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Engineering Computational Methods

1. Accuracy of Numbers

(i) Approximate numbers → 2, 4, 6, 8, 12, ... are exact numbers. $\sqrt{2} = 1.414213$ ..... & $\pi = 3.141592$ ..... are approximate numbers.

(ii) Significant figures → 7845, 3.589, 0.4758 contains four significant figures while the numbers 0.00386, 0.000587 & 0.0000296 contains only three significant figures.

(iii) Rounding off → Numbers with large no. of digits i.e. 22/7 = 3.142857143. In practice, it is desirable to limit such numbers to a manageable numbers such as 3.14 or 3.143.

(iv) Errors → In numerical computation, we came across the following type of errors:-

(a) Inherent errors
(b) Rounding errors
(c) Truncation errors
(d) Absolute, Relative & Percentage errors

(v) Binomial Expression & Binomial theorem

(vi) Probability →

(a) Certainity (b) Uncertainity

2. Solution of an Algebraic & Transcendental Equations →

If $F(x) = 0$ is a polynomial of x, then it is an algebraic equation. Ex:-

$2x^2 + 5x + 6 = 0$

$15x + 3 = 0$

$16x^3 + a^9 x + 3 = 0$

and if it is a function containing exponential, logarithmic, trignometric functions is called transcendental equations.

Ex:- $\sin x + 6 = 0$, $e^x \cos x + 15 = 0$.

3. Bisection Method → This method is given by Balzano. Suppose, we wish to locate the root of an equation $F(x)=0$ in an interval, say (x_0, x_1). Let $F(x_0)$ & $F(x_1)$ are of opposite signs such that $F(x_0).F(x_1)<0$. Then, the graph of the function $F(x)=0$ cuts the x-axis between these points, then the derived root is approximately defined by the mid-point i.e., $x_2 = \frac{x_0+x_1}{2}$ if $F(x_2)=0$, then x_2 is root of the equation $F(x)=0$.

However, if $F(x_2)\neq 0$, then the root may lies between x_0 & x_2. Now, we define the next approximation $x_3 = \frac{x_0+x_2}{2}$, provided $F(x_0).F(x_2)<0$, then the root may be found between x_0 & x_2 or $x_3 = \frac{x_1+x_2}{2}$, provided $F(x_1).F(x_2)<0$. Thus, at each step we either find the solution of the derived accuracy or narrow the range to half the previous interval.

Ex 1:- Find the root of the given equations using bisection method upto three decimal places.

(i) $x^3-4x-9=0$

Sol:- Let $x_0=1$ & $x_1=3$

$F(x_0)=-12$ & $F(x_1)=6$

$\because F(x_0).F(x_1)<0$

$\therefore$ The root lies between x_0 & x_1

$\therefore$ First approximation to the root is

$x_2 = \frac{x_0+x_1}{2} = \frac{1+3}{2} = 2$

$F(x_2) = -9$

$\because F(x_1).F(x_2)<0$

$\therefore$ The root lies between x_2 & x_1

$\therefore$ Second approximation to the root is

$x_3 = \frac{x_1+x_2}{2} = \frac{3+2}{2} = 2.5$

$F(x_3) = 15.625-10-9 = -3.375$

$\because F(x_3) . F(x_1) < 0$

$\therefore$ The root lies between x_3 & x_1. Thus, the third approximation to the root is

$x_4 = \frac{x_3 + x_1}{2} = \frac{[redacted] + 2.5}{2} = 2.75$

$F(x_4) = 20.797 - 11 - 9 = 0.797$

$\because F(x_3) . F(x_4) < 0$

$\therefore$ The root lies between x_4 & x_3. Thus, the Fourth approximation to the root is

$x_5 = \frac{x_4 + x_3}{2} = \frac{2.75 + 2.5}{2} = 2.625$

$F(x_5) = 18.088 - 10.5 - 9 = -1.413$

$\because F(x_5) . F(x_4) < 0$

$\therefore$ The root lies between x_5 & x_4. Thus, the Fifth approximation to the root is

$x_6 = \frac{x_5 + x_4}{2} = \frac{2.625 + 2.75}{2} = 2.6875$

$F(x_6) = 19.411 - 10.75 - 9 = -0.339$

$\because F(x_6) . F(x_4) < 0$

$\therefore$ The root lies between x_6 & x_4. Thus, the Sixth approximation to the root is

$x_7 = \frac{x_6 + x_4}{2} = \frac{2.6875 + 2.75}{2} = 2.71875$

Hence, the root is 2.71875.

