SYMBOLIC LOGIC FOURTH EDITION

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For Amelia

PREFACE

The general approach of this book to logic remains the same as in earlier editions. Following Aristotle, we regard logic from two different points of view: On the one hand, logic is an instrument or organon for appraising the correctness of reasoning; on the other hand, the principles and methods of logic used as organon are interesting and important topics to be themselves systematically investigated. This dual approach to logic is especially appropriate for modern symbolic logic. Through the development of its special symbols logic has become immeasurably more powerful an instrument for analysis and deduction. And the principles and methods of symbolic logic are fruitfully investigated through the study of logistic systems.

The first half of this book, Chapters 1 through 5, presents the standard notations, methods, and principles of symbolic logic for *use* in determining the validity or invalidity of arguments. It takes up successively more complex modes of argumentation: first those whose validity turns on truth-functional compounds of simple statements, next those involving the simplest kinds of quantification, then more complex kinds of multiple quantification, and finally relational arguments. The standard methods of truth tables, rules of inference, conditional and indirect modes of proof, and quantification theory by way of 'natural deduction' techniques are introduced. The logic of relations is developed in a separate chapter which includes identity theory, definite descriptions, predicates of higher type, and quantification of predicate variables. A great many exercises are provided to help the student acquire a practical mastery of the material.

The second half of the book contains a systematic treatment of the logical principles *used* in the first half. After a brief discussion of deductive systems in general, a propositional calculus is developed according to the highest modern standards of rigor, and proved to be consistent and complete. Alternative notations and axiomatic foundations for propositional calculi are presented, and then a first-order function calculus is developed. The latter is shown to be equivalent to the 'natural deduction' methods of the first half of the book, and is also proved to be consistent and complete.

There are three appendices: the first presents Boolean Expansions as an algebraic method of appraising the correctness of truth-functional arguments;

the second deals with the algebra of classes; and the third with the ramified theory of types.

This fourth edition of *Symbolic Logic* differs from its predecessors in several respects. One is purely organizational: the strengthened rule of Conditional Proof is moved forward from Chapter 4 to Chapter 3, where it can be used in working with purely truth-functional arguments. Other respects in which the new edition is different are the following. In Section 1.2 there is a somewhat more careful discussion of the distinctions among propositions, statements, and sentences. In Section 2.1 attention is paid to the roles of such words as 'either,' 'neither,' 'both,' and 'unless,' and to the significance of different places at which 'not' can occur in compound statements. In Section 2.3 there is a more explicit statement of the presuppositions involved in developing the logic of truth functions. In Section 3.2 more rules of thumb are suggested for use in devising formal proofs of validity. In Section 7.6 an improved proof is given for Metatheorem V. In Section 8.2 the danger of 'creative' definition is pointed out and a different proof presented of the deductive completeness of the Hilbert-Ackermann System, which gives the student insight into two quite different and independent methods of establishing completeness results for propositional calculi. In Section 9.1 there is a simpler consistency proof for the first-order calculus. A new Section 9.7 has been added in which Identity Theory is derived from a single additional axiom schema, following a suggestion of Professor Hao Wang.

There are two distinct skills that the first half of the book is designed to help students acquire. One is the ability to analyze statements and arguments in ordinary language, and to translate them into the notations of symbolic logic. The other is the ability to apply the techniques and methods of symbolic logic to determine the validity or invalidity of arguments already symbolized. Where both skills are required in a single problem, a mistake in translating can spoil it as an exercise in appraising validity. Consequently more exercises are provided to help the student develop these skills separately. More exercises in symbolizing have been inserted into Chapters 2, 4, and 5, and more arguments already symbolized have been inserted into Chapters 3, 4, and 5 as exercises in appraising validity. More than two hundred new exercises appear in this new edition: exercises in symbolizing and in appraising validity; some exercises aimed more at helping students recognize the forms and the specific forms of arguments and of statements; and some on different axiom systems to be proved independent, consistent, and complete.

Quite a few teachers of logic have been kind enough to suggest ways to improve this book. I have given earnest consideration to all of the advice offered, even though I was not able to incorporate all of the changes proposed. For their helpful communications I am especially grateful to Professor Alan Ross Anderson of the University of Pittsburgh, Lynn Aulick of Cedar Crest College, William F. Barr of the State University of New York College at Cortland, Walter A. Bass of Indiana State University, Robert W. Beard of Florida State University, Richard Beaulieu of Paris, James C. Bohan,

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I. M. C.

CONTENTS

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Introduction: Logic and Language

1.1 What Is Logic?

It is easy to find answers to the question 'What is Logic?' According to Charles Peirce, 'Nearly a hundred definitions of it have been given'.[1] But Peirce goes on to write: 'It will, however, generally be conceded that its central problem is the classification of arguments, so that all those that are bad are thrown into one division, and those which are good into another . . .'.

The study of logic, then, is the study of the methods and principles used in distinguishing correct (good) from incorrect (bad) arguments. This definition is not intended to imply, of course, that one can make the distinction only if he has studied logic. But the study of logic will *help* one to distinguish between correct and incorrect arguments, and it will do so in several ways. First of all, the proper study of logic will approach it as an art as well as a science, and the student will do exercises in all parts of the theory being learned. Here, as anywhere else, practice will help to make perfect. In the second place, the study of logic, especially symbolic logic, like the study of any other exact science, will tend to increase one's proficiency in reasoning. And finally, the study of logic will give the student certain techniques for testing the validity of all arguments, including his own. This knowledge is of value because when mistakes are easily detected they are less likely to be made.

Logic has frequently been defined as the science of reasoning. That definition, although it gives a clue to the nature of logic, is not quite accurate. Reasoning is that special kind of thinking called inferring, in which conclusions are drawn from premisses. As thinking, however, it is not the special province of logic, but part of the psychologist's subject matter as well. Psychologists who examine the reasoning process find it to be extremely complex and highly emotional, consisting of awkward trial and error procedures illuminated by sudden—and sometimes apparently irrelevant—flashes of insight. These are

[1]'Logic', in *Dictionary of Philosophy and Psychology*, edited by James Mark Baldwin, New York, The Macmillan Company, 1925.

all of importance to psychology. But the logician is not interested in the actual process of reasoning. He is concerned with the correctness of the completed process. His question is always: does the conclusion reached *follow* from the premisses used or assumed? If the premisses provide adequate grounds for accepting the conclusion, if asserting the premisses to be true warrants asserting the conclusion to be true also, then the reasoning is correct. Otherwise it is incorrect. The logician's methods and techniques have been developed primarily for the purpose of making the distinction clear. The logician is interested in all reasoning, regardless of its subject matter, but only from this special point of view.

1.2 The Nature of Argument

Inferring is an activity in which one proposition is affirmed on the basis of one or more other propositions accepted as the starting point of the process. The logician is not concerned with the *process* of inference, but with the propositions that are the initial and end points of that process, and the relationships between them.

Propositions are either true or false, and in this they differ from questions, commands, and exclamations. Grammarians classify the linguistic formulations of propositions, questions, commands, and exclamations as declarative, interrogative, imperative, and exclamatory sentences, respectively. These are familiar notions. It is customary to distinguish between declarative sentences and the propositions they may be uttered to assert. The distinction is brought out clearly by remarking that a declarative sentence is always part of a language, the language in which it is spoken or written, whereas propositions are not peculiar to any of the languages in which they may be expressed. Another difference between them is that the same sentence may be uttered in different contexts to assert different propositions. (For example, the sentence 'I am hungry' may be uttered by different persons to make different assertions.) The same sort of distinction can be drawn between sentences and *statements*. The same statement can be made using different words, and the same sentence can be uttered in different contexts to make different statements. The terms 'proposition' and 'statement' are not exact synonyms, but in the writings of logicians they are used in much the same sense. In this book both terms will be used. In the following chapters we will also use the term 'statement' (especially in Chapters 2 and 3) and the term 'proposition' (especially in Chapters 4 and 5) to refer to the sentences in which statements (and propositions) are expressed. In each case the context should make clear what is meant.

Corresponding to every possible inference is an *argument*, and it is with these arguments that logic is chiefly concerned. An argument may be defined as any group of propositions or statements of which one is claimed to follow from the others, which are regarded as grounds for the truth of that one. In ordinary usage the word 'argument' also has other meanings, but in logic it has the technical sense explained. In the following chapters we will use the word 'argument' also in a derivative sense to refer to any sentence or collection of sentences in which an argument is formulated or expressed. When we do we will be presupposing that the context is sufficiently clear to ensure that unique statements are made or unique propositions are asserted by the utterance of those sentences.

Every argument has a structure, in the analysis of which the terms 'premiss' and 'conclusion' are usually employed. The *conclusion* of an argument is that proposition which is affirmed on the basis of the other propositions of the argument, and these other propositions which are affirmed as providing grounds or reasons for accepting the conclusion are the *premisses* of that argument.

We note that 'premiss' and 'conclusion' are relative terms, in the sense that the same proposition can be a premiss in one argument and conclusion in another. Thus the proposition *All men are mortal* is premiss in the argument

> All men are mortal.
> Socrates is a man.
> Therefore Socrates is mortal.

and conclusion in the argument

> All animals are mortal.
> All men are animals.
> Therefore all men are mortal.

