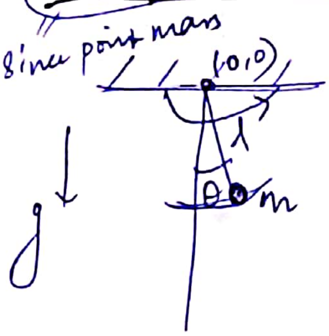
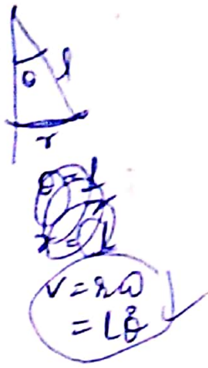
Lecture 2: Nonlinear Dynamics

Simple Pendulum





Kinetic energy

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

Potential energy

$$U = -mgl\cos\theta$$

Lagrangian

$$ml^2\ddot{\theta} + mgl\sin\theta = Q$$

generalized force (in this system, generalized force would be torque applied around the joint)

$$Q = -b\dot{\theta} + u$$

(if $\dot{\theta}$ is +ve, it will work to resist $\dot{\theta}$)

control input

$$\boxed{ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u}$$

(non-linear)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu \quad (B=1)$$

$$\boxed{\ddot{\theta} = f(\theta, \dot{\theta}, u)}$$

Solve diff. eqn but because of non-linearity not easy to solve.

Like if given: $\theta(0), \dot{\theta}(0)$ (initial cond.)

find: $\theta(t)$ (pendulum at time t)

Wrong question to ask → where is the pendulum at time t?

Good questions to ask → what is $\lim_{t\to\infty} \theta(t)$?

easy

Will my robot fall down?

Graphical Analysis

$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$ (Unit of this eqn is torque N·m, $Kg\,\frac{m^2}{s^2}$)

(2nd order system)

First order equivalent (high damped system)

damping

$b\dot{\theta} >> ml^2\ddot{\theta}$

$Kg\frac{m^2}{s}$ $\quad$ $Kg\,m^2\frac{1}{s^2}$

So, we need a characteristic time constant of pendulum in order to compare both side.

→ The behaviours we can about are the ones that are operating at this sort of time scale and the natural time scale is the natural frequency of the simple pendulum.

$\sqrt{g/l}$ has unit $\frac{1}{s}$

$\cancel{b >> ml^2}$ (since wrong units)

But,

$b\sqrt{\frac{l}{g}} >> ml^2$

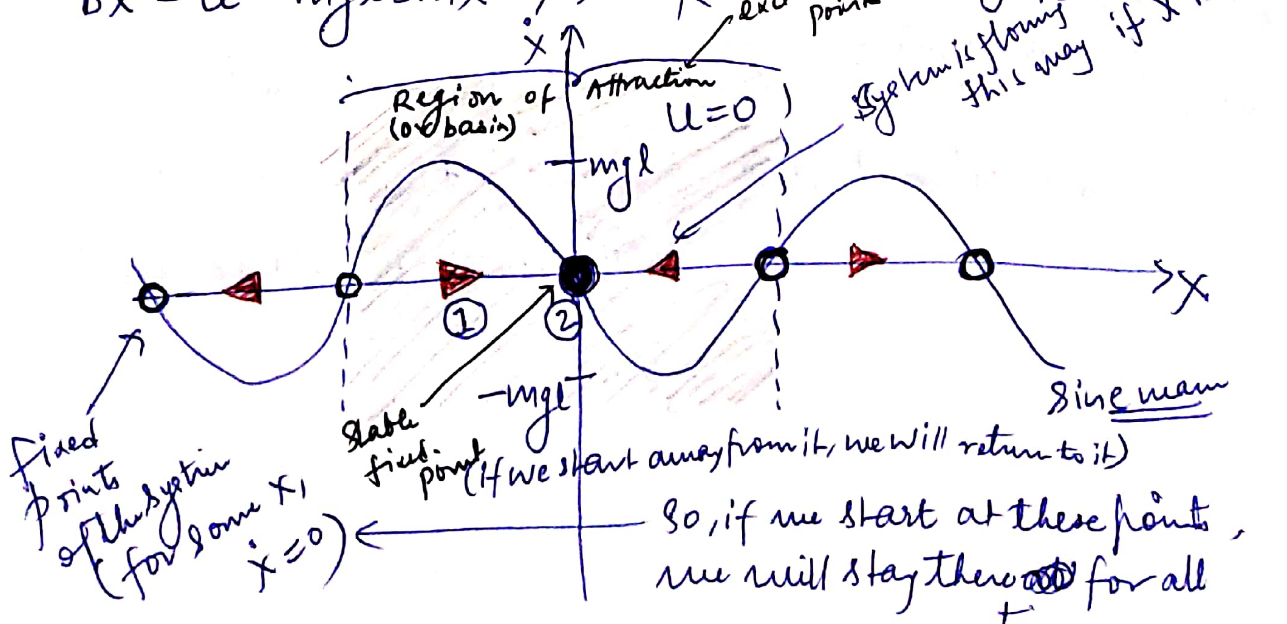
(Heavily damped regime)

