(1) $P(x = (x_1, \ldots x_n) / C_i)$

$$= \frac{m!}{x_1! \ldots x_n!} p_{i1}^{x_1} \cdots p_{in}^{x_n}$$

$$p(x) = \sum_{i=1}^{k} p(x/c_i) p(C_i)$$

$$\sum_{i=1}^{k} \pi_i = 1$$

$$L(\phi/x) = \sum_{t} \log p(x^t/\phi)$$

$$= \sum_{t} \log \sum_{i=1}^{k} \pi_i \frac{m!\ p_{11}^{x_1^t} \cdots p_n^{x_n^t}}{x_1^t! \ldots x_n^t!}$$

$$\frac{\partial L}{\partial p_{ij}} = \sum_{t} \frac{\pi_i \frac{m!\ p_{i1}^{x^t} \cdots p_n^{x_n^t}}{x^t! \ldots x_n^t!}}{\sum_{i=1}^{k} \pi_i \frac{m!\ p_{i1}^{x^t} \cdots p_n^{x_n^t}}{x_1^t \ldots x_n^t!}}$$

$$+ \alpha = 0$$

$$\Rightarrow 0 = \sum_{t} r(z_i^t) \frac{x_j^t}{p_{ij}} + \alpha$$

$$\Rightarrow \quad = \sum r(z_i)^t \sum_t x_j^t + \alpha \sum_{j=1} p_{ij}$$

$$\text{———} (1)$$

$\Rightarrow$

$$\Rightarrow 0 = \sum_{j}\sum_{t} r(z_i^t)\, \underbrace{x_j^t}_{m} + \alpha \sum_{j} \underbrace{p_{ij}}_{1}$$

$$\Rightarrow \alpha = \sum r(z_i^t)\, m$$

From eqn - ①

$$p_{ij} = \frac{\sum r(z_i^t) \cdot x_j^t}{-\alpha}$$

$$= \frac{\sum_t r(z_i^t)\, x_j^t}{m \sum_t r(z_i)^t}$$

② $$\frac{\partial}{\partial \pi_i} L = \frac{\sum_t \frac{m!}{x_1^t! \cdots x_n^t!} p_1^{x_1^t} \cdots p_{in}^{x_n^t}}{\sum \pi_i \frac{m!}{x_1^t! \cdots x_n^t!}} + \beta\pi$$

$$\Rightarrow 0 = \sum_t \frac{r(z_i^t)}{\pi_i} + \beta$$

$$\Rightarrow 0 = \sum_t r(z_i)^t + \pi_i \beta$$

—— ②

$$\Rightarrow 0 = \sum_{t} \sum_{i} \frac{\gamma(z_i)}{\beta} + \underbrace{\sum_{i} \pi_i}_{=1}$$

$$\Rightarrow -\beta = N$$

From equation 2

$$\pi_i = \sum_{t} \frac{\gamma(z_i^t)}{-\beta}$$

$$\Rightarrow \pi_i = N_i / -\beta$$

$$= N_i / N$$

For calculating complete log likelyhood we have to assume the hidden varriable.

$$P(x \mid z, P) = \prod_{i=1}^{K} P(x / P_i)^{z_i}$$

$$P(z / \pi) = \prod_{i=1}^{K} \pi_i^{z_i}$$

(1)(2) $\Rightarrow \log P(x, z / p, \pi)$

$$= \sum_{t=1}^{N} \log P(x^t, z^t / p, \pi)$$

$$= \sum_{t=1}^{N} \log P(x^t / z^t, p) \cdot P(z^t / \pi)$$

$$= \sum_{t=1}^{N} \log \prod_{z=1}^{k} \left(P(x^t / p_i)^{z_i^t} \pi_i^{z_i^t} \right)$$

$$= \sum_{t=1}^{N} \sum_{i=1}^{k} \left(z_i^t \log(\pi_i) + z_i^t \log(P(x^t / p_i)) \right)$$

$$P(x^t / p_i) = \frac{m!}{x^t! \ldots x_n^t!} p_{i_1}^{x_1 t} \ldots p_{in}^{x_n t}$$

$$\log(P(x^t / p_i)) = \log\left(\frac{m!}{x_1^t! \ldots x_n^t!} \right) + \sum_{j=1}^{n} x^t \log(p_{ij})$$

