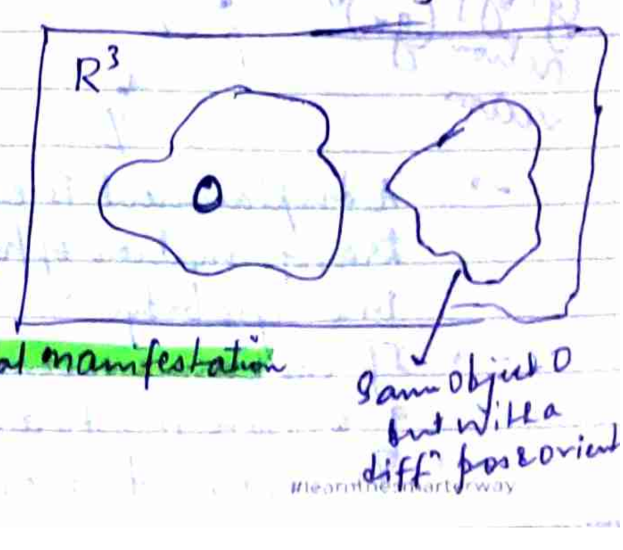
→ The key idea is that they must atleast be linearly independent.

→ we can write any vector as a linear combination of the basis vectors in either frame

$$v = v_1 a_1 + v_2 a_2 + v_3 a_3 \text{ (in frame A)}$$ → Similarly for frame B

Rigid Body Displacement

Object $O \subset \mathbb{R}^3$ (subset of $\mathbb{R}^3$)

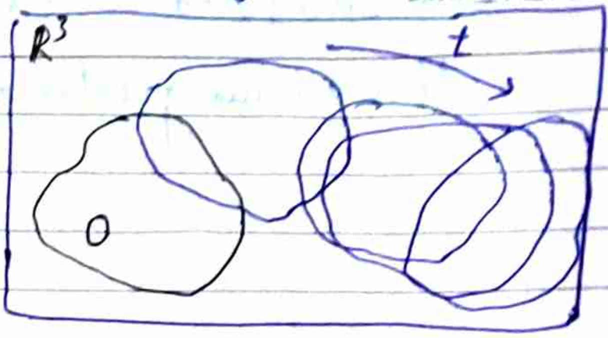


→ A rigid body displacement is nothing but a map from this collection of points in the object to its physical manifestation in $\mathbb{R}^3$ (Real Space).

$$g : O \longrightarrow \mathbb{R}^3$$

→ As time changes, this object may occupy different positions and orientations.

→ Accordingly, we have different rigid body displacements.



What is rigid body motion?

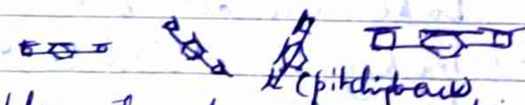
→ As time ~~passes~~ changes, we have a continuous family of maps, so g (which is a displacement) is now parametrized by time.

→ As time changes, we have the collection of points in O, moving from one position & orientation to another & so on. This is a continuous set of displacements.

Continuous family of maps
$$g(t): O \longrightarrow \mathbb{R}^3$$

Eg. this is what we see in a quadrotor, starting off from a a horizontal position, moving to another position, accelerating, changing its orientation, changing the dirⁿ of thrust & then reversing the dirⁿ of the thrust by pitching back & then slowing down to the goal position.

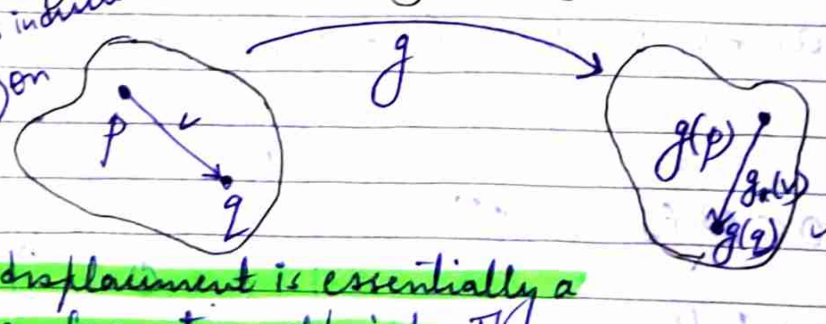
Each of these snapshots is a



displacement. This sequence of displacements represents a continuous family of displacements.

A single point on the rigid body p

Transformation (g) of points induces an action (g*) on vectors.



when this rigid body is displaced, the point p gets displaced to a new point, g(p)

→ A displacement is essentially a transformation of points. There are ∞ points in a rigid body.

→ If we have a second point q, the same displacement will take q & move it into a new point g(q).

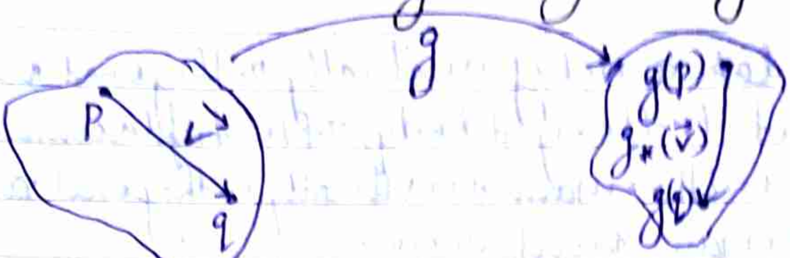
→ Every point defines a vector.

Since g moves p to $g(p)$ and $q \to g(q)$, it also moves the $\vec{v}$ to a new vector, $g_*(v)$.

So, displacement g induces a map on vectors

★ (displacement acts on points, but g^* acts on vectors)

What makes the map, g, a rigid body displacement?



Two properties the map must satisfy —

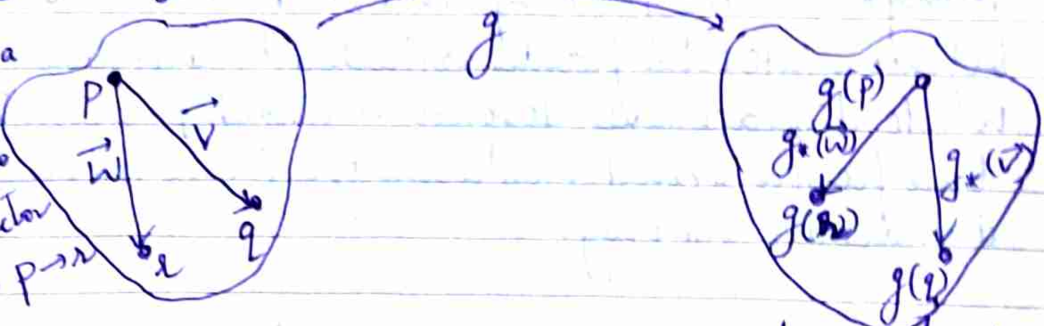
(1) the distance betⁿ any pair of points remain unchanged in a rigid body displacement, i.e.,

(length must be preserved) $\|g(p) - g(q)\| = \|p - q\|$ (defⁿ of the word "rigid")

(after the displacement) (before the displacement)

(11) (related to the cross products of vectors that are attached to the rigid body)

Let's choose a third point 'r' & generate a second vector going from $p \to r$



(cross products are preserved) $g_*(\vec{v}) \times g_*(\vec{w}) = g_*(\vec{v} \times \vec{w})$

cross product remain the same whether we do it before the displacement or after the displacement, if the displacement is rigid.

⟹ for a rigid body displacement,

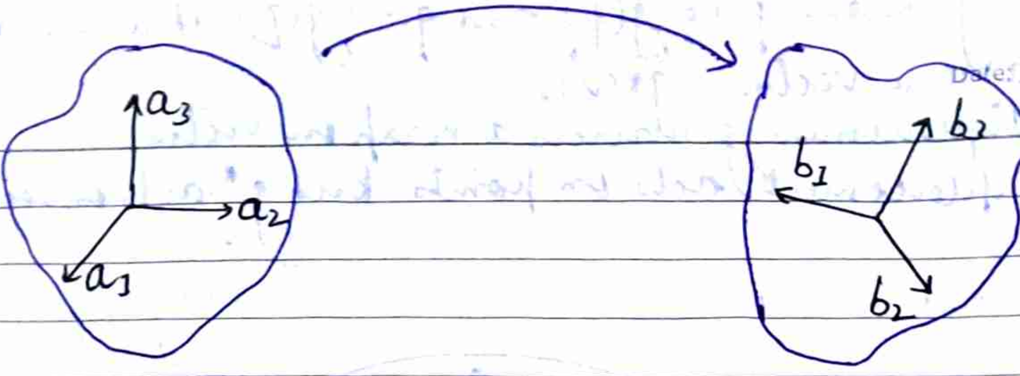
→ orthogonal vectors are mapped to orthogonal vectors.

→ g_* preserves inner products

$$g_*(\vec{v}) \cdot g_*(\vec{w}) = \vec{v} \cdot \vec{w}$$

~~(before di~~ (before displacement)

(After displacement)



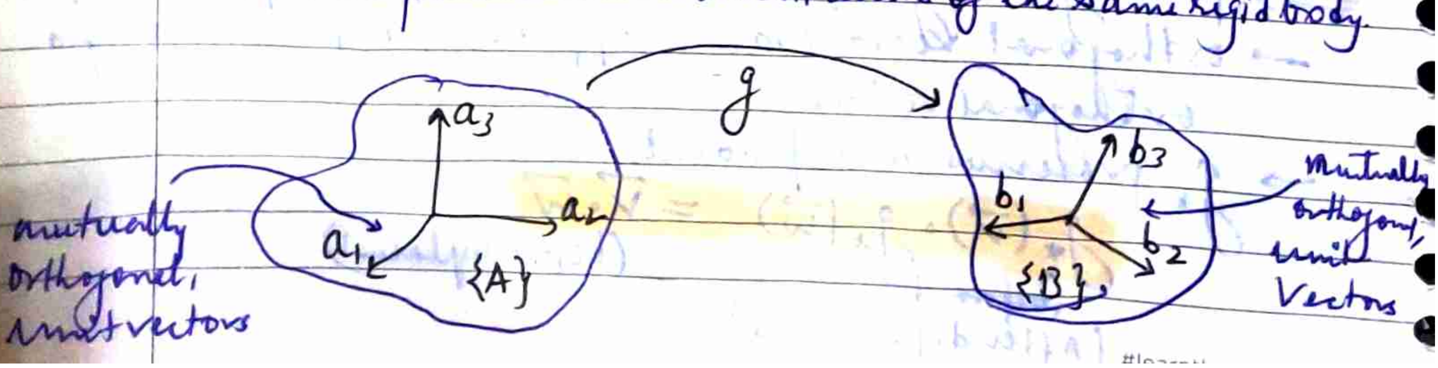
→ If we take a set of mutually orthogonal unit vectors attached to the rigid body, after displacement, those vectors will remain mutually orthogonal and they will remain unit vectors.

→ In summary, rigid body transformations or rigid body displacements satisfy two important properties. They preserve length and they preserve cross products.
(map preserves lengths) (cross products are preserved by the induced map)

<u>Note</u>: Rigid body displacements and rigid body transformation are used interchangeably. There is one important semantic difference.

Transformations generally used to describe relationship between reference frames attached to different rigid bodies while displacements describe relationships between two positions and orientation of a frame attached to a displaced rigid body.

