Lecture 11: The Poisson distribution

Sympathetic Magic,
→ Mistake of confusing a random variable with its distribution.

famous saying in Semantics

"Word is not the thing, the map is not the territory."

✓ Think of a random variable as a random house and its distribution as a blueprint of the house. If we have a blueprint, we ~~can't~~ could build many houses from the same blueprint. Just use the blueprint and build in different locations. Same way, we can have as many random variables as we want, all with the same distribution. They could be i.i.d. which would mean they are independent with the same distribution or they could be dependent, But they could have same distribution.

Poisson distribution, $X \sim \text{Pois}(\lambda)$ [Discrete distribution]

PMF: $P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$, $k \in \{0,1,2,\dots\}$

λ is the rate parameter

$\lambda > 0$

Validity check:

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

Taylor series of e^λ

$$= e^{-\lambda} e^{\lambda}$$

$$= 1$$

$$E(X) = e^{-\lambda} \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{(k-1)}}{(k-1)!} = \text{Taylor series of } e^{\lambda}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

→ Often used for application where ~~conte~~ counting number of "successes" where there are a large number of trials, each with small probability of success.

Examples

(1) # emails in an hour.

(2) # chips in chocolate chip cookie.

(3) # earthquakes in a year in some region.

Poisson paradigm/ Poisson approximation

Events $A_1, A_2, \ldots, A_n$ with

$P(A_j) = P_j$ and n is large, P_j's small.

Events independent or "weakly dependent", then number of A_j's that occur is approximately poisson,

$$\left[\lambda = \sum_{j=1}^{n} P_j \right]$$

(Even if they are dependent, by linearity, the expected no. of events that occur is the sum of P_j's)

→ should be the expected number of how many of these events occur

Binomial distribution converges to Poisson distribution

$X \sim Bin(n,p)$

Let $n \to \infty$, $p \to 0$

$\lambda = np$ is held constant

Find what happens to $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, k is fixed.

Soln: $\boxed{p = \lambda/n}$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\ldots(n-k+1)}{k!} \frac{\lambda^k}{n^k} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$\approx \frac{\lambda^k}{k!} e^{-\lambda}$$

Poisson PMF at k

$$\lim_{n\to\infty} \frac{n(n-1)\ldots(n-k+1)}{n^k}$$

$$= \lim_{n\to\infty} \frac{n}{n} \frac{(n-1)}{n} \ldots \frac{(n-k+1)}{n}$$

$$= 1$$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-k} = 1^{-k} = 1$$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

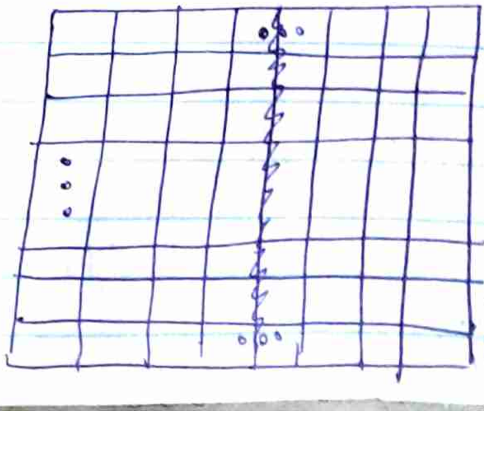
as $\lim_{n\to\infty} \left(1+\frac{\lambda}{n}\right)^n \to e^{\lambda}$

Example

Counting the number of raindrops that fall in some region.

$\to$ intuitive example for understanding the connection between binomial and Poisson

How many raindrops hit a horizontal piece of paper in say 1 minute?



Huge number of little square, each individual square is unlikely to get a raindrop hitting exactly in that. But, we are gonna get raindrops as we have huge no. of squares

In above example, λ is going to be a measure of the intensity of how hard the rain is coming.

Should we use binomial distribution for this?

→ If we assume that everyone of these squares whether it gets a drop of rain or not, is independent of all the others. And, if we assume that they all have the same probability p, then it would be exactly binomial.

But, we don't know enough about rain to answer this, but we are assuming/guessing it's not really exactly independent. But, it seems like a reasonable approximation to treat them as independent.

So, a poisson distribution seems reasonable here because we have a huge number of little squares, each one is very unlikely.

One other complication with binomial is that each one of the squares could only get 0 or 1. Now, there ~~are~~ is some tiny chance that two raindrops could fall into one of the squares. So, it's not going to be exactly binomial but even if it was a binomial, we had binomial of like a trillion, and then some tiny number. That's very very hard to work with. → even on computers

Example Have n people, find approximate prob. that there are 3 people with same birthday.

→ reasonably large (not very very large)

Soln: $\binom{n}{3}$ triplets of people, indicator r.v. for each one, I_{ijk} $(i<j<k)$

$$E(\#\text{ triple matches}) = \binom{n}{3}\frac{1}{365^2}$$

→ 1st person can have whatever birthday, 2nd person has to match the first person → $1/365$, and the 3rd person also has to match, prob. $1/365$

$X = \#$ triple matches. Approximate Poisson

$\lambda = \binom{n}{3} 1/365^2$ | I_{123}, I_{124} are not independent ↓ "weakly dependent" still we have to match 3 with 4 (in this case)

$$P(X \geq 1) = 1 - P(X=0)$$
$$\approx 1 - e^{-\lambda}\cdot\lambda^0/0!$$
$$\approx (1 - e^{-\lambda})$$