Any proposition can be either a premiss or a conclusion, depending upon its context. It is a premiss when it occurs in an argument in which it is assumed for the sake of proving some other proposition. And it is a conclusion when it occurs in an argument which is claimed to prove it on the basis of other propositions which are assumed.

It is customary to distinguish between *deductive* and *inductive* arguments. All arguments involve the claim that their premisses provide some grounds for the truth of their conclusions, but only a *deductive* argument involves the claim that its premisses provide *absolutely conclusive* grounds. The technical terms 'valid' and 'invalid' are used in place of 'correct' and 'incorrect' in characterizing deductive arguments. A deductive argument is *valid* when its premisses and conclusion are so related that it is absolutely impossible for the premisses to be true unless the conclusion is true also. The task of deductive logic is to clarify the nature of the relationship which holds between premisses and conclusion in a valid argument, and to provide techniques for discriminating the valid from the invalid.

Inductive arguments involve the claim only that their premisses provide *some* grounds for their conclusions. Neither the term 'valid' nor its opposite 'invalid' is properly applied to inductive arguments. Inductive arguments differ among themselves in the degree of likelihood or probability which their premisses confer upon their conclusions, and are studied in inductive logic. But in this book we shall be concerned only with deductive arguments, and shall use the word 'argument' to refer to deductive arguments exclusively.

1.3 Truth and Validity

Truth and falsehood characterize propositions or statements, and may derivatively be said to characterize the declarative sentences in which they are formulated. But arguments are not properly characterized as being either true or false. On the other hand, validity and invalidity characterize arguments rather than propositions or statements.[2] There is a connection between the validity or invalidity of an argument and the truth or falsehood of its premisses and conclusion, but the connection is by no means a simple one.

Some valid arguments contain true propositions only, as, for example,

All bats are mammals.
All mammals have lungs.
Therefore all bats have lungs.

But an argument may contain false propositions exclusively, and be valid nevertheless, as, for example,

All trout are mammals.
All mammals have wings.
Therefore all trout have wings.

This argument is valid because *if* its premisses were true its conclusion would have to be true also, even though in fact they are all false. These two examples show that although some valid arguments have true conclusions, not all of them do. The validity of an argument does not guarantee the truth of its conclusion.

When we consider the argument

If I am President then I am famous.
I am not President.
Therefore I am not famous.

[2]Some logicians use the term 'valid' to characterize statements which are *logically true*, as will be explained in Chapter 9, Section 9.6. For the present, however, we apply the terms 'valid' and 'invalid' to arguments exclusively.

we can see that although both premisses and conclusion are true, it is invalid. Its invalidity is made obvious by comparing it with another argument of the same form:

> If Rockefeller is President then he is famous.
> Rockefeller is not President.
> Therefore Rockefeller is not famous.

This argument is clearly invalid, since its premisses are true but its conclusion false. The two latter examples show that although some invalid arguments have false conclusions, not all of them do. The falsehood of its conclusion does not guarantee the invalidity of an argument. But the falsehood of its conclusion does guarantee that *either* the argument is invalid *or* at least one of its premisses is false.

There are two conditions that an argument must satisfy to establish the truth of its conclusion. It must be valid, and all of its premisses must be true. The logician is concerned with only one of those conditions. To determine the truth or falsehood of premisses is the task of scientific inquiry in general, since premisses may deal with any subject matter at all. But determining the validity or invalidity of arguments is the special province of deductive logic. The logician is interested in the question of validity even for arguments whose premisses might happen to be false.

A question might be raised about the legitimacy of that interest. It might be suggested that we ought to confine our attention to arguments having true premisses only. But it is often necessary to depend upon the validity of arguments whose premisses are not known to be true. Modern scientists investigate their theories by deducing conclusions from them which predict the behavior of observable phenomena in the laboratory or observatory. The conclusion is then tested directly by observation, and if it is true, this tends to confirm the theory from which it was deduced, whereas if it is false, this disconfirms or refutes the theory. In either case, the scientist is vitally interested in the validity of the argument by which the testable conclusion is deduced from the theory being investigated; for if that argument is invalid his whole procedure is without point. The foregoing is an oversimplified account of scientific method, but it serves to show that questions of validity are important even for arguments whose premisses are not true.

1.4 Symbolic Logic

It has been explained that logic is concerned with arguments, and that these contain propositions or statements as their premisses and conclusions. The latter are not linguistic entities, such as declarative sentences, but rather what declarative sentences are typically uttered to assert. However, the communication of propositions and arguments requires the use of language, and this complicates our problem. Arguments formulated in English or any other natural language are often difficult to appraise because of the vague and equivocal nature of the words in which they are expressed, the ambiguity of their construction, the misleading idioms they may contain, and their pleasing but deceptive metaphorical style. The resolution of these difficulties is not the central problem for the logician, however, for even when they are resolved, the problem of deciding the validity or invalidity of the argument remains.

To avoid the peripheral difficulties connected with ordinary language, workers in the various sciences have developed specialized technical vocabularies. The scientist economizes the space and time required for writing his reports and theories by adopting special symbols to express ideas which would otherwise require a long sequence of familiar words to formulate. This has the further advantage of reducing the amount of attention needed, for when a sentence or equation grows too long its meaning is more difficult to grasp. The introduction of the exponent symbol in mathematics permits the expression of the equation

$$A \times A \times A \times A \times A \times A \times A \times A \times A \times A \times A \times A = B \times B \times B \times B \times B \times B \times B$$

more briefly and intelligibly as

$$A^{12} = B^{7}$$

A like advantage has been obtained by the use of graphic formulas in organic chemistry. And the language of every advanced science has been enriched by similar symbolic innovations.

Logic, too, has had a special technical notation developed for it. Aristotle made use of certain abbreviations to facilitate his own investigations, and modern symbolic logic has grown by the introduction of many more special symbols. The difference between the old and the new logic is one of degree rather than of kind, but the difference in degree is tremendous. Modern symbolic logic has become immeasurably more powerful a tool for analysis and deduction through the development of its own technical language. The special symbols of modern logic permit us to exhibit with greater clarity the logical structures of arguments which may be obscured by their formulation in ordinary language. It is an easier task to divide arguments into the valid and the invalid when they are expressed in a special symbolic language, for in it the peripheral problems of vagueness, ambiguity, idiom, metaphor, and amphiboly do not arise. The introduction and use of special symbols serve not only to facilitate the appraisal of arguments, but also to clarify the nature of deductive inference.

The logician's special symbols are much better adapted than ordinary language to the actual drawing of inferences. Their superiority in this respect is comparable to that enjoyed by Arabic numerals over the older Roman kind for purposes of computation. It is easy to multiply 148 by 47, but very difficult to compute the product of CXLVIII and XLVII. Similarly, the drawing of inferences and the evaluation of arguments is greatly facilitated by the adoption of a special logical notation. To quote Alfred North Whitehead, an important contributor to the advance of symbolic logic:

> . . . by the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play the higher faculties of the brain.[3]

[3] *An Introduction to Mathematics* by A. N. Whitehead, Oxford, Eng., Oxford University Press, 1911.

2

Arguments Containing Compound Statements

2.1 Simple and Compound Statements

All statements can be divided into two kinds, simple and compound. A *simple* statement is one which does not contain any other statement as a component part, whereas every *compound* statement does contain another statement as a component part. For example, 'Atmospheric testing of nuclear weapons will be discontinued or this planet will become uninhabitable' is a compound statement that contains as its components the two simple statements 'Atmospheric testing of nuclear weapons will be discontinued' and 'this planet will become uninhabitable'. The component parts of a compound statement may themselves be compound, of course. We turn now to some of the different ways in which statements can be combined into compound statements.

The statement 'Roses are red and violets are blue' is a *conjunction*, a compound statement formed by inserting the word 'and' between two statements. Two statements so combined are called *conjuncts*. The word 'and' has other uses, however, as in the statement 'Castor and Pollux were twins', which is not compound, but a simple statement asserting a relationship. We introduce the dot '$\cdot$' as a special symbol for combining statements conjunctively. Using it, the preceding conjunction is written 'Roses are red $\cdot$ violets are blue'. Where p and q are any two statements whatever, their conjunction is written $p \cdot q$.

Every statement is either true or false, so we can speak of the *truth value* of a statement, where the truth value of a true statement is *true* and the truth value of a false statement is *false*. There are two broad categories into which compound statements can be divided, according to whether or not there is any necessary connection between the truth value of the compound statement and the truth values of its component statements. The truth value of the compound statement 'Smith believes that lead is heavier than zinc' is completely independent of the truth value of its component simple statement 'lead is heavier than zinc', for people have mistaken as well as correct beliefs. On the other hand, there is a necessary connection between the truth value of a conjunction and the truth values of its conjuncts. A conjunction is true if both its conjuncts are true, but false otherwise. Any compound statement whose truth value is completely determined by the truth values of its component statements is a *truth-functionally* compound statement. The only compound statements we shall consider here will be truth-functionally compound statements. Therefore in the rest of this book we shall use the term 'simple statement' to refer to any statement that is not truth-functionally compound.