<u>Calculation</u> :- (Assumption) we have mutually orthogonal unit vectors attached to every rigid body.
If it's a transformation, we are referring to two different rigid bodies. If it's a displacement, it's two distinct positions and orientation of the same rigid body.



mutually orthogonal unit vectors in one frame as a linear Combination of mutually orthogonal unit vectors in the other frame

$$\hat{b}_1 = R_{11}\hat{a}_1 + R_{12}\hat{a}_2 + R_{13}\hat{a}_3$$
$$\hat{b}_2 = R_{21}\hat{a}_1 + R_{22}\hat{a}_2 + R_{23}\hat{a}_3$$
$$\hat{b}_3 = R_{31}\hat{a}_1 + R_{32}\hat{a}_2 + R_{33}\hat{a}_3$$

(b_1, b_2, b_3) (a_1, a_2)

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$
(rotation matrix)

properties of a Rotation Matrix

(I) Orthogonal $\Rightarrow RR' = I = R'R$

(II) Special Orthogonal $\Rightarrow |R| = +1$

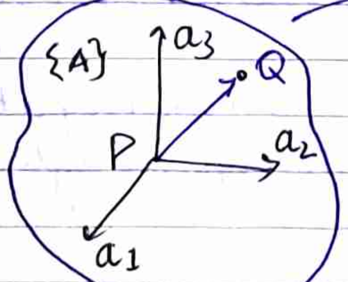
(III) Closed under multiplication

→ the product of any two rotation matrices is another rotation matrix.

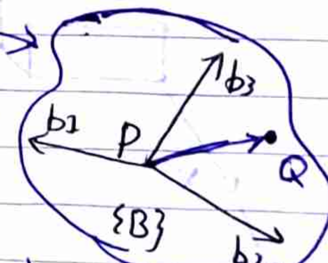
(IV) The inverse of a rotation matrix is also a rotation matrix.

Structure of a Rotation Matrix

two distinct posd. orient. of the same rigid body



g

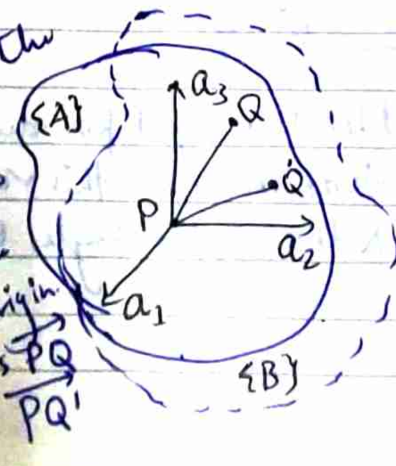


P is a reference point fixed to the object,
Q is a generic point attached to the object.

Linearly Combination

$$\vec{PQ} = q_1\hat{a}_1 + q_2\hat{a}_2 + q_3\hat{a}_3 \qquad \vec{PQ} = q_1\hat{b}_1 + q_2\hat{b}_2 + q_3\hat{b}_3$$

translate the rigid body so that the two vectors have the same origin. one vector is $\vec{PQ}$ other is $\vec{PQ'}$



$$\vec{PQ} = q_1\hat{a}_1 + q_2\hat{a}_2 + q_3\hat{a}_3$$
$$\vec{PQ'} = q_1'\hat{a}_1 + q_2'\hat{a}_2 + q_3'\hat{a}_3$$

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

How to write (q_1', q_2', q_3') as a function of (q_1, q_2, q_3)

OR $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix}$

$\Downarrow$

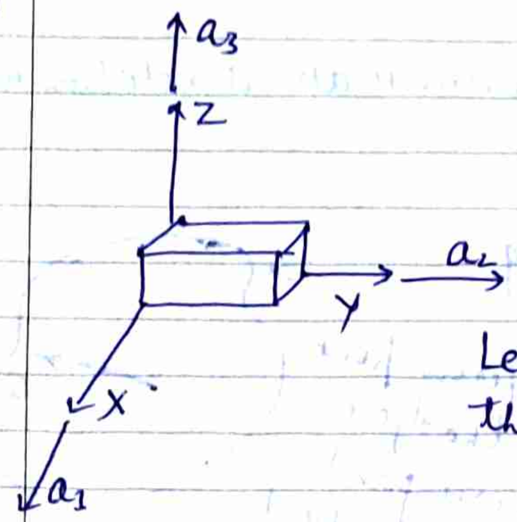
Rotation matrix.

i.e., $$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix}$$

(Rotation matrix)

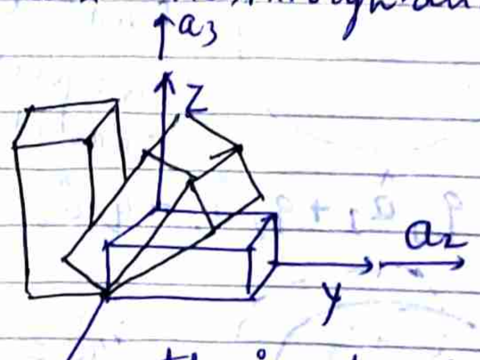
→ If we know how to write mutually orthogonal unit ~~[illegible]~~ vectors in one frame as a function of the other set of vectors (we can do this by calculating Rotation matrix). The same matrix tells us how to transform vectors in one frame to another frame.

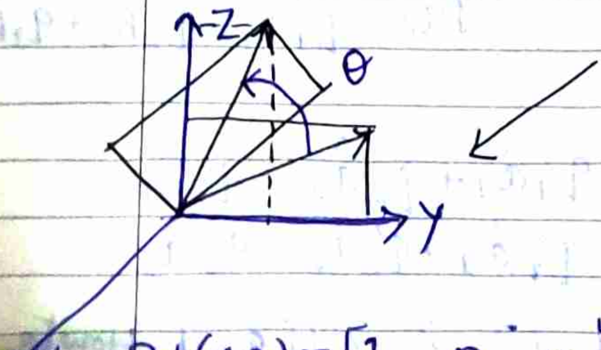
eg. Rotation – Rotation about the x-axis through θ.



Consider this rigid body, a rectangular ~~frame~~ prism, whose axes are aligned with X, Y & Z-axis or the a_1, a_2, a_3 unit vectors.

Let's rotate this rigid body about the X-axis, through an angle, θ.





X a_3

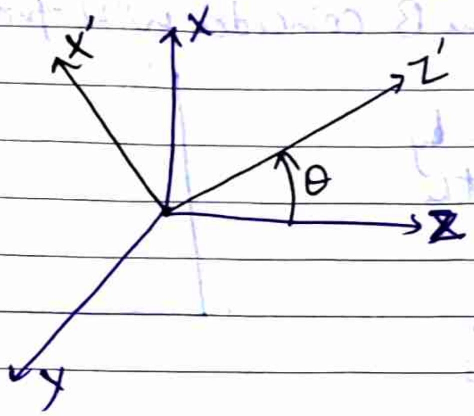
there is only 1 parameter that describes this rotation & that is the angle, θ.

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

#learnthesmarterway

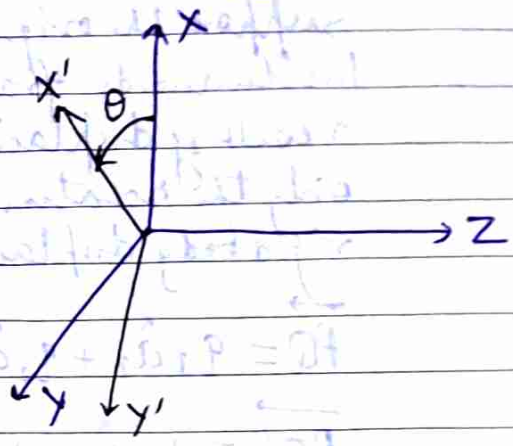
Rotation about the y-axis through θ

$$Rot(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

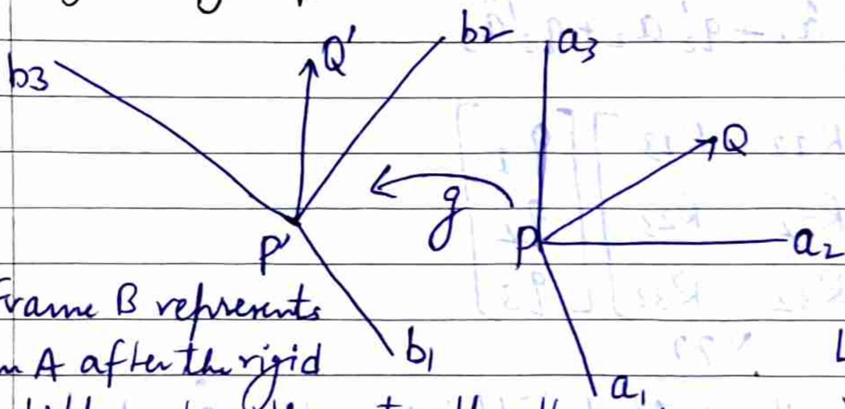


Rot. about Z-axis through θ

$$Rot(Z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Ex- Rigid Body Displacement



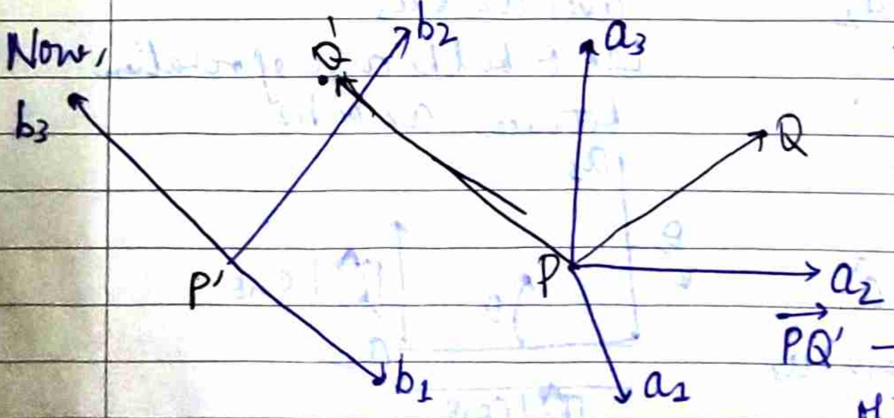
Let Frame A with mutually orthogonal unit vectors $\hat{a}_1, \hat{a}_2, \hat{a}_3$ with origin P attached to a rigid body.
Let Q be a point on this rigid body.
$\vec{PQ}$ → represents position of point Q relative to frame A.
Suppose the body undergoes a rigid body displacement 'g'.

$$\vec{PQ} = q_1\hat{a}_1 + q_2\hat{a}_2 + q_3\hat{a}_3$$

Let Frame B represents Frame A after the rigid body displacement with mutually orthogonal unit vectors $\hat{b}_1, \hat{b}_2$ & $\hat{b}_3$ with origin at P'.
point Q after the rigid body displacement is Q'.

$$\vec{P'Q'} = q_1\hat{b}_1 + q_2\hat{b}_2 + q_3\hat{b}_3$$

Now,



$\vec{PQ'}$ → gives us the position of the displaced point Q' relative to frame A.

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = R \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + d$$

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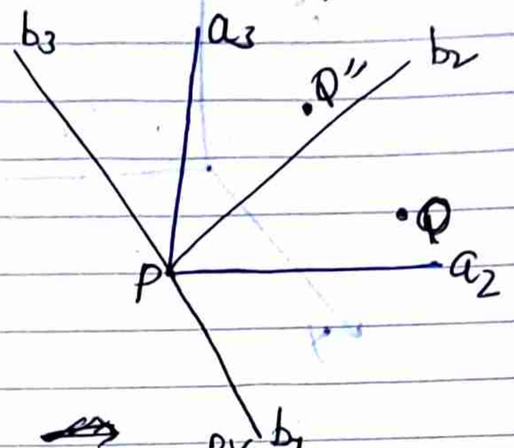
Translate {B} so reference frame share an origin.

