Lecture 14: Location, Scale and LOTUS

$Z \sim N(0,1)$

CDF Φ

$E(Z) = 0 \longrightarrow$ first moment

$Var(Z) = E(Z^2) = 1 \longrightarrow$ second moment

$E(Z^3) = 0 \longrightarrow$ Third moment

$-Z \sim N(0,1)$ (by symmetry) $\quad \underbrace{\int_{-\infty}^{\infty} Z^3 \frac{1}{\sqrt{2\pi}} e^{-Z^2/2} dz}_{\text{odd function}}$ (by Lotus) $= 0$

$\hookrightarrow$ flipping the sign changes the random variable, it makes a positive into negative and vice versa, but it doesnot change the distribution (that's what the symmetry says)

General Normal Distribution

Let $X = \mu + \sigma Z$, $\underset{\text{(real number)}}{\mu \in \mathbb{R}}$ (mean or location),

$\sigma > 0$ (SD or scale)

Then we say $X \sim N(\mu, \sigma^2)$.

because we are just rescaling everything by multiplying by a constant; that's going to affect if we draw one of the density, it's going to affect how wide or narrow that curve is.

because we are just adding a constant, it means a shift in location; We are not changing what the density looks like by adding μ; we are just moving it around left or right

$E(X) = \mu$

$Var(\mu + \sigma Z) = \sigma^2 Var(Z) = \sigma^2$] Normal distribution

$Var(X) = E\left((X - EX)^2\right)$

$= EX^2 - (EX)^2$

$Var(X+c) = Var(X)$

$Var(cX) = c^2 Var(X)$

$Var(X) \geq 0$

] Variance (in general)

$Var(X) = 0$ if and only if $P(X=a) = 1$, for some a

$Var(X+Y) \neq Var(X) + Var(Y)$ (in general)

[equal if X, Y are independent]

$Var(\underbrace{X+X}_{X \text{ is extremely dependent.}}) = Var(2X) = 4\,Var(X)$

$Z = \dfrac{X-\mu}{\sigma}$ [standardization] $\longrightarrow$ dimensionless quantity

Find PDF of $X \sim N(\mu, \sigma^2)$.

CDF: $P(X \leq x) = P\left(\dfrac{X-\mu}{\sigma} \leq \dfrac{x-\mu}{\sigma}\right)$

$= \Phi\left(\dfrac{x-\mu}{\sigma}\right)$

PDF: $\dfrac{\partial}{\partial x} CDF = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}$

$\dfrac{\partial}{\partial x} \Phi\left(\dfrac{x-\mu}{\sigma}\right) = \dfrac{1}{\sigma} \dfrac{\partial \Phi}{\partial x}$

$\downarrow$

PDF of standard Normal distribution

$$-X = -\mu + \sigma(-Z) \sim N(-\mu, \sigma^2)$$

~~Later~~

✓ If $X_j \sim N(\mu_j, \sigma_j^2)$ independent,

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$$

✓ **68–95–99.7% Rule** [How likely is it that a normal random variable will be a certain distance from its mean measured in terms of SD?]

$X \sim N(\mu, \sigma^2)$

$P(|X-\mu| \leq \sigma) \approx 0.68$ [The probability that x is within 1 standard deviation of its mean is about 68%]

$P(|X-\mu| \leq 2\sigma) \approx 0.95$ [The prob. that x is within 2 SD of its mean is about 95%]

$P(|X-\mu| \leq 3\sigma) \approx 0.997$ [The probability that x is within 3 SD of its mean is about 99.7%]

LOTUS Continued...

Prob. $p_0\ p_1\ p_2\ p_3, \ldots$

$X: 0, 1, 2, 3, \ldots$

$X^2: 0, 1^2, 4, 9, \ldots$

$E(X): \sum_x x P(X=x)$

$E(X^2): \sum_x x^2 P(X=x)$

A/c to LOTUS, this still works regardless of whether we have duplications

$X \sim Pois(\lambda)$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!}$$
$$= e^{-\lambda}e^{\lambda}(\lambda^2+\lambda)$$
$$= (\lambda^2+\lambda)$$

$$Var(X) = \cancel{\lambda^2}+\lambda - \cancel{\lambda^2}$$

$$\boxed{Var(X) = \lambda}$$

Poisson distribution, $X \sim Pois(\lambda)$ has mean λ and variance λ.

