ENPM 667 - Control of Robotic Systems

Project 2

Submitted by:

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This is the report the solution to
the double pendulum and cart problem.

Dec 22, 2020

Question:

Consider a crane that moves along an one-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m1 and m2, and the lengths of the cables are l1 and l2, respectively. The following figure depicts the crane and associated variables used throughout this project.

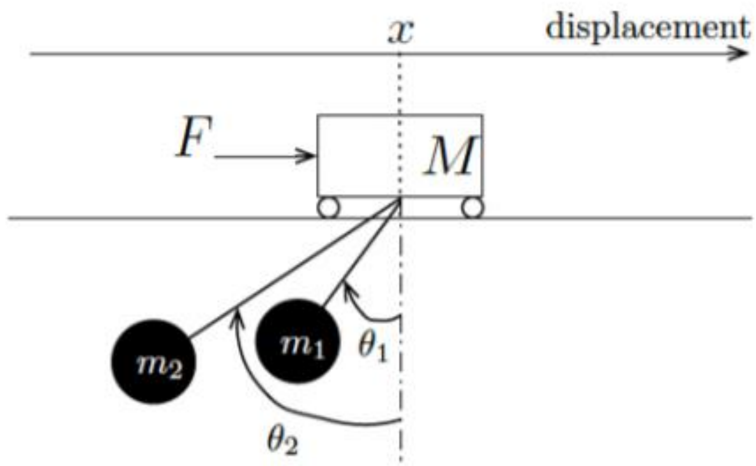


Fig1. Cart and Double Pendulum System

Solution for part A:

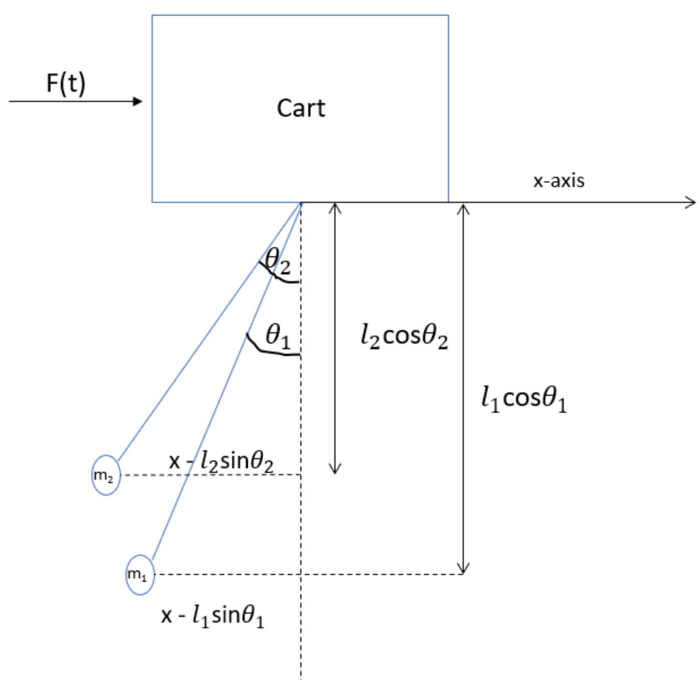


Fig 2. Free body diagram of the system

We need to derive the equations of motion of the given cart and double pendulum system.

From the figure:

$$p_1(\mathrm{t}) = (x - l_1\sin\theta_1)\mathrm{i} - l_1\cos\theta_1 j \qquad (1)$$

$$p_2(\mathrm{t}) = (x - l_2\sin\theta_2)\mathrm{i} - l_2\cos\theta_2 j \qquad (2)$$

Differentiating both terms, we get:

$$\dot{p_1}(t) = \left(\dot{x} - l_1\dot{\theta}_1 cos\theta_1\right)i + l_1\dot{\theta}_1 sin\theta_1 j \quad (3)$$
$$\dot{p_2}(t) = \left(\dot{x} - l_2\dot{\theta}_2 cos\theta_2\right)i + l_2\dot{\theta}_2 sin\theta_2 j \quad (4)$$

We know that kinetic energy of a system is given by:

$$\mathrm{K.E} = \frac{1}{2}mv^2,$$

where v can be the net velocity of the system.

Therefore,

$$\mathrm{K.E} = \frac{1}{2}\mathrm{M}\dot{x}^2 + \frac{1}{2}m_1(\dot{x} - l_1\dot{\theta}_1\, cos\theta_1)^2 + \frac{1}{2}m_1(\dot{x} - l_2\dot{\theta}_2\, cos\theta_2)^2 + \frac{1}{2}m_1(l_1\dot{\theta}_1\, sin\theta_1)^2 + \frac{1}{2}m_1(l_2\dot{\theta}_2\, sin\theta_2)^2$$

$$\mathrm{P.E} = -m_1 g l_1\cos\theta_1 - m_2 g l_2\cos\theta_2 \qquad (5)$$

Potential energy will not change since the cart is moving on the flat surface.

Lagrange Equation: **L = K.E – P.E**

$$\rightarrow \quad \mathrm{L} = \frac{1}{2}\mathrm{M}\dot{x}^2 + \frac{1}{2}m_1(\dot{x} - l_1\dot{\theta}_1\, cos\theta_1)^2 + \frac{1}{2}m_1(\dot{x} - l_2\dot{\theta}_2\, cos\theta_2)^2 + \frac{1}{2}m_1(l_1\dot{\theta}_1\, sin\theta_1)^2 + \frac{1}{2}m_1(l_2\dot{\theta}_2\, sin\theta_2)^2 + m_1 g l_1\cos\theta_1 + m_2 g l_2\cos\theta_2 \qquad (6)$$

Now,

$$\frac{\partial L}{\partial \dot{x}} = \mathrm{M}\dot{x} + m_1(\dot{x} - l_1\dot{\theta}_1\, cos\theta_1) + m_2(\dot{x} - l_2\dot{\theta}_2\, cos\theta_2) \qquad (7)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \mathrm{M}\ddot{x} + m_1(\ddot{x} - l_1\ddot{\theta}\cos\theta_1 + l_1\dot{\theta}_1{}^2 sin\theta_1) + m_2(\ddot{x} - l_2\ddot{\theta}\cos\theta_2 + l_2\dot{\theta}_2{}^2 sin\theta_2) \quad (8)$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \mathrm{M}\ddot{x} + m_1(\ddot{x} - l_1\ddot{\theta}\cos\theta_1 + l_1\dot{\theta_1}^2 sin\theta_1) + m_2(\ddot{x} - l_2\ddot{\theta}\cos\theta_2 + l_2\dot{\theta_2}^2 sin\theta_2 = \mathrm{F}$$

For θ_1

$$\frac{\partial L}{\partial \dot{\theta_1}} = m_1\left(\dot{x} - l_1\dot{\theta}_1 \cos\theta_1\right)(-l_1 cos\theta_1) + m_1(l_1\dot{\theta}_1 \sin\theta_1)(l_1 sin\theta_1) \quad (10)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_1}} = -m_1\ddot{x}l_1 cos\theta_1 + m_1 {l_1}^2\ddot{\theta}_1 + m_1\dot{x}l_1\ddot{\theta}_1 \sin\theta_1 \quad (11)$$

$$\frac{\partial L}{\partial \theta_1} = m_1 {l_1}^2\dot{\theta}_1 - m_1\dot{x}l_1 \cos\theta_1 \quad (12)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_1}} - \frac{\partial L}{\partial \theta_1} = -m_1\ddot{x}l_1 cos\theta_1 + m_1 {l_1}^2\ddot{\theta}_1 + m_1\dot{x}l_1\ddot{\theta}_1 \sin\theta_1 - m_1 {l_1}^2\dot{\theta}_1 + m_1\dot{x}l_1 \cos\theta_1 = 0 \quad (13)$$

$$\frac{\partial L}{\partial \dot{\theta_2}} = m_2(\dot{x} - l_2\dot{\theta}_2 \cos\theta_2)(-l_2 \cos\theta_2) + m_2 (l_2\dot{\theta}_2 \sin\theta_2)(l_2 \sin\theta_2) \quad (14)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_2}} = -m_2\ddot{x}l_2 cos\theta_2 + m_2 {l_2}^2\ddot{\theta}_2 + m_2\dot{x}l_2\ddot{\theta}_2 \sin\theta_2 \quad (15)$$

$$\frac{\partial L}{\partial \theta_2} = m_2 {l_2}^2\dot{\theta}_2 - m_2\dot{x}l_2 \cos\theta_2 \quad (16)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_2}} - \frac{\partial L}{\partial \theta_2} = -m_2\ddot{x}l_2 cos\theta_2 + m_2 {l_2}^2\ddot{\theta}_2 + m_2\dot{x}l_2\ddot{\theta}_2 \sin\theta_2 - m_2 {l_2}^2\dot{\theta}_2 + m_2\dot{x}l_2 \cos\theta_2 = 0$$

From above equations,

$$l_1\ddot{\theta}_1 = \ddot{x} \cos\theta_1 - \mathrm{g}\sin\theta_1 \Rightarrow \ddot{\theta}_1 = \frac{\ddot{x} \cos\theta_1 - \mathrm{g}\sin\theta_1}{l_1} \quad (18)$$

$$l_2\ddot{\theta}_2 = \ddot{x} \cos\theta_2 - g\sin\theta_2 \Rightarrow \ddot{\theta}_2 = \frac{\ddot{x} \cos\theta_2 - \mathrm{g}\sin\theta_2}{l_2} \quad (19)$$

Substitute these equations in (17) we get:

$$(M + m_2 + m_1)\ \ddot{x} = m_1(\ddot{x}\ cos\theta_1 - gsin\theta_1)\ cos\theta_1 + m_2(\ddot{x}\ cos\theta_2 - gsin\theta_2)\ cos\theta_2 - m_1 l_1 \dot{\theta}_1^{\ 2} sin\theta_1 - m_2 l_2 \dot{\theta}_2^{\ 2} sin\theta_2 + F$$

$$\ddot{x} = \frac{F - m_1 g\ cos\theta_1\ sin\theta_1 - m_2 g\ cos\theta_2\ sin\theta_2 - m_1 l_1 \dot{\theta}_1^{\ 2} sin\theta_1 - m_2 l_2 \dot{\theta}_2^{\ 2} sin\theta_2}{M + m_2 sin\theta_2^{\ 2} + m_1 sin\theta_1^{\ 2}}$$

Therefore,

$$\begin{bmatrix} \ddot{x}\ (t) \\ \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\{F - m_1 g\ cos\theta_1\ sin\theta_1 - m_2 g\ cos\theta_2\ sin\theta_2 - m_1 l_1 \dot{\theta}_1^{\ 2} sin\theta_1 - m_2 l_2 \dot{\theta}_2^{\ 2} sin\theta_2\}}{(M + m_2 sin\theta_2^{\ 2} + m_1 sin\theta_1^{\ 2})} \\ \dfrac{\ddot{x}\ cos\theta_1 - gsin\theta_1}{l_1} \\ \dfrac{\ddot{x}\ cos\theta_2 - gsin\theta_2}{l_2} \end{bmatrix} \tag{21}$$

Now, we can take the state vector to be:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\Rightarrow \dot{X} = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} \quad (22)$$

where the $\ddot{x}$, $\ddot{\theta}_1, \ddot{\theta}_2$ are obtained from equation(20). Here $f_i(x, u, t)$ are functions of state variables and making the system non-linear. We can observe terms of sin and cos in the equations.

Solution for Part B:

For small angle at equilibrium point, $x_e = 0,\ \theta_{1_e},\ \theta_{2_e} = 0$

Assume, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{F}{M} - \frac{m_1 g\theta_1}{M} - \frac{m_2 g\theta_2}{M} \\ \frac{F}{Ml_1} - \frac{m_1 g\theta_1}{Ml_1} - \frac{g\theta_1}{l_1} - \frac{m_2 g\theta_2}{Ml_1} \\ \frac{F}{Ml_2} - \frac{m_2 g\theta_2}{Ml_2} - \frac{g\theta_2}{l_2} - \frac{m_1 g\theta_2}{Ml_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{-m_1 g}{M} & \frac{-mF_2 g}{M} \\ 0 & \frac{-m_1 g}{Ml_1} - \frac{g}{l_1} & \frac{-m_2 g}{Ml_1} \\ 0 & \frac{-m_1 g}{Ml_2} & \frac{-m_2 g}{Ml_2} - \frac{g}{l_2} \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 1/M \\ 1/Ml_1 \\ 1/Ml_2 \end{bmatrix} F$$

Our state:

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M} & 0 & \frac{-m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{Ml_1} - \frac{g}{l_1} & 0 & \frac{-m_2 g}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{Ml_2} & 0 & \frac{-m_2 g}{Ml_2} - \frac{g}{l_2} & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ 1/Ml_1 \\ 0 \\ 1/Ml_2 \end{bmatrix} F$$

This same state-space representation can be obtained from the jacobian matrix calculations too. Where J = $\left[\frac{\partial f_i}{\partial x}\right]$ at the equilibrium points. On linearizing around the equilibrium point using the jacobian we get:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{M} & 0 & \frac{-m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{Ml_1} - \frac{g}{l_1} & 0 & \frac{-m_2 g}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-m_1 g}{Ml_2} & 0 & \frac{-m_2 g}{Ml_2} - \frac{g}{l_2} & 1 \end{bmatrix} \qquad (23)$$

$$B = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ 1/Ml_1 \\ 0 \\ 1/Ml_2 \end{bmatrix} \tag{24}$$

And that is same as that we have obtained using small-angle approximation of sin and $cosine$ terms.

Solution for Part C:

The given linearized system will be controllable if the controllability matrix ctrb(A, B) is full rank (here, 6) or if the determinant of the controllability matrix is not zero.

Controllability Matrix is given by:

$$CTRB(A, B) = [B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B]$$

We will be substituting the values of A and B from (23) and (24) to find the controllability matrix. We calculated this matrix in MATLAB and put the determinant to 0 to find the constraint on the values of M, m_1, m_2, l_1, l_2, g.

$$|\boldsymbol{CTRB(A, B)}| = -\frac{\boldsymbol{g^6(l_1^2 - 2l_1l_2 + l_2^2)}}{\boldsymbol{(Ml_1l_2)^6}} = -\frac{\boldsymbol{g^6(l_1 - l_2)^2}}{\boldsymbol{(Ml_1l_2)^6}}$$

From the above equation, we can infer that the determinant of the controllability matrix will be zero if $l_1 = l_2$. Therefore, for the system to be controllable, $\boldsymbol{l_1 \neq l_2}$**.**

Solution of Part D:

We are given:

$$M = 1000kg, m_1 = 100kg, m_2 = 100kg, l_1 = 20m, l_2 = 10m, g = 10ms^{-2}$$

On substituting these values in matrix A and B, we get:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -0.55 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -0.9 & 0 \end{bmatrix}, \text{ and } B = \frac{1}{1000}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0.05 \\ 0 \\ 0.1 \end{bmatrix}$$

- To check whether the system is controllable we can use the constraint derived in part C.
- We observe that $\boldsymbol{l_1} \neq \boldsymbol{l_2}$ as $l_1 = 20m, l_2 = 10m$. <u>This implies that the given system is controllable for the given parameters.</u>
- We can also check whether the open-loop system is stable or not by checking the open loop eigen values of matrix A.
- The eigen values of the open-loop system are:

$$\lambda_{1,2} = \pm 0.9408i$$

$$\lambda_{3,4} = \pm 0.7516i$$

$$\lambda_{5,6} = 0$$

- We observe that that these eigen values do not have any real part.
- All the eigen values are on imaginary axis.
- We can infer that the system will be oscillating with a fixed amplitude.
- This can be attributed to the fact that we haven't considered any damping force while deriving the equations of motion. Therefore, the absence of friction or viscous forces make the pendulums oscillate without coming to any fixed position.