<u>Rotation between frame A and frame B</u>

Suppose the origin of frame B coincides with frame A. In other words, frame B is a result of displacing frame A by only the rotation portion of the rigid body displacement.



$$\vec{PQ} = q_1\hat{a}_1 + q_2\hat{a}_2 + q_3\hat{a}_3$$

$$\vec{PQ''} = q_1\hat{b}_1 + q_2\hat{b}_2 + q_3\hat{b}_3$$

$$= q_1''\hat{a}_1 + q_2''\hat{a}_2 + q_3''\hat{a}_3$$

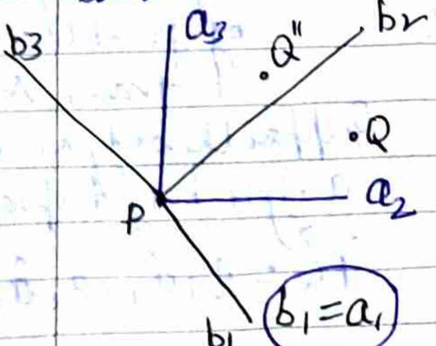
$\vec{PQ}$ is the p.v. of Q in {A}

PQ'' is the position vector of Q after the rotation in frame B / {B}

$$\begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

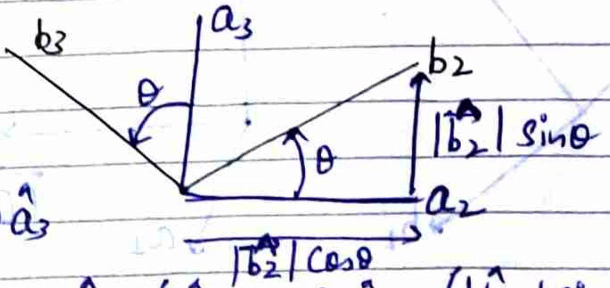
"??

we can easily see that, in this example, $\hat{a}_1$ & $\hat{b}_1$ are the same.



$\Rightarrow$ This means axes b_2 & b_3 are in the same plane as the axes a_2 & a_3.

Let θ be the angle of rotation between a_2 & b_2.



$$\hat{b}_3 = -(|\hat{b}_3|\sin\theta)\hat{a}_2 + (|\hat{b}_3|\cos\theta)\hat{a}_3$$

$$= -(\sin\theta)\hat{a}_2 + (\cos\theta)\hat{a}_3$$

$$\hat{b}_2 = (|\hat{b}_2|\cos\theta)\hat{a}_2 + (|\hat{b}_2|\sin\theta)\hat{a}_3$$

$$= (\cos\theta)\hat{a}_2 + (\sin\theta)\hat{a}_3$$

$|\hat{b}_1| = |\hat{b}_2| = |\hat{b}_3| = 1$ (unit vectors)

$$\overrightarrow{PQ}'' = q_1\hat{b}_1 + q_2\hat{b}_2 + q_3\hat{b}_3$$
$$= q_1''\hat{a}_1 + q_2''\hat{a}_2 + q_3''\hat{a}_3$$

$$\overrightarrow{PQ}'' = q_1(\hat{a}_1) + q_2[\hat{a}_2\cos\theta + a_3\sin\theta] + q_3[-a_2\sin\theta + a_3\cos\theta]$$

$$= q_1\hat{a}_1 + (q_2\cos\theta - q_3\sin\theta)\hat{a}_2 + (q_2\sin\theta + q_3\cos\theta)\hat{a}_3$$

$$\overrightarrow{PQ}'' = q_1''\hat{a}_1 + q_2''\hat{a}_2 + q_3''\hat{a}_3$$
$$\overrightarrow{PQ}'' = q_1\hat{a}_1 + (q_2\cos\theta - q_3\sin\theta)\hat{a}_2 + (q_2\sin\theta + q_3\cos\theta)\hat{a}_3$$

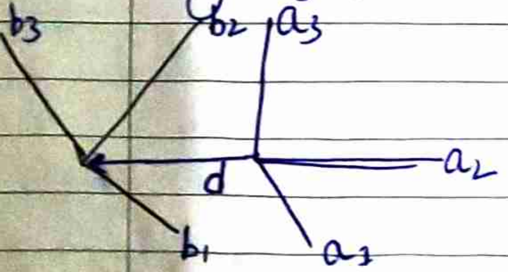
$$\therefore q_1'' = q_1$$
$$q_2'' = q_2\cos\theta - q_3\sin\theta$$
$$q_3'' = q_2\sin\theta + q_3\cos\theta$$

So, $$\begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\therefore R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = Rot(x,\theta)$$

Translation between frame A and frame B

Let d be the position vector from the origin of frame A to the origin of frame B, expressed in terms of frame A.



eg. $$\vec{d} = 1\hat{a}_1 - 3\hat{a}_2 + 1\hat{a}_3 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

→ we can characterize a rigid-body displacement with a rotation matrix and translation vector.

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Let $\theta = \pi/4 \Rightarrow R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

Suppose we know that the position vector $\vec{PQ}$ is given by component 1, 2, 1.

Components of $\vec{PQ'}$ in frame A / {A}

$$= \begin{bmatrix} 2 \\ -2.29 \\ 3.12 \end{bmatrix}$$

Qualitatively, we can see that point Q' relation to ~~point~~ {A} is in the position ~~of~~ $\hat{a}_1$, negative $\hat{a}_2$ & position $\hat{a}_3$ direction

→ If we know the components of $\vec{PQ'}$, we can invert the previous equation to find the components of $\vec{PQ}$.

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = R^T \left(\begin{bmatrix} q_1' \\ q_2' \\ q_3' \end{bmatrix} - \vec{d} \right)$$

Rotation

Special Orthogonal Matrices

$SO(3) = \{ R \in \mathbb{R}^{3\times3} \mid R^T R = R R^T = I, \det R = 1 \}$

→ Special orthogonal group in 3 dimensions (the group of rotations is called $SO(3)$)

Coordinates for $SO(3)$

1. Rotation matrices
2. Euler angles
3. Axis angle parameterization (that explicitly describes the axis of rotation & angle of rotation)
4. Exponential Coordinates
5. Quaternions

Coordinates for a Sphere

→ parameterize using a set of local coordinate charts (latitude and longitude)

Q) Given any point on the Earth surface, is there a unique combination of latitudes and longitudes that describe that point?

→ No, if we consider the two poles. The poles have a unique latitude but the longitudes are not well-defined.

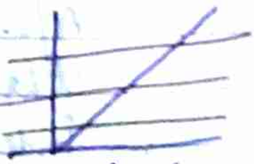
Function – A relation that assigns each element in a set of inputs, X called the domain to exactly one element in a set of output, Y called the codomain (or range).

$$f : X \to Y$$

One-to-One (injective) – $\forall a, b \in X$, if $f(a) = f(b)$, then $a = b$.
No two inputs from the domain will map to the same output in the codomain.

Onto (surjective) – $\forall y \in Y, \exists x \in X$ such that $f(x) = y$
Every output in the codomain has an input in the domain that maps to it.

→ If all imaginary lines parallel to x-axis intersect the graph at most once, then the function is 1-1

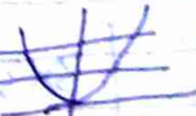


1-1

eg: $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \ln(x)$

This function is not onto

for any $y \le 0$, there is no x such that $\ln(x) = y$.



Not 1-1

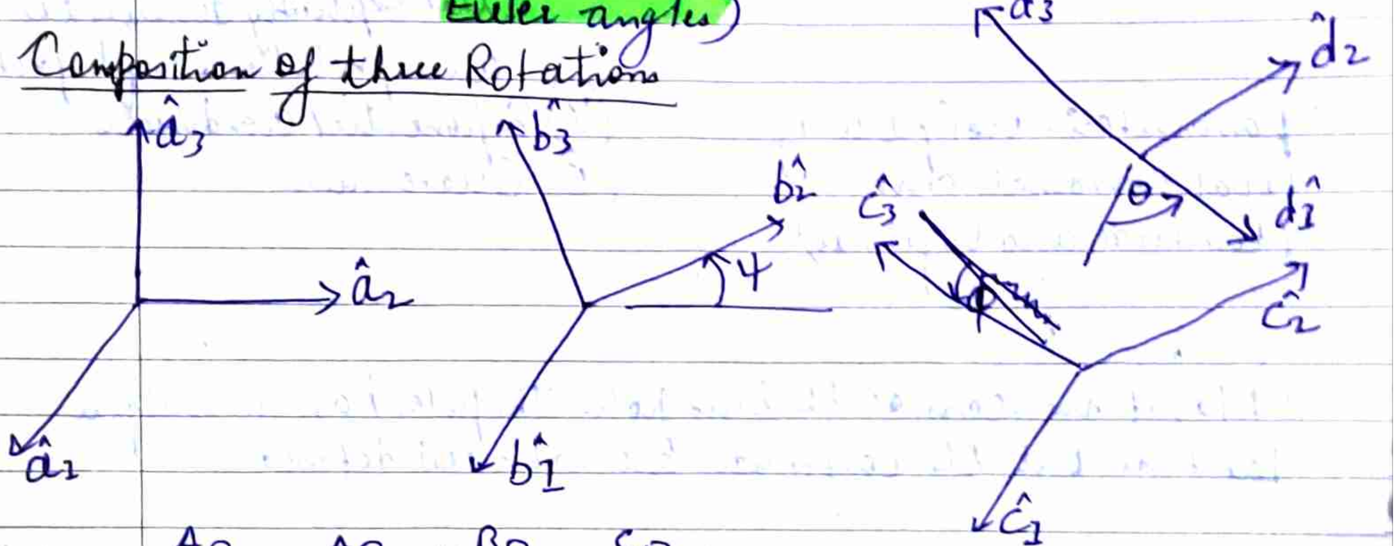
But, $f : \mathbb{R} \to (0, \infty)$ s.t. $f(x) = \ln(x)$ → is onto (the specified codomain no longer includes values $y \le 0$)

What is the minimum number of coordinates we need to parameterize $SO(3)$?

$$SO(3) = \{R \in \mathbb{R}^{3\times3} \mid R^TR = RR^T = I,\ \det R = 1\}$$

Euler Angles (Euler showed that 3 coordinates are necessary to describe a general rotation and these coordinates are called the Euler angles)

Composition of three Rotations



$${}^AR_D = {}^AR_B \times {}^BR_C \times {}^CR_D$$

$$\left[{}^AR_D = Rot(x,\underset{(Phi)}{\psi}) \times Rot(y,\underset{(Phi)}{\phi}) \times Rot(z,\theta)\right]$$

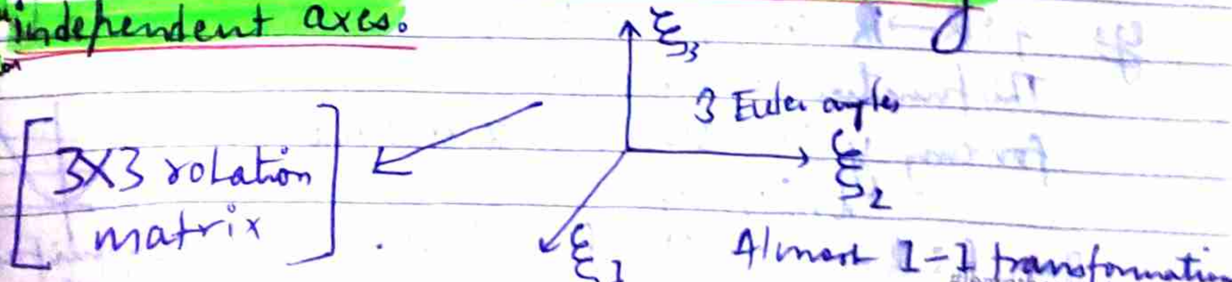
→ by simply multiplying rotations about the x-axis through ψ, the y-axis through ϕ and the z-axis through θ, we can get the rotation from frame $A \to D$.