$\log P(x, z / P, \pi)$

$$= \sum_{t=1}^{N} \sum_{i=1}^{k} z_i^t \left(\log(\pi_i) + \log(P(x^t / p_i)) \right)$$

$$= \sum_{t=1}^{N} \sum_{i=1}^{k} z_i^t \left(\log \pi_i + \log \left(\frac{m!}{x_i^t! \ldots x_n^t!} \right) \right.$$

$$+ \sum x_j^t \log(p_{ij}))$$

E step

$$\sum_{t=1}^{N}\sum_{i=1}^{k} E(z_i^t / x, p^l)(\log \pi_i + \log(P(x^t/p_i^l)))$$

$$E(z_i^t / x, p^l)$$

$$= E(z_i^t / x^t, p^l)$$

$$= P(z_i = 1 \mid x^t, p^l)$$
$$+ 0 \cdot P(z_i^t = 0 / x^t, p^l)$$

$$= P(z_i^t = 1 / x^t, p^l)$$

$$= \frac{P(x^t / z_i^t = 1, p^l)\, P(z_i^t = 1 / p^l)}{P(x^t / p^l)}$$

$$= \frac{P(x^t / p_i^l) \cdot \pi_i}{\sum_i P(x^t / p_i^l)\, \pi_i}$$

$$= \frac{\pi_i \frac{m!}{x_1^t! \cdots x_n^t!} p_{i1}^{x_1^t} \cdots p_{in}^{x_n^t}}{\sum_{r=1}^{k} \pi_r \frac{m!}{x_1^t! \cdots x_n^t!} p_{r1}^{x_1^t} \cdots p_{rn}^{x_n^t}}$$

$$= r(z_i^t)$$

M - step

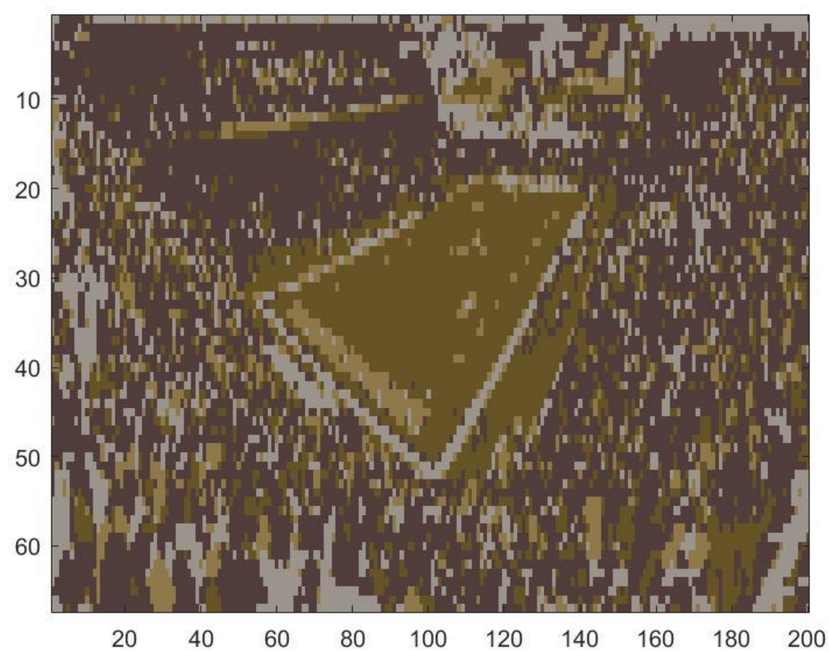
$$p^{l+1} = \underset{p}{\arg\max} \sum_t \sum_i r(z_i^t) \times [\log \pi_i + \log(p(x^t | p_i))]$$