- We know that the system is controllable and current open-loop system is oscillating. We can design a full-state feedback controller to make the system converge to a stable state.
- We will be using the LQR controller to find the feedback gain to make the system come to stable position in lesser time and with minimum effort to reach there.
- LQR is an optimal regulator for linear state-feedback control.

We need to minimize the following cost function while designing this regulator:

$$J = \int_0^T \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) \, d\tau$$

- To do this we take the value of Q such that the system comes to stable position in lesser time and that of R such that the input effort is feasible enough to not to break the dynamics of the system.

- In our case, we tried different values of Q and R.

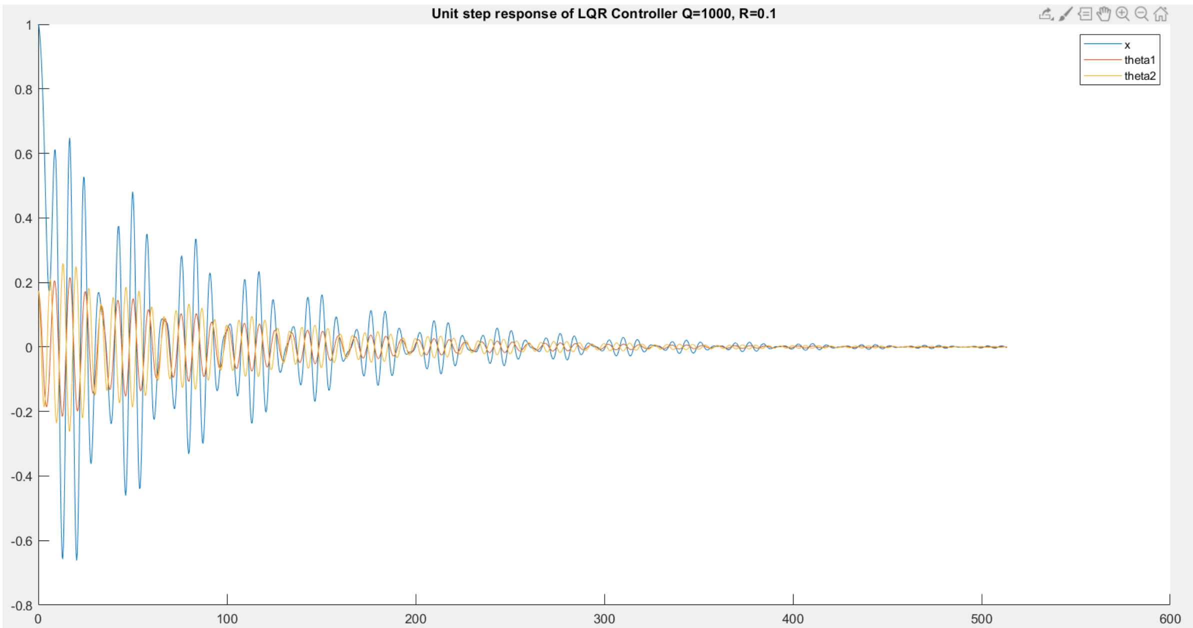


Fig3. Q = 1000 for x and θ and Q = 10 for velocities, R = 0.1

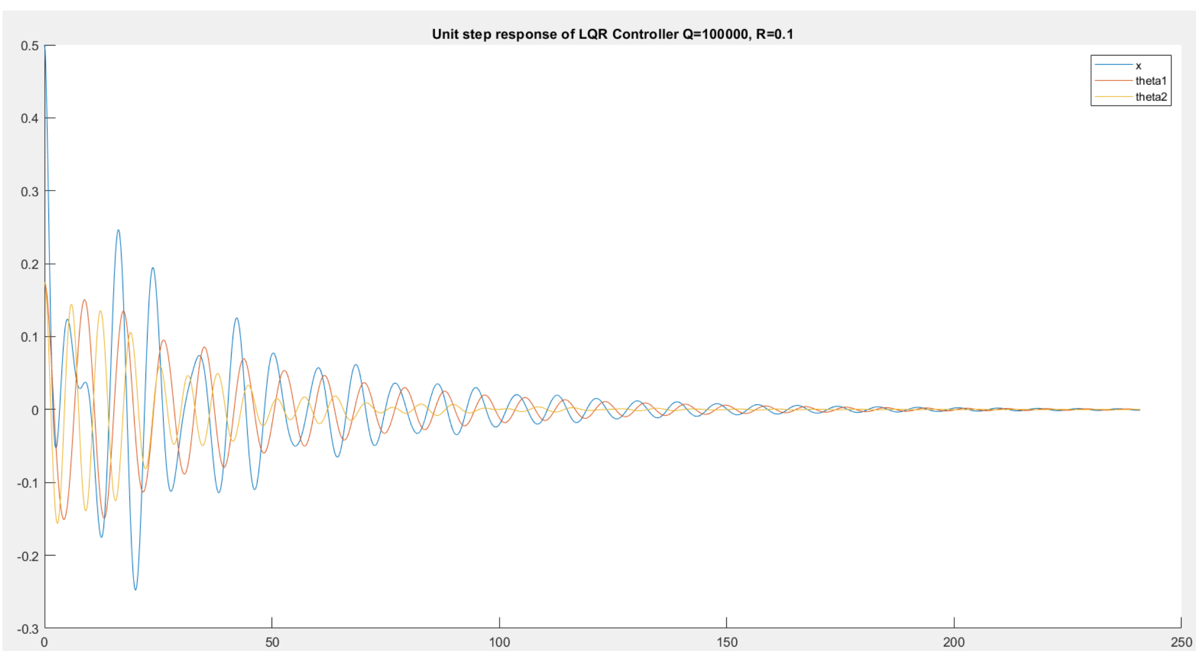


Fig4. Q = 100000 for x and θ and Q = 10 for velocities, R = 0.1

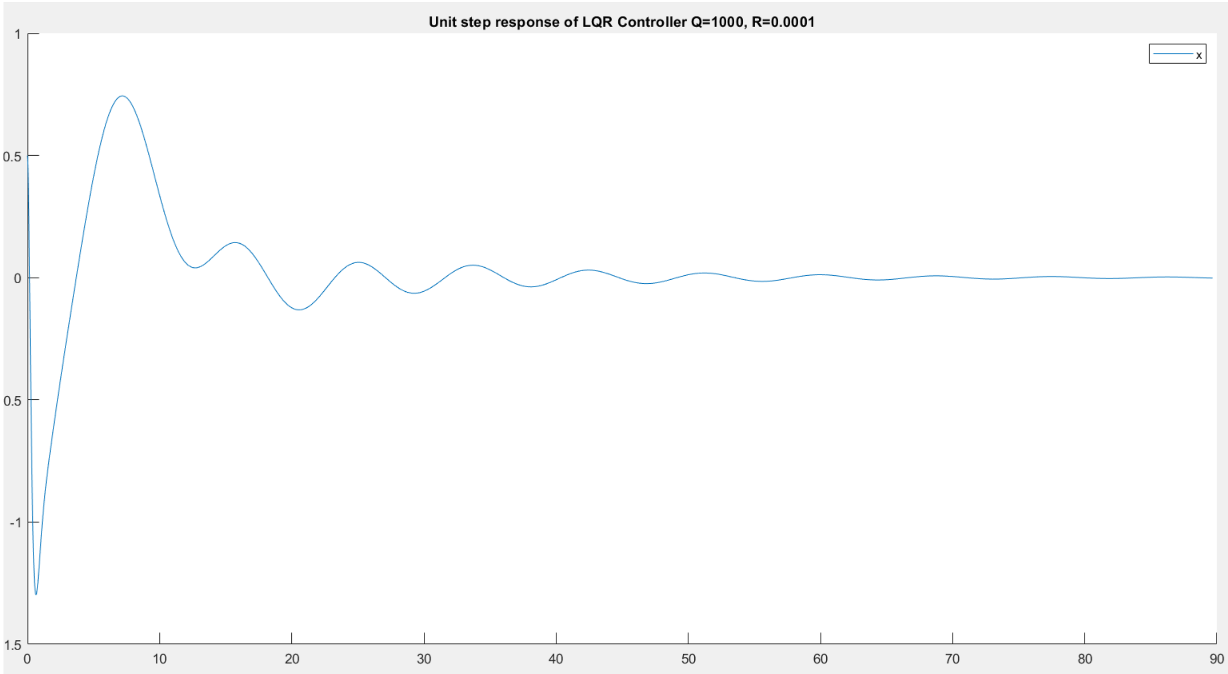


Fig5. Q = 1000 for x and Q = 10 for velocities, R = 0.001

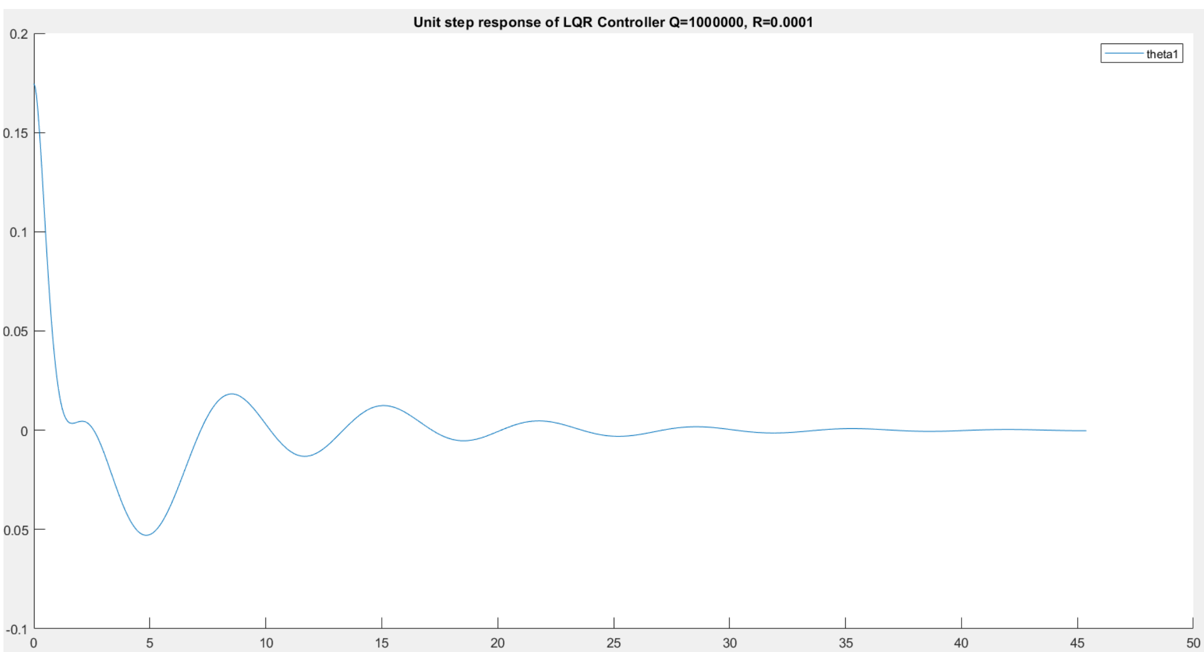


Fig6. Q = 1000000 for θ_1 and Q = 10 for velocities, R = 0.0001

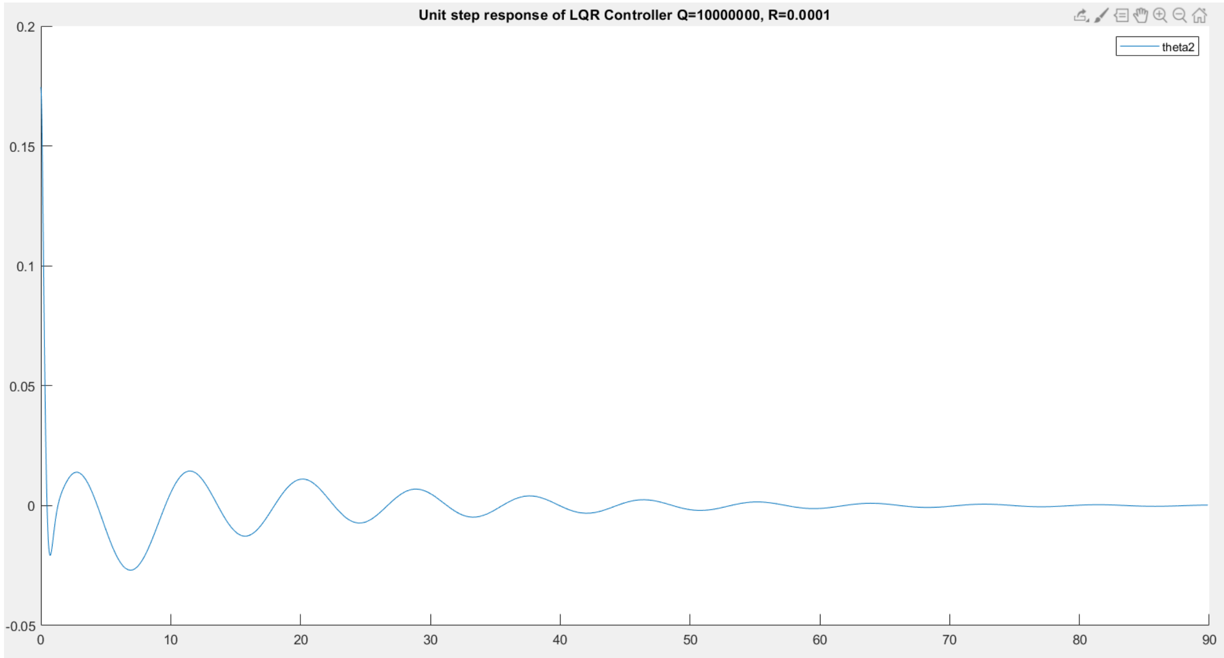


Fig7. Q = 1000000 for θ_2 and Q = 10 for velocities, R = 0.0001

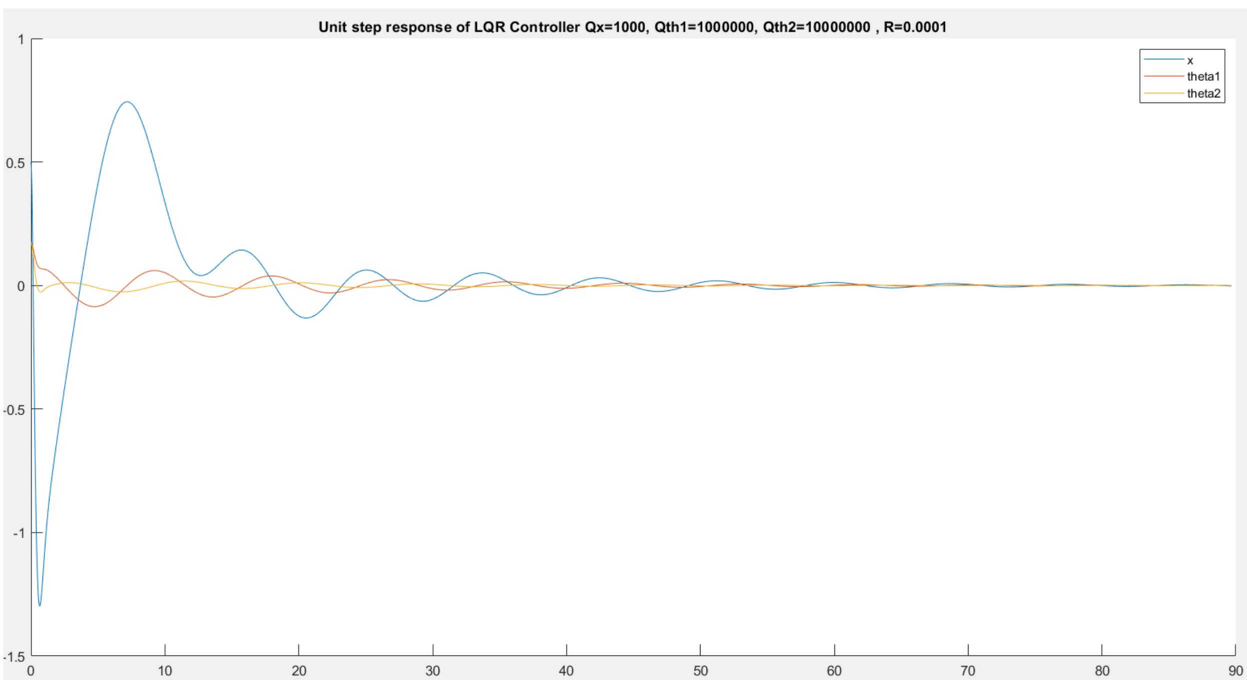


Fig8. Q = 1000 for x, Q = 1000000 for θ_1, θ_2 and Q = 10 for velocities, R = 0.0001

The final values for Q and R were the ones that gave the response in figure 8.

The states were converging to the stable position in lesser time with lesser number of oscillations.

$$Q = \begin{bmatrix} 1000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10000000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$, and R = 0.0001

The closed-loop LQR feedback gain obtained is:

K_r = 1.0e+05 *[0.0316 0.1518 0.6176 1.1011 2.9252 -1.2255]

As we have the closed-loop gain, we can check the stability of this system.

$$\lambda_{1,2} = -3.9563 \pm 4.0657i$$

$$\lambda_{3,4} = -0.0536 \pm 0.7175i$$

$$\lambda_{5,6} = -0.2063 \pm 0.2288i$$

All these closed-loop eigen values have negative real part, i.e., the poles of the closed loop system lie in the left of the s-plane. This implies that the system is stable.