$\phi \to$ Phi
$\psi \to$ Psi

$$\left[{}^AR_D = \underset{roll}{Rot(x,\psi)} \times \underset{pitch}{Rot(y,\phi)} \times \underset{yaw}{Rot(z,\theta)}\right]$$

→ A/c to Euler, any rotation can be described by three successive rotations about linearly independent axes.

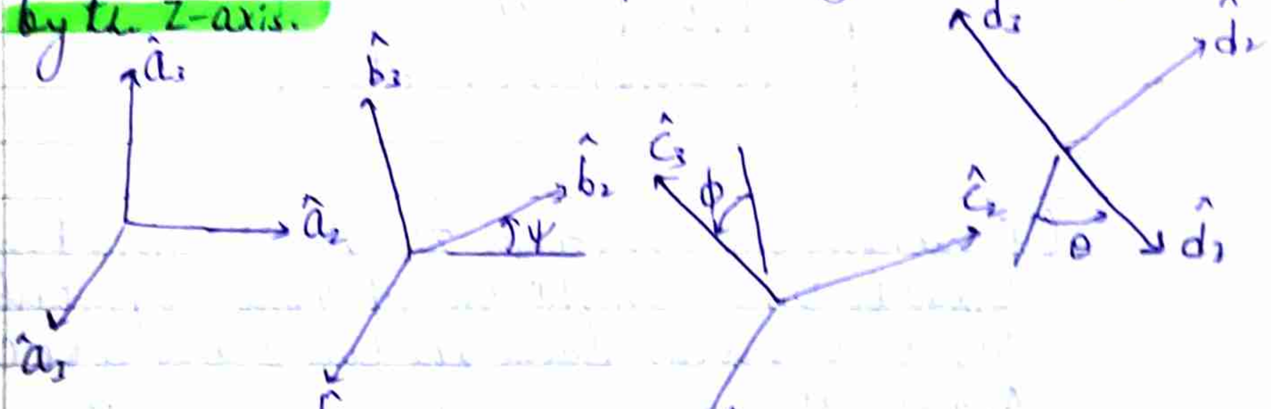
we can derive a 3x3 rotation matrix from the 3 Euler angles



Almost 1-1 transformation

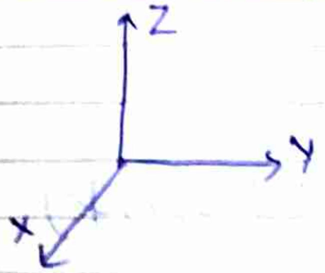
Is the reverse true? In other words, for every rotation matrix, is there a unique set of Euler angles? In other words, does this map between the 3 coordinates, the 3 Euler angles and the rotation matrix, is this map 1-1?

Ans - NO. It's almost one-to-one, but there are certain points that are analogous to the North Pole and the South Pole on the Earth's surface, at which the Euler angles are not well defined. This set of Euler angles is often called the $\boxed{\text{X-Y-Z Euler angles}}$ because the sequence in which rotation matrices are multiplied relates to a rotation about the x-axis, followed by the Y-axis, followed by the Z-axis.



$$\left[{}^{A}R_{D} = \underset{\text{roll}}{Rot(x, \psi)} \times \underset{\text{pitch}}{Rot(y, \phi)} \times \underset{\text{yaw}}{Rot(z, \phi)} \right]$$

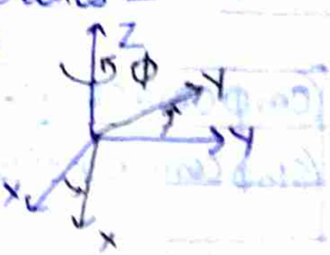
We can also have other types of Euler angles. This particular one is called a Z-Y-Z Euler angles.



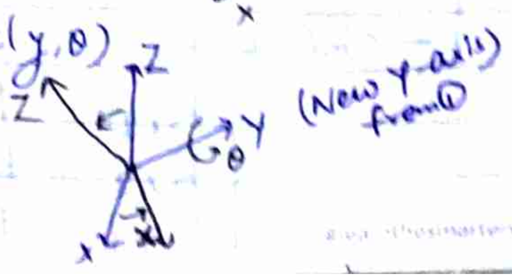
$$R = Rot(z, \phi) \times Rot(y, \theta) \times Rot(z, \psi)$$

Sequence of three rotations about body-fixed axes -

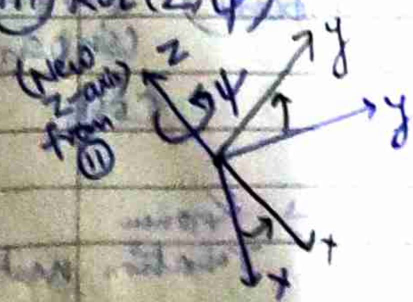
(I) $Rot(z, \phi)$



(II) $Rot(y, \theta)$



(III) $Rot(z, \psi)$



- $Rot(z, \phi)$
- $Rot(y, \theta)$
- $Rot(z, \psi)$.

— Are these linearly independent? (Although we have two rotations about z-axis, the question we have to ask is are they the same z-axis or different?)

→ we can see that the two z-axes are not collinear they are independent.

→ If this condition is satisfied, then the 3 angles are Euler angles and they can parameterize the set of rotations, i.e.,

Three Euler angles
- ϕ, θ, ψ
- parameterize rotations

<u>Note</u> - $\theta = 0$ is a special (singular) case
(if we have an Euler angle zero, in this case, θ being equal to 0, we might ~~end up~~ run into problems).
So, $\theta = 0$ would cause the two z-axes to be collinear
As a result, we will not have the condition of linear independence.
In other words, the 3 axes about which we are performing rotations are no longer independent.
So, $\theta = 0$ is analogous to be being at the North or the South pole on the Earth surface.

<u>Determination of Euler angles</u>

$$R = Rot(z, \phi) \times Rot(y, \theta) \times Rot(z, \psi)$$

$$R = \begin{bmatrix} (\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi) & (-\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi) & (\cos\phi\sin\theta) \\ (\sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi) & (-\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi) & (\sin\phi\sin\theta) \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \rightarrow \text{Known rotation matrix}$$

$R_{33} = \cos\theta \longrightarrow \theta = ?$

$R_{31} = -\sin\theta\cos\psi$
$R_{32} = \sin\theta\sin\psi$
] Anyone can be used to get ψ value, given θ is known from above

$R_{13} = \cos\phi\sin\theta$
$R_{23} = \sin\phi\sin\theta$
] $\phi = ?$, given θ (from above) (Anyone can be used)

If $|R_{33}| < 1$, $\theta = \sigma\arccos(R_{33})$, $\sigma = \pm 1$

$$\psi = \mathrm{atan2}\left(\frac{R_{32}}{\sin\theta}, \frac{-R_{31}}{\sin\theta}\right)$$

$$\phi = \mathrm{atan2}\left(\frac{R_{23}}{\sin\theta}, \frac{R_{13}}{\sin\theta}\right)$$

Two sets of Euler angles for every R for almost all R's!

(we have two pieces of information for both ψ & ϕ & because of that we don't use the standard inverse tangent function)

→ we use something called the atan2 func. The atan2 func helps us overcome the ambiguity that exists in inverse tangent function.

If $R_{33} = 1$,

$$R = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi + \sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$f(\phi+\psi)$

Infinite set of Euler angles

If $R_{33} = -1$,

$$R = \begin{bmatrix} -\cos\phi\cos\psi - \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi & 0 \\ \cos\phi\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$f(\phi+\psi)$ → If $R_{33} = \pm 1$, then in those cases θ is going to be equal to either 0 or π and we have two alternatives.

→ In both these cases, we will see from the grouping of terms that the rotation matrices are functions only of the sum of two angles, ψ and ϕ.
In other words, it's impossible to determine either ~~ϕ~~ ψ or ϕ uniquely.

~~The atan2 function~~

atan(arctangent) function

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$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

$\theta = \tan^{-1}(y/x)$ returns the angle θ for which $\tan\theta = y/x$

→ The inverse tangent function is implemented in programming languages as the 'atan' function.

$$\text{atan}(y/x) = \tan^{-1}(y/x)$$

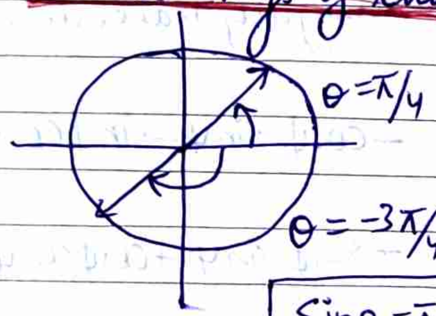
$\tan \pi/6 = 1/\sqrt{3}$

↓

$\tan^{-1}(1/\sqrt{3}) = \pi/6$

→ Shortcomings of this function –

(I)



$\sin\theta = \sqrt{2}/2$
$\cos\theta = \sqrt{2}/2$
$\tan\theta = 1$

$\tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}\left(\frac{-1}{-1}\right)$

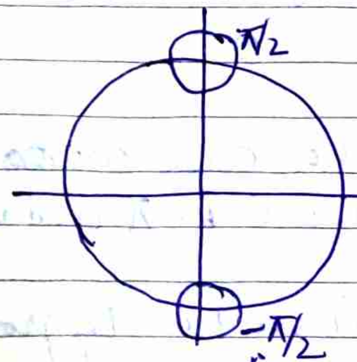
$\sin\theta = \frac{-\sqrt{2}}{2}$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\tan\theta = 1$

The atan function cannot distinguish between these two angles.
In fact, the atan function cannot distinguish between any pair of points opposite from each other on the unit circle.

(II)



→ A second problem with the atan function occurs at the angles $-\pi/2$ and $\pi/2$.

$\cos(-\pi/2) = \cos(\pi/2) = 0$

$\frac{\sin\theta}{\cos\theta} = y/x = \frac{\pm 1}{0}$ = undefined.

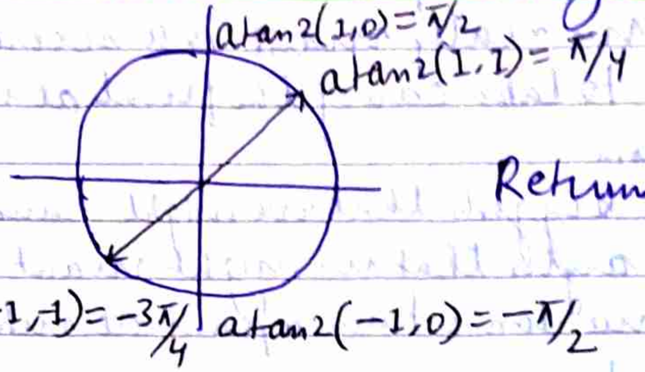
'atan' function fails at these points.

Returns values in range $(-\pi/2, \pi/2)$

To avoid these problems, the atan function can only return angles in the range $(-\pi/2, \pi/2)$ (non-inclusive)

atan2 function

→ atan2(y, x) is an implementation of the atan function that takes into account ratio and signs of y and x.