(ii) $x^3 - x - 11 = 0$

(iii) $\sin x = \frac{1}{x}$

(iv) $x^3 - 3x - 5 = 0$

(v) $3x - e^x = 0$

(vi) $x - \cos x = 0$

(vii) $3x = \sqrt{1 + \sin x}$

(viii) $x^3 - x^2 - x - 7 = 0$

(ix) $x^3 - 4x + 9 = 0$

(x) $x^4 - x - 10 = 0$

Sol (iii) $x \sin x - 1 = 0$ & $F(x) = x \sin x - 1$

Let $x_0 = 0$ & $x_1 = 36$

$F(x_0) = -1$ $F(x_1) = 14$

$\because F(x_0) . F(x_1) < 0$

$\therefore$ The root lies between x_0 & x_1. Thus the First approximation to the root is

$x_2 = \frac{x_0 + x_1}{2} = \frac{0 + 30}{2} = 15$

$F(x_2) = 3.882285677 - 1 = 2.882285677$

$\because F(x_0) . F(x_2) < 0$

$\therefore$ The root lies between x_0 & x_2. Thus, the third approximation to the root is

$x_3 = \frac{x_0 + x_2}{2} = \frac{0 + 15}{2} = 7.5$

$F(x_3) = 0.978946441 7 - 1 = -0.021053355835$

$\because F(x_3) . F(x_2) < 0$

$\therefore$ The root lies between x_3 & x_2. Thus, the fourth approximation to the root is

$x_4 = \frac{x_3 + x_2}{2} = \frac{7.5 + 15}{2} = 11.25$

$F(x_4) = 2.194766123 - 1 = 1.194766123$

$\because F(x_4) . F(x_3) < 0$

$\therefore$ The root lies between x_4 & x_3. Thus, the fifth approximation to the root is

$x_5 = \frac{x_4 + x_3}{2} = \frac{11.25 + 7.5}{2} = 9.375$

$F(x_5) = 1.527145063 - 1 = 0.527145063$

$\because F(x_5) . F(x_3) < 0$

$\therefore$ The root lies between x_5 & x_3. Thus, the sixth approximation to the root is

$x_6 = \frac{x_5 + x_3}{2} = \frac{9.375 + 7.5}{2} = 8.4375$

$F(x_6) = 1.238038378 - 1 = 0.238038378$

$\because F(x_6) . F(x_3) < 0$

$\therefore$ The root lies between x_6 & x_3. Thus, the seventh approximation to the root is

$x_7 = \frac{x_6 + x_3}{2} = \frac{8.4375 + 7.5}{2} = 7.96875$

$F(x_7) = 1.104731502 - 1 = 0.104731502$

$\because F(x_7) . F(x_3) < 0$

$\therefore$ The root lies between x_7 & x_3. Thus, the eighth approximation to the root is

$x_8 = \frac{x_7 + x_3}{2} = \frac{7.96875 + 7.5}{2} = 7.734375$

$F(x_8) = 1.040897667 - 1 = 0.040897667$

$\because F(x_8) \cdot F(x_3) < 0$

$\therefore$ The root lies between x_8 & x_3. Thus, the ninth approximation to the root is

$x_9 = \frac{x_8 + x_3}{2} = \frac{7.734375 + 7.5}{2} = 7.6171875$

$F(x_9) = 1.009686597 - 1 = 0.009686597$

$\because F(x_9) \cdot F(x_3) < 0$

$\therefore$ The root lies between x_9 & x_3. Thus, the tenth approximation to the root is

$x_{10} = \frac{x_9 + x_3}{3} = \frac{7.6171875 + 7.5}{2} = 7.55859375$

$F(x_{10}) = -0.0057423612$

$\because F(x_{10}) \cdot F(x_9) < 0$

$\therefore$ The root lies between x_{10} & x_9. Thus, the eleventh approximation to the root is

$x_{11} = \frac{x_{10} + x_9}{2} = \frac{7.55859375 + 7.6171875}{2} = 7.587890625$

$F(x_{11}) = 0.0019574$

$\because F(x_{11}) \cdot F(x_{10}) < 0$

$\therefore$ The root lies between x_{11} & x_{10}. Thus, the twelth approximation to the root is

$x_{12} = \frac{x_{11} + x_{10}}{2} = \frac{7.587890625 + 7.55859375}{2} = 7.573242188$

$F(x_{12}) = -0.0018961603$

$\because F(x_{12}) \cdot F(x_{11}) < 0$

$\therefore$ The root lies between x_{12} & x_{11}. Thus, the thirteenth approximation to the root is

$x_{13} = \frac{x_{12} + x_{11}}{2} = \frac{7.573242188 + 7.587890625}{2} = 7.580566407$

$F(x_{13}) = 0.0000297$

$\because F(x_{13}) \cdot F(x_{12}) < 0$

$\therefore$ The root lies between x_{13} & x_{12}. Thus, the fourteenth approximation to the root is

$x_{14} = \frac{x_{13} + x_{12}}{2} = \frac{7.580566407 + 7.573242188}{2} = 7.576904298$

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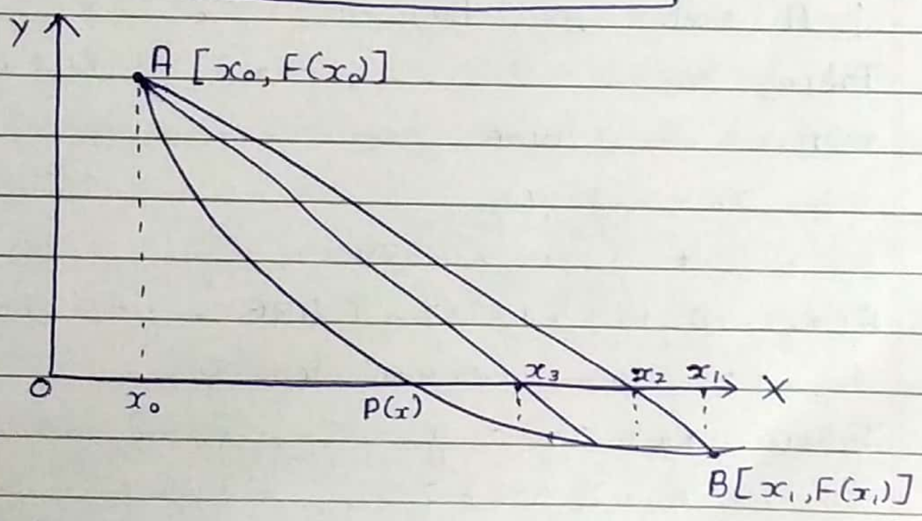
Hence, the root of the equation is 7.576904298

4. Regula - Falsi Method or Method of False Position → This is the oldest method of finding the real root of an equation $F(x)=0$ & closely resembles the bisection method.

