Lecture 24: Gamma distribution and poisson process

Stirling's formula gives the approximation for factorials

$$n! \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{n}{e}\right)^n$$

Gamma function

$$\Gamma(a) = \int_0^\infty x^a e^{-x} \frac{dx}{x}, \text{ for real } a > 0.$$

$$\Gamma(n) = (n-1)! \text{ for } n \text{ a positive integer.}$$

implies $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = 1/2\,\Gamma(1/2) = \sqrt{\pi}/2, \ldots$$

Gamma distribution

PDF: $$1 = \int_0^\infty \frac{1}{\Gamma(a)} x^a e^{-x} \frac{dx}{x} \rightarrow \text{Gamma}(a,1) \text{ PDF}$$

~~Gamma (a, λ)~~ $\lambda > 0$

Example

$Y \sim \text{Gamma}(a, \lambda)$. PDF = ?

Let $Y = X/\lambda$, $X \sim \text{Gamma}(a, 1)$

$$f_Y(y) = f_X(x)\frac{dx}{dy}$$

$Y = X/\lambda \Rightarrow X = \lambda Y$

$\frac{dx}{dy} = \lambda$

$$= \frac{1}{\Gamma(a)}(\lambda y)^a e^{-\lambda y} \cdot \frac{1}{\cancel{\lambda} y} \cancel{\lambda}$$

$$= \frac{1}{\Gamma(a)}(\lambda y)^a \frac{e^{-\lambda y}}{y}, \quad y > 0$$

→ Like exponential distribution, gamma is a continuous distribution on the positive real numbers.

→ Gamma distribution relates to the normal distribution, beta distribution, exponential distribution and the poisson distribution.

Gamma - Exponential Connection

The inter arrival times (distance between X's or time or interval between ~~two~~ emails arrivals) are i.i.d. Expo(λ).

T_n = time of nth arrival (nth email, nth phone call, etc)

$$= \sum_{j=1}^{n} X_j ,$$

X_j are i.i.d. Expo(λ) (interarrival time)

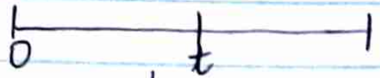
$\sim$ Gamma(n, λ), n integer.

(continuous time analog of negative binomial)

→ How long do we ~~have~~ to wait for n successes in continuous time? → in above case, it means getting an email

Stats171 Poisson process

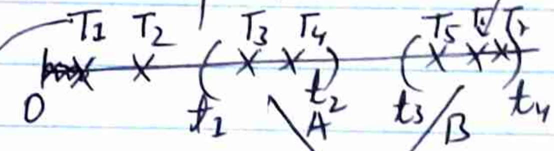
Poisson Processes discussed



N_t = # emails up to time t $\sim$ Pois(λt)

Assumption for Poisson process

Number of arrivals in the disjoint intervals are independent



(In fact, all of times between email arrivals are $\exp(\lambda)$, from memoryless property) is exponential

like number of emails we get from interval A is independent of no. of emails we get from interval B (A & B are disjoint)

T_1 → time of the first email

$P(T_1 > t)$ = same thing as saying at time t, we've n't yet got any email

$= P(N_t = 0)$ → no email up until time t

$= e^{-\lambda t}$ (Poisson)

Prove that $T_n = \sum_{j=1}^{n} X_j$, X_j i.i.d. Expo(1) is Gamma(n, 1)

<u>Proof</u>: Using MGFs:

MGF of X_1 is $\frac{1}{1-t}$, $t < 1$.

$\Rightarrow$ MGF of T_n is $\left(\frac{1}{1-t}\right)^n$, $t < 1$.

(Adding independent things and multiply the MGFs)

Let $Y \sim \text{Gamma}(n, 1)$

$$\text{MGF}(Y) = E(e^{tY})$$

$$= \int_0^\infty e^{ty}\left(\frac{1}{\Gamma(n)} y^n e^{-y} \frac{dy}{y}\right) \quad \text{(LOTUS)}$$

PDF of Gamma(n, 1)
a

$$= \frac{1}{\Gamma(n)} \int_0^\infty e^{ty} y^n e^{-y} \frac{dy}{y}$$

$$= \frac{1}{\Gamma(n)} \int_0^\infty y^n e^{-(1-t)y} \frac{dy}{y}$$

Let $x = (1-t)y$
$dx = (1-t)\,dy$

[Change of variable]

$$= \frac{1}{\Gamma(n)} \int_0^\infty \left(\frac{x}{1-t}\right)^n e^{-x} \frac{\cancel{dx}\,\cancel{(1-t)}\,dx}{\cancel{(1-t)} \cdot x}$$

$$= \frac{(1-t)^{-n}}{\Gamma(n)} \int_0^\infty x^n e^{-x} \frac{dx}{x}$$

$$= \frac{(1-t)^{-n} \cdot \cancel{\Gamma(n)}}{\cancel{\Gamma(n)}} = \left(\frac{1}{1-t}\right)^n$$

this would be a correct MGF even if n is any positive real number not just integer

Hence proved.

Example

Let $X \sim \text{Gamma}(a,1)$. find $E(X^c)$, $c=1$ (1st moment)
$c=2$ (2nd moment).

Solu: Using LOTUS:

$$\frac{1}{\Gamma(a)} \int_0^\infty x^c x^a e^{-x} \frac{dx}{x}$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty x^{a+c} e^{-x} \frac{dx}{x}$$

$$= \frac{\Gamma(a+c)}{\Gamma(a)}, \text{ if } a+c>0$$ (Since, the gamma function is only defined on positive numbers.)

$$\underset{c=1}{E(X)} = \frac{\Gamma(a+1)}{\Gamma(a)} = \frac{a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}} = a.$$

$$\underset{c=2}{E(X^2)} = \frac{\Gamma(a+2)}{\Gamma(a)} = \frac{(a+1)\Gamma(a+1)}{\Gamma(a)} = \frac{(a+1)a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}}$$

$$= a^2+a$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \cancel{a^2}+a-\cancel{a^2}$$

$$= a.$$

So, Gamma(a, 1) has mean a and variance a.

✓ $\text{Gamma}(a,\lambda)$ has mean a/λ and variance a/λ^2.