Returns values in range $[0, 2\pi)$

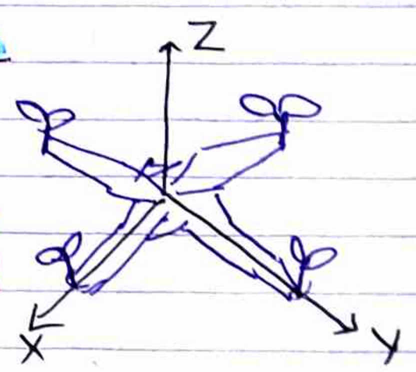
→ Given one we can determine the other but it's impossible to disambiguate between ϕ and ψ because the groupings only contain terms that combine them.

→ So, when $R_{33} = \pm 1$, we have infinite set of Euler angles.

Note – In most of our work, we use a different set of Euler angles.

Z–X–Y Euler Angles

→ so-called because the 3 rotation matrices we consider are the rotation about the Z-axis through ψ, followed by the rotation about the X-axis through ϕ and then the rotation of the y-axis through θ, i.e.,



Sequence of three rotations about body-fixed axes

- Rot (z, ψ)
- Rot (x, ϕ)
- Rot (y, θ)

Verify (by multiplying the rotations, we get a rotation matrix of below form)

$$R = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi - c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\theta s\phi c\psi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$

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→ for every set of Euler angles, we might have at least two solutions for Euler angles, for a given rotation matrix. At some points, we can have infinite solutions.

→ what we really want is a second set of Euler angles to take care of the point at which we have ∞ solutions.

So, this suggests that we might have many sets of Euler angles that we might want to consider so that we don't have a point at which we have ∞ solutions.

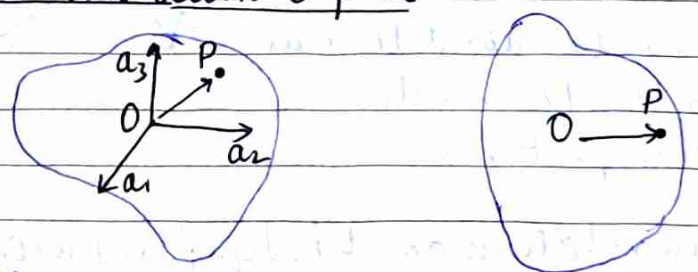
Q. What is the minimum number of set of Euler angles we need to cover all of the rotation group $SO(3)$?

$$SO(3) = \{R \in \mathbb{R}^{3\times3} \mid R^T R = R R^T = I\}$$

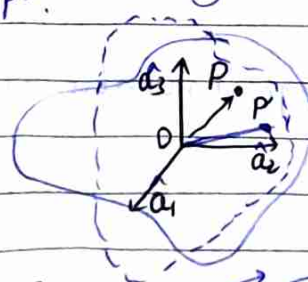
Euler's Theorem

Rotations — Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O.

Rotation with O fixed



Now, translate the rigid body from second position & orientation so that the two origins are identical so that point P moved to new point P'.



$$\vec{OP} = p_1\hat{a}_1 + p_2\hat{a}_2 + p_3\hat{a}_3$$
$$\vec{OP'} = q_1\hat{a}_1 + q_2\hat{a}_2 + q_3\hat{a}_3$$

$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{\vec{q}} = \underbrace{\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}}_{\text{Rot. matrix}} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{\vec{p}}$$

$$\vec{q} = R\vec{p}$$

Q Is there ~~any~~ an axis that remains fixed under rotation? In other words, is there a point P such that after the rotation through R, it stays fixed, i.e., Point P is invariant to the rotation?

Proof of Euler's Theorem

$$\vec{q} = R\vec{p}$$

Is there a point $\vec{p}$ that maps onto itself?

→ If there were such a point p, then $\vec{p} = R\vec{p}$ (this is nothing but a statement that 1 is the eigenvalue of the rotation matrix R)

Solve eigenvalue problem

$$R\vec{p} = \lambda\vec{p}$$

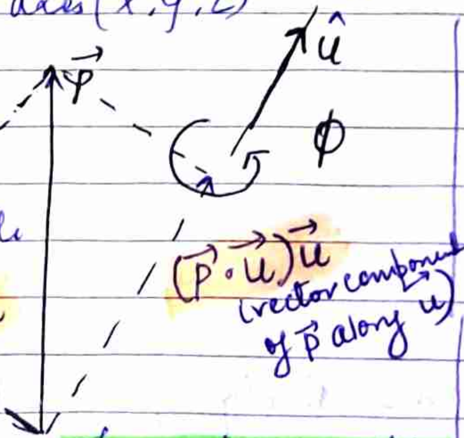
(verify $\lambda = 1$ is an eigenvalue for any R)

How does one find the rotation matrix for a general axis and angle of rotation?

Note - we already know the answer if the axis of rotation is one of the coordinate axes $(\hat{x}, \hat{y}, \hat{z})$

Suppose we have an arbitrarily oriented axis given by the unit vector $\hat{u}$ and an arbitrary angle of rotation, ϕ. Let's consider a generic vector $\vec{p}$

$\vec{p} - (\vec{p}\cdot\vec{u})\vec{u}$



$\vec{p}\cdot\vec{u} = p \cos\theta$, $|\vec{u}| = 1$

proj. of $\vec{p}$ on $\hat{u} = |\vec{p}|\cos\theta = |\vec{p}|\dfrac{\cos\theta\,|\vec{u}|}{|\vec{u}|} = \dfrac{\vec{p}\cdot\vec{u}}{|\vec{u}|}$

Let's see how the vector $\vec{p}$ gets rotated if the rigid body to which it's attached gets rotated about the axis $\hat{u}$ through an angle ϕ

The component of the vector that's along the axis is never changed by the rotation - ①

→ In other words, the vector $(\vec{p}\cdot\vec{u})\vec{u}$ remains unchanged by the rotation R.

→ But the ⊥ component which is along $\vec{v}$ will get rotated

Let's use $\vec{u}, \vec{v}$ & orthogonal vector $\vec{w}$, where $\vec{w} = \vec{u} \times \vec{v}$ as local coordinate system or local basis

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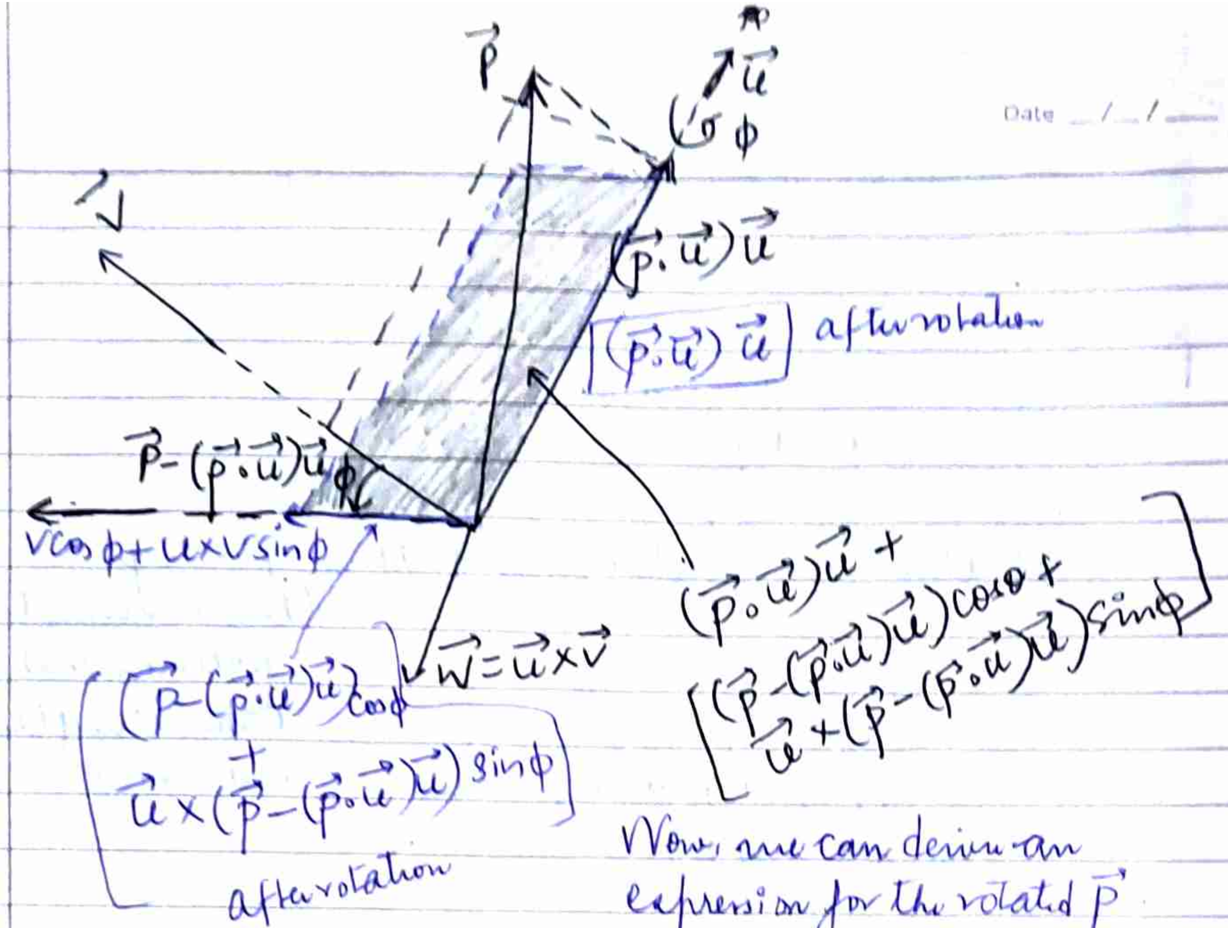

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Now, we can derive an expression for the rotated $\vec{p}$

$$\vec{p}\cos\phi + \vec{u}\vec{u}^T(1-\cos\phi)\vec{p} + \vec{u}\times\vec{p}\sin\phi$$

(the expression holds for any vector of $\vec{p}$ since it's a generic vector)

Let $\hat{u} = \boxed{\vec{u}\times}$

↳ skewed symmetric matrix whose components are the same as components of $\vec{u}$

(matrix ⇒ { Symmetric, $A^T = A$; Skew-symmetric, $A^T = -A$ })

must have all zeros on the diagonal

General expression for 3×3 Skew-Symmetric matrix

$$A = \begin{bmatrix} 0 & -A_{21} & A_{13} \\ A_{21} & 0 & -A_{32} \\ -A_{13} & A_{32} & 0 \end{bmatrix}$$

(only has 3 independent parameters)

We can concisely represent a skew-symmetric matrix as a 3×1 vector:

$$A = \begin{bmatrix} 0 & -A_{21} & A_{13} \\ A_{21} & 0 & -A_{32} \\ -A_{13} & A_{32} & 0 \end{bmatrix}$$

$$\Downarrow$$

$$a = \begin{bmatrix} A_{32} \\ A_{13} \\ A_{21} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

We use the hat operator to switch between these two representation

$$\hat{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

<u>Vector Cross Product</u> — The hat operator is also used to denote the cross product between two vectors —

$$\vec{u} \times \vec{v} = \hat{u}\vec{v}$$

$$= \underbrace{\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}}_{[u \times]} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

<u>Representation of Angular Velocities</u>

angular velocity vector: $\hat{w}^b = R^T\dot{R}$

$\hat{w}^s = \dot{R}R^T$

$R^T\dot{R}$ and $\dot{R}R^T$ are skew-symmetric.