Since conjunctions are truth-functionally compound statements, our symbol is a truth-functional connective. Given any two statements p and q there are just four possible sets of truth values they can have, and in every case the truth value of their conjunction $p \cdot q$ is uniquely determined. The four possible cases can be exhibited as follows:

in case p is true and q is true, $p \cdot q$ is true;
in case p is true and q is false, $p \cdot q$ is false;
in case p is false and q is true, $p \cdot q$ is false;
in case p is false and q is false, $p \cdot q$ is false.

Representing the truth values true and false by the capital letters '**T**' and '**F**', respectively, the way in which the truth value of a conjunction is determined by the truth values of its conjuncts can be displayed more briefly by means of a *truth table* as follows:

p	q	$p \cdot q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Since it specifies the truth value of $p \cdot q$ in every possible case, this truth table can be taken as *defining* the dot symbol. Other English words such as 'moreover', 'furthermore', 'but', 'yet', 'still', 'however', 'also', 'nevertheless', 'although', etc., and even the comma and the semicolon, are also used to conjoin two statements into a single compound one, and all of them can be indifferently translated into the dot symbol so far as truth values are concerned.

The statement 'It is not the case that lead is heavier than gold' is also compound, being the *negation* (or *denial* or *contradictory*) of its single component statement 'lead is heavier than gold'. We introduce the symbol '$\sim$', called a *curl* (or a *tilde*) to symbolize negation. There are often alternative formulations in English of the negation of a given statement. Thus where L symbolizes the statement 'lead is heavier than gold', the different statements 'it is not the case that lead is heavier than gold', 'it is false that lead is heavier than gold', 'it is not true that lead is heavier than gold', 'lead is not heavier than gold' are all indifferently symbolized as $\sim L$. More generally, where p is any statement whatever, its negation is written $\sim p$. Since the negation of a true statement is false and the negation of a false statement is true, we can take the following truth table as defining the curl symbol:

p	$\sim p$
T	F
F	T

When two statements are combined disjunctively by inserting the word 'or' between them, the resulting compound statement is a *disjunction* (or *alternation*), and the two statements so combined are called *disjuncts* (or *alternatives*). The word 'or' has two different senses, one of which is clearly intended in the statement 'Premiums will be waived in the event of sickness or unemployment'. The intention here is obviously that premiums are waived not only for sick persons and for unemployed persons, but also for persons who are both sick *and* unemployed. This sense of the word 'or' is called *weak* or *inclusive*. Where precision is at a premium, as in contracts and other legal documents, this sense is made explicit by use of the phrase 'and/or'.

A different sense of 'or' is intended when a restaurant lists 'tea or coffee' on its table d'hôte menu, meaning that for the stated price of the meal the customer can have one or the other, but *not both*. This second sense of 'or' is called *strong* or *exclusive*. Where precision is at a premium and the exclusive sense of 'or' is intended, the phrase 'but not both' is often added.

A disjunction which uses the inclusive 'or' asserts that *at least one disjunct is true*, while one which uses the exclusive 'or' asserts that *at least one disjunct is true but not both are true*. The *partial common meaning*, that at least one disjunct is true, is the whole meaning of an inclusive disjunction, and a part of the meaning of an exclusive disjunction.

In Latin, the word 'vel' expresses the inclusive sense of the word 'or', and the word 'aut' expresses the exclusive sense. It is customary to use the first letter of 'vel' to symbolize 'or' in its inclusive sense. Where p and q are any two statements whatever, their weak or inclusive disjunction is written $p \vee q$. The symbol 'v', called a *wedge* (or a *vee*), is a truth-functional connective, and is defined by the following truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

An obviously valid argument containing a disjunction is the following Disjunctive Syllogism:

> The United Nations will be strengthened or there will be a third world war.
> The United Nations will not be strengthened.
> Therefore there will be a third world war.

It is evident that a Disjunctive Syllogism is valid on *either* interpretation of the word 'or', that is, regardless of whether its first premiss asserts an inclusive or exclusive disjunction. It is usually difficult, and sometimes impossible, to discover which sense of the word 'or' is intended in a disjunction. But the typical valid argument that has a disjunction for a premiss is, like the Disjunctive Syllogism, valid on either interpretation of the word 'or'. Hence we effect a simplification by translating any occurrence of the word 'or' into the logical symbol 'v'—*regardless of which sense of 'or' is intended.* Of course where it is explicitly stated that the disjunction is exclusive, by use of the added phrase 'but not both', for example, we do have the symbolic apparatus for symbolizing that sense, as will be explained below.

The use of parentheses, brackets, and braces for punctuating mathematical expressions is familiar. No number is uniquely denoted by the expression '$6 + 9 \div 3$', although when punctuation makes clear how its constituents are to be grouped, it denotes either 5 or 9. Punctuation is needed to resolve ambiguity in the language of symbolic logic too, since compound statements may themselves be combined to yield more complicated compounds. Ambiguity is present in $p \cdot q \vee r$, which could be either the conjunction of p with $q \vee r$, or else the disjunction of $p \cdot q$ with r. These two different senses are unambiguously given by different punctuations: $p \cdot (q \vee r)$ and $(p \cdot q) \vee r$. In case p and q are both false and r is true, the first punctuated expression is false (since its first conjunct is false) but the second punctuated expression is true (since its second disjunct is true). Here a difference in punctuation makes all the difference between truth and falsehood. In symbolic logic, as in mathematics, we use parentheses, brackets, and braces for punctuation. To cut down on the number of punctuation marks required, however, we establish the symbolic convention that in any expression the curl will apply to the smallest component that the punctuation permits. Thus the ambiguity of $\sim p \vee q$, which might mean either $(\sim p) \vee q$ or $\sim(p \vee q)$, is resolved by our convention to mean the first of these, for the curl can (and therefore by our convention *does*) apply to the first component p rather than to the larger expression $p \vee q$.

The word 'either' has a variety of different uses in English. It has conjunctive force in 'The Disjunctive Syllogism is valid on either interpretation of the word 'or'.' It frequently serves merely to introduce the first disjunct in a disjunction, as in 'Either the United Nations will be strengthened or there will be a third world war'. Perhaps the most useful function of the word 'either' is to punctuate some compound statements. Thus the sentence

> More stringent anti-pollution measures will be enacted and the laws will be strictly enforced or the quality of life will be degraded still further.

can have its ambiguity resolved in one direction by placing the word 'either' at its beginning, or in the other direction by inserting the word 'either' right after the word 'and'. Such punctuation is effected in our symbolic language by parentheses. The ambiguous formula $p \cdot q \vee r$ discussed in the preceding paragraph corresponds to the ambiguous sentence considered in this one. The two different punctuations of the formula correspond to the two different punctuations of the sentence effected by the two different insertions of the word 'either'.

Not all conjunctions are formulated by explicitly placing the word 'and' between complete sentences, as in 'Charlie's neat and Charlie's sweet'. Indeed the latter would more naturally be expressed as 'Charlie's neat and sweet'. And the familiar 'Jack and Jill went up the hill' is the more natural way of expressing the conjunction 'Jack went up the hill and Jill went up the hill'. It is the same with disjunctions: 'Either Alice or Betty will be elected' expresses more briefly the proposition alternatively formulated as 'Either Alice will be elected or Betty will be elected'; and 'Charlene will be either secretary or treasurer' expresses somewhat more briefly the same proposition as 'Either Charlene will be secretary or Charlene will be treasurer'.

The negation of a disjunction is often expressed by using the phrase 'neither-nor'. Thus the disjunction 'Either Alice or Betty will be elected' is denied by the statement 'Neither Alice nor Betty will be elected'. The disjunction would be symbolized as $A \vee B$ and its negation as either $\sim(A \vee B)$ or as $(\sim A)\cdot(\sim B)$. (The logical equivalence of these two formulas will be discussed in Section 2.4.) To deny that at least one of two statements is true is to assert that both of the two statements are false.

The word 'both' serves various functions. One is simply a matter of emphasis. To say 'Both Jack and Jill went up the hill' is only to emphasize that the two of them did what they are asserted to have done by saying 'Jack and Jill went up the hill'. A more useful function of the word 'both' is punctuational, like that of 'either'. 'Both . . . and -·- are not ---' is used to make the same statement as 'Neither . . . nor -·- is ---. In such sentences the *order* of the words 'both' and 'not' is very significant. There is a great difference between

> Alice and Betty will not both be elected.

and

> Alice and Betty will both not be elected.

The former would be symbolized as $\sim(A\cdot B)$, the latter as $(\sim A)\cdot(\sim B)$.

Finally, it should be remarked that the word 'unless' can also be used in expressing the disjunction of two statements. Thus 'Our resources will soon be exhausted unless more recycling of materials is effected' and 'Unless more recycling of materials is effected our resources will soon be exhausted' can equally well be expressed as 'Either more recycling of materials is effected or our resources will soon be exhausted' and symbolized as $M \vee E$.