approximate

So, $b\dot{\theta} \approx u - mgl\sin\theta$

($\because b\dot{\theta} >> ml^2\ddot{\theta}$)

$b\dot{x} = u - mgl\sin x$, $x \in \mathbb{R}$



So, if we start at these points, we will stay there for all time.

So, as time goes to ∞, any initial conditions that started at 1 are going to end up at fixed point 2. (numbers are just for understanding purpose)

→ Every stable fixed point will have a region of attraction.

⇒ A linear system can be stable, marginally stable or unstable

→ A linear system is stable at the origin.

Definition of stability

Local Stability
(In what sense is a fixed point of a non-linear system locally stable?)

→ In the sense of Lyapunov (i.s.L.) → if we start near a region, we won't go too far away from that region.

→ Locally attractive (says we will get to the region, converge to the region)

→ ~~Locally~~ Asymptotically stable. → attractive + i.s.L.

→ Exponentially stable. (says there is some rate at which we get there. eg. we are going to get there faster than a linear system with a particular const.)

→ An undamped pendulum started near the bottom will never converge. It will sit there oscillating forever.

→ So, it's not asymptotically stable, it's not attractive. But, it's stable in the sense of Lyapunov.

In the sense of Lyapunov (i.s.L.)

ϵ, δ are small position constants

for every ϵ, $\exists\, \delta$ s.t.

$$\|x(0) - x^*\| < \delta \Rightarrow \forall t \ \|x(t) - x^*\| < \epsilon$$

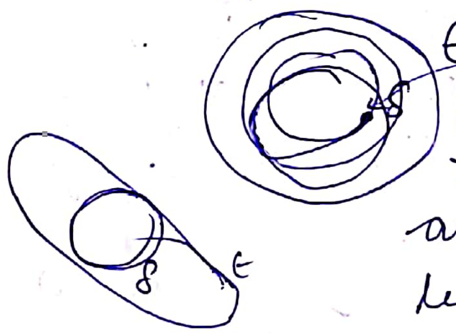
euclidean distance between x at $t=0$ & x^* in our state space

fixed point

it's like a ball in state space

we will never go too far

for every ball in state space, we can find some smaller balls which must be strictly constrained.



trajectories that start they might walk around but they'll never leave ϵ-ball.

Locally Attractive

$$\lim_{t \to \infty} x(t) \Rightarrow x^*$$

Asymptotically stable (isL + attractive)

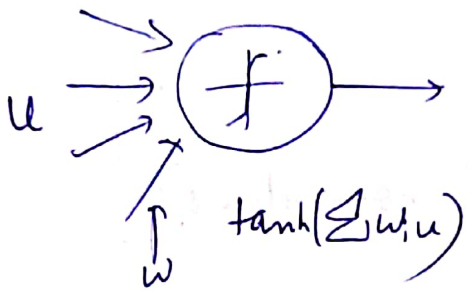
Exponentially stable

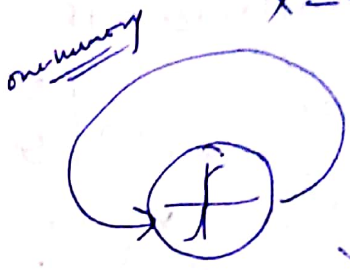
$$\forall t, \ \|x(t) - x^*\| < Ce^{-at}, (a > 0)$$

(will converge faster than linear system)

