Lecture 8 : Computational Approaches to Lyapunov functions

Recap :- DP view of the world – minimizing some cost-to-go.

$$\int_0^\infty l(x,u)\,dt \quad , \text{s.t. } \dot{x} = f(x,u)$$

Cond. for Continuous time $J^*(x) \Rightarrow \quad 0 = \min_u \left[l(x,u) + \frac{\partial J}{\partial x} f(x,u) \right]$

$$\frac{dJ^*}{dt} = -l(x,u^*)$$

(Cost-to-go must go down the hill ~~at~~ exactly ~~at~~ the rate of the loss when we are taking optimal controller)

Lyapunov view of the world

→ Instead of find a J that's going down the hill at exactly ~~the~~ above rate $(-l(x,u^*))$

find a function $V(x) > 0$

s.t. $\frac{dV}{dt} \leq 0$ (going down the hill at all)

eventually but not exactly

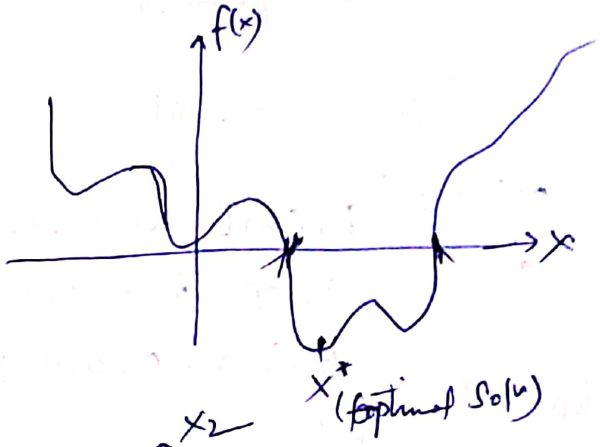
(There are more functions that will satisfy this inequality constraints and because of that it would be easier to search over them)

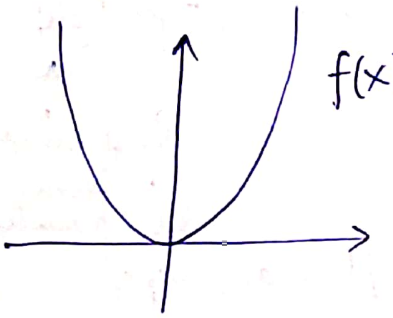
Optimization Crash Course

$$\min_{x} f(x)$$

decision variable — scalar objective

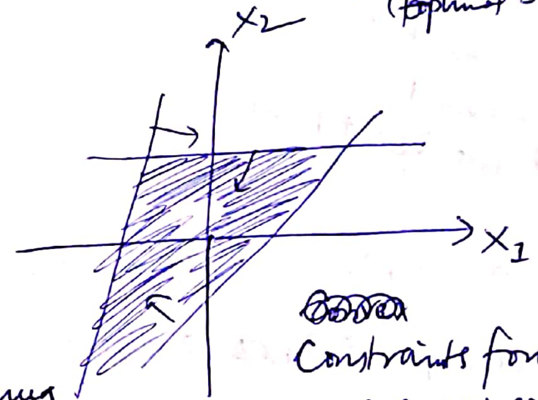
subject to $\forall x,\ g(x) \le 0$





$f(x) = x^TQx$

Convex func^n



Constraints form a convex set.

When the objective function is convex and constraints form a convex set, then we are in the land of convex optimization.

Linear Programming (LP)

Linear cost function, Linear constraints

$$\min_{x} c^Tx$$
$$\text{s.t. } Ax \le b$$

Second order cone programming (SOCP)

Quadratic Programming (QP)

$$\min \sqrt{Q_x}$$
$$\text{st. } Ax \le b$$

If Q_x is +ve definite, then it's a convex quadratic programming but still have linear constraints.

Semidefinite Programming (SDP)

$$\min_{x} c^Tx$$
$$Ax \le b$$
$$P(x) \succeq 0 \leftarrow \text{Semidefinite Constraint}$$

Pendulum dynamics

Drake Tutorial - for diff. kind of programing

$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = 0$$

$$\phi(x) = [1, \cos\theta, \sin\theta, \dot{\theta}, \cos^2\theta, \sin\theta\cos\theta, \dot{\theta}\sin\theta, \dot{\theta}\cos\theta, \dot{\theta}^2]$$

basis of the trignometric polynomial that we expect to see

$$\left(V(x) = \alpha_0 + \alpha_1\cos\theta + \alpha_2\sin\theta + \alpha_3\dot{\theta} + \dots\right.$$

find α_i's (coeffs. of above polynomials) such that the conditions hold.

Search over a class of functions that could be lyapunov function

Use linear function approximators

$$V(x) = \sum_i \alpha_i \phi(x) = \alpha^T\phi(x)$$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = \left(\alpha^T \frac{\partial \phi}{\partial x}\right) f(x)$$

(Still linear in these same parameters)

Idea - Sample many x_i's

find α, so that $\forall_i \; V(x_i) \geq 0$

$\dot{V}(x_i) \leq 0$

$V(x_i) > 0$ $\quad V(x_i) \geq \epsilon x_i^T x_i$ $\quad$ 1e-3

How can we certify that lyapunov condition are true $\forall x$?

