Lecture 23: Beta distribution

→ It's a generalization of the uniform distribution.

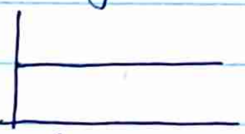
$Beta(a,b)$, $a>0, b>0$

PDF: $f(x) = c\,x^{a-1}(1-x)^{b-1}$, $0<x<1$

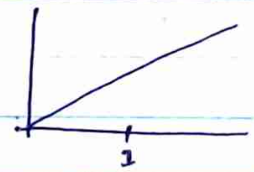
c is a normalization constant.

→ flexible family of continuous distributions on $(0,1)$.
↳ as we vary a and b, we can make this take different shapes.

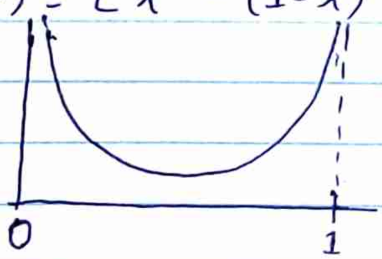
$a=b=1$, $f(x)=c$ (generalization of uniform)



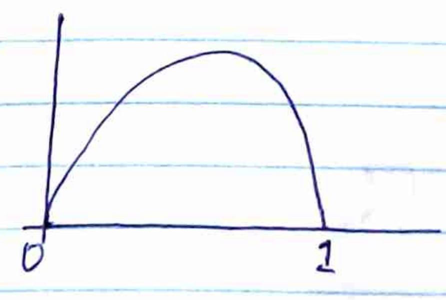
$a=2, b=1$, $f(x)=cx$



$a=\frac{1}{2}, b=\frac{1}{2}$, $f(x) = c\,x^{-1/2}(1-x)^{-1/2} = \dfrac{c}{\sqrt{x(1-x)}}$



$a=b=2$, $f(x) = cx(1-x)$



→ Often used as prior for a parameter in (0,1).
So, if we have a parameter that's a probability, we know it's between 0 and 1, and want to give it a prior distribution. The beta distribution is, by far, the most widely used choice in practice, because it has a lot of nice properties. like

$\boxed{\text{Stat 111}}$

→ "Conjugate prior to Binomial".
→ Connections to other distributions.

<u>Conjugate prior for Binomial</u> [(Generalization of Laplace rule of succession) we used Beta(1,1) i.e. uniform

$X|p \sim Bin(n,p)$, $\underbrace{p \sim Beta(a,b)\ [\text{prior}]}$

data, (what we get to observe)

This is our reflection of our uncertainty about p, i.e., we don't know the true probability, so we are just treating it as a random variable.

Before we observe X, we have a prior on p, that's our prior uncertainty. So, prior on p is not based on data. After we observe X, then we want to update our probabilities, using Bayes' rule. Updating our beliefs based on evidences.

Find posterior distribution $p|X$.

$$f(p|X=k) = \frac{P(X=k|p)\,f(p)}{P(X=k)}$$

p is continuous so, it has PDF

$P(X=k)$ does not depend on p because this is just going to be a constant w.r.t. p

$$= \frac{\binom{n}{k} p^k (1-p)^{n-k}\, c\, p^{a-1}(1-p)^{b-1}}{P(X=k)}$$

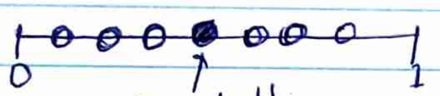
$\propto p^{a+k-1}(1-p)^{b+n-k-1}$ (ignore terms not dependent on p)

$\Rightarrow p|X \sim \text{Beta}(a+X, b+n-X)$

✓ Find $\int_0^1 \binom{n}{k} x^k (1-x)^{n-k}\,dx$ without using calculus, using a story (Bayes' Billiards)

Story 1:

$(n+1)$ billiards balls, all white, paint one pink, throw them on $(0,1)$ independently.



(we have i.i.d. uniforms.)

this

Story 2: First throw, then paint one pink.

Above two ways of generating a picture that looks like are completely equivalent.