Poisson doesnot have units, it's just counting number of things, so it doesnot have that some dimensional interpretation.

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$$ (Taylor's series)

$$\sum_{k=1}^{\infty} \frac{k \cdot \lambda^{k-1}}{k!} = e^{\lambda}$$ (take derivative of both side)

$$\lambda \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{k!} = \lambda e^{\lambda}$$

$$\sum_{k=1}^{\infty} \frac{k\lambda^{k}}{k!} = \lambda e^{\lambda}$$

take derivate again,

$$\sum_{k=1}^{\infty} \frac{k^2\lambda^{k-1}}{k!} = \lambda e^{\lambda} + e^{\lambda} = e^{\lambda}(\lambda+1)$$

$$\sum_{k=1}^{\infty} \frac{k^2\lambda^{k}}{k!} = e^{\lambda}(\lambda^2+\lambda)$$

$X \sim Bin(n,p)$, find $var(X)$.

$X = I_1 + \ldots + I_n$, $I_j \overset{i.i.d.}{\sim} Bern(p)$

(Indicator of success on the jth trial, add up those and we get a binomial distribution)

$\cancel{E(X^2)} \equiv X^2 = I_1^2 + \ldots + I_n^2 + 2I_1I_2 + 2I_1I_3 + \ldots + 2I_{n-1}I_n$

$$E(X^2) = nE(I_1^2) + 2\underbrace{\binom{n}{2}}_{\binom{n}{2} \text{ cross terms}}E(I_1I_2)$$ [These are just i.i.d., so by symmetry n x anyone of them]

(Using symmetry, indicator random variables and linearity)

$I_1^2 = I_1$

$E(I_1) = E(X \sim Bern(p)) = p$

$E(X^2) = np + n(n-1)p^2$
$= np + n^2p^2 - np^2$

[$E(I_1 I_2) \longrightarrow$ The probability of success on both the first trial and the second trial, because the trials are independent $= p^2$]

$\therefore Var(X) = E(X^2) - (E(X))^2$
$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$
[$\because E(X) = np$]
$= np(1-p)$
$= npq$

Product of indicator random variables is an indicator random variable.

$I_1 I_2 \rightarrow$ Indicator of success on both the first and second trial.

$$I_1 I_2 = \begin{cases} 0, & \text{if atleast one of } I_1 \text{ or } I_2 \text{ is } 0 \\ 1, & \text{if both } 1 \end{cases}$$

Prove LOTUS for discrete sample space.

Show $E(g(X)) = \sum_x g(x)P(X=x)$.

Proof

$$\underbrace{\sum_x g(x)P(X=x)}_{\text{grouped}} = \underbrace{\sum_{s \in S} g(X(s)) \underbrace{P(\{s\})}_{\text{mass of that pebble}}}_{\text{Ungrouped}}$$

Think s as a pebble

S → Sample space

$$\sum_{s \in S} g(X(s))P(\{s\}) \longrightarrow \text{ungrouped case}$$

This says take each pebble compute function $g(X(s))$ and then take a weighted average. $P(\{s\}) \rightarrow$ weight of a given pebble s.

$$\sum_{x} g(x)\, P(X=x) \longrightarrow \text{grouped case}$$

This says, first combine all of the pebbles that have the same value of x in a super-pebble. A super-pebble means we grouped together all pebbles with the same x value (not the same $g(x)$ value), then ~~t~~ average them.

Think of above as double sum

$$\sum_{x} \sum_{s:X(s)=x} \overbrace{g(X(s))}^{g(x)} P(\{s\}) = \sum_{x} g(x) \underbrace{\sum_{s:X(s)=x} P(\{s\})}$$

first sum over x values, group together and sum over all the pebbles that have that value.

$$= \sum_{x} g(x) P(X=x)$$

Add up the masses of all the pebbles labeled x. Sum of masses of the little pebbles form the super-pebble