Here, we chose two points x_0 & x_1 such that $F(x_0)$ & $F(x_1)$ are of opposite signs i.e, the graph of $y=F(x)$ crosses the x-axis between these points. This indicates that a root lies between x_0 & x_1 & consequently $F(x_0).F(x_1)<0$.

Equation of the chord joining the points $A[x_0, F(x_0)]$ & $B[x_1, F(x_1)]$ is

$$\boxed{y-F(x_0)=\frac{F(x_1)-F(x_0)}{x_1-x_0}(x-x_0)}$$



This method consists in replacing the curve AB by means of the chord AB & taking the point of intersection of the chord with the X-axis as an approximation to the root. So the abscissa of the point where the chord cuts the X-axis ($y=0$) is given by

$$\boxed{x_2=x_0-\left(\frac{x_1-x_0}{F(x_1)-F(x_0)}\right)F(x_0)} \quad \text{— eq①}$$

which is an approximation to the root.

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If now $F(x_0)$ & $F(x_2)$ are of opposite signs, then the root lies between x_0 & x_2. So replacing x_1 by x_2 in eq ①, we obtain the next approximation x_3. This process is repeated till the root is found to the desired accuracy. The iteration process based on eq ① is known as the method of false position & the rate of convergence is faster than that of the bisection method.

Ex 2:- Find a real root of $x^3-2x-5=0$ by the regula-Falsi method correct to three decimal places.

Sol:- Let $F(x) = x^3-2x-5$

Let $x_0 = 2$ & $x_1 = 3$

$F(x_0) = 8-4-5 = -1$ $\quad F(x_1) = 27-6-5 = 16$

$\because F(x_0) \cdot F(x_1) < 0$

$\therefore$ A root lies between x_0 & x_1

Taking $x_0 = 2$ & $x_1 = 3$, $F(x_0) = -1$ & $F(x_1) = 16$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{F(x_1) - F(x_0)} F(x_0) \quad \text{— eq ①}$$

$$x_2 = 2 + \frac{1}{17} = 2.0588$$

$F(x_2) = 8.7265447937 - 4.1176 - 5 = -0.3910$

$\because F(x_1) \cdot F(x_2) < 0$

$\therefore$ root lies between x_2 & x_1

Taking $x_3 = x_2 - \frac{x_1 - x_2}{F(x_1) - F(x_2)} F(x_2)$

$$x_3 = 2.0588 + \frac{0.368}{16.3910} = 2.0826$$

$F(x_3) = 9.0327 - 4.1652 - 5 = -0.1325$

$\because F(x_3) \cdot F(x_2) < 0$

$\therefore$ the root lies between x_3 & x_4

$$x_4 = x_3 - \frac{x_2 - x_3}{F(x_2) - F(x_3)} F(x_3)$$

$$x_4 = 2.0826 + \frac{0.0031}{0.2585} = 2.0946$$

$F(x_4) = 9.1897 - 4.1892 - 5 - 0$

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$x_5 = 2.0915$, $x_6 = 2.0934$
$x_7 = 2.0941$, $x_8 = 2.0943$
Hence, the root is 2.094 correct to 3 decimal places

5. <u>Secant Method</u> → This method is an improvement over the method of False position as it does not require the condition $F(x_0).F(x_1) < 0$ of that method. Here also the graph of the function $y = F(x)$ is approximated by a secant line but at each iteration, two most recent approximations to the root are used to find the next approximation. Also, it is not necessary that the interval must contain the root.
Taking x_0, x_1 as the initial limits of the interval, we write the equation of the chord joining these as

$$\boxed{y - F(x_1) = \frac{F(x_1) - F(x_0)}{x_1 - x_0}(x - x_1)}$$

Then, the abscissa of the point where it crosses the x-axis $(y=0)$ is given by

$$\boxed{x_2 = x_1 - \left(\frac{x_1 - x_0}{F(x_1) - F(x_0)}\right) F(x_1)}$$

which is an approximation to the root. The general formula for successive approximations is, therefore, given by

$$\boxed{x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{F(x_n) - F(x_{n-1})}\right) F(x_n)}, \quad n \geq 1$$

If at any iteration $F(x_n) = F(x_{n-1})$, this method fails & shows that it does not converge necessarily.