Simple Recurrent Neural Network

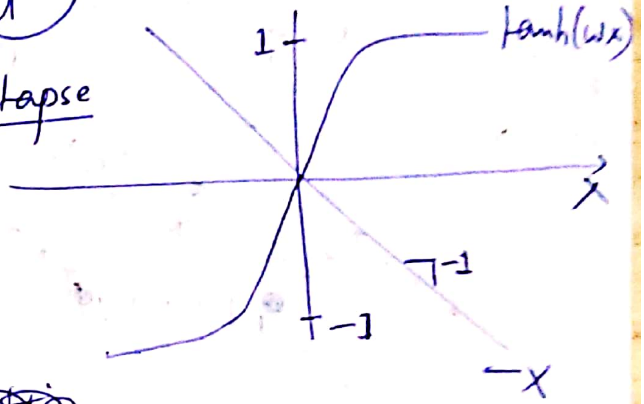
$$\dot{x} = -x + \tan(wx + u)$$





Autapse

$$\dot{x} = -x + \tanh(wx + u)$$



LSTM (Long Short-term memory)

JANET model

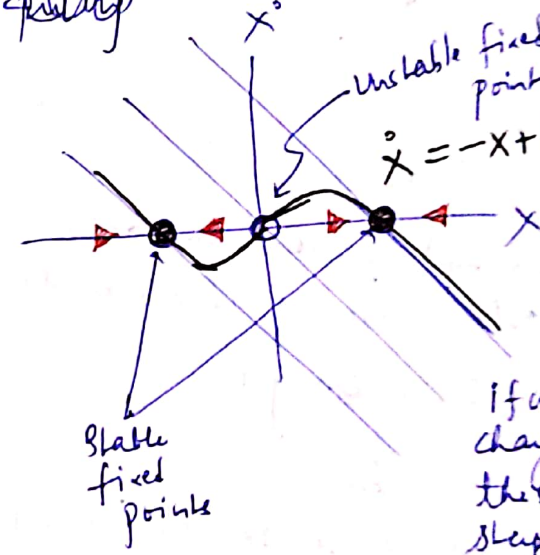
Transformer → replacing recurrent NNs in many applications.

Hopfield networks as associated memory

X_i represents activation (e.g. firing rate) of neuron i.

"Memory" is fixed point.

Dynamics can fill in the details.



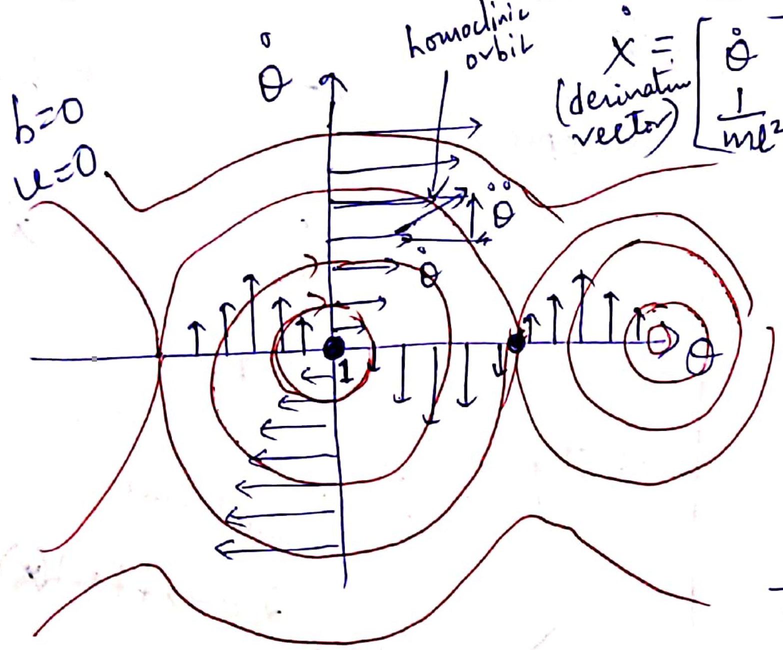
If we change w, the sharpness/steepness of the curve changes

2nd order System

$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$$

$$\dot{x} = f(x, u) \qquad \underset{\text{(State vector)}}{x} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\underset{\text{(derivative vector)}}{\dot{x}} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{ml^2}\left(u - b\dot{\theta} - mgl\sin\theta\right) \end{bmatrix}$$



1 is fixed point which is stable in the sense of Lyapunov but it's not asymptotically stable

→ trajectories remain in the vicinity of it but it never converge.
"Undamped Pendulum"

→ In simple pendulum, above orbits are defined as levels of const. energy.

One idea of control is feedback cancellation

Eq. of motion for pendulum

$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u \quad (1)$$

Our goal is to pick u.