———— x ———— x ————

<u>Axis/Angle to Rotation Matrix</u>

Rotation of a generic vector $\vec{p}$ about $\vec{u}$ through ϕ. | Axis of rotation $\vec{u}$ | Rotation angle ϕ

$$Rp = p\cos\phi + uu^T(1-\cos\phi)p + \hat{u}p\sin\phi$$

Rodrigues' formula

$$\boxed{Rot(u,\phi) = I\cos\phi + uu^T(1-\cos\phi) + \underbrace{\hat{u}}_{\text{skew-symmetric } 3\times3}\sin\phi}$$

So, given a generic vector $\vec{u}$ and a rotation angle ϕ, we can derive the rotation matrix using this closed form expression.

④ Set $\hat{u}$ to be a unit vector along x (or y or z). Verify result is the same as $Rot(x,\phi)$.

② Is the ~~ang~~ (axis, angle) to rotation matrix map onto? ~~?~~ 1-1?

Euler's theorem

$Rot(u, \phi)$ and $Rot(-u, 2\pi - \phi)$?

restrict ϕ to the interval $[0, \pi]$?

☆ for a given rotation matrix, can we extract $\vec{u}$ and $\vec{v}$ from this matrix?

Let's extract this axis and the angle from the rotation matrix, R.

verify, $\cos\phi = \dfrac{\tau - 1}{2}$, τ is the trace of the rotation matrix.

$\tau = (R_{11} + R_{22} + R_{33})$

If above is true, then ~~we~~.

from Rodrigues's formula,

$$\hat{u} = \frac{1}{2\sin\phi}(R - R^T)$$ (we get u, without solving for eigenvector)

→ (axis, angle) to rotation matrix map is many to 1.

→ restricting angle to the interval $[0, \pi]$ makes it 1-1

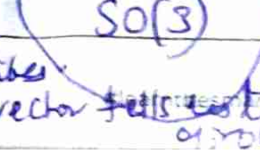
except for

$\tau = 3 \Rightarrow \phi = 0 \Rightarrow$ no unique axis of rotation

$\tau = -1 \Rightarrow \phi = \pi \Rightarrow$ two axis of rotation (either the axis we obtained using the formula above ~~we~~ or the opposite) i.e., u or $-u$

✎ The best way to think about the rotation group, SO(3) is by imagining a solid ball of radius π.

Any point on the ball's surface or inside the ball essentially represents a rotation whose angle is given by the radius and the vector that describes the point with the origin of sphere as the origin of the vector ~~tells~~ tells the axis of rotation.



Again, it seems like there is 1-1 map between points on the surface of the ball or inside the ball & the set of rotations.
This is true, except for points that are on the surface of the ball.

So, if we look at any pair of points on the ball, but they are diametrically opposite, they essentially correspond to the same rotation.

Rotations and angular Velocities

Rate derivatives of rotations

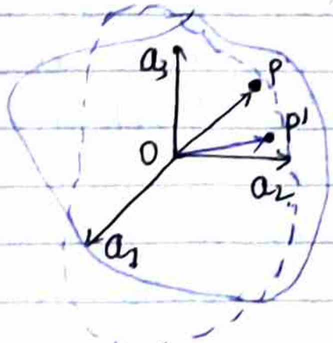
Rotation matrix, $R(t)$

Orthogonality

$$R^T(t)R(t) = I \xrightarrow{\frac{d}{dt}(\cdot)} \dot{R}^T R + R^T \dot{R} = 0$$
$$R(t)R^T(t) = I \qquad\qquad \dot{R}R^T + R\dot{R}^T = 0$$

$R^T\dot{R}$ and $\dot{R}R^T$ are skew symmetric.
(In other words, there ~~are~~ are only 3 independent elements in these skew symmetric matrices).

Rotation with O fixed



$$\vec{OP} = p_1\hat{a}_1 + p_2\hat{a}_2 + p_3\hat{a}_3$$
$$\vec{OP'} = q_1\hat{a}_1 + q_2\hat{a}_2 + q_3\hat{a}_3$$

$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{q} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{12} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{p}$$

$$\boxed{q(t) = R(t)p}$$

Constant coordinates of P in body-fixed frame

Changing coordinates of P as the rigid body rotates

diff. both sides, we get

$$\dot{q} = \dot{R}p \quad (p = \text{const.})$$

(velocity in inertial frame)

position in body-fixed frame

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$\underbrace{R^T\dot{q}}$ = $\underbrace{R^T\dot{R}}p$ (premultiply both sides by the transpose of the rotation matrix)

velocity in body fixed frame

$\hat{\omega}^b$ encodes angular velocity in body-fixed frame

$\dot{q} = \underbrace{\dot{R}R^T}q$ $\hat{\omega}^s$ → not in body frame, but spatial angular velocity

(velocity in inertial frame)

encodes angular velocity in inertial frame

→ we have seen that skew-symmetric matrices encode cross products. So, above two equations is essentially our ability to generate velocities by taking the cross product of an angular velocity vector with a position vector.

(I) In the first eqn. it's the angular velocity in the body-fixed frame which yields the velocity in the body-fixed frame.

(II) In the second eqn, it's the angular velocity in the inertial frame which then yields the velocity in the inertial frame.

So, we have two different representations of the same angular velocity vector.

The first one is written in term of basis vectors on the body-fixed frame, and the second one is written in terms of basis vectors in inertial frame.

Q) What's the angular velocity for a rotation about Z-axis?

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \dot{R} = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\theta}$$

Angular velocity for a rotation about the Z-axis

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T \dot{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

$$= \dot{R}R^T = \dot{\theta} \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}$$

$R^T\dot{R} = \dot{R}R^T$ this happens in this special case when axis of rotation is const.

In this particular case, this symmetric matrix corresponds to the (0, 0, 1) vector which is the Z-axis. If we rotate a rigid body about the Z-axis and the axis of rotation is constant, then the angular velocity vector will also be along the Z-axis.

Two Rotations

What happens if our rotation is obtained by composing two rotations?

$$R = R_Z(\theta) R_x(\phi)$$

(In this case the rotations are about the Z-axis through θ, about the x-axis through ϕ)

$$\hat{\omega}^b = R^T \dot{R} = (R_Z R_x)^T (\dot{R}_Z R_x + R_Z \dot{R}_x)$$

$$= \underbrace{R_x^T R_Z^T \dot{R}_Z R_x}_{\text{depends only on the rate of change of } R_Z}, \underbrace{+ R_x^T \dot{R}_x}_{\text{depends only on the rate of change of } R_x},$$

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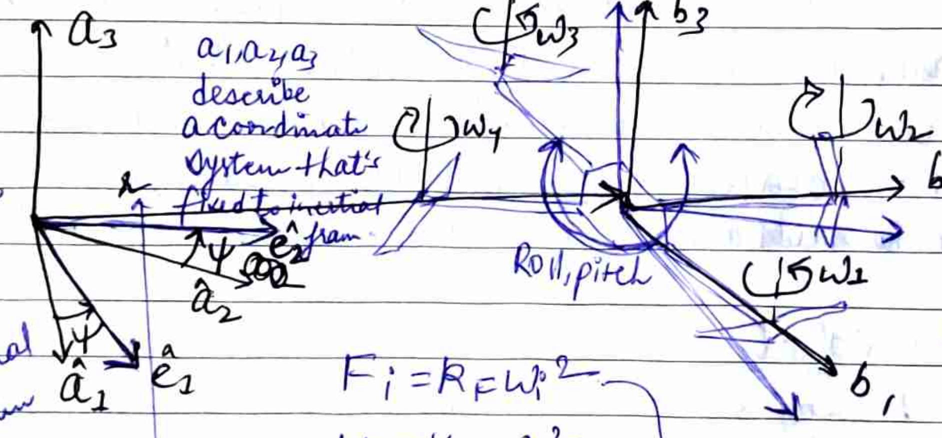
Spatial angular velocity

$$\hat{\omega}^s = \dot{R}R^T = (\dot{R}_Z R_x + R_Z \dot{R}_x)(R_Z R_x)^T$$
$$= \underbrace{\dot{R}_Z R_Z^T}_{\text{depends on the rate of change of } R_Z} + \underbrace{R_Z \dot{R}_x R_x^T R_Z^T}_{\text{depends on the rate of change of } R_x}$$

Above two expression are different. In both cases, they consist of two terms. One depends on $\dot{\theta}$ and the second depends on $\dot{\phi}$ but because the axis of rotation is not fixed, the two expressions are different.
The body fixed angular velocity and the spatial angular velocity have different expressions.

Dynamics of a Quadrotor

two coordinate systems, one attached to the moving robot & the other, the inertial coordinate system



b_1, b_2, b_3 constitute the set of unit vectors that describe the body fixed coordinate system.

position vector of the center of mass

$F_i = k_F \omega_i^2$ → expression of the thrust that the motors produce on the airframe

$M_i = k_M \omega_i^2$ (reaction moments)

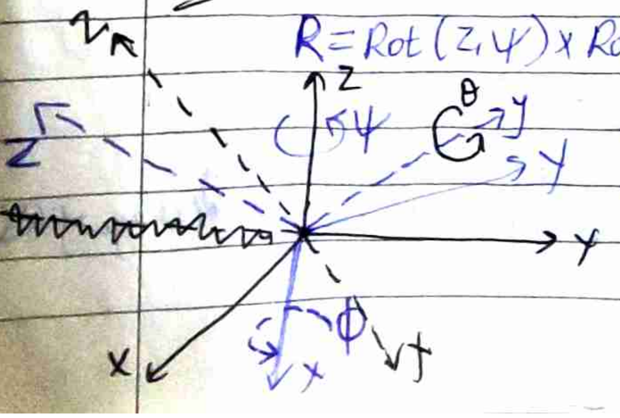
$\omega_i \Rightarrow$ angular speed of the rotor

Euler Angles

ZXY convention used

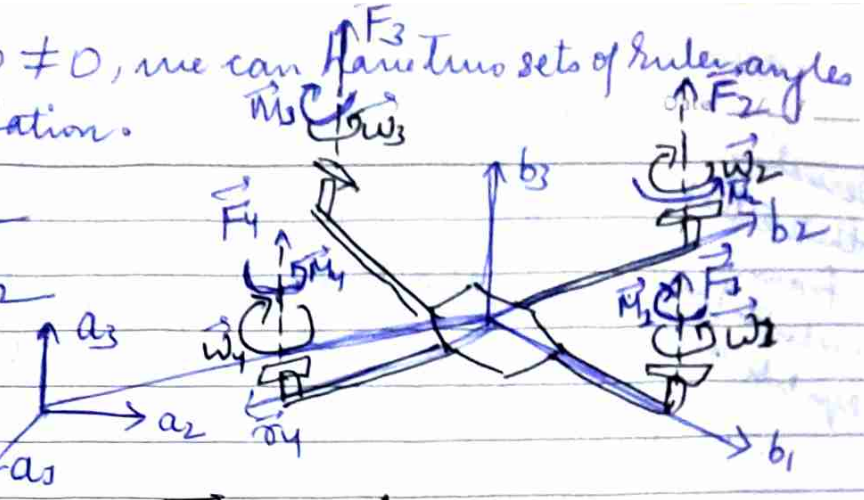
Note Z-X-Y Euler Angles
- Rot(z, ψ)
- Rot(x, ϕ)
- Rot(y, θ)

$R = Rot(z,\psi) \times Rot(x,\phi) \times Rot(y,\theta)$



→ Singularities occur when the roll angle is equal to 0 (i.e. $\phi = 0$)

Even when $\phi \neq 0$, we can have two sets of Euler angles for every rotation.