Ex 3. Find the root of the equation $xe^x = \cos x$ using the secant method correct to four decimal places.
Sol: $F(x) = \cos x - xe^x$
Let $x_0 = 0$ & $x_1 = 1$

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$F(x_0) = 1$ $\qquad F(x_1) = \cos 1 - e = -2.17798$

$\therefore x_2 = x_1 - \left(\frac{x_1 - x_0}{F(x_1) - F(x_0)}\right) F(x_1) = 1 - \frac{2.17798}{3.17798} = 0.31467$

$F(x_2) = 0.51987$

$\therefore x_3 = x_2 - \left(\frac{x_2 - x_1}{F(x_2) - F(x_1)}\right) F(x_2) = 0.44673$

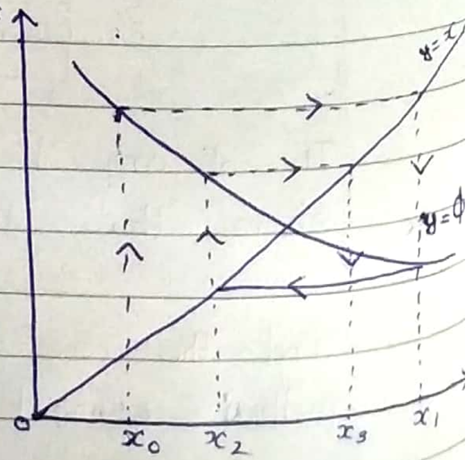
$F(x_3) = 0.20354$

$\therefore x_4 = x_3 - \left(\frac{x_3 - x_2}{F(x_3) - F(x_2)}\right) F(x_3) = 0.53171$

$\therefore x_5 = 0.51690$, $x_6 = 0.51775$ & $x_7 = 0.51776$

Hence, the root is 0.5177

6. Iteration Method → To find the roots of the equation $F(x) = 0$ —eq①
by successive approximations, we rewrite eq ① in the form
$x = \phi(x)$ —— eq②
The roots of eq ① are the same as the point of intersection of the straight line $y = x$ & the curve representing $y = \phi(x)$.
Let $x = x_0$ be an initial approximation of the desired root α. Then the first approximation x_1 is given by
$x_1 = \phi(x_0)$.
Now treating x_1 as the initial value, the second approximation is $x_2 = \phi(x_1)$
Procedding in the same way,
the n^{th} approximation is given by
$\boxed{x_n = \phi(x_{n-1})}$



Sufficient condition for convergence of iterations → Now it is not sure whether the sequence of approx. $x_1, x_2, \ldots, x_n$ always converges to the same number which is a root of eq ① or not. As such we have to choose the initial approximation x_0 suitably so that the successively approximations $x_1, x_2, \ldots x_n$ converge to the root α. The following theorem helps in making the right choice of x_0.

Theorem → If

(i) α be a root of $F(x)=0$ which is equivalent to $x=\phi(x)$

(ii) I, be any interval containing the point $x=\alpha$.

(iii) $|\phi'(x)| < 1$ for all x in I,

then the sequence of approximations $x_0, x_1, x_2, \ldots x_n$ will converge to the root α provided the initial approx. x_0 is chosen in I.

Ex4:- Find a real root of the equation $F(x) = x^3+x^2-1=0$ by the method of iterations.

Sol:- $F(0) = -1$ & $F(1)=1$, a root lies between 0 & 1.

Rewriting the given equation as

$x^2(x+1)=1$

$x^2 = \frac{1}{x+1}$

$x = (x+1)^{-1/2}$

$\Rightarrow \phi(x) = (x+1)^{-1/2}$

Now, $\phi'(x) = -\frac{1}{2}(x+1)^{-3/2}$

$\because |\phi'(x)| < 1$ for $x<1$

Hence, iteration method can be applied. Starting with $x_0 = 0.75$, the successive approximations are

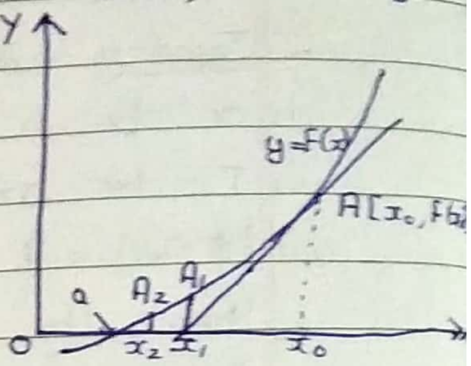
$x_1 = \phi(x_0) = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{1.75}} = 0.7559$

$x_2 = \phi(x_1) = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{1.7559}} = 0.75466$

$x_3 = \phi(x_2) = \frac{1}{\sqrt{x_2+1}} = 0.75492$, $x_4 = 0.75487$, $x_5 = 0.75988$

Hence, the root is 0.7548

7 Newton - Raphson Method → Let the curve $F(x)=0$ meets the x-axis at $x=\alpha$. It means that α is the root of the function $F(x)=0$. Let x_0 be a point near to the root α, then the equation of the tangent at $P[x_0, F(x_0)]$ is



$$\boxed{y - F(x_0) = F'(x_0)(x - x_0)}$$

It cuts the x-axis at $\boxed{x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}}$

which is the first approximation to the root α. If $A_1[x_1, F(x_1)]$ is the point corresponding to x_1 on the curve, then the tangent at A_1 is

$$\boxed{y - F(x_1) = F'(x_1)(x - x_0)}$$

It cuts the X-axis at $\boxed{x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}}$

which is the second approximation to the root α. Continue, this process till the root is found to the desired accuracy.