→ what if we choose a different basis function.

$$V(x) = \sum_i \alpha_i \phi_i^2(x), \quad \alpha \geq 0$$

(constraint)

$$= \phi^T(x) \begin{bmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \ddots \end{bmatrix} \phi(x) \quad \text{(matrix form)}$$

Nice generalization → $\phi^T(x) G \phi(x)$, $G \succeq 0$

(+ve semidefinite matrix)

(convex set)

$\forall x, \; x^T G x \geq 0$

G be +ve semidefinite

$x^T G_1 x \geq 0$ $\quad$ $x^T G_2 x \geq 0$

$\Uparrow$ $\quad$ $\nearrow$

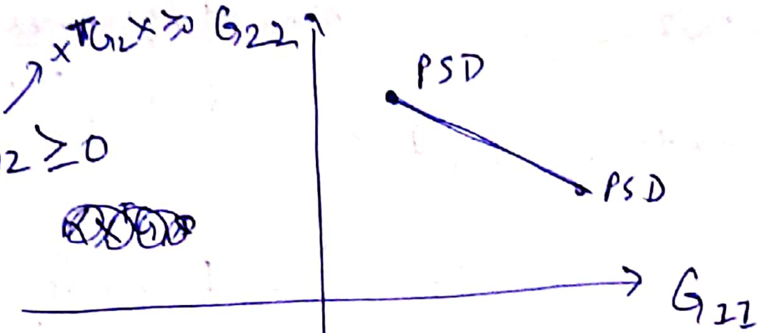
$G_1 \geq 0, G_2 \geq 0$

$\Downarrow$

$\alpha G_1 + (1-\alpha) G_2 \geq 0$

$\forall \alpha \in [0, 1]$

$\alpha x^T G_1 x$



PSD - positive semi-definite matrix

All of the positive semi-definite matrix that can be made by convex combination of them (end points) are also PSD.

So, the set of PSD is a convex set.

Note - If instead of writing a bunch of x_i's $V(x_i) \geq 0$, we would have parametrised $V(x)$ like below

$$V(x) = \sum_i \alpha_i \phi_i^2(x) \quad , \alpha \geq 0$$

$$= \phi^T(x) \begin{bmatrix} \alpha & & \\ & \ddots & \\ & & \ddots \end{bmatrix} \phi(x) \quad \forall x, x^T G x \geq 0$$

$\phi^T(x) G \phi(x)$ and just add the constraint $G \geq 0$, then that guarantees for all X's, V is positive definite function.

No sampling required.

What about $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = -P(x)$? basis function

$P(x) = m^T(x) G_2 m(x), G_2 \geq 0$ P for polynomial

when $P(x)$ is polynomial, this requires finite linear constraints.

$P(x) = 1 + 2x^2 \overset{?}{\geq} 0$ (Sum of squares decomposition of a polynomial)

$P(x) = 2 - 4x + 5x^2 = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ - & G & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$

Is $P(x)$ positive for all x?

If $G \geq 0$ (Positive definite) and if above equality holds, then $P(x)$ is positive $\forall x$.

For polynomial, coeff. matching is linear and can be written as linear constraints & is sufficient

$G_{11} = 2$

$G_{12} + G_{21} = -4$

Square terms coeff. $= 5$

If we can find a matrix G constrained to match above polynomial $P(x) = 2 - 4x + 5x^2$ such that $G \geq 0$ (positive semi definite), then $2 - 4x + 5x^2$ must have been positive.

→ General optimizing over polynomials using "sum of squares" decomposition trick is called "Sum of Squares (SOS) optimization".

$\dot{x} = Ax$ is it stable?

$V(x) = x^T P x$, $P > 0$

$\dot{V}(x) = x^T A^T P x + x^T P A x < 0$

$x^T (A^T P + PA) x$ $\quad A^T P + PA < 0$

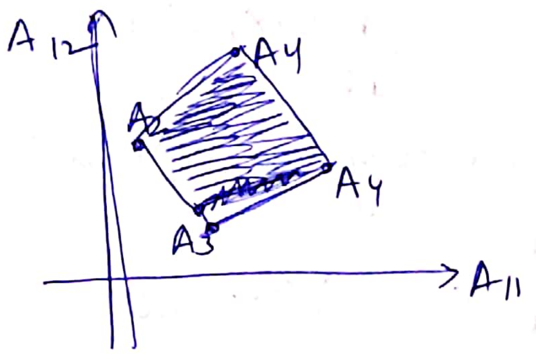
Find a P such that P is positive definite and $\dot{V}(x)$, i.e., $A^T P + PA$ is negative definite

$\underbrace{A^T P + PA = -P_2}_{\text{(Linear Constraint)}}$, $P_2 > 0$ (Positive semidefinite Constraint)

Robust Stability (for linear Systems) via Common Lyapunov fun^c

$$\dot{x} = Ax$$

$$A = \sum_i \beta_i A_i \quad , \; \beta_i \geq 0 \, , \; \sum_i \beta_i = 1$$



Find a P s.t. $P > 0$.

$V(x) = x^T P x$

$\forall i$, $PA_i + A_i^T P < 0$

when the optimization fails in the land of convex optimization it means there is no solutions to the problem. There is no quadratic common Lyapunov function for those systems

if this is true, then

$$P\left(\sum \beta_i A_i\right) + \left(\sum_i \beta_i A_i\right)^T P < 0$$

(anything in the convex hull is also going downhill)

$$\dot{x}_1 = -x_1 - 2x^2$$

$$\dot{x}_2 = -x_2 - x_1 x_2 - 2x_2^3$$

$$V(x) = \phi^T(x)\, G\, \phi(x)$$

$$\dot{V}(x) < 0$$

find a Lyapunov fun that ~~satisfy for any optimality~~ certify that optimality we can do this

$$G = \begin{bmatrix} x_1 \\ x_2 \\ x_1 x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix} \text{ (generic)}$$

find a G such than so that
$V(x) = \phi^T(x) G \phi(x)$ is true
and $\dot{V}(x) < 0$