Since an exclusive disjunction asserts that at least one of its disjuncts is true but they are not both true, we can symbolize the exclusive disjunction of any two statements p and q quite simply as $(p \vee q) \cdot \sim(p \cdot q)$. Thus we are able to symbolize conjunctions, negations, and both inclusive and exclusive disjunctions. Any compound statement which is built up out of simple statements by repeated use of truth-functional connectives will have its truth value completely determined by the truth values of those simple statements. For example, if A and B are true statements and X and Y are false, the truth value of the compound statement $\sim[(\sim A \vee X) \vee \sim(B \cdot Y)]$ can be discovered as follows. Since A is true, $\sim A$ is false, and since X is false also, the disjunction $(\sim A \vee X)$ is false. Since Y is false, the conjunction $(B \cdot Y)$ is false, and so its negation $\sim(B \cdot Y)$ is true. Hence the disjunction $(\sim A \vee X) \vee \sim(B \cdot Y)$ is true, and its negation, which is the original statement, is false. Such a stepwise procedure, beginning with the inmost components, always permits us to determine the truth value of a truth-functionally compound statement from the truth values of its component simple statements.

EXERCISES[1]

I. If A and B are true statements and X and Y are false statements, which of the following compound statements are true?

*1. $\sim(A \vee X)$
2. $\sim A \vee \sim X$
3. $\sim B \cdot \sim Y$
4. $\sim(B \cdot Y)$
*5. $A \vee (X \cdot Y)$
6. $(A \vee X) \cdot Y$
7. $(A \vee B) \cdot (X \vee Y)$
8. $(A \cdot B) \vee (X \cdot Y)$
9. $(A \cdot X) \vee (B \cdot Y)$
*10. $A \cdot [X \vee (B \cdot Y)]$
11. $A \vee [X \cdot (B \vee Y)]$
12. $X \vee [A \cdot (Y \vee B)]$
13. $\sim\{\sim[\sim(A \cdot \sim X) \cdot \sim A] \cdot \sim X\}$
14. $\sim\{\sim[\sim(A \cdot \sim B) \cdot \sim A] \cdot \sim A\}$
*15. $[(A \cdot X) \vee \sim B] \cdot \sim[(A \cdot X) \vee \sim B]$
16. $[(X \cdot A) \vee \sim Y] \vee \sim[(X \cdot A) \vee \sim Y]$
17. $[A \cdot (X \vee Y)] \vee \sim[(A \cdot X) \vee (A \cdot Y)]$
18. $[X \vee (A \cdot Y)] \vee \sim[(X \vee A) \cdot (X \vee Y)]$
19. $[X \cdot (A \vee B)] \vee \sim[(X \vee A) \cdot (X \vee B)]$
20. $[X \vee (A \cdot Y)] \vee \sim[(X \vee A) \vee (X \vee Y)]$

II. Using the letters A, B, C, and D to abbreviate the simple statements: 'Atlanta wins their conference championship', 'Baltimore wins their conference championship', 'Chicago wins the superbowl', and 'Dallas wins the superbowl', symbolize the following:

*1. Either Atlanta wins their conference championship and Baltimore wins their conference championship or Chicago wins the superbowl.
2. Atlanta wins their conference championship and either Baltimore wins their conference championship or Dallas does not win the superbowl.
3. Atlanta and Baltimore will not both win their conference championships but Chicago and Dallas will both not win the superbowl.
4. Either Atlanta or Baltimore will win their conference championships but neither Chicago nor Dallas will win the superbowl.
*5. Either Chicago or Dallas will win the superbowl but they will not both win the superbowl.

[1]Solutions to starred exercises will be found on pages 311–335.

6. Chicago will win the superbowl unless Atlanta wins their conference championship.
7. It is not the case that neither Atlanta nor Baltimore wins their conference championship.
8. Either Chicago or Dallas will fail to win the superbowl.
9. Either Chicago or Dallas will win the superbowl unless both Atlanta and Baltimore win their conference championships.
10. Either Chicago will win the superbowl and Dallas will not win the superbowl or both Atlanta and Baltimore will win their conference championships.

III. Using capital letters to abbreviate simple statements, symbolize the following:

*1. The words of his mouth were smoother than butter, but war was in his heart. (Psalm 55:21)
2. Promotion cometh neither from the east, nor from the west, nor yet from the south. (Psalm 75:6)
3. As for man, his days are as grass: as a flower of the field, so he flourisheth. (Psalm 103:15)
4. Wine is a mocker, strong drink is raging. (Proverbs 20:1)
*5. God hath made man upright; but they have sought out many inventions. (Ecclesiastes 7:29)
6. The race is *not* to the swift, nor the battle to the strong . . . (Ecclesiastes 9:11)
7. Love is strong as death; jealousy is cruel as the grave. (The Song of Solomon 8:6)
8. A bruised reed shall he not break, and the smoking flax shall he not quench. (Isaiah 42:3)
9. Saul and Jonathan were lovely and pleasant in their lives . . . (2 Samuel 1:23)
10. His eye was not dim, nor his natural force abated. (Deuteronomy 34:7)
11. The voice is Jacob's voice, but the hands are the hands of Esau. (Genesis 27:22)
12. He shall return no more to his house, neither shall his place know him any more. (Job 7:10)

2.2 Conditional Statements

The compound statement 'If the train is late then we shall miss our connection' is a *conditional* (or a *hypothetical*, an *implication*, or an *implicative statement*). The component between the 'if' and the 'then' is called the *antecedent* (or the *implicans* or *protasis*), and the component which follows the 'then' is the *consequent* (or the *implicate* or *apodosis*). A conditional does not assert either that its antecedent is true or that its consequent is true; it asserts only that *if* its antecedent is true then its consequent is true also, that is, that its antecedent *implies* its consequent. The key to the meaning of a conditional is the relation of *implication* asserted to hold between its antecedent and consequent, in that order.

If we examine a number of different conditionals we can see that there are different kinds of implications they may assert. In the conditional 'If all cats like liver and Dinah is a cat then Dinah likes liver', the consequent follows *logically* from its antecedent. On the other hand, in the conditional 'If the figure is a triangle then it has three sides', the consequent follows from the antecedent by the very *definition* of the word 'triangle'. But the truth of the conditional 'If gold is placed in *aqua regia* then the gold dissolves' is not a matter of either logic or definition. The connection asserted here is *causal*, and must be discovered empirically. These examples show that there are different kinds of implications which constitute different senses of the 'if-then' phrase. Having noted these differences, we now seek to find some identifiable common meaning, some partial meaning that is common to these admittedly different types of conditionals.

Our discussion of 'if-then' will parallel our previous discussion of the word 'or'. First, we pointed out two different senses of that word. Second, we noted that there was a common partial meaning: that *at least one disjunct is true* was seen to be involved in both the inclusive and the exclusive 'or'. Third, we introduced the special symbol 'v' to represent this common partial meaning (which was the whole meaning of 'or' in its inclusive sense). Fourth, we observed that since arguments like the Disjunctive Syllogism are valid on either interpretation of the word 'or', symbolizing *any* occurrence of the word 'or' by the wedge symbol preserves the validity of such arguments. And since we are interested in arguments from the point of view of determining their validity, this translation of the word 'or' into 'v', which may abstract from or ignore part of its meaning in some cases, is wholly adequate for our present purposes.

A common partial meaning of these different kinds of conditional statements emerges when we ask what circumstances would suffice to establish the *falsehood* of a conditional. Under what circumstances would we agree that the conditional 'If gold is placed in this solution then the gold dissolves' is false? Clearly the statement is false in case gold is actually placed in this solution and *does not dissolve*. Any conditional with a true antecedent and a false consequent must be false. Hence any conditional *if* p *then* q is known to be false in case the conjunction $p \cdot \sim q$ is known to be true, that is, in case its antecedent is true and its consequent false. For the conditional to be true, the indicated conjunction must be false, which means that the negation of that conjunction must be true. In other words, for any conditional *if* p *then* q to be true, $\sim(p \cdot \sim q)$, the negation of the conjunction of its antecedent with the negation of its consequent, must be true also. We may, then, regard the latter as a *part* of the meaning of the former.

We introduce the new symbol '$\supset$', called a *horseshoe*, to represent the partial meaning common to all conditional statements, defining '$p \supset q$' as an abbreviation for '$\sim(p \cdot \sim q)$'. The horseshoe is a truth-functional connective, whose exact significance is indicated by the following truth table:

p	q	$\sim q$	$p \cdot \sim q$	$\sim(p \cdot \sim q)$	$p \supset q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Here the first two columns represent all possible truth values for the component statements p and q, and the third, fourth, and fifth represent successive stages in determining the truth value of the compound statement $\sim(p \cdot \sim q)$ in each case. The sixth column is identically the same as the fifth since the formulas which head them are defined to express the same proposition. The horseshoe symbol must not be thought of as representing *the* meaning of 'if-then', or *the* relation of implication, but rather a common partial factor of the various different kinds of implications signified by the 'if-then' phrase.

We can regard the horseshoe as symbolizing a special, extremely weak kind of implication, and it is expedient for us to do so, since convenient ways to read '$p \supset q$' are 'if p then q', 'p implies q', or 'p only if q'. The weak implication symbolized by '$\supset$' is called *material implication*, and its special name indicates that it is a special notion, not to be confused with the other more usual kinds of implication. Some conditional statements in English do assert merely material implications, as for example 'If Russia is a democracy then I'm a Dutchman'. It is clear that the implication asserted here is neither logical, definitional, nor causal. No 'real connection' is alleged to hold between what the antecedent asserts and what is asserted by the consequent. This sort of conditional is ordinarily intended as an emphatic or humorous method of denying the truth of its antecedent, for it typically contains a notoriously or ridiculously false statement as consequent. Any such assertion about truth values is adequately symbolized using the truth-functional connective '$\supset$'.

