Lecture 7 – Lyapunov Analysis I

Recap Dynamic Programming view of optimal control

like having sample points at very regular intervals over the entire state space (very significant constraint)

exact DP in some sense
- Tabular setting (on a discrete graph) ~ mesh over continuous states (Curse of dimensionality as #states ↑)
- LQR
  restricted to linearized dynamics

Approx DP
- Function Approximators (Neural nets)
  Least-squares value iteration

$$\text{loss} = \sum_i \| \hat{J}^*_\alpha(x_i) - J^d(x_i) \|^2$$

via gradient descent

$$\Delta\alpha = -\eta \frac{\partial \text{loss}}{\partial \alpha}$$

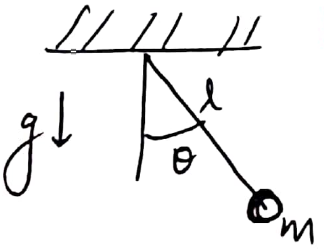
η: learning rate

Lyapunov Analysis

- "Good enough" solutions.
- Certify approximate optimal controllers

– The general questions of if our controllers is good enough can often be framed in the language of Stability analysis.

Ex Non-linear Stability Analysis of the Pendulum



$$ml^2\ddot{\theta} + mgl\sin\theta = -b\dot{\theta}, \quad b > 0$$

$-b\dot{\theta}$ (damping term) — think of this as joint friction.

$$E = \underbrace{\tfrac{1}{2}ml^2\dot{\theta}^2}_{\text{Kinetic}\ (v = l\dot{\theta})} \underbrace{- mgl\cos\theta}_{\text{potential}} \geq -mgl$$

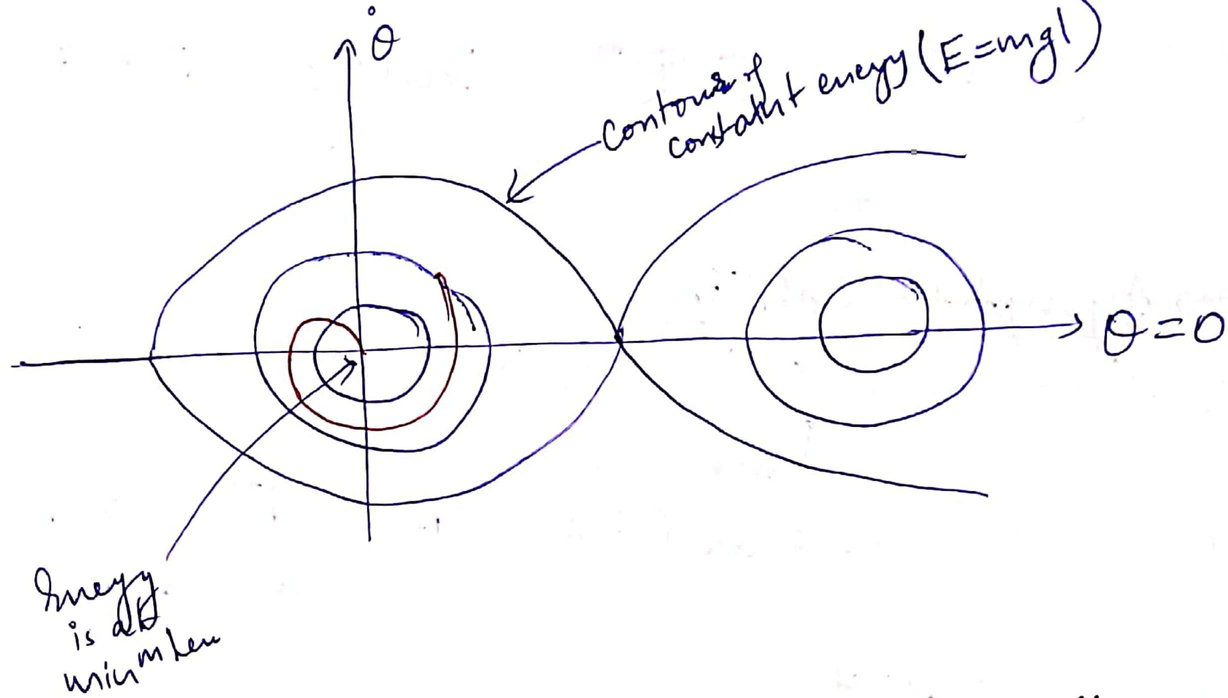
(Rate of change of Energy)

$$\frac{dE}{dt} = ml^2\ddot{\theta}\dot{\theta} + \dot{\theta}mgl\sin\theta = -b\dot{\theta}^2 \leq 0$$