Let x_0 be an approximate root of the equation $F(x)=0$. If $x_1 = x_0 + h$ be the exact root, then $F(x_1) = 0$.

Expanding $F(x_0+h)$ by Taylor's series

$$F(x_0) + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \dots\dots = 0$$

Since, h is small, neglecting h^2 & higher power of h

$$F(x_0) + hF'(x_0) = 0$$

$$\Rightarrow h = -\frac{F(x_0)}{F'(x_0)} \quad \text{—— eq (1)}$$

∴ A closer approximation to the root is given by

$$\boxed{x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}}$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$$

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In general, $\boxed{x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}}$ $(n = 0, 1, 2, 3, \ldots)$

which is known as Newton - Raphson Formula.

Ex 5:- Find the positive root of $x^4 - x = 10$ correct to three decimal places by Newton - Raphson method.

Sol:- Let $x_0 = 1$ & $x_1 = 2$ $F'(x) = 4x^3 - 1$

$F(x_0) = -10$ & $F(x_1) = 4$

$\because F(x_0) . F(x_1) < 0$

$\therefore$ Root lies between 1 & 2.

Let $x_0 = 2.$

From Newton - Raphson Formula

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \quad \text{— eq (1)}$$

Put $n = 0$, the First approximation is given by

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)} = 2 - \frac{4}{31} = 1.871$$

Put $n = 1$, the second approximation is given by

$$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)} = 1.871 - \frac{0.3835}{25.199} = 1.856$$

Putting $n = 2$ in eq (1), the third approximation is given by

$$x_3 = x_2 - \frac{F(x_2)}{F'(x_2)} = 1.856 - \frac{0.010}{24.574} = 1.856$$

Hence, the desired root is 1.856.

Ex 6:- Find the real root of Following by Newton - Raphson method.

(i) $3x = \cos x + 1$

(ii) $x \log_{10} x = 1.2$

(iii) $x^3 - x - 11 = 0$

(iv) $x^3 - x^2 + x - 7 = 0$

(v) $x^3 - 2x - 5 = 0$

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(vi) $x - \cos x = 0$

8. Some deductions From Newton - Raphson Formula → We can derive the following results from Newton - Raphson Formula.

(i) Iterative Formula to Find $1/N$ is $x_{n+1} = x_n(2 - Nx_n)$

Proof :- Let $x = \frac{1}{N}$ or $\frac{1}{x} - N = 0$

$\therefore F(x) = \frac{1}{x} - N$

$F'(x) = -x^{-2}$

Now, by Newton Raphson Method

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

$$x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - N\right)}{-x_n^{-2}}$$

$$x_{n+1} = x_n + \left(\frac{1}{x_n} - N\right) x_n^2$$

$$x_{n+1} = x_n + x_n - Nx_n^2$$

$$x_{n+1} = 2x_n - Nx_n^2$$

$$\boxed{x_{n+1} = x_n(2 - Nx_n)}$$

(ii) Iterative Formula to Find $\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$

Proof :- Let $x = \sqrt{N}$ or $x^2 - N = 0$

$\therefore F(x) = x^2 - N$ or $F'(x) = 2x$

Now, by Newton Raphson Method

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^2 - N)}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x}\right)}$$

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(iii) Iterative Formula to Find $1/\sqrt{N}$ is $x_{n+1} = \frac{1}{2}(x_n + 1/Nx_n)$

Proof :- Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$

$\therefore F(x) = x^2 - \frac{1}{N}$ or $F'(x) = 2x$

Now, by Newton Raphson method

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

$$x_{n+1} = x_n - \frac{\left(x_n^2 - \frac{1}{N}\right)}{2x_n}$$

$$x_{n+1} = \frac{1}{2} \frac{\left(x_n^2 + \frac{1}{N}\right)}{x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)}$$

(iv) Iterative Formula to find $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + N/x_n^{k-1}\right]$

Proof :- Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

$\therefore F(x) = x^k - N$ or $F'(x) = kx^{k-1}$

Now, by Newton Raphson method

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^k - N)}{kx_n^{k-1}}$$

$$x_{n+1} = \frac{kx_n^k - x_n^k + N}{kx_n^{k-1}}$$

$$x_{n+1} = \frac{1}{k}\left[\frac{(k-1)x_n^k}{x_n^{k-1}} + \frac{N}{x_n^{k-1}}\right]$$

$$\boxed{x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]}$$

Ex 7 :- Evaluate the following correct to four decimal places by Newton - Raphson method

(i) $1/31$

Sol :- Taking $N = 31$ in $x_{n+1} = x_n(2 - Nx_n)$ becomes

$$x_{n+1} = x_n(2 - 31x_n)$$

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Since, an approximate value of $\frac{1}{31} = 0.03$, we take $x_0 = 0.03$

$x_1 = x_0(2 - 31x_0) = 0.03(2 - 31 \times 0.03) = 0.0321$

$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$

$x_3 = x_2(2 - 31x_2) = 0.032257(2 - 31 \times 0.032257) = 0.03226$

Hence, the root is 0.0323.

(ii) $\sqrt{5}$ (iii) $1/\sqrt{14}$

(iv) $\sqrt[3]{24}$ (v) $(30)^{-1/5}$

9. Solution of simultaneous algebraic equations → We describe below some such methods of solutions:-

(i) Gauss elimination method → In this method, the unknowns are eliminated successively by transforming the given system with upper triangular matrix into an equivalent system by means of elementary row operation From which the unknown are found by back substitution.