$$\pi_i = \frac{\sum_t r(z_i^t)}{N}$$

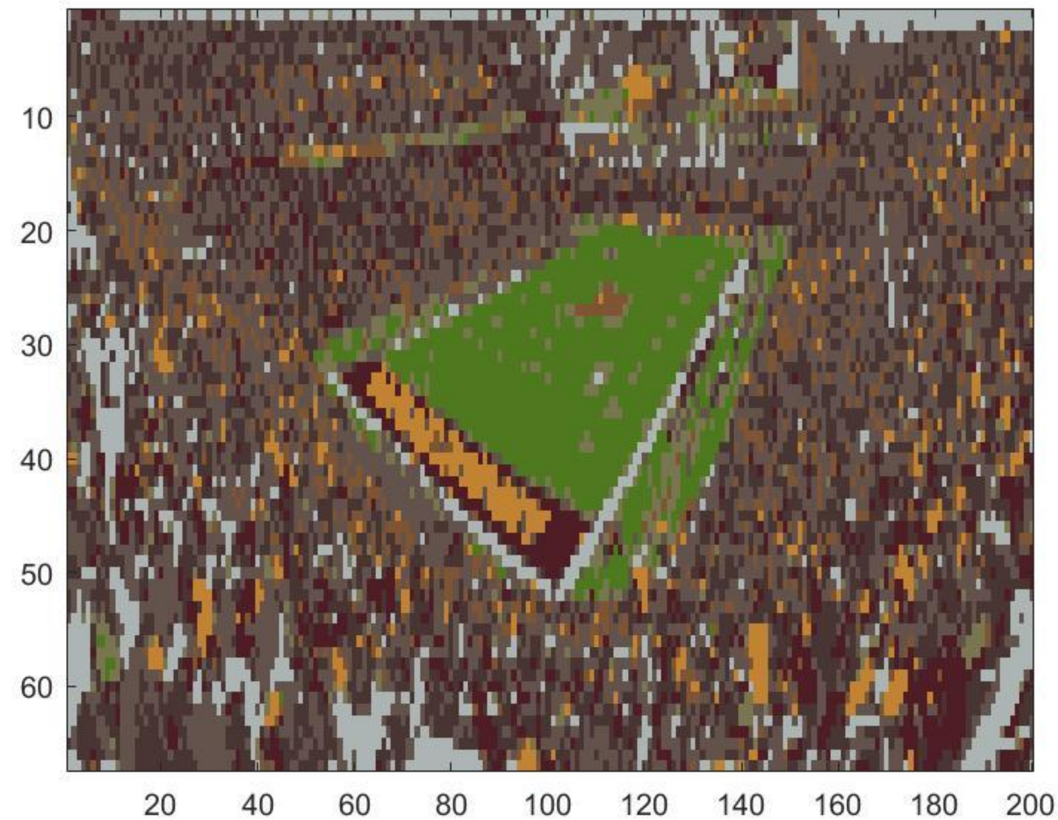
$$= \frac{N_i}{N}$$

Question 2

K=4



K=8


10
20
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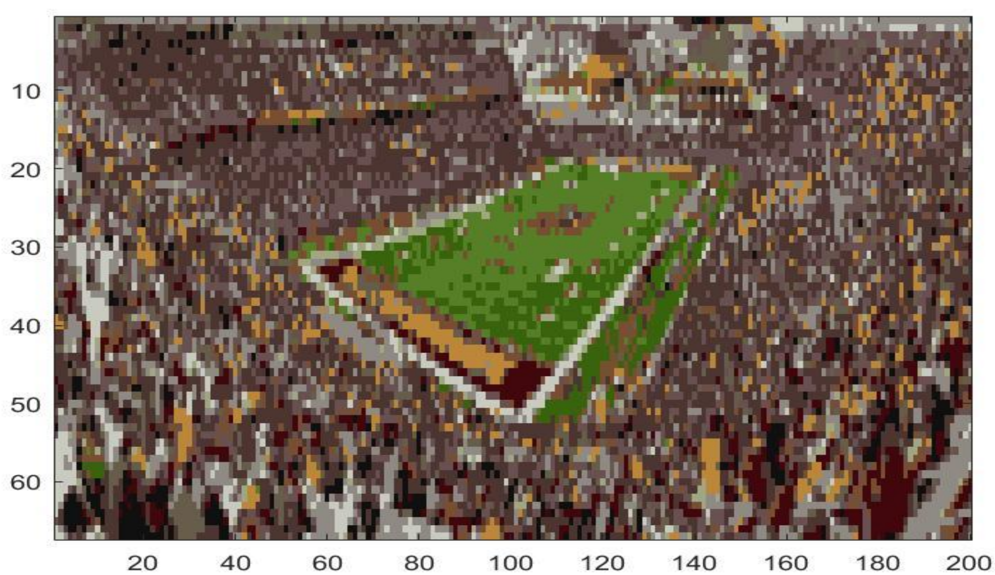