$ml^2\ddot{\theta}\dot{\theta} = (-b\dot{\theta} - mgl\sin\theta)\dot{\theta}$

(torque × velocity)

Scanned with OKEN Scanner



Generalize energy argument w/ Lyapunov functions

$\dot{x} = f(u)$

want to prove $x^* = 0$ is a stable fixed point.

Find a differentiable function $V(x)$, such that $V(0)=0$, $\forall x \neq 0,\ V(x) > 0$

$\frac{\partial V}{\partial x} f(x) = \frac{dV}{dt} = \dot{V}(0) = 0$, $\forall x = 0,\ \dot{V}(x) \leq 0$ → going downhill

Then x^* is stable in the sense of Lyapunov.

$\Rightarrow V(x) > 0$ (Positive definite function) — 0 at 0 & positive for all other x's

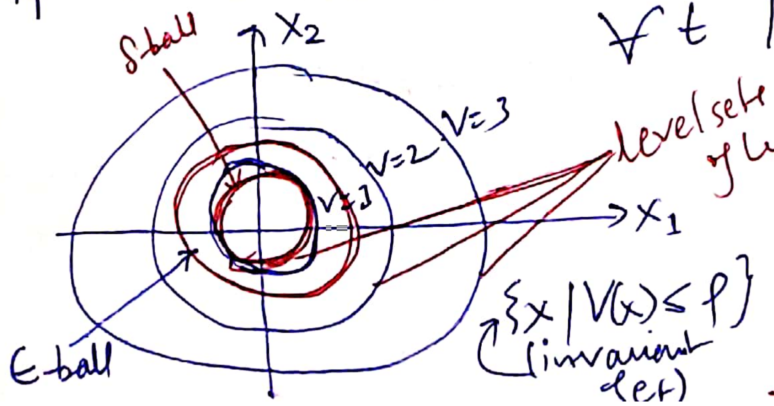
$\dot{V}(x) \leq 0$

x^* is stable i.s.L

if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\|x(0) - x^*\| < \delta$

$\forall t \quad \|x(t) - x^*\| < \epsilon$

<u>Invariant set</u>
If we start in that set, we will never leave that set because in order to leave we have to go up & we know $\dot{V}$ is going down (≤ 0)



If we know $\dot{V}(x) \leq 0$

Then energy level set of our Lyapunov function is an <u>invariant set</u>.

$\dot{V}(x) > 0$ $\quad \forall x \in D \subseteq \mathbb{R}^n$ (local & global stability)

$\dot{V}(x) \leq 0$

domain

Asymptotic Stability

$V(x) > 0$, $\dot{V}(x) < 0$ $\quad \forall x \in D$.

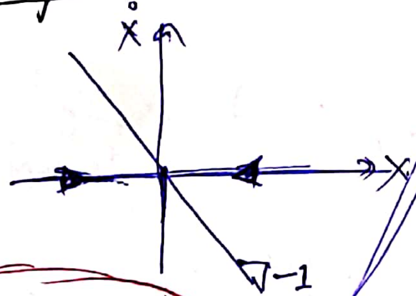
more $<$ vs $\leq$

globally asymptotic stable (G.A.S.) if $D = \mathbb{R}^n$, + radially unbounded

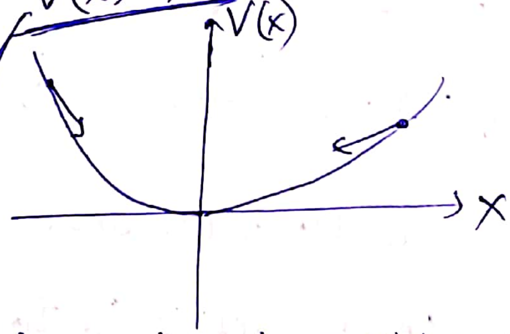
Exponential Stability

$\dot{V}(x) > 0$, $\dot{V}(x) \leq -\alpha V(x)$ $\quad \Big| \quad V(x(t)) \leq V(x(0)) e^{-\alpha t}$