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \text{— eq ①}$$

The system is in matrix Form $AX = B$

where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

We consider the augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \quad \text{— eq ②}$$

PIVOT → An element with help we can make other element 0

Two types of pivoting are:-

(a) Complete Pivoting

(b) Partial Pivoting

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Let $a_1 \neq 0$, then

$R_2 \to R_2 - \frac{a_2}{a_1} R_1$

$R_3 \to R_3 - \frac{a_3}{a_1} R_1$ is $\left[\begin{array}{ccc:c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array}\right]$

Here, a_1 is called the First pivot & $b_2, c_2, d_2, b_3, c_3, d_3$ are the transformed elements.

Now, we take b_2 as the pivot, then,

$R_3 \to R_3 - \frac{b_3}{b_2} R_2$ is $\left[\begin{array}{ccc:c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right]$

Now, if $c_3'' \neq 0$, then the given system of equations

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ b_2'y + c_2'z = d_2' \\ c_3''z = d_3'' \end{array}\right\} \quad \text{—— eq ③}$$

The value of x, y & z are found from the reduced system eq ③ by back substitution

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Ex 8:- Solve the following equations by Gauss elimination theorem

(i) $x + 4y - z = 5$

$x + y - 6z = -12$

$3x - y - z = 4$

Sol:- $A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ -12 \\ 4 \end{bmatrix}$

$[A|B] = \left[\begin{array}{ccc:c} 1 & 4 & -1 & 5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right]$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 3R_1$

$[A|B] = \left[\begin{array}{ccc:c} 1 & 4 & -1 & 5 \\ 0 & -3 & -5 & -4 \\ 0 & -13 & 2 & 19 \end{array}\right]$

$R_3 \to R_3 - \left(\frac{-13}{-3}\right) R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & 0 & 71/3 & 148/3 \end{array}\right]$$

$\therefore \quad x + 4y - z = -5$

$-3y - 5z = -7$

$\frac{71}{3} z = \frac{148}{3}$

$\Rightarrow \quad z = \frac{148}{71}$

$z = 2.08$

$\Rightarrow \quad y = 1.14 \quad \& \quad x = -7.48$

(ii) $5x + 4y = 15$

$3x + 7y = 12$

(iii) $3x - y + 2z = 12$

$x + 2y + 3z = 11$

$2x - 2y - z = 2$

(ii) Gauss Jordan Method → This is a modification of Gauss elimination method. In this method, unknown are obtained by transforming the augmented matrix to diagonal matrix form.

Ex 9:- Solve the following equation by Gauss Jordan method.

(i) $x + y + z = 9$

$2x - 3y + 4z = 13$

$3x + 4y + 5z = 40$

Sol:- $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

$R_2 \to R_3 - R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 6 & 0 & 18 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

$R_2 \to \frac{1}{6} R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 10 \end{array}\right]$$

$R_3 \to \frac{1}{2} R_3$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 \to R_1 - R_3$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\Rightarrow x_1 = 1, \quad y = 3, \quad z = 5$

ii) $5x + 4y = 15$

$3x + 7y = 12$

(iii) $3x - y + 2z = 12$

$x + 2y + 3z = 11$

$2x - 2y - z = 2$

(iii) Iterative Method (Indirect method) → In this method, we start from an approximation to the true solution & if convergent derive a sequence of closer approximations, we repeat the cycle of computations till the required accuracy is obtained. But, the method is not applicable to all system of equations. For this, each equation of the system must contain a largest coefficient & the larger coefficient must be attached to a different unknown in that equation, i.e,

$a_1x + b_1y + c_1z = d_1$ $\qquad |a_1| > |b_1| + |c_1|$

$a_2x + b_2y + c_2z = d_2$ $\qquad$ if $\qquad |b_2| > |a_2| + |c_2|$

$a_3x + b_3y + c_3z = d_3$ $\qquad |c_3| > |a_3| + |b_3|$

(a) Jacobi Method of iteration → This method is also known as Gauss - Jacobi Method. Consider

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} \quad \text{— eq ①}$$

Solve for x, y & z, we get

$$\left.\begin{array}{l} x = \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y = \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{array}\right\} \quad \text{— eq ②}$$

Let $|a_1| > |b_1| + |c_1|$

$|b_2| > |a_2| + |c_2|$

$|c_3| > |a_3| + |b_3|$

Let us start with initial approximations x_0, y_0, z_0 for the value of x, y, z by substituting in eq ②, we get

$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$

$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$

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$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$

Substituting the value of x_1, y_1 & z_1 in eq ②

$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$

$y_2 = \frac{1}{b_2}(d_2 - a_3 y_1 - c_2 z_1)$

$z_3 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$

Proceding the same way if x_r, y_r, z_r are the r^{th} iterates, then

$x_{r+1} = \frac{1}{a_1}(d_1 - b_1 y_r - c_1 z_r)$

$y_{r+1} = \frac{1}{b_2}(d_2 - a_2 x_r - c_2 z_r)$

$z_{r+1} = \frac{1}{c_3}(d_3 - a_3 x_r - b_3 y_r)$

REMARK :- In the absence of better estimates, the initial approximations are taken as $x_0 = 0$, $y_0 = 0$ & $z_0 = 0$.