The eigen values have an imaginary part too, that suggests that there will still be some oscillations (damped in this case) while the states converge to equilibrium state.

Thus, the state-feedback can stabilize the un-damped oscillations of the cart-pendulum system.

Lyapunov's indirect method to check the stability of this system suggests us to check the eigen value of the closed loop dynamics,

$$A_{cl} = A - B * Kr$$

The eigen values mentioned above are those of A_{cl}. Therefore, the system is stable locally around the equilibrium point where the system was linearized.

Response of the LQR controller to the original non-linear system:

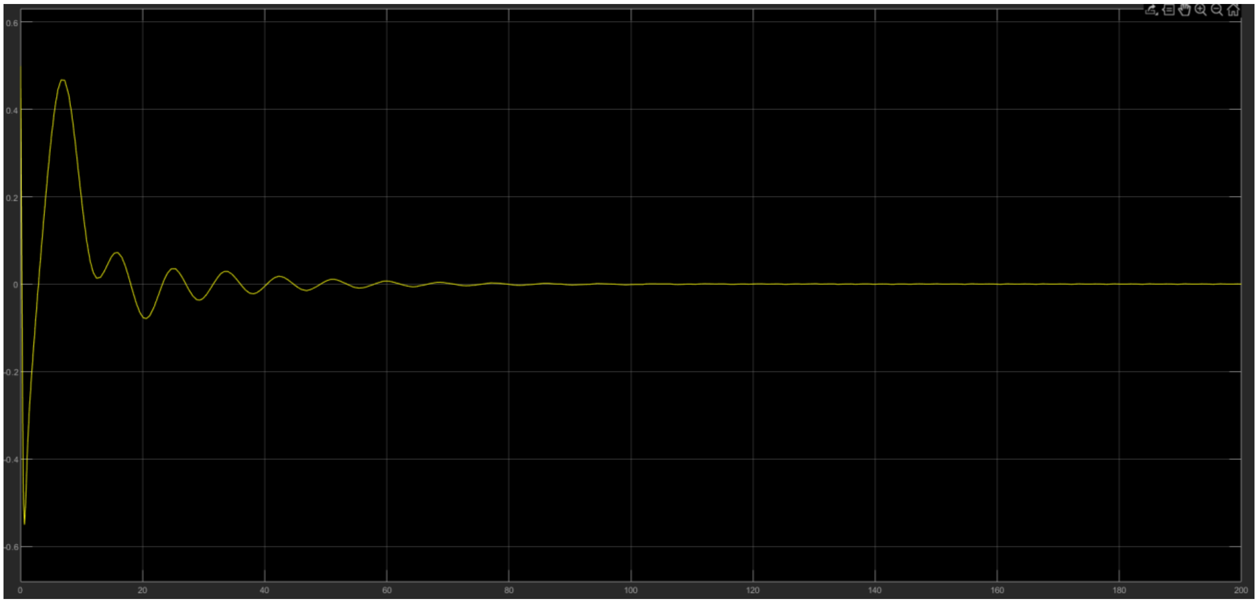


Fig9. LQR controller response, x(t)

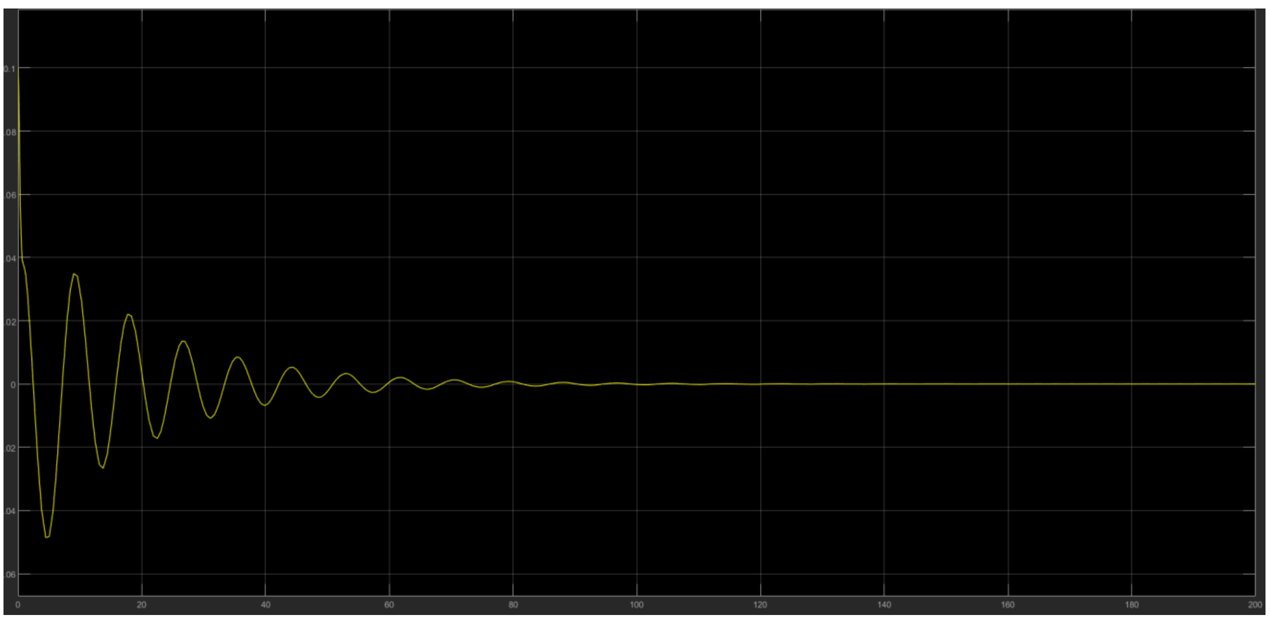


Fig10. LQR controller response, θ_1(t)

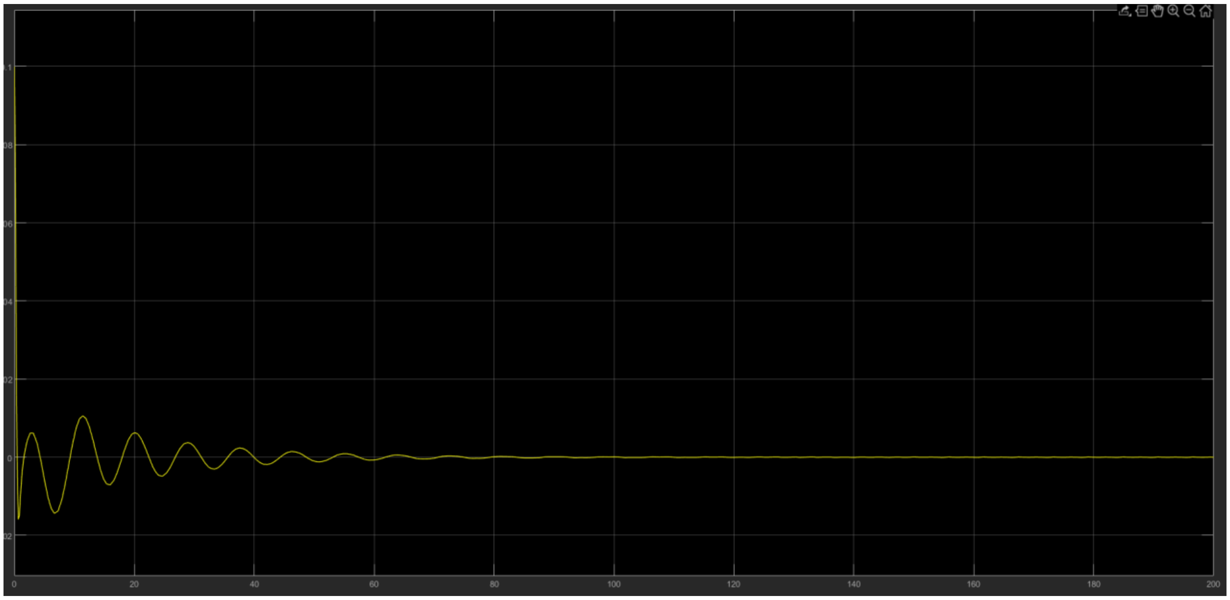


Fig11. LQR controller response, θ_2(t)

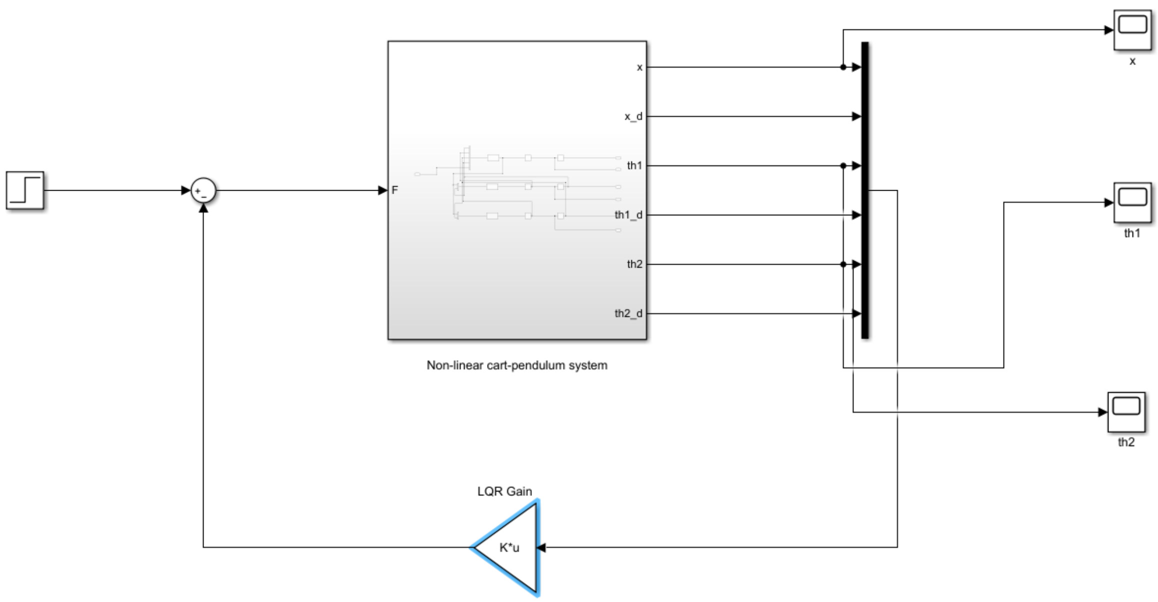


Fig11. LQR controller design for non-linear system

Solution of Part E:

To check if the system is observable or not, we check the rank of the observability matrix.

The observability matrix is given by:

$$OBSV(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix}$$

If this observability matrix is full-rank (here, 6) then the system is said to be observable and we can estimate the full state of the system.

C matrix gives us the information about the states that are being measured.

Checking observability for different cases:

<u>**Case 1)**</u> Output = $x(t)$

In this case $C = [1\ 0\ 0\ 0\ 0\ 0]$.

The rank of $OBSV(A,C)$ is 6 in this case. Hence, the system is observable if information about the displacement of the cart is given.

<u>**Case 2)**</u> Output = $\theta_1(t), \theta_2(t)$

In this case $C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

The rank of $OBSV(A,C)$ is 4 in this case. Hence, the system is NOT observable with this limited information.

<u>**Case 3)**</u> Output = $x(t), \theta_2(t)$

In this case $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

The rank of $OBSV(A,C)$ is 6 in this case. Hence, the system is observable with this limited information.