Although most conditional statements assert more than a merely material implication between antecedent and consequent, we now propose to symbolize *any* occurrence of 'if-then' by the truth-functional connective '$\supset$'. It must be admitted that such symbolizing abstracts from or ignores part of the meaning of most conditional statements. But the proposal can be justified on the grounds that the validity of valid arguments involving conditionals is preserved when the conditionals are regarded as asserting material implications only, as will be established in the following section.

Conditional statements can be expressed in a variety of ways. A statement of the form 'if p then q' could equally well be expressed as 'if p, q', 'q if p', 'that p implies that q', 'that p entails that q', 'p only if q', 'that p is a sufficient condition that q', or as 'that q is a necessary condition that p', and any of these formulations will be symbolized as $p \supset q$.

EXERCISES

I. If A and B are true statements and X and Y are false statements, which of the following compound statements are true?

*1. $X \supset (X \supset Y)$
2. $(X \supset X) \supset Y$
3. $(A \supset X) \supset Y$
4. $(X \supset A) \supset Y$
*5. $A \supset (B \supset Y)$
6. $A \supset (X \supset B)$
7. $(X \supset A) \supset (B \supset Y)$
8. $(A \supset X) \supset (Y \supset B)$
9. $(A \supset B) \supset (\sim A \supset \sim B)$
*10. $(X \supset Y) \supset (\sim X \supset \sim Y)$
11. $(X \supset A) \supset (\sim X \supset \sim A)$
12. $(X \supset \sim Y) \supset (\sim X \supset Y)$
13. $[(A \cdot X) \supset Y] \supset (A \supset Y)$
14. $[(A \cdot B) \supset X] \supset [A \supset (B \supset X)]$
*15. $[(X \cdot Y) \supset A] \supset [X \supset (Y \supset A)]$
16. $[(A \cdot X) \supset B] \supset [A \supset (B \supset X)]$
17. $[X \supset (A \supset Y)] \supset [(X \supset A) \supset Y]$
18. $[X \supset (X \supset Y)] \supset [(X \supset X) \supset X]$
19. $[(A \supset B) \supset A] \supset A$
20. $[(X \supset Y) \supset X] \supset X$

II. Symbolizing 'Amherst wins its first game' as A, 'Colgate wins its first game' as C, and 'Dartmouth wins its first game' as D, symbolize the following compound statements:

*1. Both Amherst and Colgate win their first games only if Dartmouth does not win its first game.
2. Amherst wins its first game if either Colgate wins its first game or Dartmouth wins its first game.
3. If Amherst wins its first game then both Colgate and Dartmouth win their first games.
4. If Amherst wins its first game then either Colgate or Dartmouth wins its first game.
*5. If Amherst does not win its first game then it is not the case that either Colgate or Dartmouth wins its first game.
6. If it is not the case that both Amherst and Colgate win their first games then both Colgate and Dartmouth win their first games.
7. If Amherst wins its first game then not both Colgate and Dartmouth win their first games.
8. If Amherst does not win its first game then both Colgate and Dartmouth do not win their first games.
9. Either Amherst wins its first game and Colgate does not win its first game or if Colgate wins its first game then Dartmouth does not win its first game.
*10. If Amherst wins its first game then Colgate does not win its first game, but if Colgate does not win its first game then Dartmouth wins its first game.
11. If Amherst wins its first game then if Colgate does not win its first game then Dartmouth wins its first game.
12. Either Amherst and Colgate win their first games or it is not the case that if Colgate wins its first game then Dartmouth wins its first game.
13. Amherst wins its first game only if either Colgate or Dartmouth does not win its first game.
14. If Amherst wins its first game only if Colgate wins its first game, then Dartmouth does not win its first game.
15. If Amherst and Colgate both do not win their first games, then Amherst and Colgate do not both win their first games.

2.3 Argument Forms and Truth Tables

In this section we develop a purely mechanical method for testing the validity of arguments containing truth-functionally compound statements. That method is closely related to the familiar technique of *refutation by logical analogy* which was used in the first chapter to show the invalidity of the argument

> If I am President then I am famous.
> I am not President.
> Therefore I am not famous.

That argument was shown to be invalid by constructing another argument of the same form:

> If Rockefeller is President then he is famous.
> Rockefeller is not President.
> Therefore Rockefeller is not famous.

which is obviously invalid since its premisses are true but its conclusion false. Any argument is proved invalid if another argument *of exactly the same form* can be constructed with true premisses and a false conclusion. This reflects the fact that validity and invalidity are purely *formal* characteristics of arguments: any two arguments having the same form are either both valid or both invalid, regardless of any differences in their subject matter.[2] The notion of two arguments having *exactly the same form* is one that deserves further examination.

It is convenient in discussing forms of arguments to use small letters from the middle part of the alphabet, 'p', 'q', 'r', 's', ... as *statement variables*, which are defined simply to be letters for which, or in place of which, statements may be substituted. Now we define an *argument form* to be any array of symbols which contains statement variables, such that when statements are substituted for the statement variables—the same statement replacing the same statement variable throughout—the result is an argument. For definiteness, we establish the convention that in any argument form, 'p' shall be the first statement variable that occurs in it, 'q' shall be the second, 'r' the third, and so on.

Any argument which results from the substitution of statements for the statement variables of an argument form is said to *have* that form, or to be a *substitution instance* of that argument form. If we symbolize the simple

[2] Here we assume that the simple statements involved are neither logically true (e.g. 'All equilateral triangles are triangles') nor logically false (e.g. 'Some triangles are nontriangles'). We assume also that the only logical relations among the simple statements involved are those asserted or entailed by the premisses. The point of these restrictions is to limit our considerations in Chapters 2 and 3 to truth-functional arguments alone, and to exclude other kinds of arguments whose validity turns on more complex logical considerations to be introduced in Chapters 4 and 5.

statement 'The United Nations will be strengthened' as U, and the simple statement 'There will be a third world war' as W, then the Disjunctive Syllogism presented earlier can be symbolized as

(1)
$$\begin{array}{l} U \vee W \\ \sim U \\ \therefore W \end{array}$$

It has the form

(2)
$$\begin{array}{l} p \vee q \\ \sim p \\ \therefore q \end{array}$$

from which it results by replacing the statement variables p and q by the statements U and W, respectively. But that is not the only form of which it is a substitution instance. The same argument is obtained by replacing the statement variables p, q, and r in the argument form

(3)
$$\begin{array}{l} p \\ q \\ \therefore r \end{array}$$

by the statements $U \vee W$, $\sim U$, and W, respectively. We define *the specific form* of a given argument as that argument form from which the argument results by replacing each distinct statement variable by a different *simple* statement. Thus the specific form of the argument (1) is the argument form (2). Although the argument form (3) is *a* form of the argument (1), it is not *the specific form* of it. The technique of refutation by logical analogy can now be described more precisely. If the specific form of a given argument can be shown to have any substitution instance with true premisses and false conclusion, then the given argument is invalid.

The terms 'valid' and 'invalid' can be extended to apply to argument forms as well as arguments. An *invalid* argument form is one which has at least one substitution instance with true premisses and a false conclusion. The technique of refutation by logical analogy presupposes that any argument of which the specific form is an invalid argument form is an invalid argument. Any argument form is *valid* which is not invalid; a *valid* argument form is one which has no substitution instance with true premisses and false conclusion. Any given argument can be proved valid if it can be shown that the specific form of the given argument is a valid argument form.

To determine the validity or invalidity of an argument form we must examine all possible substitution instances of it to see if any of them have true premisses and false conclusions. The arguments with which we are here concerned contain only simple statements and truth-functional compounds of them, and we are interested only in the truth values of their premisses and conclusions. We can obtain all possible substitution instances whose premisses and conclusions have different truth values, by considering all possible arrangements of truth values for the statements substituted for the distinct statement variables in the argument form to be tested. These can be set forth most conveniently in the form of a truth table, with an initial or guide column for each distinct statement variable appearing in the argument form. Thus to prove the validity of the Disjunctive Syllogism form

$$p \vee q$$
$$\sim p$$
$$\therefore q$$

we construct the following truth table:

p	q	$p \vee q$	$\sim p$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	T

Each row of this table represents a whole class of substitution instances. The **T**'s and **F**'s in the two initial columns represent the truth values of statements which can be substituted for the variables p and q in the argument form. These determine the truth values in the other columns, the third of which is headed by the first 'premiss' of the argument form and the fourth by the second 'premiss'. The second column's heading is the 'conclusion' of the argument form. An examination of this truth table reveals that whatever statements are substituted for the variables p and q, the resulting argument cannot have true premisses and a false conclusion, for the third row represents the only possible case in which both premisses are true, and there the conclusion is true also.

