Lecture 7 : Gambler's Ruin and Random Variables

Most Important topics of stats

(1) Conditioning : the soul of statistics

(2) Random variables and their distributions

Gambler's Ruin → bankruptcy

Two gamblers, A and B, sequence of rounds bet \$1,

$p = P(\text{A wins a certain round})$

$q = 1 - p$

Keep repeating the round (~~so~~ every round is independent) until one go bankruptcy. Find the prob. that A wins entire game (so B is "ruined") ?

Assuming A starts with $\$i$; B starts with $\$(N-i)$.

Another way to think of above problem as Random walk:

$0 \quad i-1 \quad i \quad i+1 \quad N$

let $P_i = P$(A wins game | A starts at \$i)

P = prob. of going right

Absorbing states $\rightarrow 0, N$

strategy: condition on first step

LOTP

$$\left[P_i = p P_{i+1} + q P_{i-1}, \quad 1 \leq i \leq N-1\right]$$

$\rightarrow$ difference equation (not differential equation)

Boundary conditions

$P_0 = 0$

$P_N = 1$

$P_i = p P_{i+1} + q P_{i-1}$, p, q are known

guess $P_i = x^i$

$$x^i = p x^{i+1} + q x^{i-1}$$

$$p x^2 - x + q = 0$$

$$x = \frac{1 \pm \sqrt{1-4pq}}{2p}$$

$$= \frac{1 \pm (2p-1)}{2p}$$

$$\frac{\cancel{1}+2p-\cancel{1}}{2p}, \quad \frac{1-(2p-1)}{2p}$$

$$1 \qquad \frac{-p+1}{p} = q/p$$

A difference equation is a discrete analog of a differential equation

$1-4pq$

$1-4p(1-p)$

$4p^2-4p+1$

$(2p-1)^2$

$$x = 1, q/p$$

General solution

$$P_i = A 1^i + B (q/p)^i, \quad p \neq q$$

if the roots are distinct, then the general solution is the linear combination of value of root in the guess (in this ~~gu~~ case, $P_i = x^i$)

$P_0 = 0 \Rightarrow B = -A$

$P_N = 1 \Rightarrow 1 = A\left(1 - q/p\right)^N$

So, $P_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq q \\ i/N, & \text{if } p = q \end{cases}$

let $x = q/p$

$\lim_{x \to 1} \dfrac{1-x^i}{1-x^N}$

$\lim_{x \to 1} \dfrac{i\,x^{i-1}}{N\,x^{N-1}}$ (L'Hospital Rule)

i/N

Unfair case

Let $i = N - i$, $P = 0.49$

A and B starts with equal Amt. of money

game is slightly unfair towards gambler A (by 0.01 in each round)

if $N = 20 \Rightarrow 0.40$ (chance that A wins is only 40%)

if $N = 100 \Rightarrow 0.12$ (each player starts with \$50, there is only 12% chance that A wins)

if $N = 200 \Rightarrow 0.02$ (2% chance that A wins)

Fair case

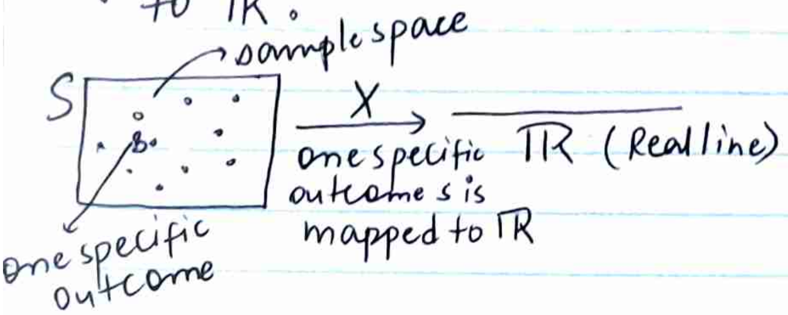
$\underbrace{\dfrac{i}{N}}_{\text{prob. that A wins and B goes bankrupt}} + \underbrace{\dfrac{N-i}{N}}_{\text{prob. that B wins and A goes bankrupt}} = 1$ (this means there is no prob. left for oscillation.)

Hence, prob. is 0 ~~for~~ that the game goes forever.

Random Variables

→ It is a function from a sample space S to $\mathbb{R}$.

↳ it's an abstraction of a quantity that can take different values



$X + 2 = 9$

$X = 7$

↳ x is a symbol that stands for constant. It's not a variable.

→ Think of as a numerical "summary" of an aspect of the experiment.

↳ does not have to summarize the entire experiment, but just an aspect of the experiment

→ Randomness comes from the random experiments. After we done the experiment, we observed a specific outcome, s, then we map that to a real number.

Definition (Bernoulli): A random variable X is said to have Bernoulli(p) distribution, if X has only 2 possible values, 0 and 1, and $P(\underbrace{X=1}_{\text{event } \{s : X(s)=1\}}) = p$, $P(X=0) = 1-p$

So, no matter what the outcome is, it is only allowed to mapped to 0 or 1.

Binomial(n,p): The distribution of the number of successes in an n independent Bernoulli(p) trials is called Bin(n,p). Its distribution is given by

↓ n different exp^n

$$P(X=k) = {}^{n}C_{k}\, p^{k} (1-p)^{n-k} \quad \Big| \quad X = \text{no. of successes}$$

$$0 \le k \le n,\ k = \text{integer}$$

↳ PMF → prob. mass function

$X \sim Bin(n,p)$, $Y \sim Bin(m,p)$, independent,

then $X+Y \sim Bin(n+m, p)$

proof - Consider n trials, m more trials, all independent trials.