<u>Case 4)</u> Output = $x(t), \theta_1(t), \theta_2(t)$

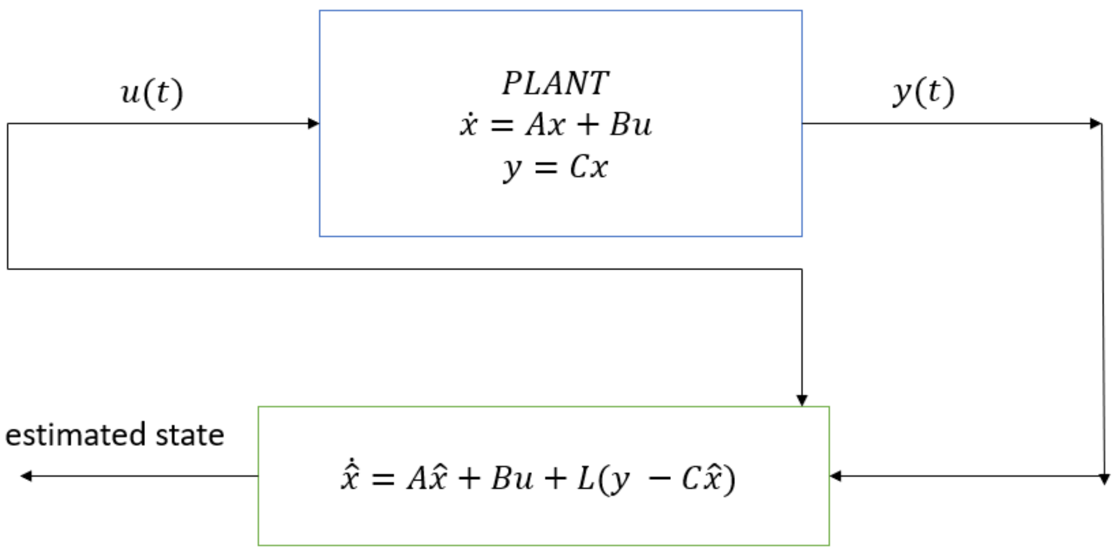
In this case $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

The rank of $OBSV(A, C)$ is 6 in this case. Hence, the system is observable with this limited information.

Solution to Part F:

It is not always possible to measure the complete state of the state that can be used for the full-state feedback. Hence, we try to estimate the remaining states with the help of an observer/estimator. We can do this iff $OBSV(A, C)$ if full-rank.

We designed a simple Luenberger observer to estimate the state of the system.



Luenberger Observer

The dynamics of the observer are same as that of the system.

This observer tries to make the estimated state converge to the actual state and thus reducing the error (ideally 0). The error dynamics are same as that of the system and are considered to be stable if the eigen values of $A - L * C$ lie in the left half of the s-plane. We can make the error dynamics stable iff the pair (A, C) is observable. Here L, is the feedback gain matrix just like we have for a normal controller.

Luenberger observers were designed for each of the observable output combinations mentioned in part E of the solution. The observer was made to track the open-loop state response of the system.

While designing a (good/best) observer to find the appropriate gain matrix L, we should take care of the fact that the observer dynamics is faster than the state-feedback controller.
In other words, the estimated states should converge to the actual states in minimum amount of time, i.e. the error should converge to zero as soon as possible.

For that we can try to place the poles of the observer to the left of the controller's poles.

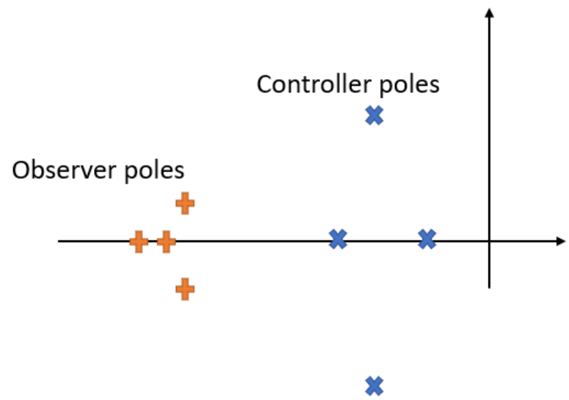


Fig12. Observer poles being placed to the left side of the controller's poles

Case1) Output = $x(t)$

In this case $C = [1\ 0\ 0\ 0\ 0\ 0]$.

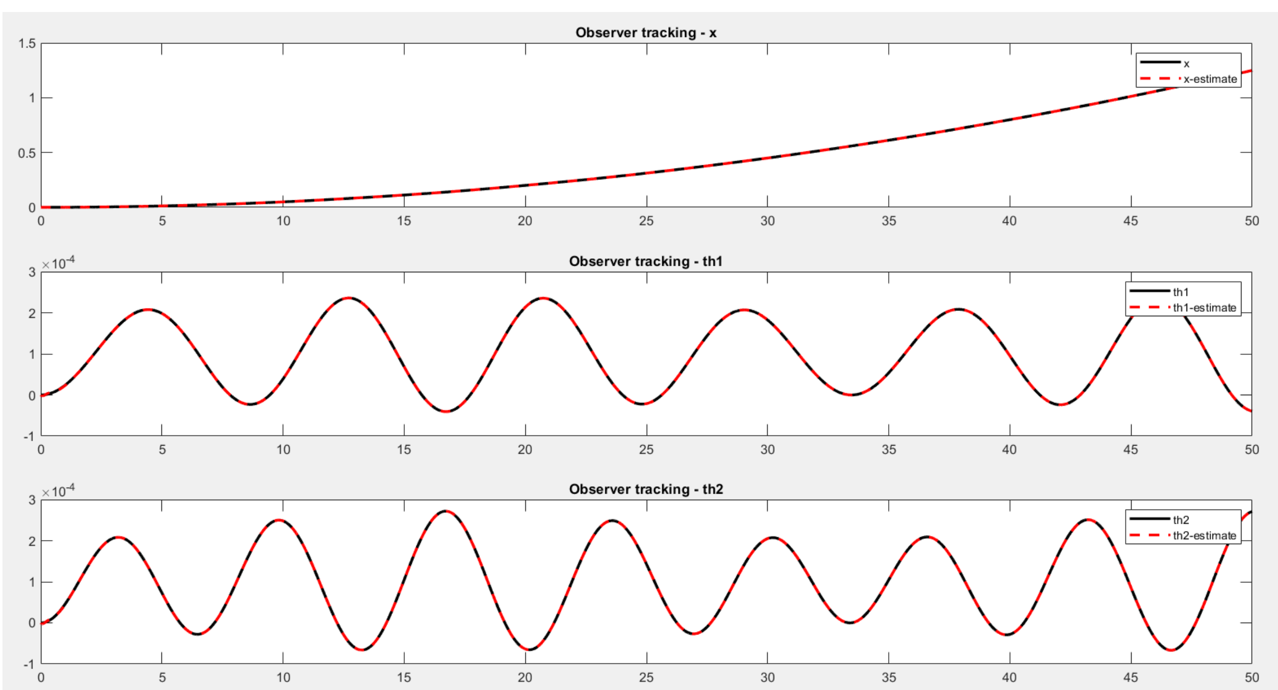


Fig13. Open loop tracker of full-states when only x(t) is measured

<u>Case 3)</u> Output = $x(t), \theta_2(t)$

In this case $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

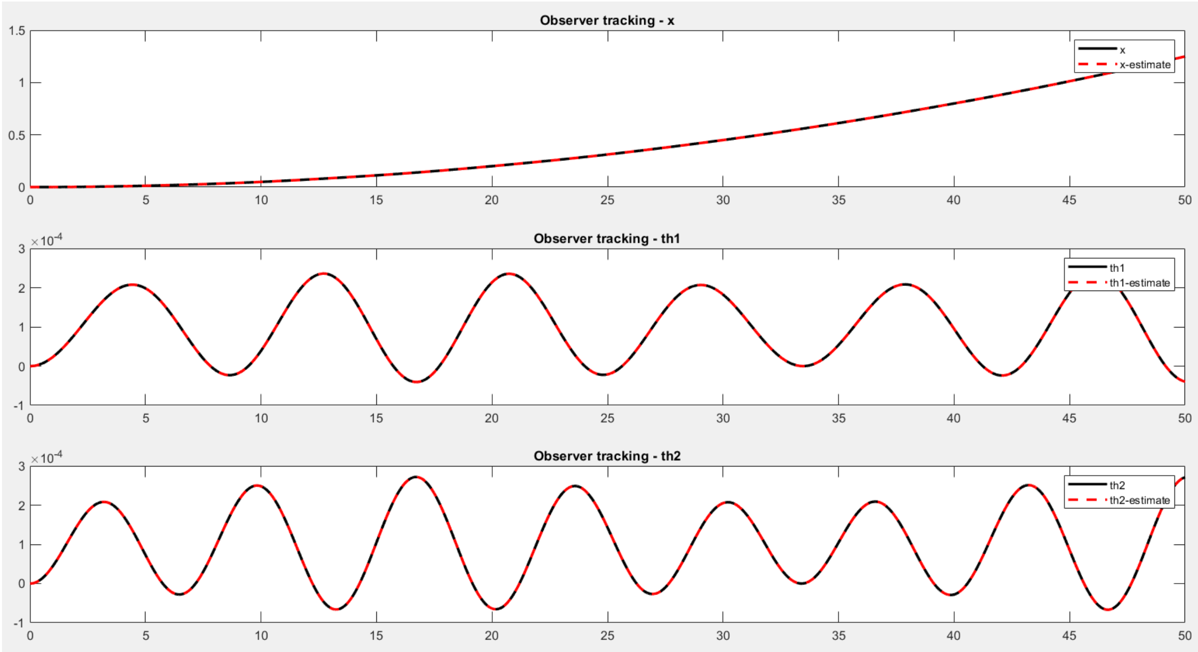


Fig14. Open loop tracker of full-states when $x(t), \theta_2(t)$ are measured

<u>Case 4)</u> Output = $x(t), \theta_1(t), \theta_2(t)$

In this case $C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

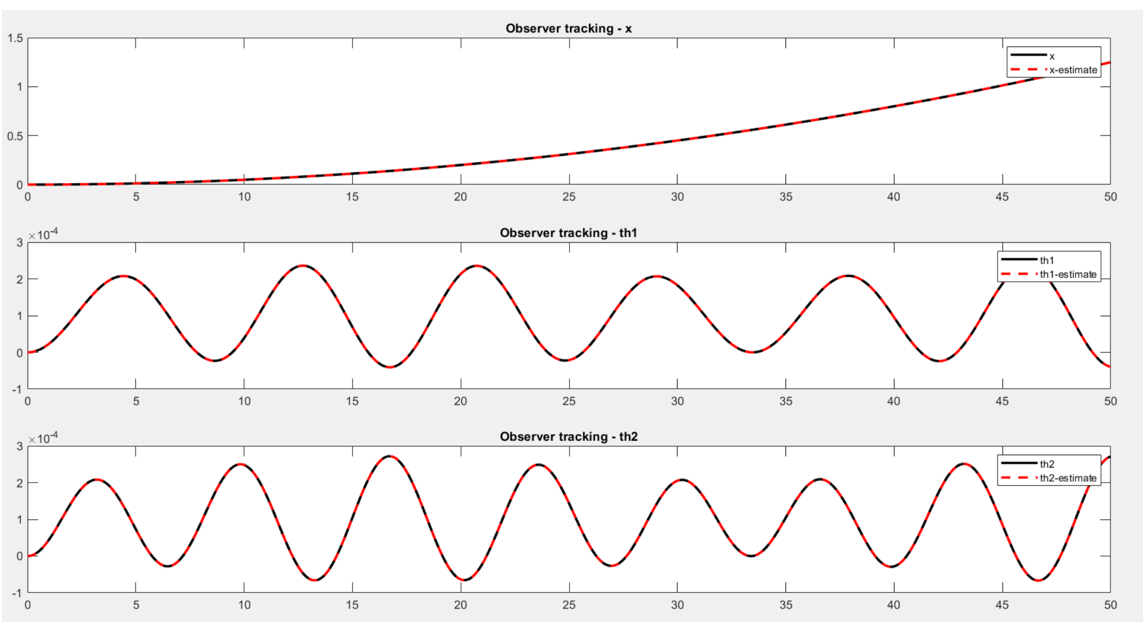


Fig15. Open loop tracker of full-states when $x(t), \theta_1(t), \theta_2(t)$ are measured

Observer design for non-linear system.

As we know that the observer dynamics and the error dynamics are same as that of the system dynamics, we will be using the same linearized observer dynamics in the case of estimating the states for non-linear systems.