Since truth tables provide a purely mechanical or *effective* method of deciding the validity or invalidity of any argument of the general type here considered, we can now justify our proposal to symbolize all conditional statements by means of the truth-functional connective '$\supset$'. The justification for treating all implications as though they were mere material implications is that valid arguments containing conditional statements remain valid when those conditionals are interpreted as asserting material implications only. The three simplest and most intuitively valid forms of argument involving conditional statements are

Modus Ponens If p then q
p
$\therefore q$

Modus Tollens If p then q
$\sim q$
$\therefore \sim p$

and the

Hypothetical Syllogism   If p then q
If q then r
$\therefore$ If p then r

That they all remain valid when their conditionals are interpreted as asserting material implications is easily established by truth tables. The validity of *Modus Ponens* is shown by the same truth table that defines the horseshoe symbol:

p	q	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

Here the two premisses are represented by the third and first columns, and the conclusion by the second. Only the first row represents substitution instances in which both premisses are true, and in that row the conclusion is true also. The validity of *Modus Tollens* is shown by the truth table:

p	q	$p \supset q$	$\sim q$	$\sim p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Here only the fourth row represents substitution instances in which both premisses (the third and fourth columns) are true, and there the conclusion (the fifth column) is true also. Since the Hypothetical Syllogism form contains three distinct statement variables, the truth table for it must have three initial columns and will require eight rows for listing all possible substitution instances:

p	q	r	$p \supset q$	$q \supset r$	$p \supset r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

In constructing it, the three initial columns represent all possible arrangements of truth values for the statements substituted for the statement variables p, q, and r, the fourth column is filled in by reference to the first and second, the fifth by reference to the second and third, and the sixth by reference to the first and third. The premisses are both true only in the first, fifth, seventh, and eighth rows, and in these rows the conclusion is true also. This suffices to demonstrate that the Hypothetical Syllogism remains valid when its conditionals are symbolized by means of the horseshoe symbol. Any doubts that remain about the claim that valid arguments containing conditionals remain valid when their conditionals are interpreted as asserting merely material implication can be allayed by the reader's providing, symbolizing, and testing his own examples by means of truth tables.

To test the validity of an argument form by a truth table requires one with a separate initial or guide column for each different statement variable, and a separate row for every possible assignment of truth values to the statement variables involved. Hence testing an argument form containing n distinct statement variables requires a truth table having 2^n rows. In constructing truth tables it is convenient to fix upon some uniform pattern for inscribing the **T**'s and **F**'s in their initial or guide columns. In this book we shall follow the practice of simply alternating **T**'s and **F**'s down the extreme right-hand initial column, alternating pairs of **T**'s with pairs of **F**'s down the column directly to its left, next alternating quadruples of **T**'s with quadruples of **F**'s, . . . , and finally filling in the top half of the extreme left-hand initial column with **T**'s and its bottom half with **F**'s.

There are two invalid argument forms that bear a superficial resemblance to the valid argument forms *Modus Ponens* and *Modus Tollens*. These are

$$\begin{array}{l} p \supset q \\ q \\ \therefore p \end{array} \qquad \text{and} \qquad \begin{array}{l} p \supset q \\ \sim p \\ \therefore \sim q \end{array}$$

and are known as the Fallacies of Affirming the Consequent and of Denying the Antecedent, respectively. The invalidity of both can be shown by a single truth table:

p	q	$p \supset q$	$\sim p$	$\sim q$
T	**T**	**T**	**F**	**F**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**T**	**F**
F	**F**	**T**	**T**	**T**

The two premisses in the Fallacy of Affirming the Consequent head the second and third columns, and are true in both the first and third rows. But the conclusion, which heads the first column, is false in the third row, which shows that the argument form does have a substitution instance with true premisses and a false conclusion, and is therefore invalid. Columns three and four are headed by the two premisses in the Fallacy of Denying the Antecedent, which are true in both the third and fourth rows. Its conclusion heads the fifth column, and is false in the third row, which shows that the second argument form is invalid also.

It must be emphasized that although a valid argument form has only valid arguments as substitution instances, an invalid argument form can have both valid and invalid substitution instances. So to prove that a given argument is invalid we must prove that *the specific form* of that argument is invalid.

EXERCISES

I. For each of the following arguments indicate which, if any, of the argument forms in Exercise II below have the given argument as a substitution instance, and indicate which, if any, is the specific form of the given argument:

*a. A
$\therefore A \vee B$

b. $C \cdot D$
$\therefore C$

c. $E \supset (F \cdot G)$
$\therefore \sim(F \cdot G) \supset \sim E$

d. H
I
$\therefore H \cdot I$

*e. $J \supset (K \cdot L)$
$J \vee (K \cdot L)$
$\therefore K \cdot L$

f. $M \supset (N \supset O)$
$O \supset \sim M$
$\therefore O \supset \sim N$

g. $(P \supset Q) \cdot (R \supset S)$
$\therefore P \supset Q$

h. $T \supset U$
$\therefore (T \supset U) \vee (V \cdot T)$

i. $W \supset X$
$\therefore X \supset (W \supset X)$

*j. $Y \vee (Z \cdot \sim Y)$
Y
$\therefore \sim(Z \cdot \sim Y)$

k. $(A \supset B) \cdot (C \supset D)$
$A \vee C$
$\therefore B \vee D$

l. $(E \supset F) \cdot (G \supset H)$
$\sim F \vee \sim G$
$\therefore \sim E \vee \sim H$

m. $I \supset J$
$\therefore (I \supset J) \supset (I \supset J)$

n. $K \supset (L \supset M)$
$K \supset L$
$\therefore K \supset M$

o. $N \supset (N \supset O)$
$N \supset N$
$\therefore N \supset O$

II. Use truth tables to determine the validity or invalidity of each of the following argument forms:

*1. $p \cdot q$
$\therefore p$

2. p
$\therefore p \cdot q$

3. $p \vee q$
$\therefore p$

4. p
$\therefore p \vee q$

*5. p
$\therefore p \supset q$

6. p
$\therefore q \supset p$

7. $p \supset q$
$\therefore \sim q \supset \sim p$

8. $p \supset q$
$\therefore \sim p \supset \sim q$

9. $p \supset (q \cdot r)$
$\therefore \sim(q \cdot r) \supset \sim p$

*10. $p \vee q$
p
$\therefore \sim q$

11. p
q
$\therefore p \cdot q$

12. $p \supset q$
$q \supset p$
$\therefore p \vee q$

13. $p \supset q$
    $p \vee q$
    $\therefore q$

14. $p \supset (q \supset r)$
    $p \supset q$
    $\therefore p \supset r$

*15. $(p \supset q) \cdot (p \supset r)$
    p
    $\therefore q \vee r$

16. $p \supset (q \vee r)$
    $p \supset \sim q$
    $\therefore p \vee r$

17. $(p \supset q) \cdot (r \supset s)$
    $p \vee r$
    $\therefore q \vee s$

18. $(p \supset q) \cdot (r \supset s)$
    $\sim q \vee \sim s$
    $\therefore \sim p \vee \sim r$

19. $(p \vee q) \supset (p \cdot q)$
    $p \cdot q$
    $\therefore p \vee q$

20. $p \vee (q \cdot \sim p)$
    p
    $\therefore \sim (q \cdot \sim p)$

21. $(p \vee q) \supset (p \cdot q)$
    $\sim (p \vee q)$
    $\therefore \sim (p \cdot q)$

III. Use truth tables to determine the validity or invalidity of each of the following arguments:

*1. If Alice is elected class president then either Betty is elected vice-president or Carol is elected treasurer. Betty is elected vice-president. Therefore if Alice is elected class president then Carol is not elected treasurer.

2. If Alice is elected class president then either Betty is elected vice-president or Carol is elected treasurer. Carol is not elected treasurer. Therefore if Betty is not elected vice-president then Alice is not elected class president.

3. If Alice is elected class president, then Betty is elected vice-president and Carol is elected treasurer. Betty is not elected vice-president. Therefore Alice is not elected class president.

4. If Alice is elected class president then if Betty is elected vice-president then Carol is elected treasurer. Betty is not elected vice-president. Therefore either Alice is elected class president or Carol is elected treasurer.

*5. If the seed catalog is correct then if the seeds are planted in April then the flowers bloom in July. The flowers do not bloom in July. Therefore if the seeds are planted in April then the seed catalog is not correct.

6. If the seed catalog is correct then if the seeds are planted in April then the flowers bloom in July. The flowers bloom in July. Therefore if the seed catalog is correct then the seeds are planted in April.

7. If the seed catalog is correct then if the seeds are planted in April then the flowers bloom in July. The seeds are planted in April. Therefore if the flowers do not bloom in July then the seed catalog is not correct.

8. If the seed catalog is correct then if the seeds are planted in April then the flowers bloom in July. The flowers do not bloom in July. Therefore if the seeds are not planted in April then the seed catalog is not correct.

9. If Ed wins first prize then Fred wins second prize, and if Fred wins second prize then George is disappointed. Either Ed wins first prize or George is disappointed. Therefore Fred does not win second prize.

*10. If Ed wins first prize then either Fred wins second prize or George is disappointed. Fred does not win second prize. Therefore if George is disappointed then Ed does not win first prize.

11. If Ed wins first prize then Fred wins second prize, and if Fred wins second prize then George is disappointed. Either Fred does not win second prize or George is not disappointed. Therefore Ed does not win first prize.

12. If Ed wins first prize then Fred wins second prize, and if Fred wins second prize then George is disappointed. Either Ed does not win first prize or

Fred does not win second prize. Therefore either Fred does not win second prize or George is not disappointed.