Example $\quad \dot{x} = -x$



If we choose $V(x) = x^2$



Linear systems always have quadratic Lyapunov functions if they are stable

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x)$$

$$\dot{V}(x) = 2x(-x) = -2x^2 < 0$$

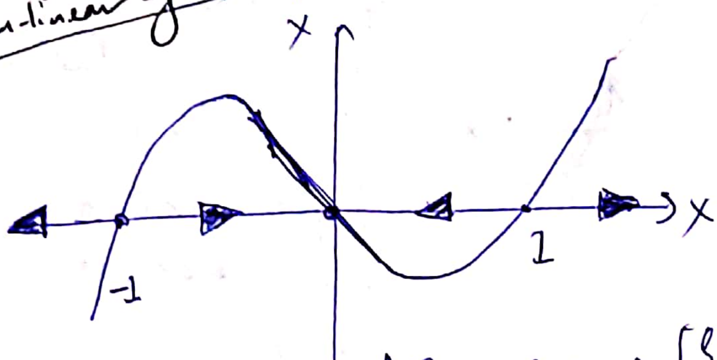
dynamics

(a negative definite function)

$$\dot{V}(x) \leq -\alpha V(x), \quad \alpha = 2$$

(exponentially stable & rate is given by 2)

Example (non-linear System) $\dot{x} = -x + x^3$



ROA (Region of attraction) $\forall x \in (-1, 1)$ [Every initial condition but not including -1 & 1 will get to that fix point]

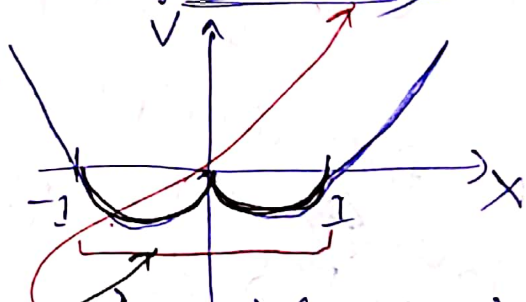
choose $V(x) = x^2$

$$\dot{V}(x) = 2x(-x+x^3) = -2x^2 + 2x^4 = 2x^2(x^2-1)$$

→ $x \in (-1,1)$ is a sublevel set of $V(x)$

$V(x) < 1$ called an invariant set.

We know once we are inside $(-1,1)$, we won't leave and we know that $\dot{V}(x) < 0$. Together they gives us condition that satisfies that this region is inside the R.O.A.



$< 0 \quad \forall x \in (-1,1)$

$\Rightarrow \forall x \in (-1,1)$ is inside the R.O.A.

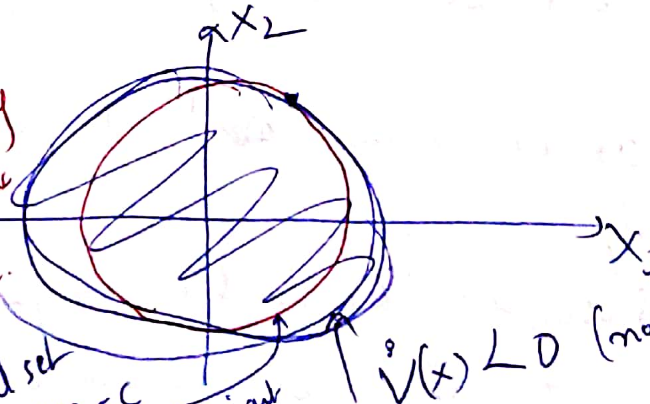
For region of attraction (R.O.A.)

$V(x) > 0, \ \dot{V}(x) < 0 \quad \forall x \in D$ often use $D = \{x \mid V(x) \le \rho\}$ scalar level set

We are gonna get to the origin (That's inside R.O.A.)

an invariant set (if a trajectory that starts inside D stays inside D for all time)

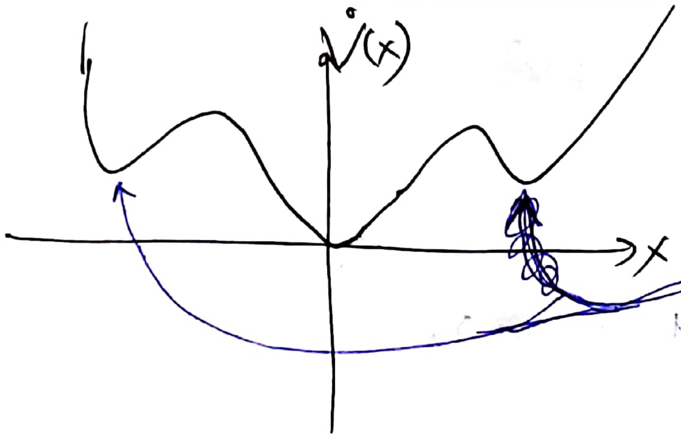
2D State Space



Any sublevel of a lyapunov func is an invariant set.