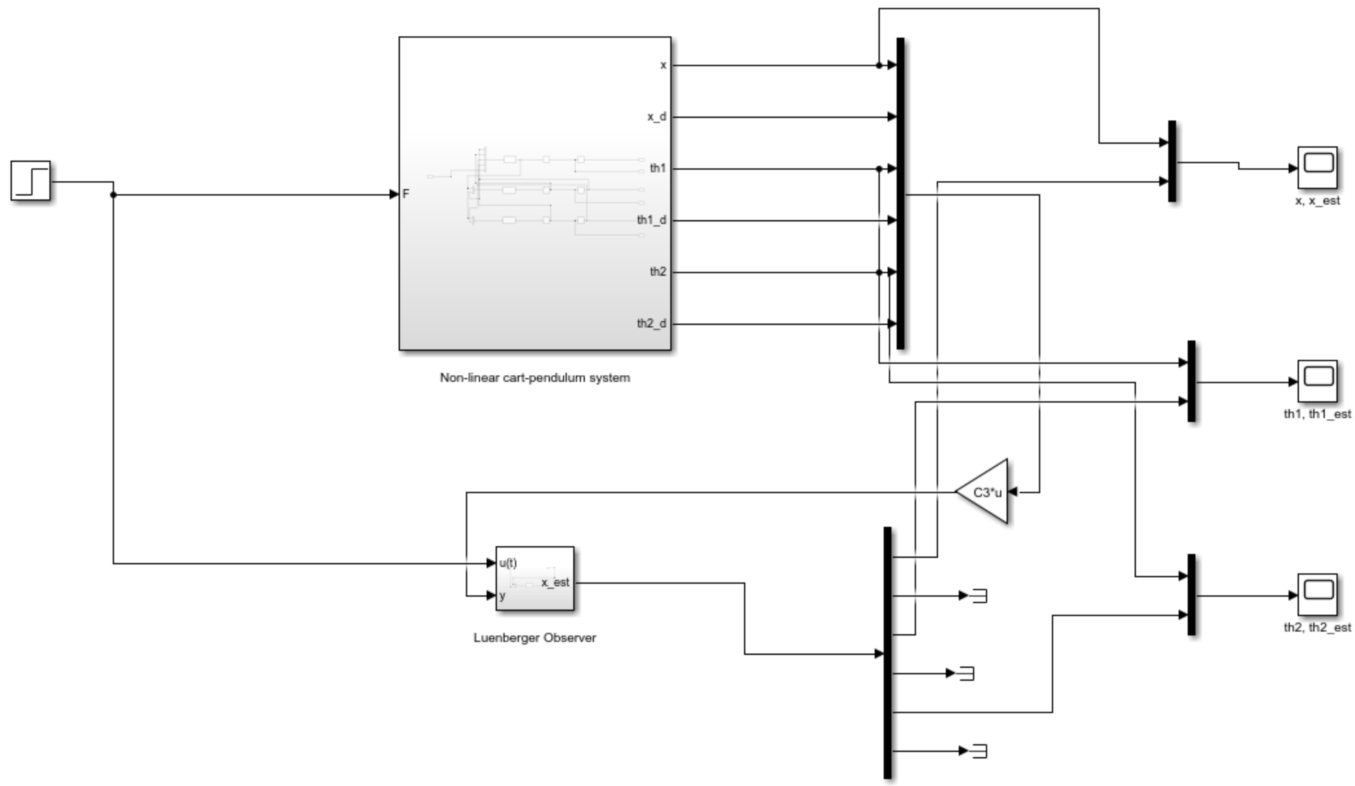


Fig16. Open loop tracker of non-linear system

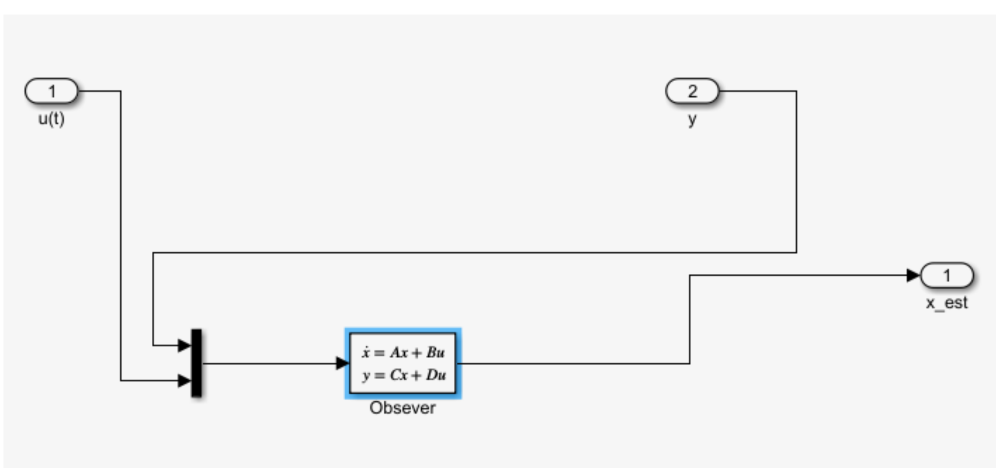


Fig17. Luenberger observer state space model

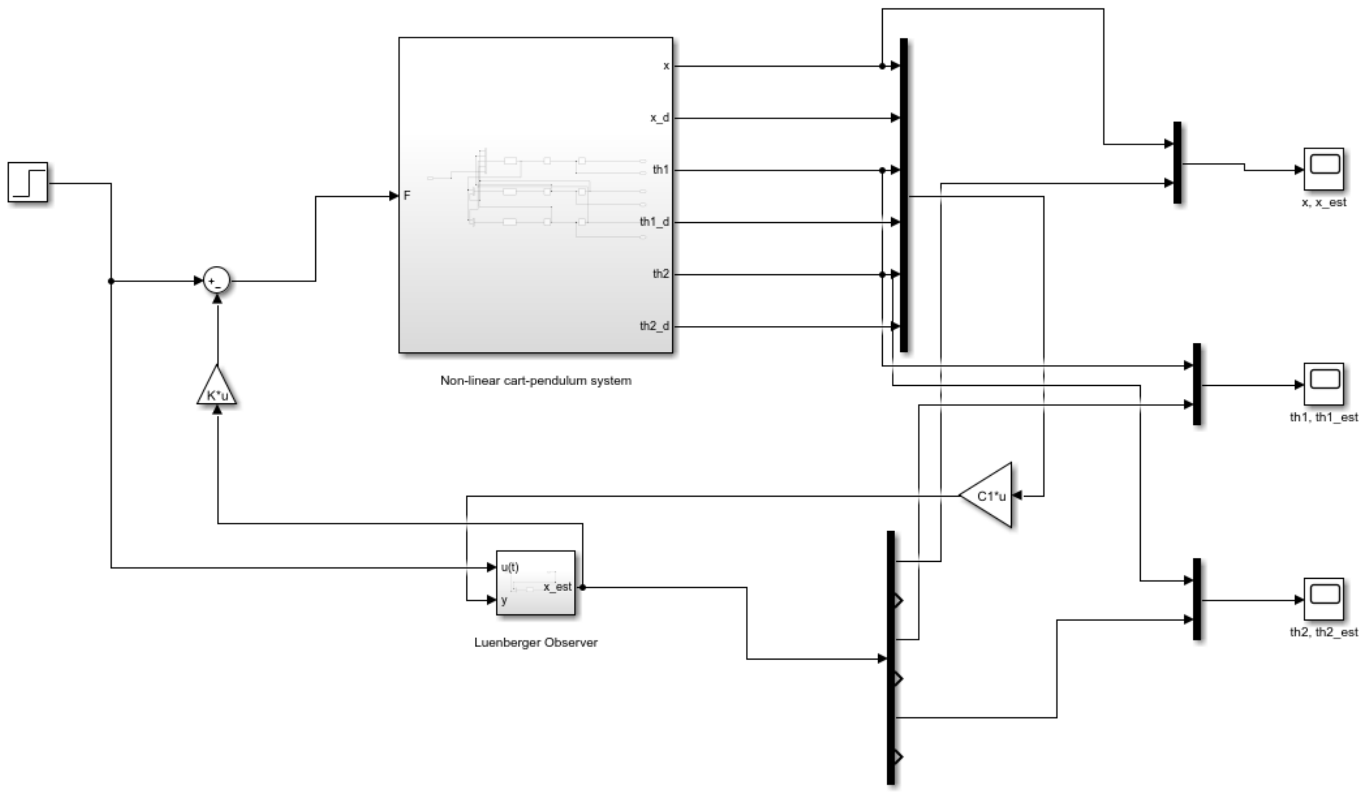


Fig18. Luenberger observer for non-linear system– Closed loop

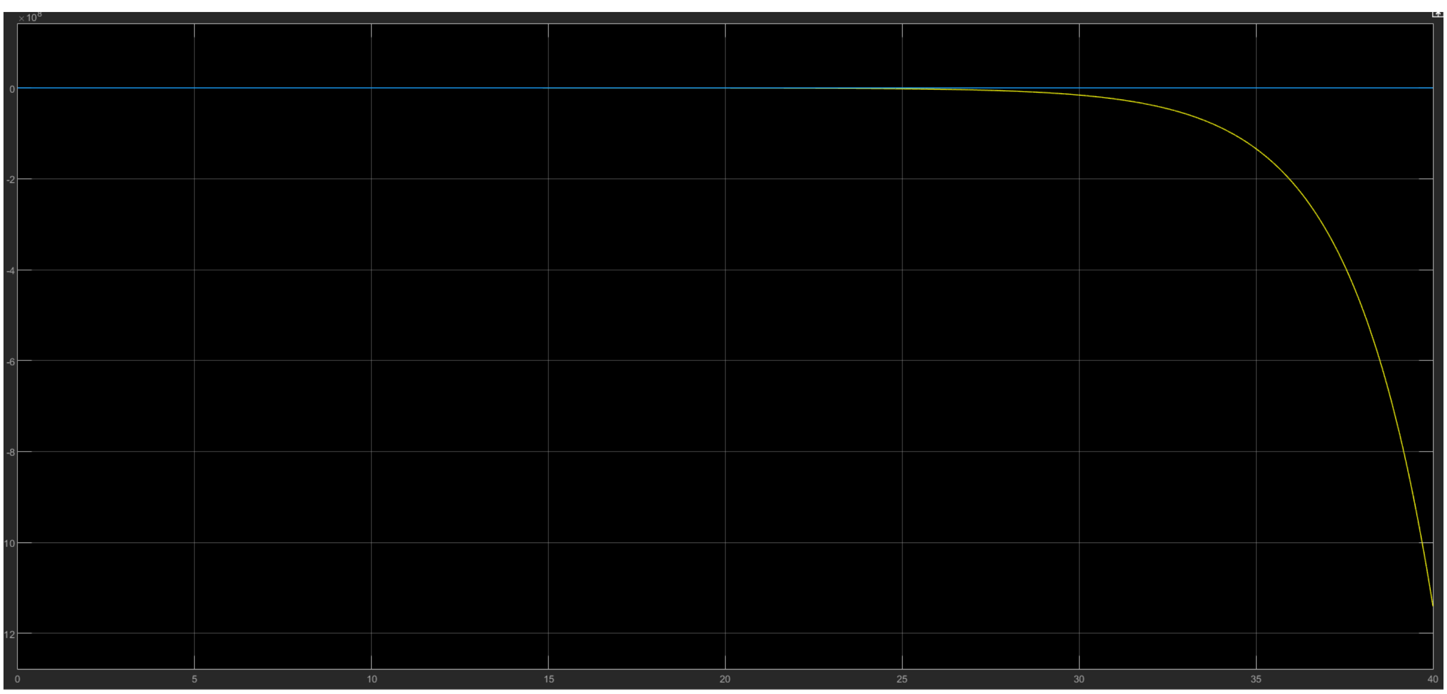


Fig19.Case 1 - Luenberger observer response for non-linear system x(t), $x_{est}(t)$ – Closed loop

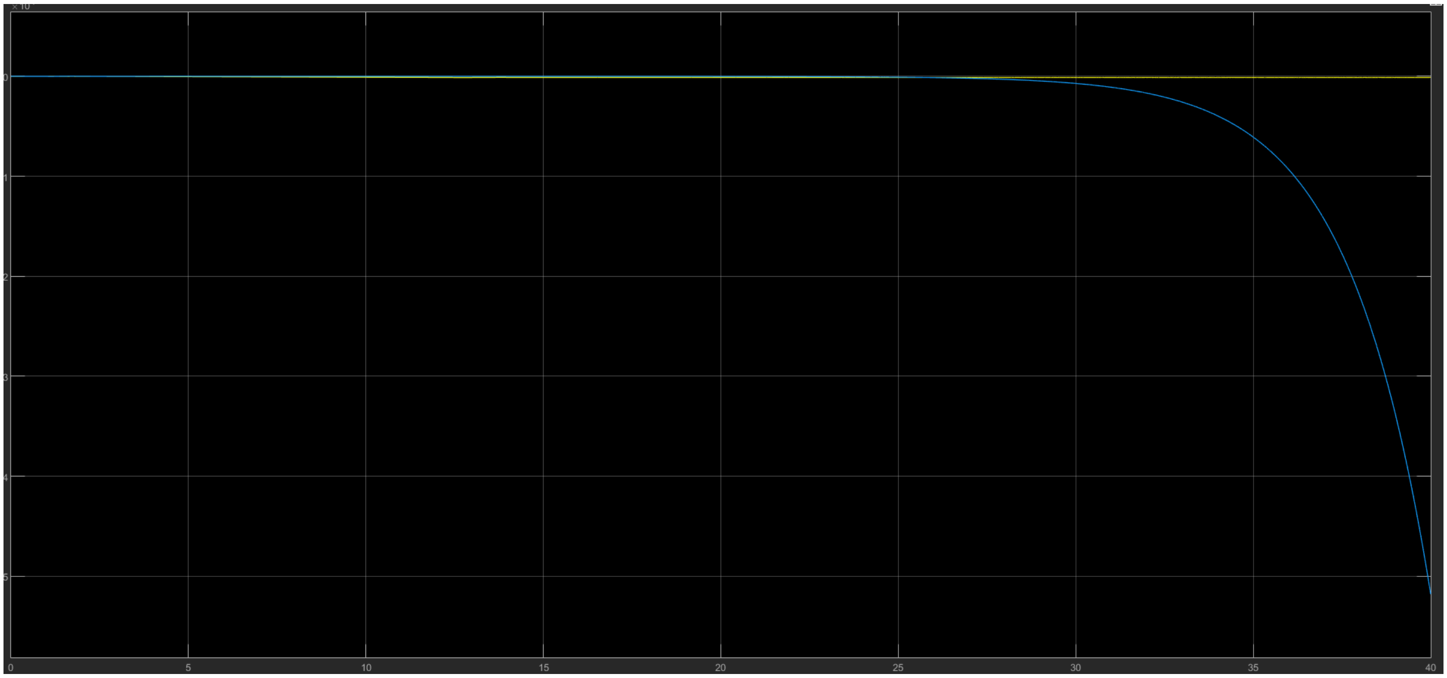


Fig20. Case 1 - Luenberger observer response for non-linear system θ_1(t), $\theta_{1_{est}}$(t) – Closed loop

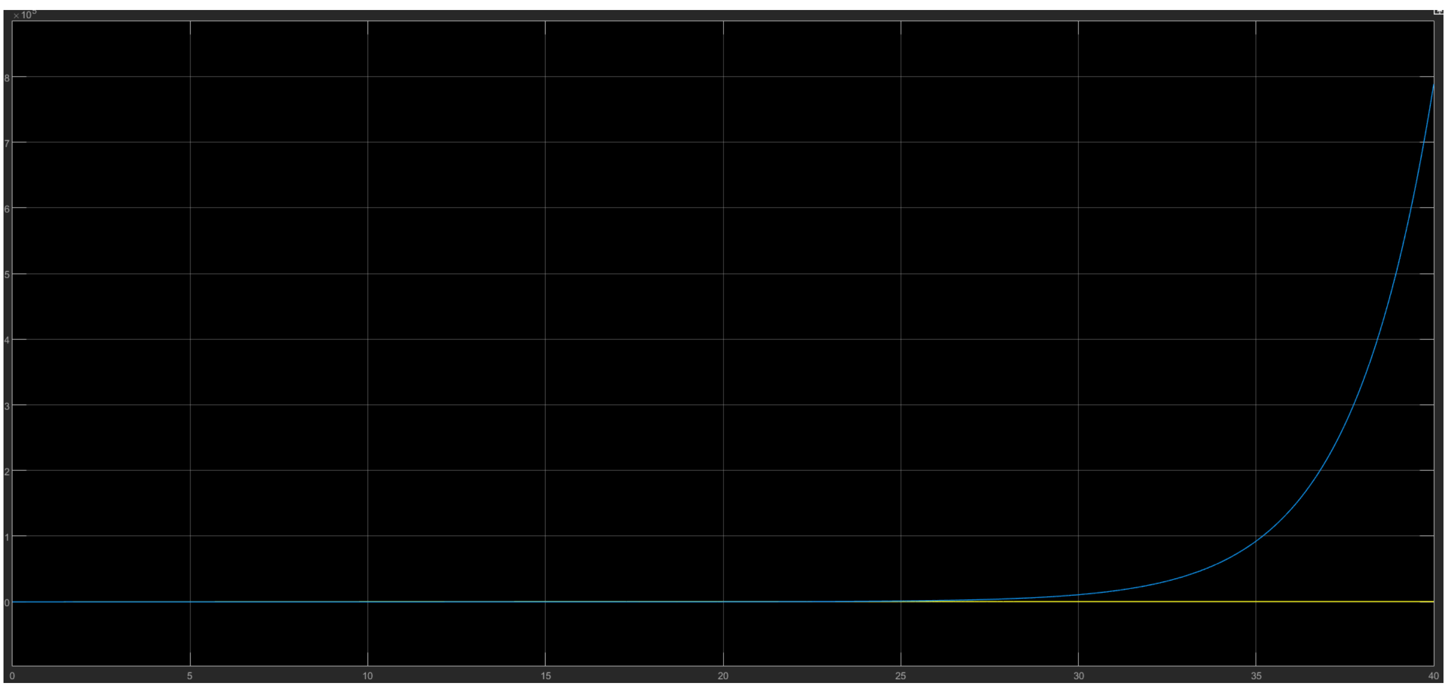


Fig21. Case 1 - Luenberger observer response for non-linear system θ_2(t), $\theta_{2_{est}}$(t) – Closed loop