13. If the weather is warm and the sky is clear then we go swimming and we go boating. It is not the case that if the sky is clear then we go swimming. Therefore the weather is not warm.
14. If the weather is warm and the sky is clear then either we go swimming or we go boating. It is not the case that if the sky is clear then we go swimming. Therefore if we do not go boating then the weather is not warm.
*15. If the weather is warm and the sky is clear then either we go swimming or we go boating. It is not the case that if we do not go swimming then the sky is not clear. Therefore either the weather is warm or we go boating.

2.4 Statement Forms

The introduction of statement variables in the preceding section enabled us to define both argument forms in general and the specific form of a given argument. Now we define a *statement form* to be any sequence of symbols containing statement variables, such that when statements are substituted for the statement variables—the same statement replacing the same statement variable throughout—the result is a statement. Again for definiteness, we establish the convention that in any statement form 'p' shall be the first statement variable that occurs in it, 'q' shall be the second, 'r' the third, and so on. Any statement which results from substituting statements for the statement variables of a statement form is said to *have* that form, or to be a *substitution instance* of it. Just as we distinguished the specific form of a given argument, so we distinguish *the specific form* of a given statement as that statement form from which the given statement results by replacing each distinct statement variable by a different simple statement. For example, where A, B, and C are different simple statements, the compound statement $A \supset (B \vee C)$ is a substitution instance of the statement form $p \supset q$, and also of the statement form $p \supset (q \vee r)$, but only the latter is the specific form of the given statement.

Although the statements 'Balboa discovered the Pacific Ocean' (B) and 'Balboa discovered the Pacific Ocean or else he didn't' ($B \vee {\sim}B$) are both true, we discover their truth in quite different ways. The truth of B is a matter of history, and must be learned through empirical investigation. Moreover, events might possibly have been such as to make B false; there is nothing *necessary* about the truth of B. But the truth of the statement $B \vee {\sim}B$ *can* be known independently of empirical investigation, and no events could possibly have made it false, for it is a necessary truth. The statement $B \vee {\sim}B$ is a formal truth, a substitution instance of a statement form *all* of whose substitution instances are true. A statement form that has only true substitution instances is said to be *tautologous*, or a *tautology*. The specific form of $B \vee {\sim}B$ is $p \vee {\sim}p$, and is proved a tautology by the following truth table:

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

That there are only T's in the column headed by the statement form in question shows that all of its substitution instances are true. Any statement that is a substitution instance of a tautologous statement form is formally true, and is itself said to be tautologous, or a tautology.

Similarly, although the statements 'Cortez discovered the Pacific' (C) and 'Cortez discovered the Pacific and Cortez did not discover the Pacific' ($C \cdot \sim C$) are both false, we discover their falsehood in quite different ways. The first simply *happens* to be false, and that must be learned empirically; whereas the second is necessarily false, and that can be known independently of empirical investigation. The statement $C \cdot \sim C$ is formally false, a substitution instance of a statement form *all* of whose substitution instances are false. One statement is said to contradict, or to be a contradiction of, another statement when it is logically impossible for them both to be true. In this sense, *contradiction* is a relation between statements. But there is another, related sense of that term. When it is logically impossible for a particular statement to be true, that statement itself is said to be self-contradictory, or a self-contradiction. Such statements are also said more simply to be contradictory, or contradictions, and we shall follow the latter usage here. A statement form that has only false substitution instances is said to be *contradictory*, or a *contradiction*, and the same terms are applied to its substitution instances. The statement form $p \cdot \sim p$ is proved a contradiction by the fact that in its truth table only F's occur in the column which it heads.

Statements and statement forms which are neither tautologous nor contradictory are said to be *contingent*, or *contingencies*. For example, p, $\sim p$, $p \vee q$, $p \cdot q$, and $p \supset q$ are contingent statement forms; and B, C, $\sim B$, $\sim C$, $B \cdot C$, $B \vee C$ are contingent statements. The term is appropriate, since their truth values are not formally determined but are dependent or contingent upon what happens to be the case.

It is easily proved that $p \supset (q \supset p)$ and $\sim p \supset (p \supset q)$ are tautologies. When expressed in English as 'A true statement is implied by any statement whatever', and as 'A false statement implies any statement whatever', they seem rather strange. They have been called by some writers the *paradoxes of material implication*. But when it is kept in mind that the horseshoe symbol is a truth-functional connective which stands for *material* implication rather than either 'implication in general' or more usual kinds such as logical or causal, then the tautologous statement forms in question are not at all surprising. And when the misleading English formulations are corrected by inserting the word 'materially' before 'implied' and 'implies', then the air of paradox vanishes. Material implication is a special, technical notion, and the logician's motivation for introducing and using it is the tremendous extent to which it simplifies his task of discriminating valid from invalid arguments.

Two statements are said to be *materially equivalent* when they have the same truth value, and we symbolize the statement *that* they are materially equivalent by inserting the symbol '$\equiv$' between them. Being a truth-functional connective, the three bar symbol is defined by the following truth table:

p	q	$p \equiv q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

To say that two statements are materially equivalent is to say that they materially imply each other, as is easily verified by a truth table. Hence the three bar symbol may be read either 'is materially equivalent to' or 'if and only if'. A statement of the form $p \equiv q$ is called a *biconditional.* Two statements are said to be *logically equivalent* when the biconditional expressing their material equivalence is a tautology. The 'principle of Double Negation', expressed as $p \equiv {\sim}{\sim}p$ is proved to be tautologous by a truth table.

There are two logical equivalences which express important interrelations of conjunctions, disjunctions, and negations. Since a conjunction asserts that both its conjuncts are true, its negation need assert only that at least one is false. Thus negating the conjunction $p \cdot q$ amounts to asserting the disjunction of the negations of p and q. This statement of equivalence is symbolized as ${\sim}(p \cdot q) \equiv ({\sim}p \vee {\sim}q)$, and proved to be a tautology by the following truth table:

p	q	$p \cdot q$	${\sim}(p \cdot q)$	${\sim}p$	${\sim}q$	${\sim}p \vee {\sim}q$	${\sim}(p \cdot q) \equiv ({\sim}p \vee {\sim}q)$
T	**T**	**T**	**F**	**F**	**F**	**F**	**T**
T	**F**	**F**	**T**	**F**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**F**	**T**	**T**
F	**F**	**F**	**T**	**T**	**T**	**T**	**T**

Similarly, since a disjunction asserts merely that at least one disjunct is true, to negate it is to assert that both are false. Negating the disjunction $p \vee q$ amounts to asserting the conjunction of the negations of p and q. It is symbolized as ${\sim}(p \vee q) \equiv ({\sim}p \cdot {\sim}q)$, and is easily proved tautologous by a truth table. These two equivalences are known as De Morgan's Theorems, after the English mathematician-logician Augustus De Morgan (1806–1871), and can be stated compendiously in English as: The negation of the $\left\{\begin{matrix}\text{conjunction}\\ \text{disjunction}\end{matrix}\right\}$ of two statements is logically equivalent to the $\left\{\begin{matrix}\text{disjunction}\\ \text{conjunction}\end{matrix}\right\}$ of their negations.

Two statement forms are said to be logically equivalent if no matter what statements are substituted for their statement variables—the same statement replacing the same statement variable in both statement forms—the resulting pairs of statements are equivalent. Thus De Morgan's Theorem asserts that $\sim(p \vee q)$ and $\sim p \cdot \sim q$ are logically equivalent statement forms. By De Morgan's Theorem and the principle of Double Negation $\sim(p \cdot \sim q)$ and $\sim p \vee q$ are logically equivalent, hence either can be taken as defining $p \supset q$; the second is the more usual choice.

To every argument corresponds a conditional statement whose antecedent is the conjunction of the argument's premisses and whose consequent is the argument's conclusion. That corresponding conditional is a tautology if and only if the argument is valid. Thus to the valid argument form

$$\begin{array}{l} p \vee q \\ \sim p \\ \therefore q \end{array}$$

corresponds the tautologous statement form $[(p \vee q) \cdot \sim p] \supset q$; and to the invalid argument form

$$\begin{array}{l} p \supset q \\ q \\ \therefore p \end{array}$$

corresponds the nontautologous statement form $[(p \supset q) \cdot q] \supset p$. An argument form is valid if and only if its truth table has a **T** under its conclusion in every row in which there are **T**'s under all of its premisses. Since an **F** can occur in the column headed by its corresponding conditional only where there are **T**'s under all of those premisses and an **F** under the conclusion, it is clear that there can be only **T**'s under a conditional that corresponds to a valid argument form. If an argument is valid, the statement that the conjunction of its premisses implies its conclusion is a tautology.

An alternative version of the truth table test of a statement form is the following, which corresponds to the preceding truth table.

$\sim$	$(p$	$\cdot$	$q)$	$\equiv$	$(\sim$	p	$\vee$	$\sim$	$q)$
F	**T**	**T**	**T**	**T**	**F**	**T**	**F**	**F**	**T**
T	**T**	**F**	**F**	**T**	**F**	**T**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**T**	**F**	**T**	**F**	**T**
T	**F**	**F**	**F**	**T**	**T**	**F**	**T**	**T**	**F**
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)

Here columns (2), (4), (7), (10) are the initial or guide columns. Column (3) is filled in by reference to columns (2) and (4), and column (1) by reference to column (3). Column (6) is filled in by reference to column (7), column (9) is filled in by reference to column (10), and then column (8) by reference to columns (6) and (9). Finally column (5) is filled in by reference to columns

(1) and (8). That its main connective has only **T**'s under it in the truth table establishes that the statement form being tested is a tautology.