$F_i = k_F \omega_i^2$

$M_i = k_M \omega_i^2$

External forces & Moments

$$\vec F = \vec F_1 + \vec F_2 + \vec F_3 + \vec F_4 - mg\,\hat a_3$$

$$\vec M = \vec r_1 \times \vec F_1 + \vec r_2 \times \vec F_2 + \vec r_3 \times \vec F_3 + \vec r_4 \times \vec F_4 + \vec M_1 + \vec M_2 + \vec M_3 + \vec M_4$$

reaction moments

moment of thrust force F_4

→ To predict the net acceleration, we have to write down the equations of motion. These come from Newton & Euler and they are called the Newton-Euler Equation

Newton-Euler Equation

Newton's Eqn of Motion for a Single Particle of mass m

Resulting force acting on a single particle

$\vec F = m\vec a$ (Newton's Second Law)

~~System of Particles~~

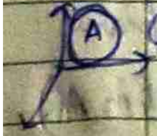
Newton's Second Law for a System of particles

Centre of mass →

$$r_C = \frac{1}{m} \sum_{i=1,N} m_i \vec p_i$$

(Weighted sum of all the position vectors)

→ The center of mass for a system of particles, S, accelerates in an inertial frame (A) as if it were a single particle with mass m (equal to the total mass of the system) acted upon by a force equal to the net external force.



$$\vec F = \sum_{i=1}^{N} \vec F_i = m \frac{{}^A d\; {}^A\vec v^{\,C}}{dt}$$

the Superscript A refers to the fact that we are computing all these quantities in an inertial frame A

the Superscript C refers to the fact that we are computing the velocity of the Center of mass

rate of change of linear momentum

Rate of Change of Linear momentum

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Derivation in the inertial frame (in order for this eqn to be valid)

$$\vec{F} = \frac{{}^{A}d\vec{L}}{dt}$$, Linear momentum of the system of particles in the inertial frame A.

(Also true for a rigid body) → nothing but a set of infinite particles, all glued together rigidly.

So, if above eqn is valid for a set of particles, it must also be valid for a collection of ∞ particles.

What's the equivalent for rotational motion?

Rotational Equations of motion for a rigid body.

The rate of change of angular momentum of the rigid body B relative to C in A is equal to the resultant moment of all external forces acting on the body relative to C

* (Differentiation must be done in an inertial frame)

$$\frac{{}^{A}d\ {}^{A}\vec{H}_C^{B}}{dt} = \vec{M}_C^{B}$$, $\vec{H}$ is the angular momentum of the rigid body B, with the origin C, the centre of mass, in the inertial frame A.

Net moment applied to the rigid body.

Angular momentum (a 3D vector)

$$\boxed{{}^{A}\vec{H}_C^{B} = \vec{I}_C \circ {}^{A}\vec{\omega}^{B}}$$

(these computation have to be done in 3D)

$\vec{\omega}$ → a 3D vector

inertial tensor with C as the origin $\vec{I}$ → a 3D vector.

${}^{A}\vec{\omega}^{B}$ → also obtained in the inertial frame (angular velocity of B in A)

Net moment from all ext. forces & torques about reference C (Take all the ext. forces, compute their moments & add these moments to all ext. couples or torques.

→ Here also the trailing subscript C tell us that we are computing with C as an origin.

Principal Axes and Principal Moments

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Principal axis of inertia

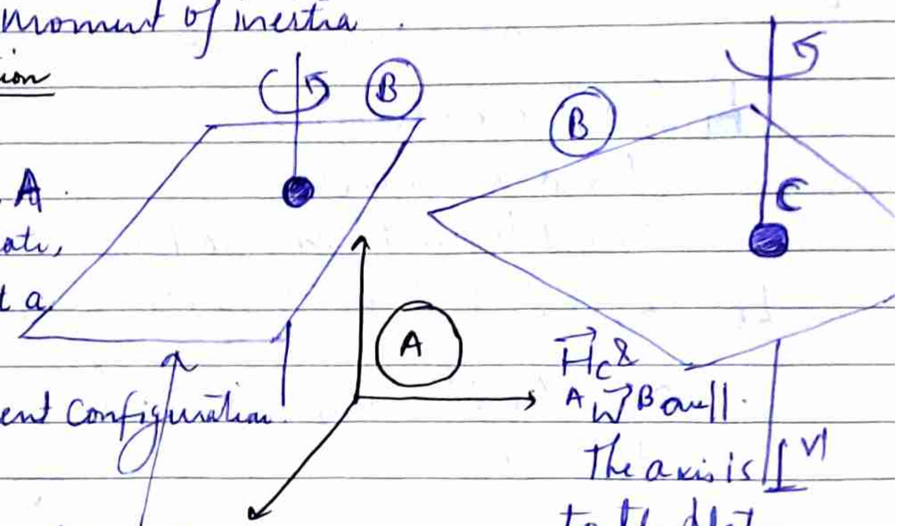
$\hat{u}$ is a unit vector along a principal axis if $\vec{I}\cdot\hat{u}$ is parallel to $\hat{u}$. There are 3 independent principal axes (In other words, there are 3 independent axes such that $\vec{I}$ times the unit vector along that axis will give us a vector that's parallel to that axis.)
The moment of Inertia is essentially the scaling term

Principal moment of inertia

The moment of inertia w.r.t. a principal axis, $\vec{u}\cdot\vec{I}\cdot\vec{u}$, is called a principal moment of inertia.

Physical Interpretation

we have a rigid frame, A.
we have a parallel plate, that's spinning about a vertical axis, but there are two different configuration.

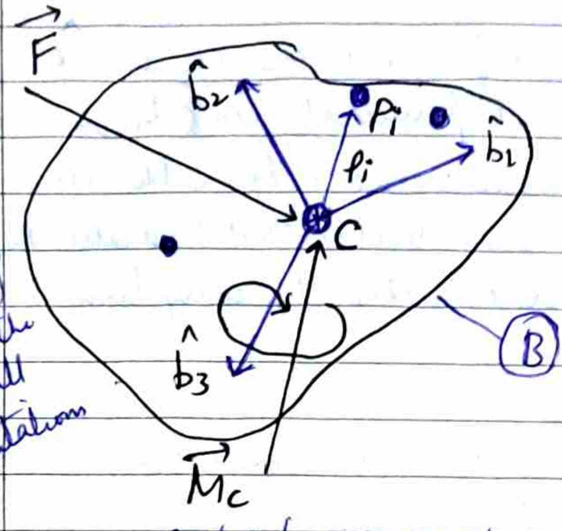


the configuration is symmetric but the axis is not $\perp^r$ to the plate.
→ angular momentum vector & angular velocity vector are not parallel because the axis of rotation does not coincide with any of the principal axes.

$\vec{H}_C$ & A, B all ∥.
The axis is $\perp^r$ to the plate.
In fact this configuration is symmetric.
→ If we compute the angular momentum we will find that the angular momentum vector & angular velocity vectors are parallel

Euler's Equation (tells us the rotational equation of motion)

$$\frac{{}^{A}d\vec{H}_c}{dt} = \vec{M}_c \quad \text{--- } (1)$$



we take C, the Center of mass as the origin for all our calculations

The rate of change of angular momentum is equal to the net moment applied to the rigid body.

$\hat{b}_1, \hat{b}_2$ & $\hat{b}_3$ are a set of body fixed unit vectors that define a body fixed frame.

Let $\hat{b}_1, \hat{b}_2$ & $\hat{b}_3$ be along the principal axes

and ${}^{A}\vec{\omega}^{B} = \omega_1\hat{b}_1 + \omega_2\hat{b}_2 + \omega_3\hat{b}_3$

diff. (1),

$$\frac{{}^{B}d\vec{H}_c}{dt} + \underbrace{{}^{A}\vec{\omega}^{B} \times \vec{H}_c}_{\downarrow} = \vec{M}_c \text{ (Net moment)}$$

(involves the derivative in a body fixed frame)

(Correction factor, takes into account the fact that the differentiation is done in the body fixed frame)

Correction factor is simply the angular velocity of the moving body fixed frame crossed with the angular momentum.

So therefore the first term which involves $\vec{I}\times\vec{\omega}$, (the inertia tensor times the angular velocity) consists of 3 terms → diagonal terms I_{11}, I_{22}, I_{33} which multiply $\vec{\omega}_1$, $\vec{\omega}_2$ & $\vec{\omega}_3$ yield 3 terms that we see on the RHS.

* → first term can be written in terms of inertia matrix times the angular velocity vector. Because we have chosen principal axes, it turns out that the off diagonal elements in the inertia tensor are zero.

→ Above correction factor is a well-known fact in mechanics.

Anytime we differentiate a vector in a moving frame, it's derivative is different from the derivative in a fixed frame.

* The difference is obtained by simply factoring the cross product of angular velocity with that vector ←

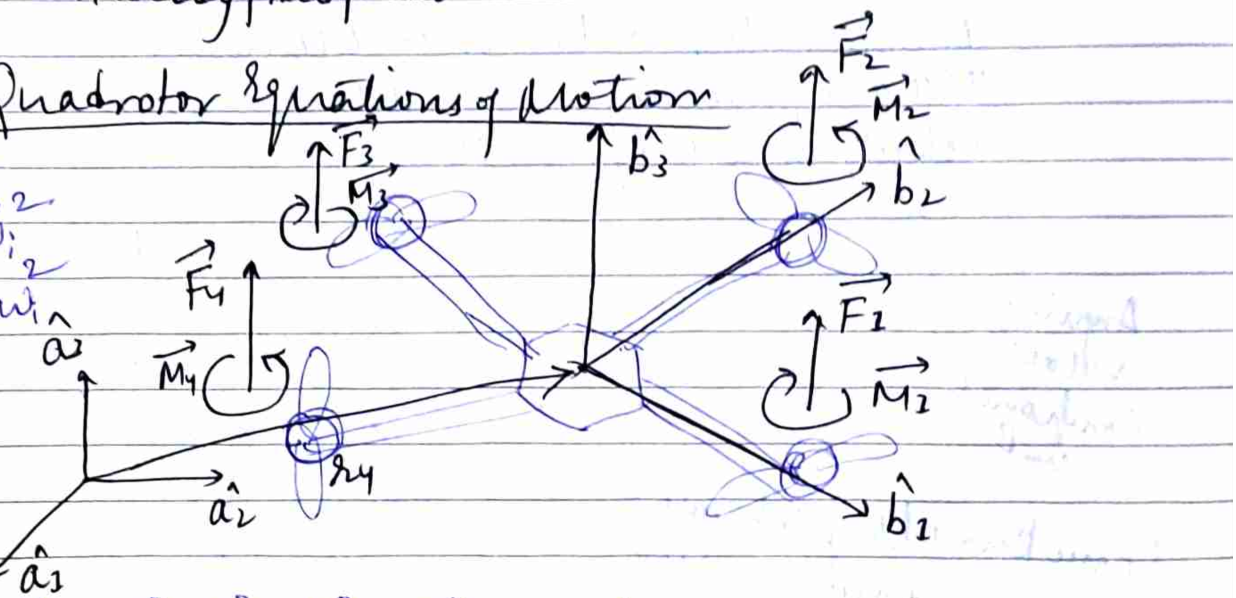
$$ {}^B\frac{d\vec{H}_c}{dt} = I_{11}\dot{w}_1\hat{b}_1 + I_{22}\dot{w}_2\hat{b}_2 + I_{33}\dot{w}_3\hat{b}_3 $$

Second term $\rightarrow$ $({}^A\vec{w}^B \times \vec{H}_c)$ can be written in the component form as

(Euler's Equation of motion)

$$ \underbrace{\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix}}_{\text{derivative of angular momentum in a body fixed frame}} + \underbrace{\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{\text{(Correction term)}} = \underbrace{\begin{bmatrix} M_{c,1} \\ M_{c,2} \\ M_{c,3} \end{bmatrix}}_{\text{net moment.}} $$

Quadrotor Equations of Motion



$\vec{F}_i = k_F \vec{w}_i^2$

$M_i = k_M w_i^2$

Net force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 - mg\hat{a}_3$

Net Moment $\vec{M} = \vec{r}_1 \times \vec{F}_1 + \dots + \vec{r}_4 \times \vec{F}_4 + \vec{M}_1 + \dots + \vec{M}_4$

If we combine the net force and the net moment with the Newton Euler Equations, we get these two sets of eqn

${}^A\vec{w}^B = p\hat{b}_1 + q\hat{b}_2 + r\hat{b}_3$ | p, q, r are the component of angular velocity vector along $\hat{b}_1, \hat{b}_2$ & $\hat{b}_3$ resp.