160
180
200

K=12


10
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100
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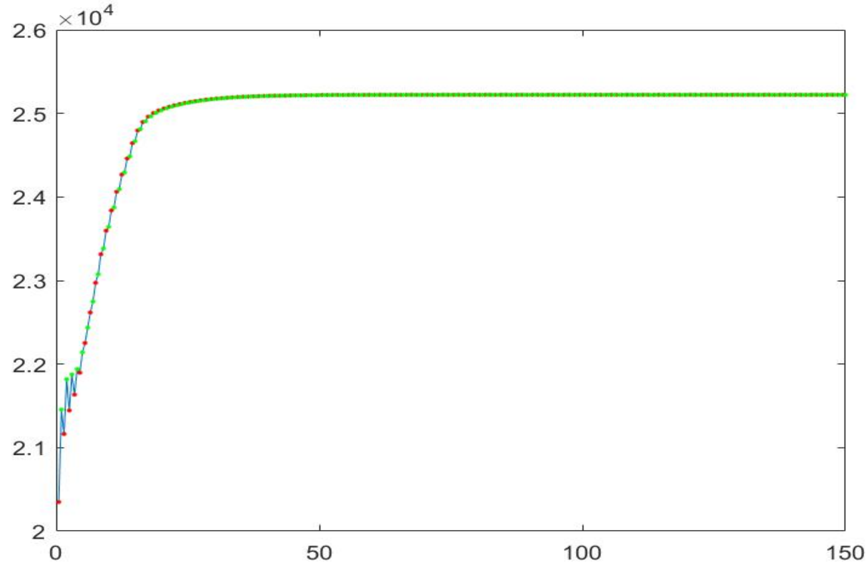
Question 2b

