Lecture 30: Chi-Square, Student-t, Multivariate Normal

Chi-Square distribution (Univariate distribution)

$\chi^2(n)$ (chi-square) (n is degree of freedom)

How many normals squared did we add up?

Let $V = Z_1^2 + Z_2^2 + \dots + Z_n^2$, Z_i i.i.d. $N(0,1)$

Then (by defn.) $V \sim \chi^2(n)$.

Fact: $\chi^2(1)$ is Gamma$(1/2, 1/2)$.

So, $\chi^2(n)$ is Gamma$(n/2, 1/2)$.

Student-t distribution (Univariate distribution)

(Gosset, 1908)

Let $T = \dfrac{Z}{\sqrt{V/n}}$, where $Z \sim N(0,1)$

$V \sim \chi^2(n)$ indep. Z, V.

Then $T \sim t_n$ (n is degree of freedom)

properties:

1. Symmetric distribution, i.e., $-T \sim t_n$.
2. $n = 1 \Rightarrow$ Cachy, mean doesn't exist.
3. $n \geq 2 \Rightarrow E(T) = E(Z) E\left(\dfrac{1}{\sqrt{V/n}}\right)$

$= 0$

$E(Z^2) = 1$, $E(Z^4) = 3$, $E(Z^6) = 5$, ...

Use MGFs.

Another way:

$E(Z^{2n}) = E(\underbrace{(Z^2)}^n)$ After finding nth moment of Gamma(1/2, 1/2) ↓ then,

$\chi_1^2 \sim \text{Gamma}(1/2, 1/2)$ (Use LOTUS)

(4) Heavier-tailed than Normal, i.e., extreme values are relatively more likely than they would be for the normal. That's especially true if n is small, like in the Cauchy has very heavy tales. Normal is decaying much much faster than a Cachy.

(5) For n large (like 30, 40, 50 or larger), t_n looks very much like $N(0,1)$.
distribution of t_n (either CDF or pdf) goes to $N(1,0)$ as $n \to \infty$.

Let $T_n = Z / \sqrt{V_n/n}$, with $Z_1, Z_2, \ldots$ i.i.d. $N(0,1)$
$V_n = Z_1^2 + \ldots + Z_n^2$.
$Z \sim N(0,1)$ indep. of Z_j's.

$\underline{n \to \infty}$ $\frac{V_n}{n} \to 1$ with prob. 1 by Law of Large numbers since $E Z_1^2 = 1$ (variance of Z_1)

$\sqrt{\frac{V_n}{n}} \to 1$ with prob. 1

(LLNs says that the average of the same converges to the true theoretical average, which is just the expected value of one of $\{Z_1^2, Z_2^2 \ldots Z_n^2\}$)

So, $T_n \to Z$ with prob. 1. (Random variable)

Hence, t_n Converges to $N(0,1)$ distribution.

distribution

Multivariate Normal (MVN)

Defn. Random vector $(X_1, X_2, \ldots, X_K) = \vec{X}$ is Multivariate Normal if every linear combination $t_1X_1 + t_2X_2 + \cdots + t_KX_K$ is Normal.

Example Let Z, W be i.i.d. $N(0,1)$. Then $(Z+2W, 3Z+5W)$ is MVN, since

$$s(Z+2W)+t(3Z+5W), \quad s,t \text{ contants}$$
$$= \underbrace{(s+3t)Z + (2s+5t)W}_{\text{is Normal.}},$$

Sum of two indep. normals is normal.

Example (Not a Multivariate Normal) Let $Z \sim N(0,1)$, let S be random sign indep. of Z.

Then Z, SZ are marginally $N(0,1)$
(look at Z on its own, look at SZ on its own)

But, (Z, SZ) pair is not MVN.

Look at $Z+SZ$ (linear combination)

Half the time, S is -1, we get 0. The other half the time, we get some continuous thing. So, $(Z+SZ)$ is actually a mixture of discrete and continuous. We will never find a normal distribution that equals 0 with prob. equal to $1/2$. That's not a property of the normal distribution.

MGF of $\vec{X}$ (MVN) is $E(e^{\vec{t}'\vec{X}})$ → dot ~~vector product~~ of vector of $\vec{t}$ and $\vec{X}$

Let $EX_j = \mu_j$

$= E(e^{t_1X_1 + \dots + t_kX_k})$

$= e^{(t_1\mu_1 + \dots + t_k\mu_k) + \frac{1}{2}Var(t_1X_1 + \dots + t_kX_k)}$

$X \sim N(\mu, \sigma^2)$

$MGF(X) = E(e^{tX}) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$

(MGF of any univariate normal)

Theorem: Within MVN, uncorrelated implies indep.

(Note:- In general, indep implies uncorrelation, not otherway around)

$\vec{X} = \begin{pmatrix} \vec{X_1} \\ \vec{X_2} \end{pmatrix}$ MVN, if every component of $\vec{X_1}$ is uncorrelated with every component of $\vec{X_2}$, then $\vec{X_1}$ is independent of $\vec{X_2}$.

Example Let X, Y be i.i.d. $N(0,1)$. Then, $\underbrace{(X+Y, X-Y)}_{\text{taking together as a vector}}$ is MVN. → 2D (bivariate normal)

They are uncorrelated:

$$Cov(X+Y, X-Y) = Var(X) + \cancel{Cov(X,Y)} - \cancel{Cov(X,Y)} - Var(Y) = 0$$

So, $X+Y$, $X-Y$ are independent.