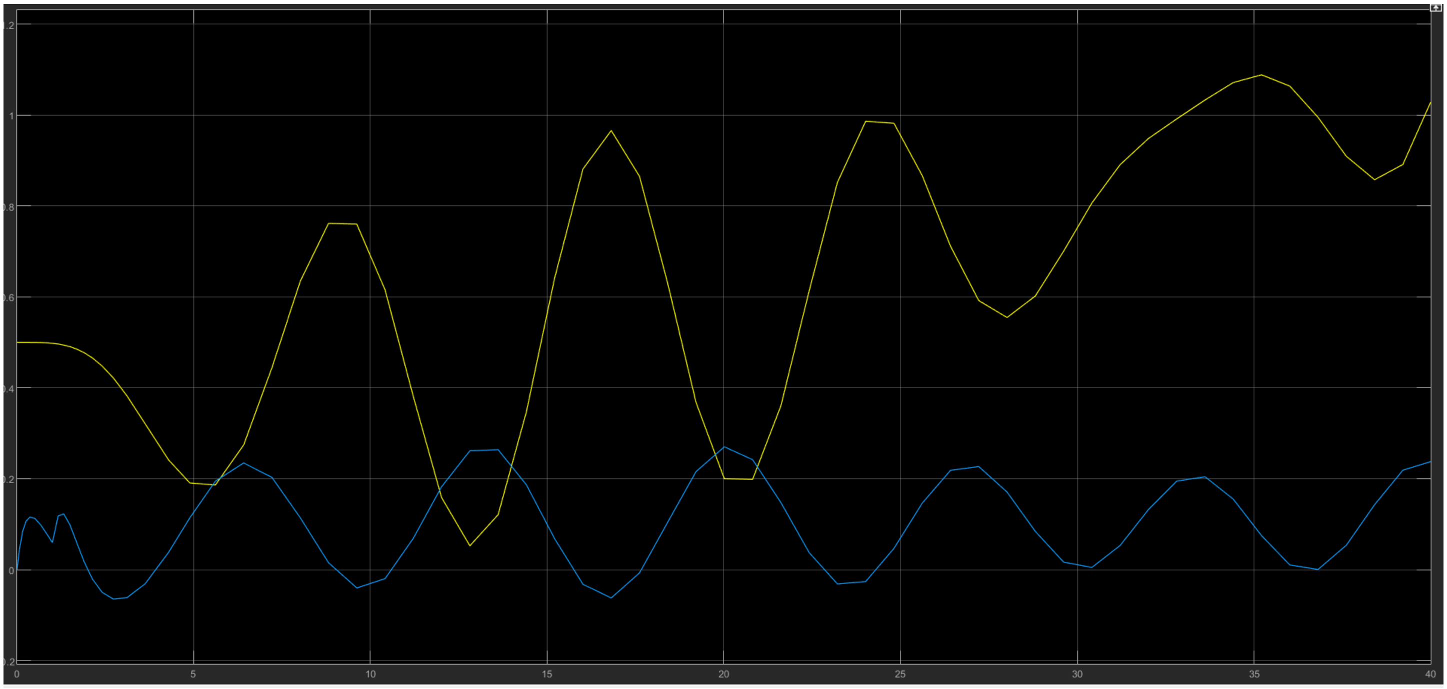


Fig22. Case 3 - Luenberger observer response for non-linear system x(t), x_{est}(t) – Open loop

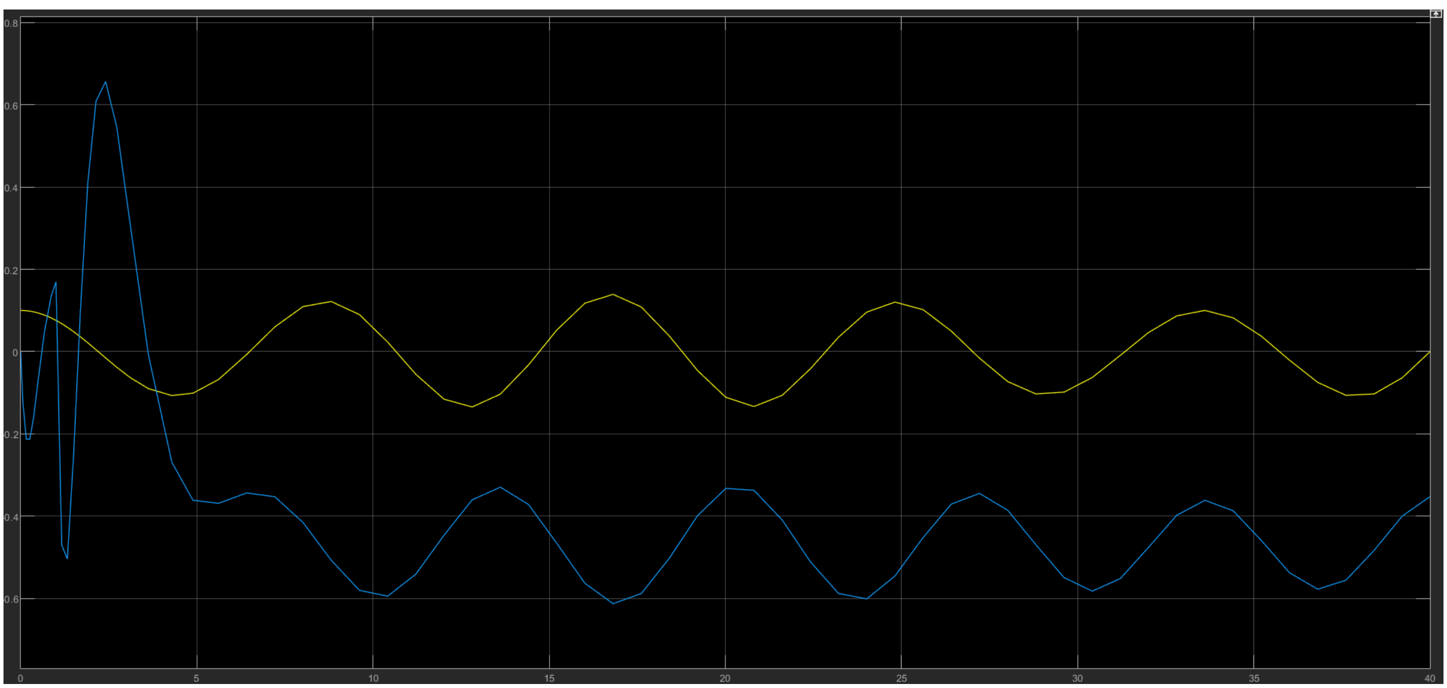


Fig23. Case 3 - Luenberger observer response for non-linear system θ_1(t), $\theta_{1_{est}}$(t) – Open loop

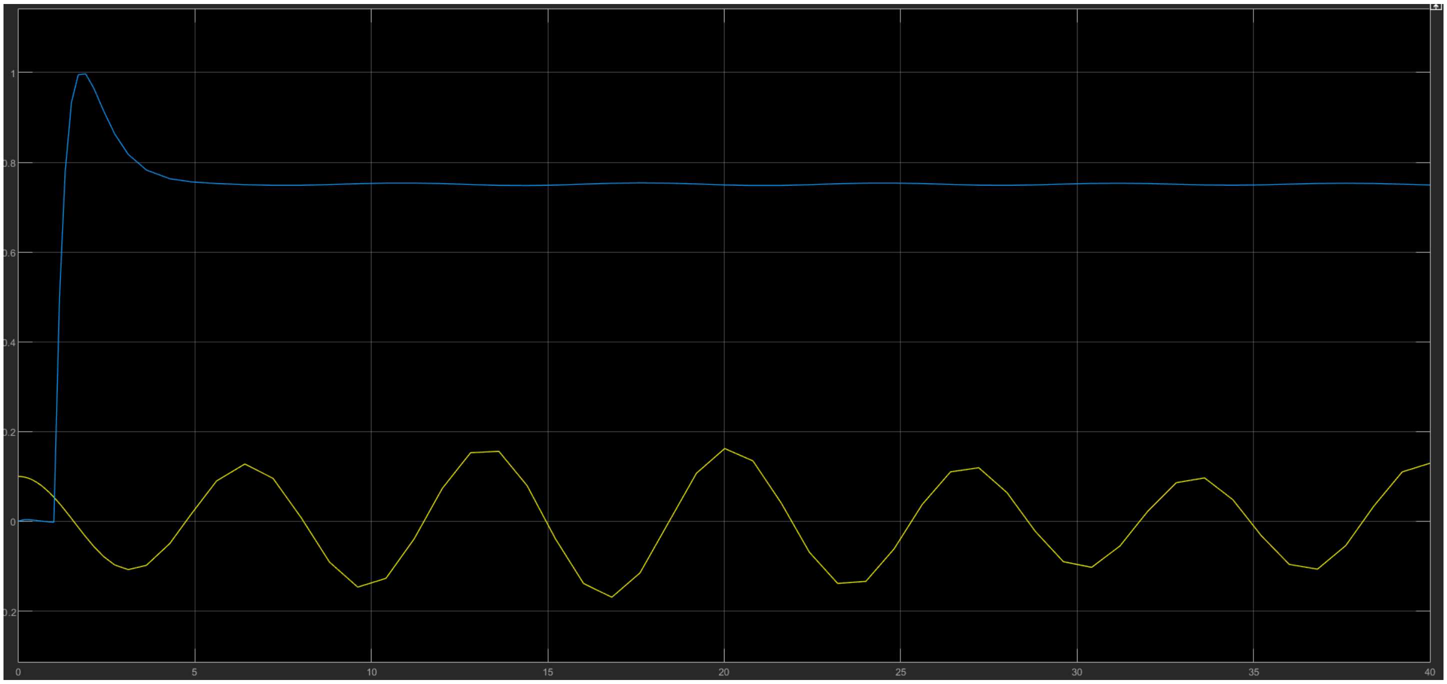


Fig24. Case 3 - Luenberger observer response for non-linear system θ_2(t), $\theta_{2_{est}}$(t) – Open loop

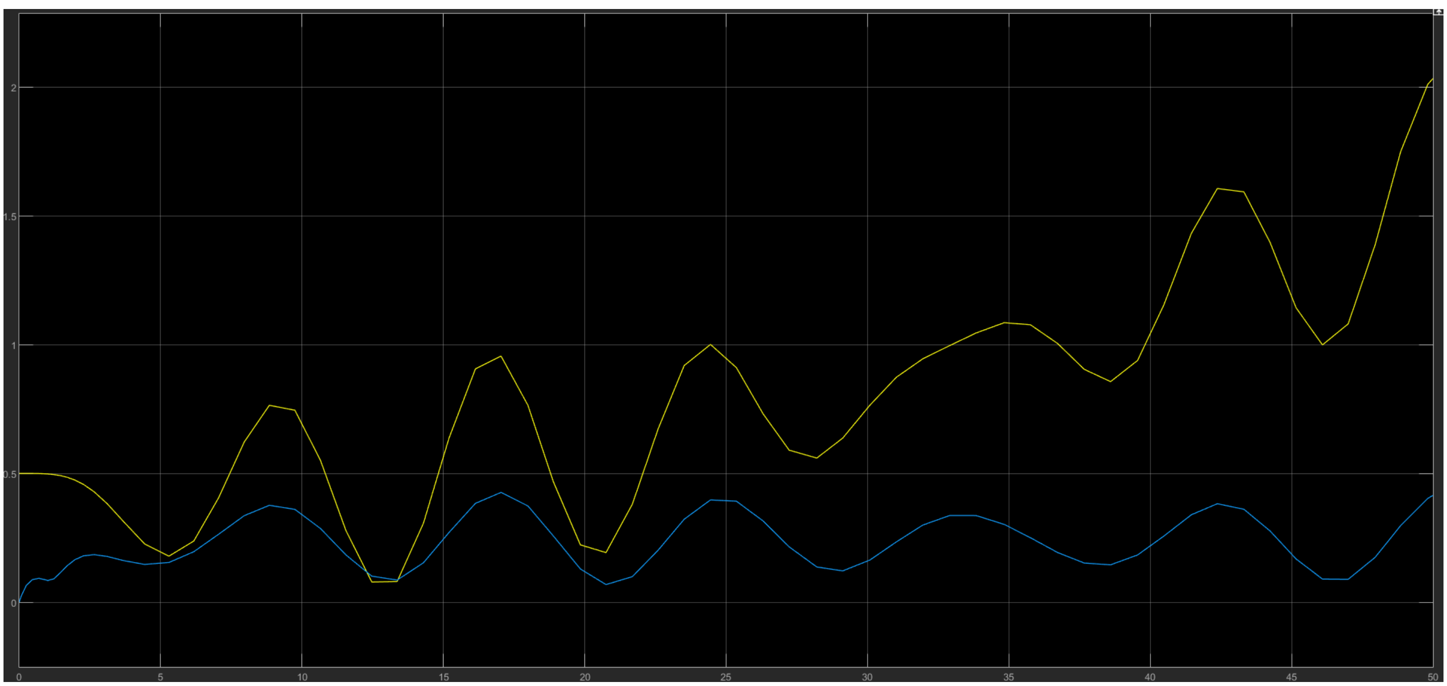


Fig25. Case 4 - Luenberger observer response for non-linear system x(t), x_{est}(t) – Open loop