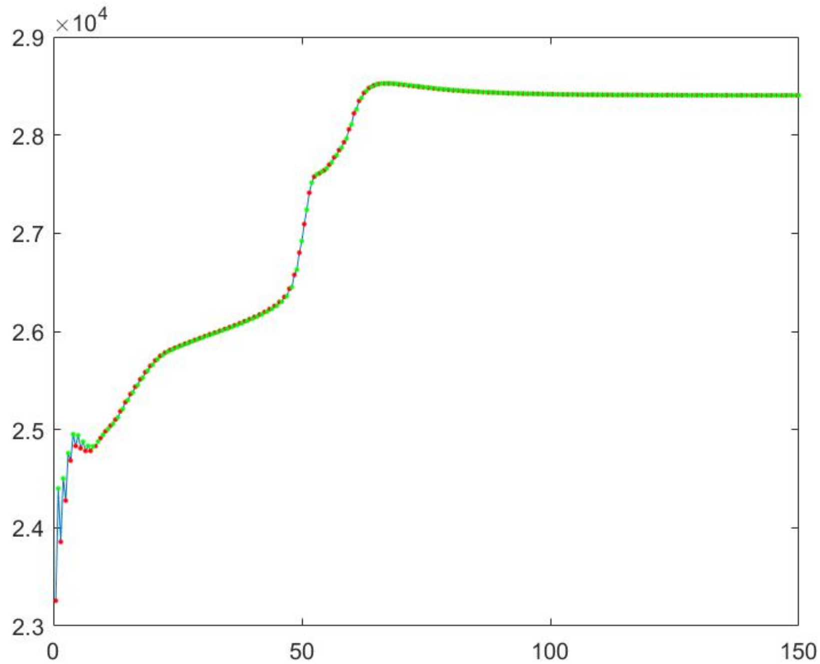
Red=After expectation

Green=After maximization   Xaxis=iterations Yaxis=loglikelihood value

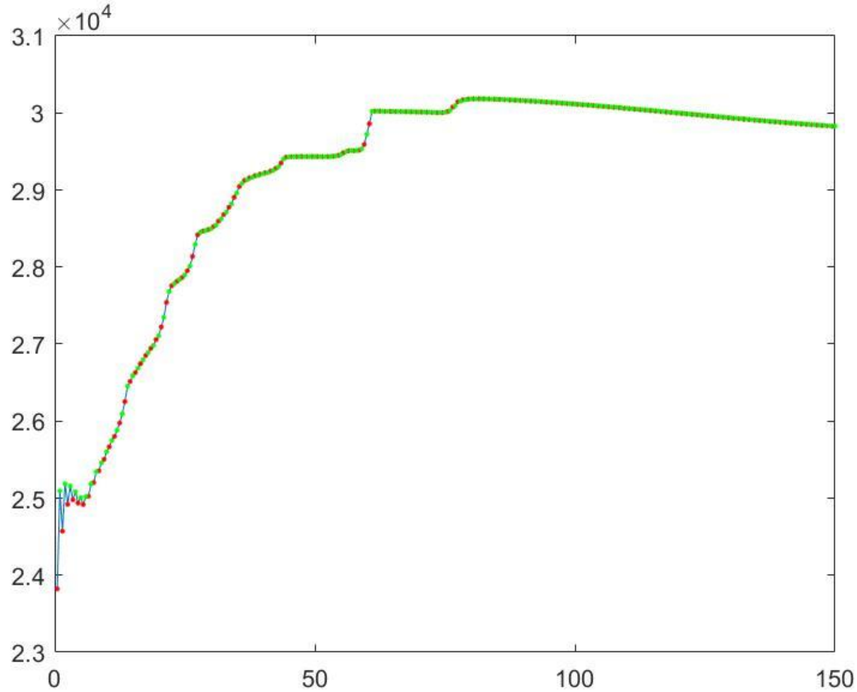
K=4



K=8

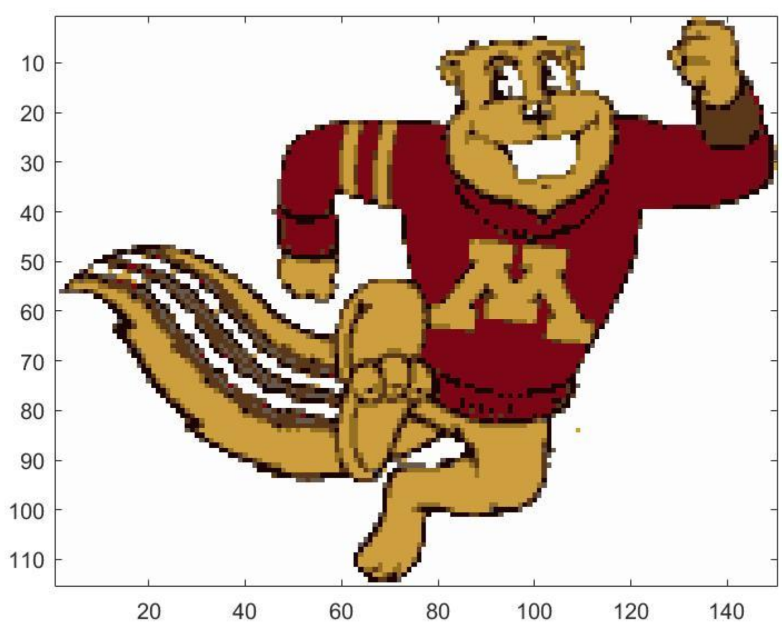


K=12



If we increase the value of K the value of complete log-likelihood increases.

2c



Here the kmeans (it does not required to calculate the covariance matrix and it uses Euclidian Distance)algorithm ran successfully but the Em implementation failed .

since the Em using gaussians distributions uses covariance matrix and sigma matrix(covariance matrix) in this case is not positive definite so calculation of the square root of sigma matrix is impossible .thats why it was failed.

