Lecture 16: Exponential Distribution

rate parameter λ → Rate at which some type of event occurs

$X \sim Expo(\lambda)$ has PDF $\lambda e^{-\lambda x}$, $x > 0$ (0 otherwise)

Valid PDF: $\int_0^\infty \lambda e^{-\lambda x} dx = 1$

CDF: $F(x) = \int_0^x \lambda e^{-\lambda t} dt = \underbrace{1 - e^{-\lambda x}}_{\text{increasing function}}$, $x > 0$.

Let $Y = \lambda X$, then $Y \sim Expo(1)$. ← proves that this statement is true

Since $P(Y \leq y) = P(X \leq y/\lambda)$

$= 1 - e^{-y}$

Let $Y \sim Expo(1)$, find $E(Y)$, $Var(Y)$.

Solu: $E(Y) = \int_0^\infty \underbrace{y}_{u} \underbrace{e^{-y} dy}_{dv}$

First Norm

$du = dy$, $v = -e^{-y}$

$= (-ye^{-y})\Big|_0^\infty + \int_0^\infty e^{-y} dy$

$= 1$.

$Var(Y) = E(Y^2) - (E(Y))^2$

Second Norm $= \int_0^\infty y^2 e^{-y} dy - 1$ (By LOTUS)

$= 1$.

In general
$Y = \lambda X$
so, $X = Y/\lambda$ has $E(X) = 1/\lambda$, $Var(X) = 1/\lambda^2$

Memoryless Property (Continuous case)

$$\underbrace{P(X \geq s+t \mid X \geq s)} = P(X \geq t)$$ [General distribution]

Think like this: {We already waited s minutes and we haven't gotten our phone call (so we know $X \geq s$), then what's the probability we would have to wait at least an additional t minutes.

& (Because we've started over with the fresh exponential distribution with the same parameter)

Proof that above equation is satisfied by the exponential distribution.

- Here $\underbrace{P(X \geq s)}_{\text{Survival function}} = 1 - \overset{\text{CDF}}{P(X \leq s)}$
$= 1 - (1 - e^{-\lambda s})$
$= e^{-\lambda s}$

Think of X as how long someone is going to live? (prob. that they will live more than at least s seconds)

$$P(X \geq s+t \mid X \geq s) = \frac{P(X \geq s+t, X \geq s)}{P(X \geq s)}$$ [defn. of Conditional prob.]

$$= \frac{P(X \geq s+t)}{P(X \geq s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X \geq t)$$

proved.

$X \sim Expo(\lambda)$

$E(X|X>a) \longrightarrow$ given the information that we have waited atleast a minutes, what's the expected value of X given that information?

Conditional Expectation

$E(X|x>a) = a + E(X-a|X>a)$ (by linearity)

$= a + 1/\lambda$ (by memoryless property)

Cool thing about the memoryless property is, given that $X>a$, $(X-a)$ just becomes a fresh exponential. This is the additional waiting time that we waited a but it's started over again.

Note — Exponential is the only memoryless distribution in continuous time. In discrete time, we have the geometric.
The geometric is the discrete analog of the exponential. And, the exponential is the continuous analog of the geometric.