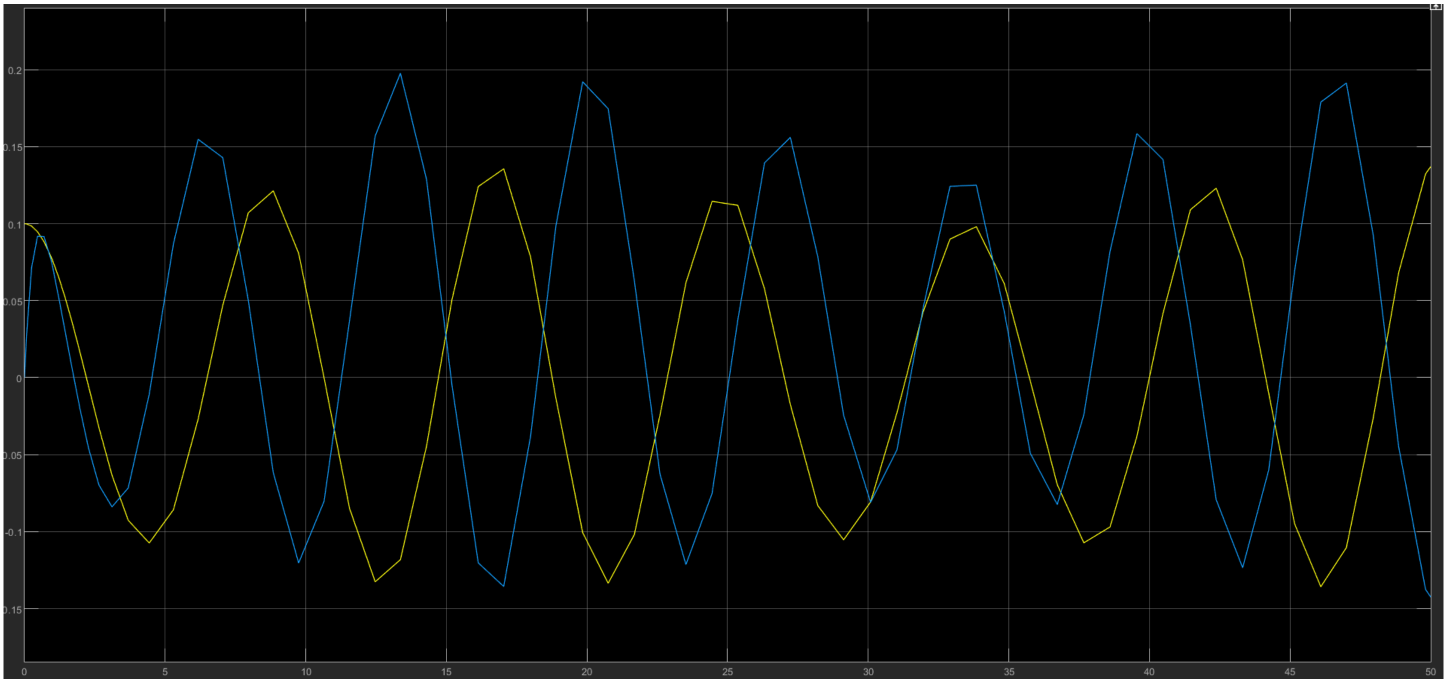


Fig26. Case 4 - Luenberger observer response for non-linear system θ_1(t), $\theta_{1_{est}}$(t) – Open loop

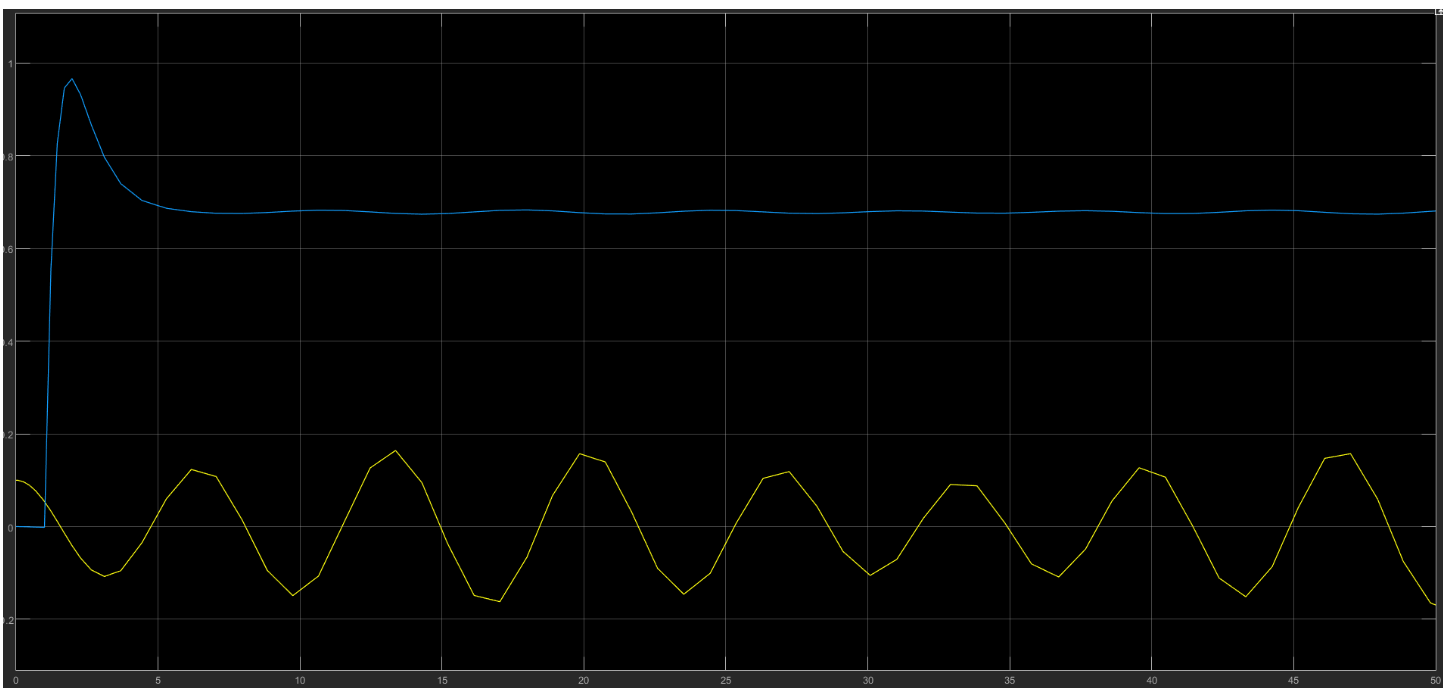


Fig27. Case 4 - Luenberger observer response for non-linear system θ_2(t), $\theta_{2_{est}}$(t) – Open loop

Solution to Part G:

We need to design an output feedback controller that will be used to estimate the full state of the system and then the estimated states will be used as a feedback to control the system.

To optimally estimate the states keeping in account the system disturbances and the measurement noise, we will design a linear quadratic estimator aka, Kalman Filter to optimally make the error converge to zero.

The state-space model considering the process and measurement noise can be modelled as:

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + \mathbf{v}(t),$$

$$\mathbf{y}(t) = C(t)\mathbf{x}(t) + \mathbf{w}(t)$$

Here, v(t) is the process noise (system disturbances) and w(t) is the measurement noise. They are assumed to be white gaussian noise.

Kalman Filter Design:

1) Find the Kalman filter gain –
   a. Define the noise co-variance matrix,
   b. Minimize the cost function that we used in LQR controller considering a trade-off between system and measurement noise/belief.
   c. Obtain the gain K_f.
2) Dynamics of Kalman filter – Optimal state estimator/observer –

$$\dot{\hat{x}} = (A - K_f * C)\hat{x} + [B\ K_f]\begin{bmatrix} u \\ y \end{bmatrix}$$
$$\hat{y} = C * \hat{x}$$

Combining state estimator with feedback controller:

   a. The separation principle allows us to design the optimal full-state feedback controller and the state estimator separately and then combining them together to form another optimal controller, ie, LQG controller.

Dynamics of LQG controller:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - B * K_{lqr} & B * K_{lqr} \\ 0 & A - K_f * C \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}[u\ y]$$
$$y = [C\ 0]x$$