Ex 10 :- Solve the Following equations by Jacobi method of iteration

$8x - 3y + 2z = 20$

$4x + 11y - z = 33$

$6x + 3y + 12z = 35$

Sol :- Here, $|8| > |-3| + |2|$

$|11| > |4| + |-1|$

$|12| > |6| + |3|$

$$\left.\begin{aligned} x &= \frac{1}{8}(20 + 3y - 2z) \\ y &= \frac{1}{11}(33 - 4x + z) \\ z &= \frac{1}{12}(35 - 6x - 3y) \end{aligned}\right\} \quad \text{—— eq ①}$$

Let $x_0 = y_0 = z_0 = 0$

$x_1 = \frac{20}{8} = \frac{5}{2}$, $\quad y_1 = \frac{33}{11} = 3$, $\quad z_1 = \frac{35}{12}$

Put the value of x_1, y_1 & z_1 in eq ①, we get

$x_2 = \frac{1}{8}\left(20 + 9 - \frac{35}{6}\right) = \frac{1}{8}\left(29 - \frac{35}{6}\right) = \frac{139}{48} = 2.89$

$y_2 = \frac{1}{11}\left(33 - 10 + \frac{35}{12}\right) = \frac{311}{132} = 2.35$

$z_2 = \frac{1}{12}(35 - 15 - 9) = \frac{11}{12} = 0.91$

Put the value of x_2, y_2 & z_2 in eq ①, we get

$$x_3 = \frac{1}{8}(20 + 7.05 - 1.82) = 3.153$$
$$y_3 = \frac{1}{11}(33 + 0.91 - 11.56) = 2.032$$
$$z_3 = \frac{1}{12}(35 - 17.34 - 7.05) = 0.884$$

Put the value of x_3, y_3 & z_3 in eq ②, we get

$$x_4 = \frac{1}{8}(20 + 6.096 - 1.768) = 3.041$$
$$y_4 = \frac{1}{11}(33 + 0.884 - 12.612) = 1.934$$
$$z_4 = \frac{1}{12}(35 - 18.918 - 6.096) = 0.832$$

Put the value of x_4, y_4 & z_4 in eq ②, we get

$$x_5 = \frac{1}{8}(20 + 5.802 - 1.664) = 3.017$$
$$y_5 = \frac{1}{11}(33 + 0.832 - 12.164) = 1.969$$
$$z_5 = \frac{1}{12}(35 - 18.246 - 5.802) = 0.913$$

(b) Gauss Seidal Iteration Method → As before,

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} \quad \text{— eq ①}$$

The above equation is written as

$$\left.\begin{array}{l} x = \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y = \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z = \frac{1}{c_2}(d_3 - a_3x - b_3y) \end{array}\right\} \quad \text{— eq ②}$$

Here, we also take initial approximation as $x_0 = y_0 = z_0 = 0$

Substituting, $y = y_0$; $z = z_0$ in First equation of — eq②

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Then putting $x = x_1$, $z = z_0$ in the second equation of eq②

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Next substituting $x = x_1$, $y = y_1$ in the third equation of eq②

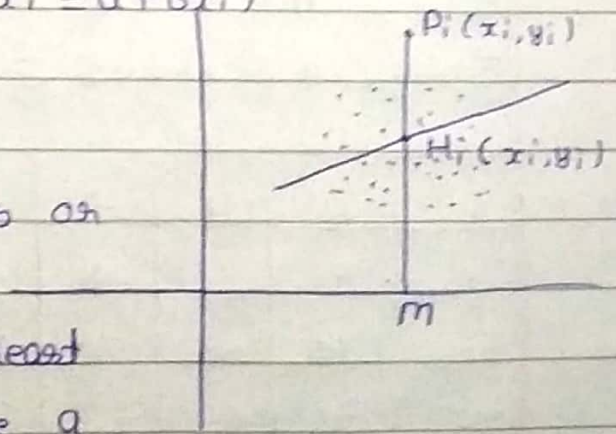
$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

& so on i.e., as soon as a new approximation for an unknown is found, it is immediately used in the next step. This process of iteration is repeated till the values of x, y & z are obtained to the desired accuracy.

10. Curve Fitting →

(i) Principle of Least squares → Let us choose to represent a set of data (x_i, y_i), $i = 1, 2, 3, \ldots n$ by some relation $y = F(x)$ containing m unknowns parameters $a_1, a_2, a_3, \ldots a_n$ & we consider the deviation
$\varepsilon_i = \pm [F(x_i) - y_i]$
The Functions S_i, the Sum of squares of residuals viz.
$S = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} [F(x) - y_i]^2$ is clearly a Functions of the parameters $a_1, a_2, a_3, \ldots a_n$.
We can find these parameters so that $S(a_1, a_2, \ldots a_n)$ is minimum For this
$\left(\frac{\partial S}{\partial a_k}\right) = 0$, $k = 1, 2, 3, \ldots m$

(ii) Fitting of straight lines → Let $y = a + bx$ be a set of points (x_i, y_i) $i = 1, 2, 3, \ldots n$ For different values of a & b, we have different straight lines.
The problem is to determine a & b, so that the line is the line of base Fit.
Let $P_i(x_i, y_i)$ be any general point in the scatter diagram Draw $P_iM \perp$ to x-axis meeting the line at the co-ordinates of the H_i are $(x_i = a + bx_i)$