Let X = # balls to left of pink one ●
(that is going to be some integer between 0 and n)

$$P(X=k) = \int_0^1 \underbrace{P(X=k|p)}_{\text{binomial}}\, f(p)\, dp \quad \text{(By LOTP)}$$

exactly k white balls to the left of pink ball

denoting where is pink ball.

$$= \int_0^1 \binom{n}{k} p^k (1-p)^{(n-k)}\, dp$$

1 (we are assuming it's uniform between 0 & 1)

$$= \frac{1}{(n+1)}$$

(from second story) → Threw the balls down, paint a random one pink equally likely

So, it's equally likely that there's any valid no of white balls still left to the left of pink one anywhere

All integers from 0 to n are equally likely

Hence, $(n+1)$ possible options

Stats123 Applied ~~Quantitative Finance~~ on Wall Street
Financial Derivative

A financial derivative is a contract (it's a bet, an agreement) between two people whose payout at some maturity date is a function of or derives from the value of some other random variable.

for instance, if person A enters into a contract with person B where person A pays person B $1 if some event occurs (like ~~be~~ person B wins a game ~~etc.~~), or total snowfall in Boston is above 80 inches this winter, etc) → weather derivative

The $1 is a function of some other random variable, in the case of snowfall, the number of inches of snow in Boston.

→ In financial, usually means the underlying random variable that the payoff is a function of, is a financial asset. It's a price of a financial asset.

Stock → S_T → Random variable is ~~donted~~ denoted by S in finance, not X.

like S_T is a price of google stock in one year, is a random variable.

$g(S_T)$ → financial derivative (something that pays off a function of that)

like $g(S_T)$ might be the indicator function above 500.

So, if google stock is above $500 in a year's time, ~~so~~ we pay $1 or we get from our contract.

→ A function of a random variable is another random variable.

Fundamental theorem of finance

The price that we would pay for a derivative contract is very very closely related to the expected value of $g(S_T)$ or $E[g(S_T)]$.

There is a powerful result that says if we choose the random variable correctly and choose the distribution correctly, it is exactly an expected value.

Example

FX

foreign exchange

$$\$1/€ \quad \begin{cases} \xrightarrow{1/2} \$1.25/€ \quad (€0.80/\$) \\ \xrightarrow{1/2} \$0.80/€ \quad (€1.25/\$) \end{cases}$$

1€|\$ 1/2

Where do we expect it to be in a years time?

Soln:- $E(€) = \frac{1}{2} \times \overset{0.625}{\cancel{1.25}} + \cancel{\frac{1}{2} \times \overset{0.4}{0.80}}$

$= \$1.025$

So, $\$1 = €0.9756$

$E(\$) = €1.025$

~~Simple~~ probabilistic model

TARP (Troubled Asset Release Program)

Warrant → A type of financial derivative, called a call option.

warrant ≡ call option

[US govt. paid \$ 450mm to Goldman Sachs. In return, US govt. has right to buy 10mm GS shares for \$ 125 each ~~dollar~~ in 10 years time.

Oct 2008, GS shares were \$ 95.

So, the question is what was the govt. doing paying all that money to buy GS shares for \$ 125 when they could have just gone out and bought them for \$ 95?

If in 10 years time, what is the value of one of these warrants to the US? → \$ 25 (we can buy something that worth \$ 150 by paying \$ 125 for it)

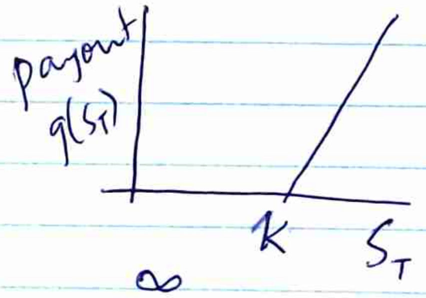
GS \$ 150 option worth \$ 25

Here
$K = \$125$ (strike price)

GS \$ 100 option worth \$ 0

for all option $g(S_T) = \max\{\underbrace{S_T}_{\text{maturity } T} - K, 0\}$

Here, it's 10 years



$$\text{Price} = \int_0^\infty \max\{S_T - K, 0\} f(S_T)\, dS_T$$

(LOTUS)

$$= \int_K^\infty (S_T - K) f(S_T)\, dS_T$$

Black-scholes formula