Sometimes the em using gaussian distribution fails because the sigma matrix(covariance matrix) is singular so calculation of inv(sigma)is not possible.

Formula to check a matrix is positive definite or not : all(eig(S(:,:,4)) > eps)
2 d

2d expectation Log Likelyhood:

$$Q = \sum_t \sum_i h_i^t \left(-\frac{1}{2} \log|\Sigma_i| - \frac{1}{2} (x^t - \mu_i)^T \Sigma_i^{-1} (x^t - \mu_i) - \frac{\lambda}{2} \sum_j (\Sigma^{-1})_{jj} \right)$$

$$\frac{\partial Q}{\partial \Sigma_i^{-1}} = 0$$

$$\Rightarrow \sum_t \frac{h_i^t}{2} \Sigma_i - \frac{1}{2} \sum_t h_i^t (x^t - \mu_i)^T (x^t - \mu_i) - \sum_t h_i^t \cdot \frac{\lambda I}{2} = 0$$

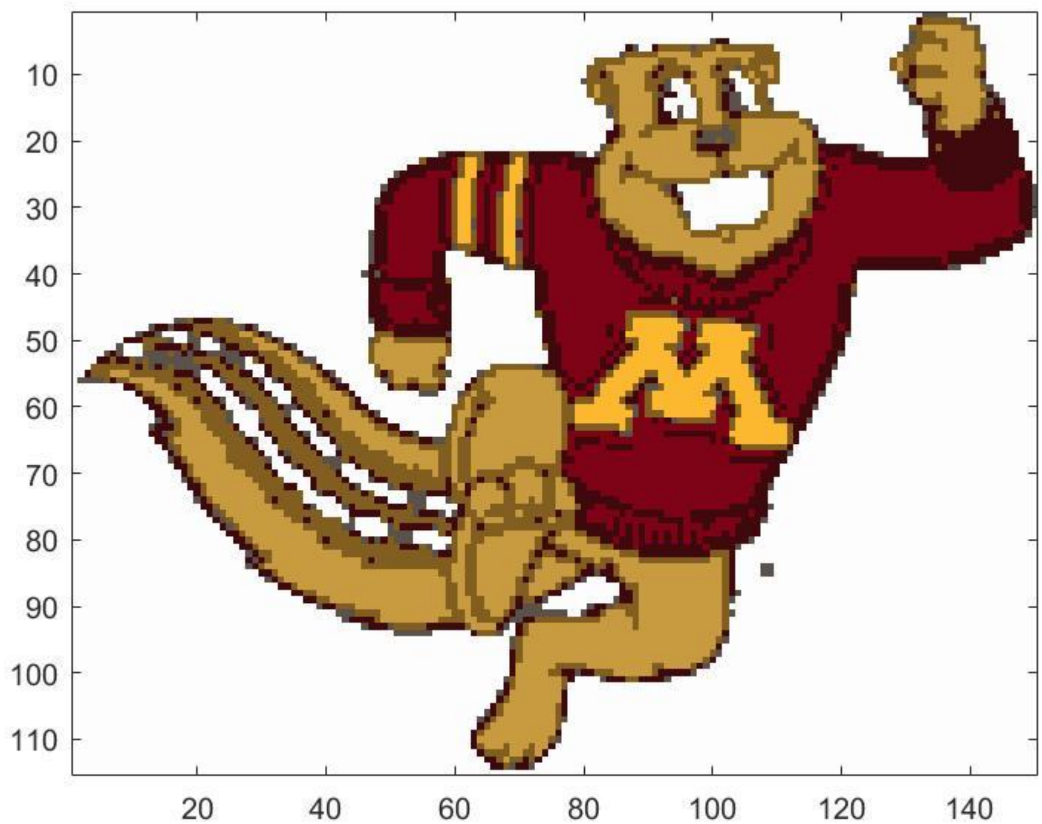
$$\Rightarrow \Sigma_i = \frac{\sum_t h_i^t \left((x^t - \mu_i)^T (x^t - \mu_i) + \lambda I \right)}{\sum h_i^t}$$