$P_iH_i = P_iM - H_iM$
$= y_i - (a + bx_i)$
called the error of estimates or residual of y_i.
According to the Principal of Least squares we have to determine a & b so that
$E = \sum_{i=1}^{n} (P_iH_i)^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$ is minimum.
By the principle of Minima & Maxima, the partial derivates of the E w.r.t a & b variables respectively
$\frac{\partial E}{\partial a} = 0$

$\Rightarrow 2\sum_{i=1}^{n}(y_i - a - bx_i)(-1) = 0$

$\Rightarrow \sum_{i=1}^{n}(y_i - a - bx_i) = 0$

$\Rightarrow \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - b\sum_{i=1}^{n} x_i = 0$

$\Rightarrow \sum_{i=1}^{n} y_i - na - b\sum_{i=1}^{n} x_i = 0$

$\Rightarrow \sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$ —— eq ①

Now, $\frac{\partial E}{\partial b} = 0$

$\Rightarrow 2\sum_{i=1}^{n}(y_i - a - bx_i)(-x_i) = 0$

$\Rightarrow \sum_{i=1}^{n} x_i y_i - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^2 = 0$

$\Rightarrow \sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$ —— eq ②

By Solving eq ① & eq ②, we find the estimated values of a & b, then $y = a + bx$

Ex 11:- Fit a straight line to the following data.

x	6	7	7	8	8	8	9	9	10
y	5	5	4	5	4	3	4	3	3

Sol:- $y_i = a + bx_i$

$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$

$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$

x	6	7	7	8	8	8	9	9	10	$\Sigma x_i = 72$
y	5	5	4	5	4	3	4	3	3	$\Sigma y_i = 36$
$x_i y_i$	30	35	28	40	32	24	36	27	30	$\Sigma x_i y_i = 282$
x_i^2	36	49	49	64	64	64	81	81	100	$\Sigma x_i^2 = 588$

$\therefore 282 = a.72 + b.588$

$\Rightarrow 72a + 588b = 282$

Also, $36 = 9a + b.72$

$\Rightarrow 9a + 72b = 36$

Ex12. Fit a second degree parabola to the following data

(i)

x	0	1	2	3	4
y	1	18	13	25	63

(ii)

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

(iii)

x	1989	1990	1991	1992	1993	1994	1995	1996	1997
y	352	356	357	358	360	361	361	360	359

Sol:- $X = x - 1993$, $Y = y - 360$

X	-4	-3	-2	-1	0	1	2	3	4	$\Sigma X_i = 0$
Y	-8	-4	-3	-2	0	1	1	0	-1	$\Sigma Y_i = -16$
XY	32	12	6	2	0	1	2	0	-4	$\Sigma X_i Y_i = 51$
X^2	16	9	4	1	0	1	4	9	16	$\Sigma X_i^2 = 60$

$Y = a + bX$

$y - 360 = a + b(x - 1993)$

$\Rightarrow y = a + bx$

$\sum_{i=1}^{n} Y_i = na + b\sum_{i=1}^{n} X_i$

$\sum_{i=1}^{n} X_i Y_i = a\sum_{i=1}^{n} X_i + b\sum_{i=1}^{n} X_i^2$

$\Rightarrow$

(iii) Fitting of a curve of the types

- $y = a + bx^2$

Put $x^2 = X$

$\therefore y = a + bX$

$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} X$

$\sum_{i=1}^{n} X y_i = a\sum_{i=1}^{n} X_i + b\sum_{i=1}^{n} X_i^2$

- $y = ax + bx^2$

$\frac{y}{x} = a + bx$

Put $\frac{y}{x} = Y$

$\therefore Y = a + bx$

- $y = ax + \frac{b}{x}$

$xy = ax^2 + b$

Put $xy = Y$ & $x^2 = X$

$\therefore Y = aX + b$

- $y = ax^2 + \frac{b}{x}$

$xy = ax^3 + b$

Put $xy = Y$ & $x^3 = X$

(iv) Logarithmic, exponential & Power curve →

- $y = ax^b$

$\log y = \log a + b \log x$

Put $\log y = Y$, $\log a = A$, $b \log x = BX$

$\therefore Y = A + BX$

$\Rightarrow \sum_{i=1}^{n} Y_i = nA + B \sum_{i=1}^{n} X_i$

$\sum_{i=1}^{n} X_i Y_i = a \sum_{i=1}^{n} X_i + b \sum_{i=1}^{n} X_i^2$

- $y = ab^x$

$\log y = \log a + x \log b$

Put $\log y = Y$, $\log a = A$, $x \log b = XB$

$\therefore Y = A + XB$

- $y = ae^{bx}$

$\log y = \log a + bx$

Put $\log y = Y$, $\log a = A$, $bx = Bx$

$\therefore Y = A + Bx$

Ex12:- Find a curve of the type $y = ab^x$.

x	50	450	780	1200	4400	4800	5300
y	28	30	32	36	51	58	69

Sol:- $y = ab^x$

$\log y = \log a + x \log b$

Put $\log y = Y$, $\log a = A$, $x \log b = XB$

$\therefore Y = A + Bx$