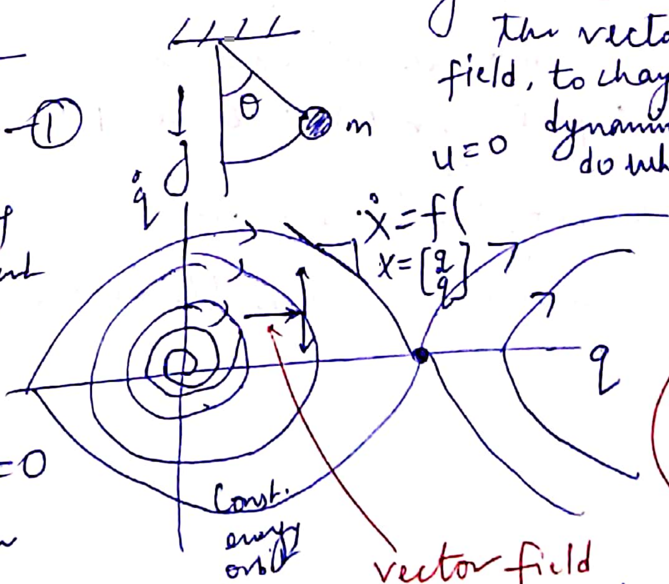
$u = \pi(q, \dot{q})$ ← (function of my current state)

$= 2mgl\sin\theta$ (we make this choice)

Putting $u = 2mgl\sin\theta$ in (1)

$$ml^2\ddot{\theta} + b\dot{\theta} - mgl\sin\theta = 0$$

(eqn of upside down pendulum)



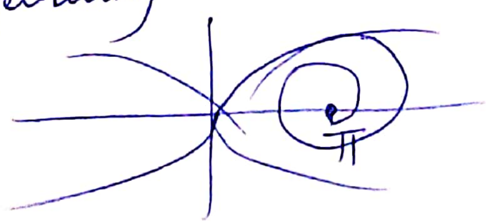
Goal is to change the vector field, to change the dynamics to do what we want

u=0

Second order System

vector field is everywhere

u gives us the ability to vector field by moving it up/down



~~One idea is Feedback Cancellation~~

what if

$$|u| \le \frac{mgl}{q}$$

$u = \text{sat}\,(2mgl\sin\theta)$ $\nearrow \frac{mgl}{q}\ \frac{gmgl}{q}$

saturate

first order system plot

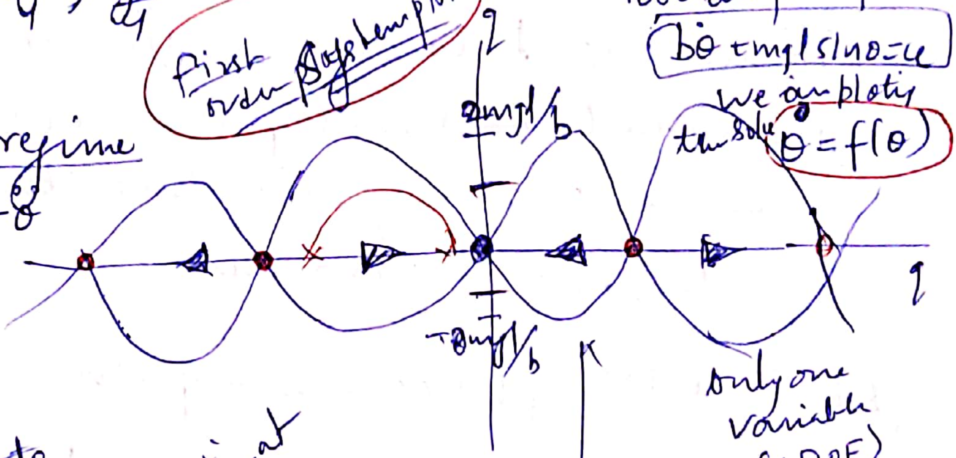
we are ploting overdamped

$\boxed{b\dot{\theta} + mgl\sin\theta = u}$

we are ploting the sol $\dot{\theta} = f(\theta)$

overdamped regime

$$b\dot{\theta} \gg ml^2\ddot{\theta}$$



If we apply complete torque cancelling control, then we changed it to bean upside down pendulum, then position at red dot become stable fixed point while origin become unstable fixed point.

only one variable (1 DOF) going around.

These are not orbits anymore unlike plots on previous page. The system is only evolving along the x-axis.