Sigma is only modified and all the others calculation will be same.Now the covariance matrix calculation is modified in the em algorithm (flag=1) by adding a factor lambda*eye();

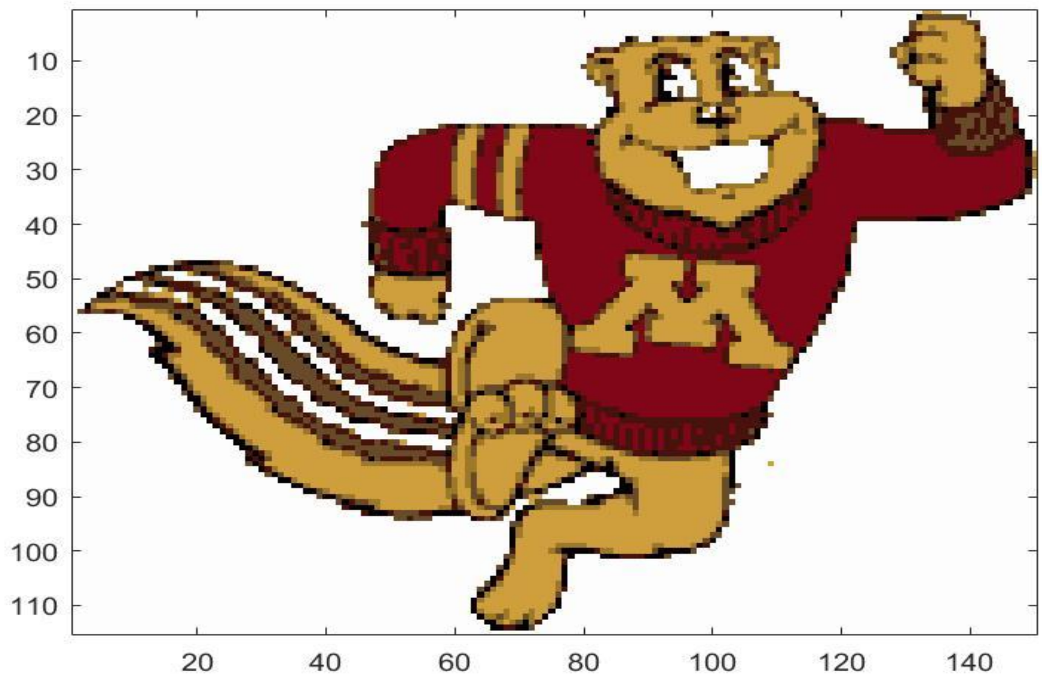
E

So after adding the lambda*I parameter we have removed the probability of the covariance matrix being singular and decreased the probability of matrix not being positive definite so in this case the EM worked successfully in goldy image.

Em goldy.bmp



Kmeans goldy



Log likely hood em