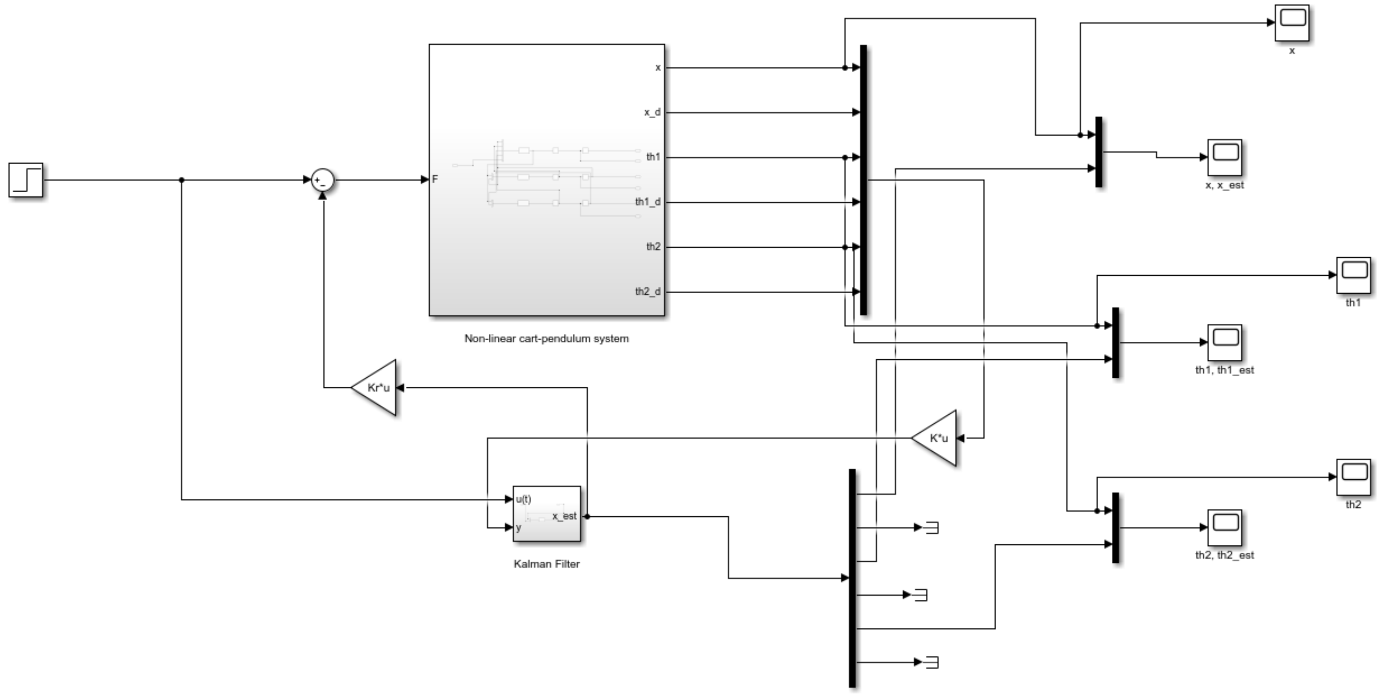


Fig 28. LQG Controller Non-linear model

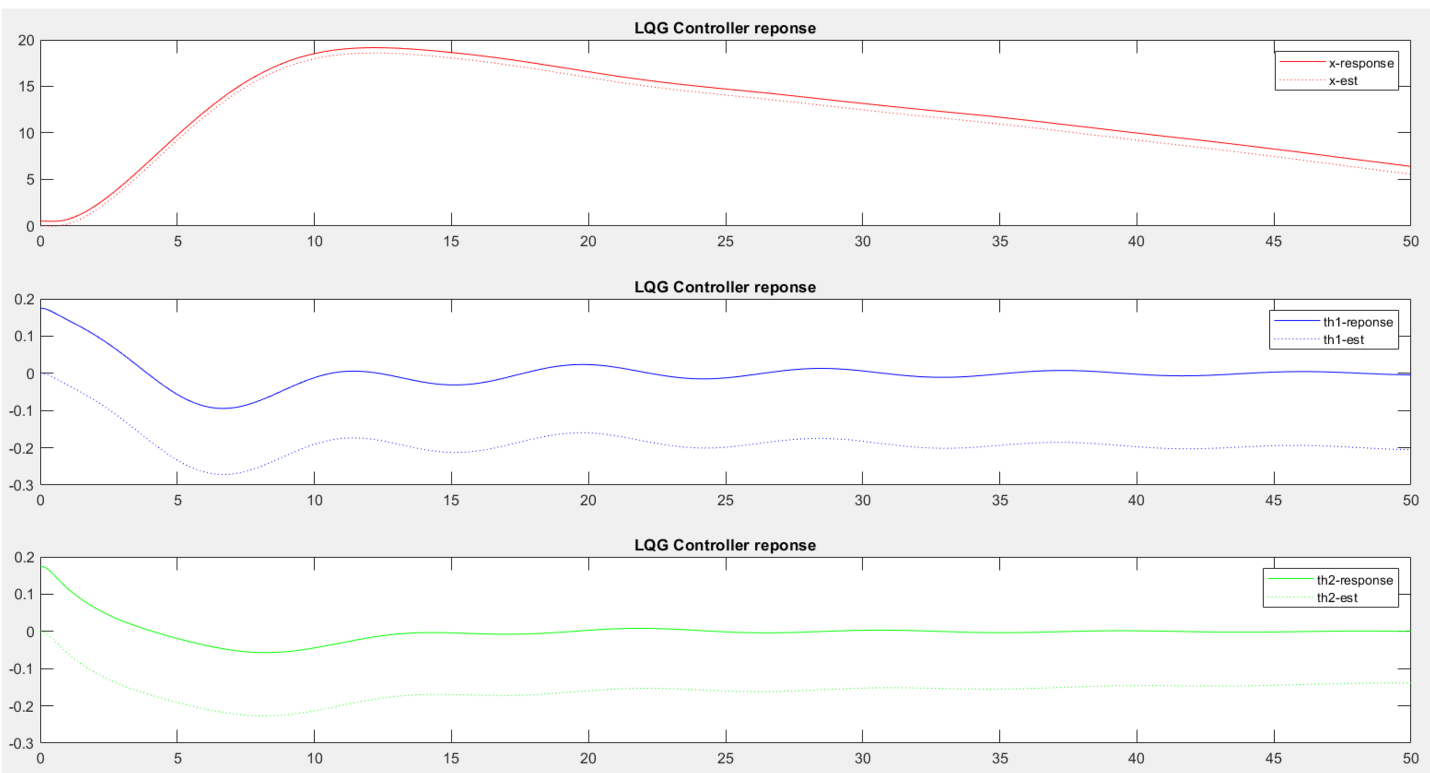


Fig 29. LQG Controller response – Linear system

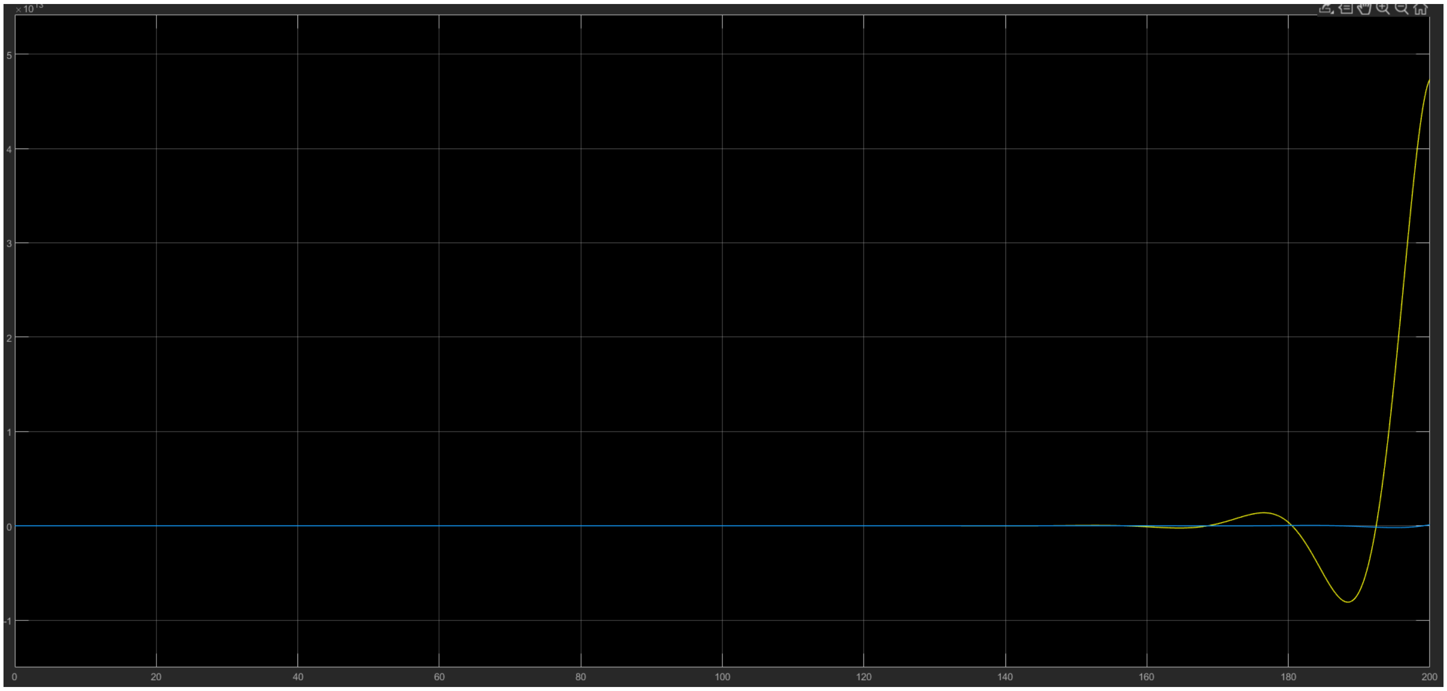


Fig 30. LQG Controller response – Non-linear system – x(t), x_{est}(t)

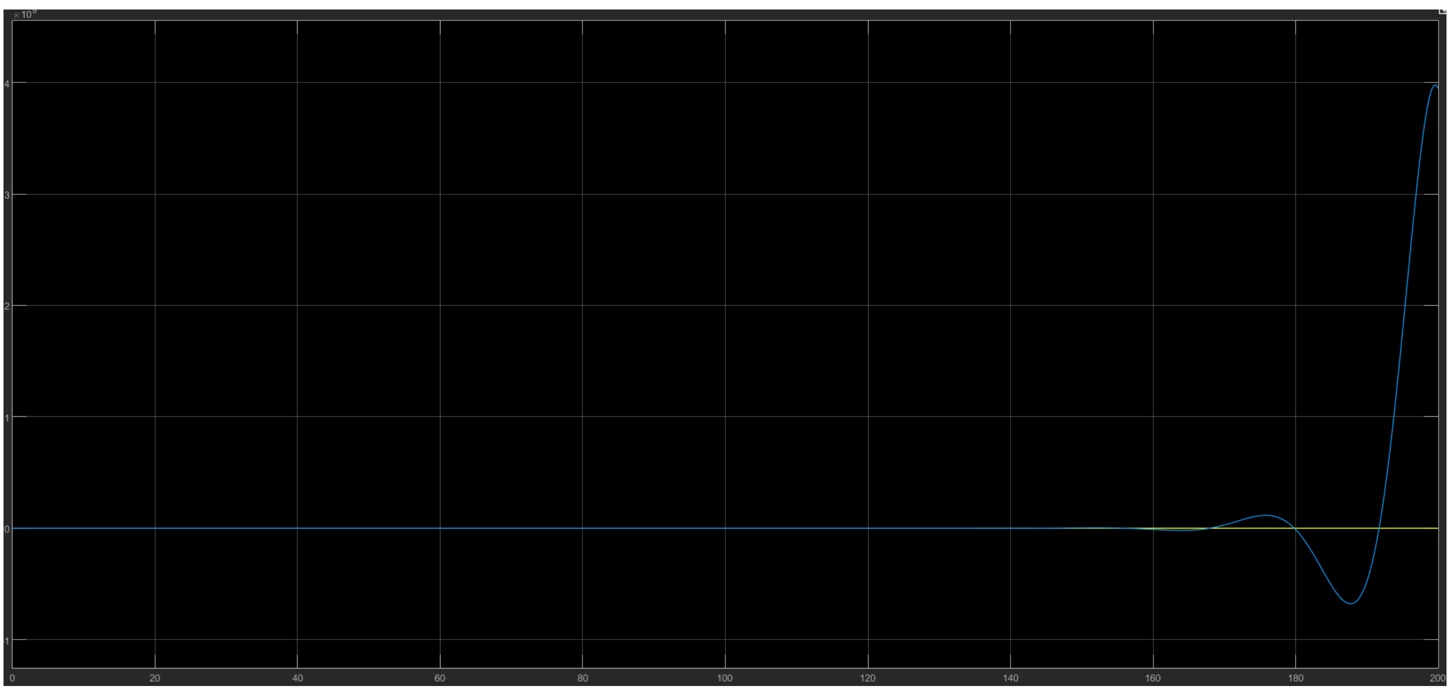


Fig 31. LQG Controller response – Non-linear system – θ_1(t), ${\theta_1}_{est}$(t)

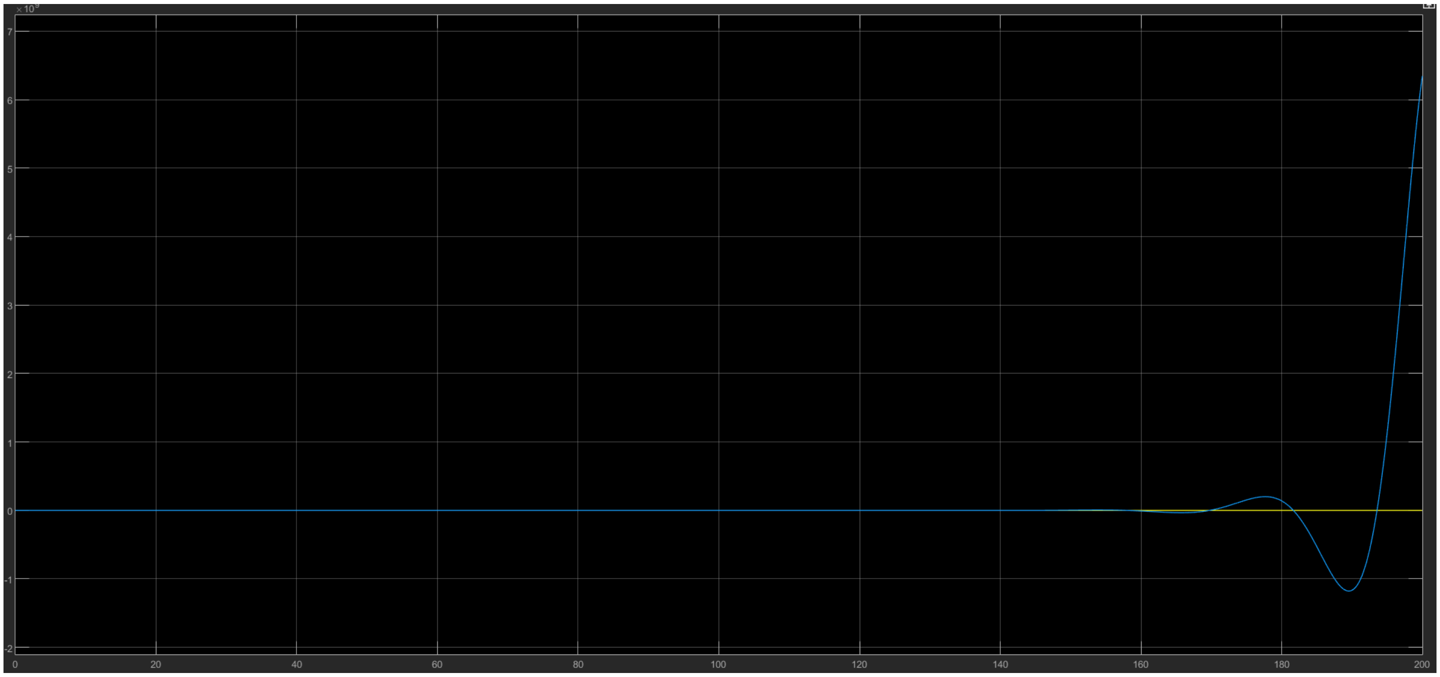


Fig 32. LQG Controller response – Non-linear system – θ_2(t), $\theta_{2_{est}}$(t)

To track a constant reference in this output feedback controller, we can have an integral gain/action at the input side hence eliminating any off-set.

Here, we have modified (in blue) the existing LQG-Kalman filter controller to add an integral action (highlighted in yellow).

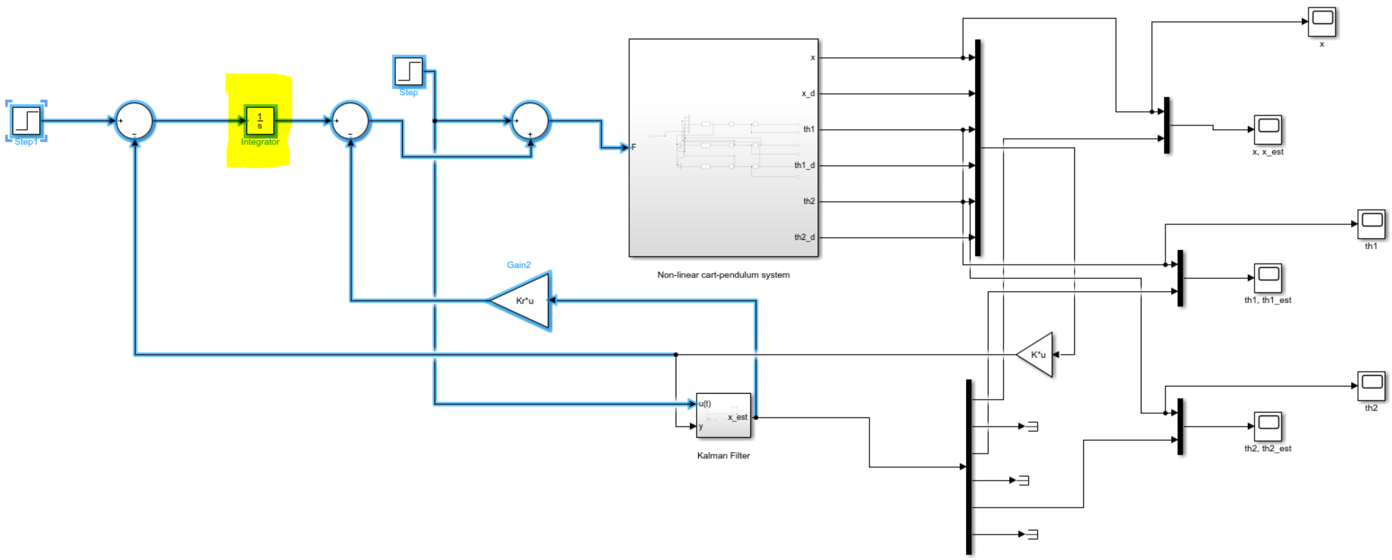


Fig 33. LQG Controller with integral action

This LQG controller is not robust to disturbances and noise. LQG controllers do not guarantee robustness although they might be stable at some point of time. Unlike LQR controllers (that have a large gain margin), LQG controllers do not have a gain margin at all. So, they MIGHT not be able to reject the external system disturbances that might come. It might be able to reject some, but this might not be the case every time.

Reference Model:

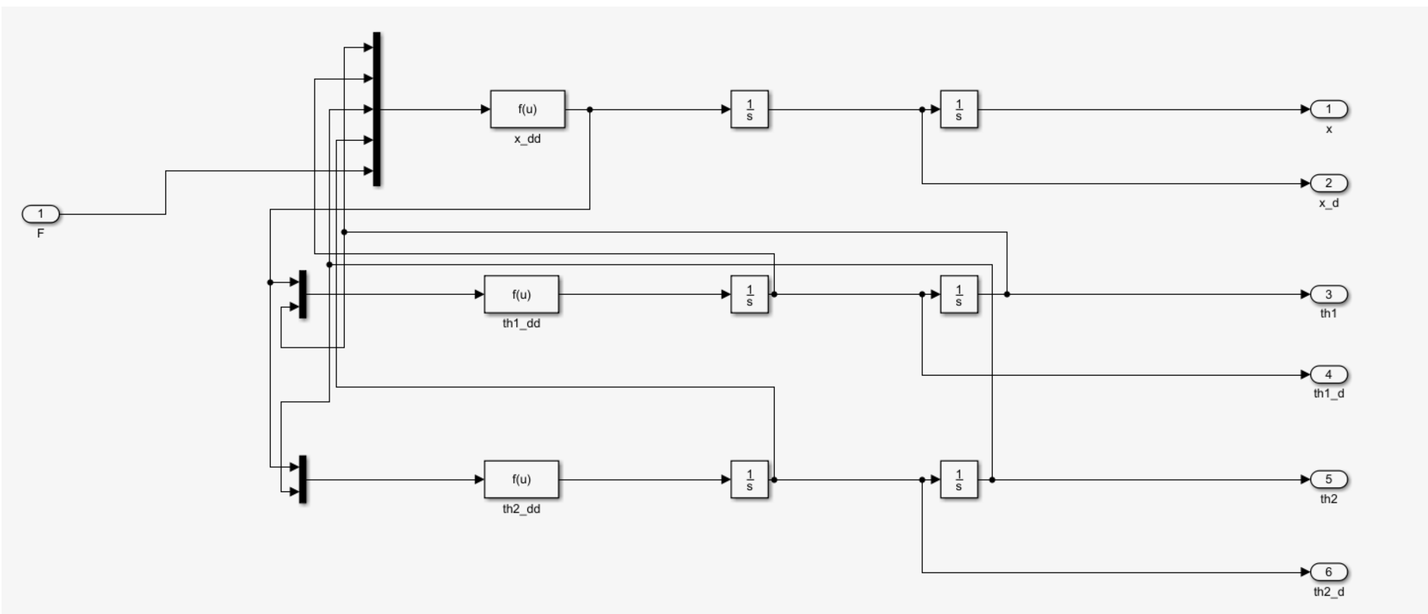


Fig 34. Non-linear system model