Level set $V(x) = c$ (an invariant set)

$\dot{V}(x) \le 0$ (not quite enough on its own to guarantee that we will get to the origin)



if we have local minima then we can satisfies the lyapunov condition over these areas.

We need strictly decreasing to get to the origin and that rules out non convex lyapunov functions like this.

What's the relationship to DP?

(DP) $$0 = \min_u \Big[l(x,u) + \underbrace{\frac{\partial J^*}{\partial x} f(x,u)}_{\frac{dJ}{dt}} \Big]$$ (Bellman's eqn)

for the minimizing u,

for the optimal policy (optimal u) $u^* = \pi^*(x)$

$$\underbrace{\frac{dJ^*}{dt} = -l(x, u^*)}_{\text{partial diff. eqn}}$$

(Hard to find a J^* that satisfies that everywhere — that's going downhill exactly the rate.)

(Lyapunov) $$\frac{dV}{dt} \leq 0$$ (much easier version)

Rather than saying it's going down at exactly some rate, we are saying tell us it's going down at all.

Think of this as relaxation of hard eqn in DP into a soft equality constraint.

And if this satisfied, then that's enough to say that we are gonna get to the origin eventually but it gives up on optimality.

there is exactly one J that satisfies $\left(\frac{dJ^*}{dt} = -\ell(x, u^*)\right)$

but there are many V's that satisfies $\left(\frac{dV}{dt} \leq 0\right)$

we can have this condition satisfied for multiple task at the same time.

softer condition.

if we choose a cost that's only +ve, then the optimal cost-to-go is also a lyapunov function

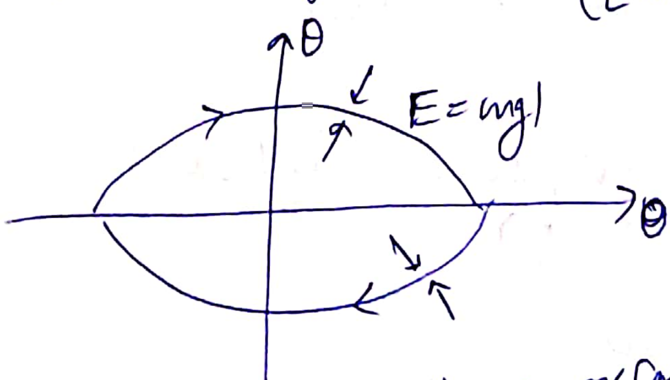
La Salle's Theorem (for Asymptotic Stability)

$V(x) \succ 0, \dot{V}(x) \preceq 0$

$\dot{V} \to 0$ as $t \to \infty \Rightarrow x \to$ largest invariant set w/ (with) $\dot{V} = 0$

Swing-up of the pendulum

Recall the homoclinic orbit (because it visits the fixed point) $(E = mgl)$



$ml^2\ddot{\theta} + mgl\sin\theta = u$

$$\frac{dE}{dt} = u\dot{\theta}$$

(when $u = -b\dot{\theta}$)

Lyapunov Candidate

$$V(x) = \frac{1}{2}\left(E(x) - E^{desired}\right)^2$$

energy at some state

mgl

we want to find a controller that guides downhill so that the energy goes towards the desired energy

$$\tilde{E} = E - E^d$$

$$\frac{dV}{dt} = \dot{E}(x)\tilde{E} = u\dot{\theta}\tilde{E}$$

Choose $u = -K\dot{\theta}\tilde{E}$

$K > 0$ (Control gain)

$$\boxed{\frac{dV}{dt} = -K\dot{\theta}^2\tilde{E}^2 \leq 0}$$

And, the above Controller obtain its minimum at the homoclinic orbit.