Components in the inertial frame along $\hat{a}_1, \hat{a}_2, \hat{a}_3$

$$ m\ddot{\vec{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \overset{\text{Rot. of thrust vector from B}\to\text{A.}}{R}\begin{bmatrix} 0 \\ 0 \\ \underbrace{\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4}_{u_1 \text{ (total thrust)}} \end{bmatrix} $$

$$ I\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \underbrace{\begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}}_{\text{Net moment } u_2 \text{ (in the body fixed frame)}} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \vec{I}\begin{bmatrix} p \\ q \\ r \end{bmatrix} $$

components in the body frame along $\hat{b}_1, \hat{b}_2$ & $\hat{b}_3$, the principal axes

How do we estimate/calculate all the parameters in this model? (online setting)

$$m\ddot{\vec{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R\begin{bmatrix} 0 \\ 0 \\ \vec{F_1}+\vec{F_2}+\vec{F_3}+\vec{F_4} \end{bmatrix}$$

$$\vec{I}\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L(\vec{F_2}-\vec{F_4}) \\ L(\vec{F_3}-\vec{F_1}) \\ \vec{M_1}-\vec{M_2}+\vec{M_3}-\vec{M_4} \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \vec{I}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

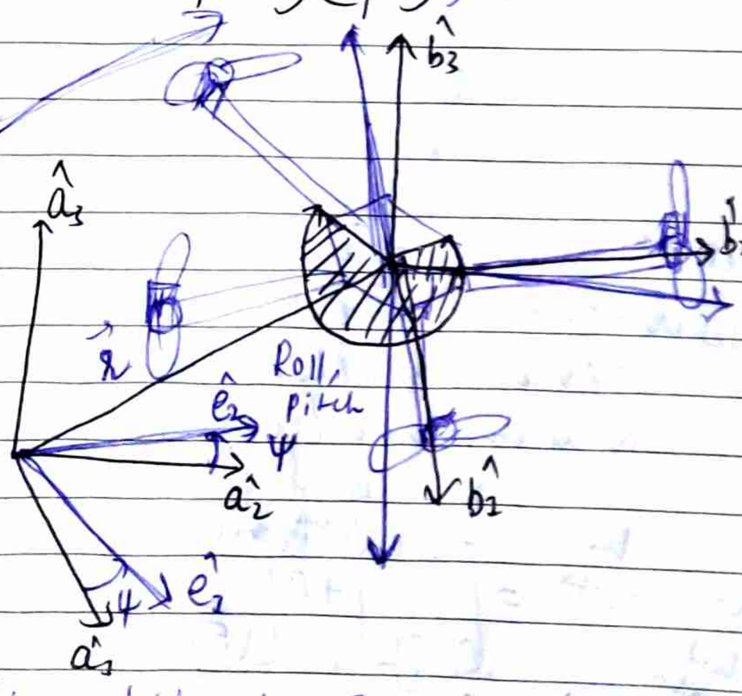
Length

If we have a system that allowed to measure positions, velocities & acc^n, it's actually not too hard to estimate lengths, the masses & the inertias.
It's also quite easy to calculate the angular velocity in the body fixed frame.

Angular velocity Components in B

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} C\theta & 0 & -C\phi S\theta \\ 0 & 1 & S\phi \\ S\theta & 0 & C\phi C\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \begin{matrix} \text{Pitch} \\ \text{Roll} \\ \text{Yaw} \end{matrix}$$

If we know the pitch, roll & yaw and we know the rate of change of the pitch, roll & yaw angles on the right side, a simple transformation yields the angular velocity components p, q & r along $\hat{b}_1$, $\hat{b}_2$ & $\hat{b}_3$.



This model involves 3 components of position, velocity & acc^n, three components of rotation, angular velocities & angular accelerations

Planar Quadrotor Model

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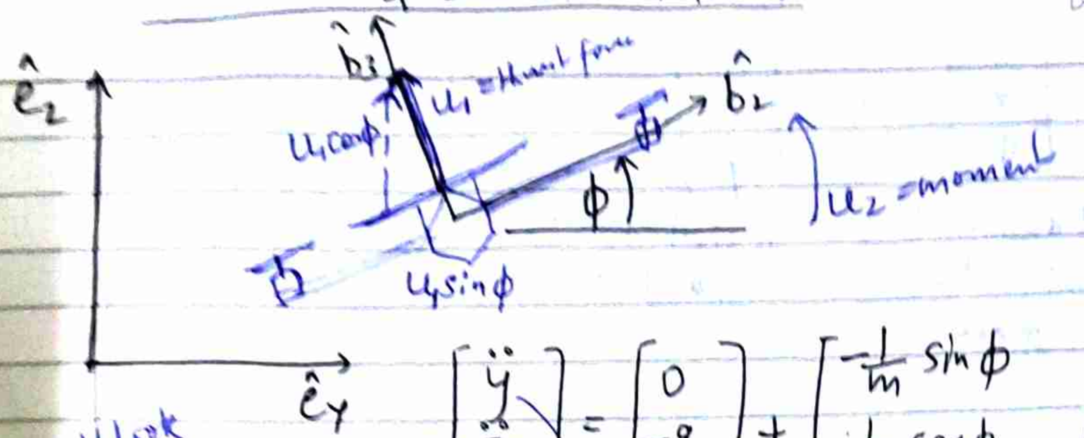

$$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

rate of change of velocity in y & z dir.

rate of change of angular velocity in the roll dir.

we will look at the eqn of motion in YZ plane. we will assume that the robot cannot move out of this plane (in other words, there is no motion in the X dirn) & we also assume there is no yaw or pitch motion

Planar Quadrotor

$$q = \begin{bmatrix} y \\ z \\ \phi \end{bmatrix}, \quad x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

3 dimensional configuration space

6 dimensional state space

To describe the dynamics of system, it's useful to define a state vector

State Space for Quadrotors

State Vectors

$$q = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{bmatrix}, \quad X = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

→ q describes the configuration (position) of the system

→ X describes the state of the system

state vector includes the configuration & its derivative

Equilibrium Configuration

$$q_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 0 \\ 0 \\ \psi_0 \end{bmatrix}, \quad X_e = \begin{bmatrix} q_e \\ \hline 0 \end{bmatrix}$$

→ q_e describes the eqb configuration of the system

→ X_e describes the eqb state of the system

0 roll angle

0 pitch angle

any yaw angle

this gives a 12D vector when bottom 6 elements are all 0.

0 velocity vector

State vector for planar Quadrotor

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$$q = \begin{bmatrix} y \\ z \\ \phi \end{bmatrix}, \quad x = \begin{bmatrix} q \\ \hline \dot{q} \end{bmatrix}$$

3 dimensional Configuration space

a six dimensional state space

Equilibrium state/configuration

$$q_e = \begin{bmatrix} y_0 \\ z_0 \\ 0 \end{bmatrix}, \quad x_e = \begin{bmatrix} q_e \\ \hline 0 \end{bmatrix}$$

Dynamical Systems in State-Space form

↳ System when the effects of action do not occur immediately

→ Evolution of the system's states is governed by a set of ordinary differential equations

→ Ordinary differential equations are often rearranged into state-space form.

$$\dot{x} = f(x, u)$$

x is a matrix of states
u is a matrix of inputs

matrices

State space form

Given an ODE:

① Identify the order, n, of the system

② Define the states

$(n-1)^{st}$ derivative

$$x_1 = y(t),\ x_2 = \dot{y}(t),\ \ldots,\ x_n = y^{(n-1)}(t)$$

③ Create the state vector $x = [x_1\ x_2\ \ldots\ x_n]^T$

$$= [y\ \dot{y}\ \ldots\ y^{(n-1)}]^T$$

#learnthesmarterway

④ write the coupled first-order differential equations —

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_2$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}\dot{y} = \ddot{y} = x_3$$

$$\cdots$$

$$\frac{d}{dt}x_n = \frac{d}{dt}y^{(n-1)} = g(y, \dot{y}, \ldots, y^{(n-1)}, u) =$$

$$g(x_1, x_2, \ldots, x_n, u)$$

from governing ODEs

⑤ write system of first-ODEs as matrix.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ g(x_1, x_2, \ldots, x_n, u) \end{bmatrix}$$

$$\dot{x} = f(x, u)$$

Eg. ① Mass-Spring System $m\ddot{y}(t) + ky(t) = u(t)$

① Identify $n = 2$ (2nd order)

② Define states $x_1 = y$, $x_2 = \dot{y}$

③ Create the state vector $x = [x_1 \ x_2]^T = [y \ \dot{y}]^T$

④ write the coupled first-ODEs:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_2$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}\dot{y} = \ddot{y} = \frac{u(t) - ky(t)}{m} = \frac{u(t) - kx_1}{m}$$

⑤ Write system of first-ODEs as matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{u(t) - kx_1}{m} \end{bmatrix}$$

This system is actually linear in states's x & inputs u

This system is actually linear in states & inputs!
So, we can write the eqns in the following manner.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\dot{x} = Ax + Bu$$

Eg ② (Higher order system) Planar Quadrotor Model

$$m\ddot{y} = (\sin\phi) u_1$$
$$m\ddot{z} = (\cos\phi) u_1 + mg$$
$$I_{xx}\ddot{\phi} = u_2$$

① Identify $n = 2$

② Define states $x_1 = y$, $x_2 = z$, $x_3 = \phi$, $x_4 = \dot{y}$, $x_5 = \dot{z}$, $x_6 = \dot{\phi}$

③ Define the state vector

$$x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^T = [y\ z\ \phi\ \dot{y}\ \dot{z}\ \dot{\phi}]^T$$

④ Define the system of first-order diff^n eqn:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_4 \qquad \frac{d}{dt}x_4 = \frac{d}{dt}\dot{y} = \ddot{y} = \frac{(\sin\phi)u_1}{m} = \frac{(\sin x_3)u_1}{m}$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}z = \dot{z} = x_5 \qquad \frac{d}{dt}x_5 = \frac{d}{dt}\dot{z} = \ddot{z} = \frac{(\cos\phi)u_1}{m} - g = \frac{u_1(\cos x_3)}{m} - g$$

$$\frac{d}{dt}x_3 = \frac{d}{dt}\phi = \dot{\phi} = x_6$$

$$\frac{d}{dt}x_6 = \frac{d}{dt}\dot{\phi} = \ddot{\phi} = \frac{u_2}{I_{xx}}$$

⑤ Write system of first ODEs as matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \frac{\sin(x_3)u_1}{m} \\ \frac{\cos(x_3)u_1}{m} - g \\ u_2 \end{bmatrix}$$

these eqn are nonlinear because of the func sine & cosine of the state x_3.