EXERCISES

I. Use truth tables to characterize the following statement forms as tautologous, contradictory, or contingent:

*1. $p \supset \sim p$
2. $(p \supset \sim p)\cdot(\sim p \supset p)$
3. $p \supset (p \supset p)$
4. $(p \supset p) \supset p$
*5. $p \supset (p\cdot p)$
6. $(p\cdot q) \supset p$
7. $(p \supset q) \supset [\sim(q\cdot r) \supset \sim(r\cdot p)]$
8. $(\sim p\cdot q)\cdot(q \supset p)$
9. $[(p \supset q) \supset q] \supset q$
10. $[(p \supset q) \supset p] \supset p$

II. Use truth tables to decide which of the following are logical equivalences:

*1. $(p \supset q) \equiv (\sim p \supset \sim q)$
2. $(p \supset q) \equiv (\sim q \supset \sim p)$
3. $[(p\cdot q) \supset r] \equiv [p \supset (q \supset r)]$
4. $[p \supset (q \supset r)] \equiv [(p \supset q) \supset r]$
*5. $[p\cdot(q \vee r)] \equiv [(p\cdot q) \vee (p\cdot r)]$
6. $[p \vee (q\cdot r)] \equiv [(p \vee q)\cdot r]$
7. $[p \vee (q\cdot r)] \equiv [(p \vee q)\cdot(p \vee r)]$
8. $(p \equiv q) \equiv [(p\cdot q) \vee (\sim p\cdot \sim q)]$
9. $p \equiv [p\cdot(p \supset q)]$
10. $p \equiv [p\cdot(q \supset p)]$

3

The Method of Deduction

3.1 Formal Proof of Validity

When arguments contain more than two or three different simple statements as components, it becomes cumbersome and tedious to use truth tables to test their validity. A more convenient method of establishing the validity of some arguments is to *deduce* their conclusions from their premisses by a sequence of shorter, more elementary arguments already known to be valid. Consider, for example, the following argument, in which five different simple statements occur:

Either the Attorney General has imposed a strict censorship or if Black mailed the letter he wrote then Davis received a warning.
If our lines of communication have not broken down completely then if Davis received a warning then Emory was informed about the matter.
If the Attorney General has imposed a strict censorship then our lines of communication have broken down completely.
Our lines of communication have not broken down completely.
Therefore if Black mailed the letter he wrote then Emory was informed about the matter.

It may be translated into our symbolism as

$$\begin{array}{l} A \vee (B \supset D) \\ \sim C \supset (D \supset E) \\ A \supset C \\ \sim C \\ \therefore B \supset E \end{array}$$

To establish the validity of this argument by means of a truth table would require a table with thirty-two rows. But we can prove the given argument valid by deducing its conclusion from its premisses by a sequence of just four arguments whose validity has already been remarked. From the third and fourth premisses, $A \supset C$ and $\sim C$, we validly infer $\sim A$ by *Modus Tollens*. From $\sim A$ and the first premiss, $A \vee (B \supset D)$, we validly infer $B \supset D$ by a

Disjunctive Syllogism. From the second and fourth premisses, $\sim C \supset (D \supset E)$ and $\sim C$, we validly infer $D \supset E$ by *Modus Ponens*. And finally, from these last two conclusions (or subconclusions), $B \supset D$ and $D \supset E$, we validly infer $B \supset E$ by a Hypothetical Syllogism. That its conclusion can be deduced from its premisses using valid arguments exclusively *proves* the original argument to be valid. Here the elementary valid argument forms *Modus Ponens* (M.P.), *Modus Tollens* (M.T.), Disjunctive Syllogism (D.S.), and Hypothetical Syllogism (H.S.) are used as *Rules of Inference* by which conclusions are validly deduced from premisses.

A more formal and more concise way of writing out this proof of validity is to list the premisses and the statements deduced from them in one column, with the latter's "justifications" written beside them. In each case the "justification" for a statement specifies the preceding statements from which, and the Rule of Inference by which, the statement in question was deduced. It is convenient to put the conclusion to the right of the last premiss, separated from it by a slanting line which automatically marks all of the statements above it to be premisses. The formal proof of validity for the given argument can be written as

1.	$A \vee (B \supset D)$	
2.	$\sim C \supset (D \supset E)$	
3.	$A \supset C$	
4.	$\sim C$	$/ \therefore B \supset E$
5.	$\sim A$	3, 4, M.T.
6.	$B \supset D$	1, 5, D.S.
7.	$D \supset E$	2, 4, M.P.
8.	$B \supset E$	6, 7, H.S.

A *formal proof of validity* for a given argument is defined to be a sequence of statements each of which is either a premiss of that argument or follows from preceding statements by an elementary valid argument, and such that the last statement in the sequence is the conclusion of the argument whose validity is being proved. This definition must be completed and made definite by specifying what is to count as an 'elementary valid argument'. We first define an *elementary valid argument* to be any argument that is a substitution instance of an elementary valid argument form, and then present a list of just nine argument forms that are sufficiently obvious to be regarded as elementary valid argument forms and accepted as Rules of Inference.

One matter to be emphasized is that *any* substitution instance of an elementary valid argument form is an elementary valid argument. Thus the argument

$$\begin{array}{l} \sim C \supset (D \supset E) \\ \sim C \\ \therefore D \supset E \end{array}$$

is an elementary valid argument because it is a substitution instance of the elementary valid argument form *Modus Ponens* (M.P.). It results from

$$p \supset q$$
$$p$$
$$\therefore q$$

by substituting $\sim C$ for p and $D \supset E$ for q, therefore it is of that form even though *Modus Ponens* is not *the specific form* of the given argument.

We begin our development of the method of deduction by presenting a list of just nine elementary valid argument forms that can be used in constructing formal proofs of validity:

RULES OF INFERENCE

1. *Modus Ponens* (M.P.)
   $p \supset q$
   p
   $\therefore q$
2. *Modus Tollens* (M.T.)
   $p \supset q$
   $\sim q$
   $\therefore \sim p$
3. Hypothetical Syllogism (H.S.)
   $p \supset q$
   $q \supset r$
   $\therefore p \supset r$
4. Disjunctive Syllogism (D.S.)
   $p \vee q$
   $\sim p$
   $\therefore q$
5. Constructive Dilemma (C.D.)
   $(p \supset q) \cdot (r \supset s)$
   $p \vee r$
   $\therefore q \vee s$
6. Destructive Dilemma (D.D.)
   $(p \supset q) \cdot (r \supset s)$
   $\sim q \vee \sim s$
   $\therefore \sim p \vee \sim r$
7. Simplification (Simp.)
   $p \cdot q$
   $\therefore p$
8. Conjunction (Conj.)
   p
   q
   $\therefore p \cdot q$
9. Addition (Add.)
   p
   $\therefore p \vee q$

These nine Rules of Inference are elementary valid argument forms whose validity is easily established by truth tables. They can be used to construct formal proofs of validity for a wide range of more complicated arguments. The names listed are for the most part standard, and the use of their abbreviations permits formal proofs to be set down with a minimum of writing.

EXERCISES

I. For each of the following arguments state the Rule of Inference by which its conclusion follows from its premiss or premisses:

*1. $(A \supset \sim B) \cdot (\sim C \supset D)$
$\therefore A \supset \sim B$

2. $E \supset \sim F$
$\therefore (E \supset \sim F) \vee (\sim G \supset H)$

3. $(I \equiv \sim J) \cdot (I \equiv \sim J)$
$\therefore I \equiv \sim J$

4. $K \vee (L \vee M)$
$\therefore [K \vee (L \vee M)] \vee [K \vee (L \vee M)]$

*5. $N \supset (O \equiv \sim P)$
$(O \equiv \sim P) \supset Q$
$\therefore N \supset Q$

6. $(R \equiv \sim S) \supset (T \supset U)$
$R \equiv \sim S$
$\therefore T \supset U$

7. $(V \supset W) \vee (X \supset Y)$
$\sim(V \supset W)$
$\therefore X \supset Y$

8. $(A \supset \sim B) \cdot [C \supset (D \cdot E)]$
$\sim\sim B \vee \sim(D \cdot E)$
$\therefore \sim A \vee \sim C$

9. $(F \supset \sim G) \supset (\sim H \vee \sim I)$
$F \supset \sim G$
$\therefore \sim H \vee \sim I$

*10. $[\sim(J \cdot K) \supset \sim L] \cdot (M \supset \sim N)$
$\sim(J \cdot K) \vee M$
$\therefore \sim L \vee \sim N$

11. $O \supset \sim P$
$\sim P \supset Q$
$\therefore (O \supset \sim P) \cdot (\sim P \supset Q)$

12. $(\sim R \equiv S) \vee (T \vee U)$
$\sim(\sim R \equiv S)$
$\therefore T \vee U$

13. $[(V \cdot \sim W) \supset X] \cdot [(W \cdot \sim Y) \supset Z]$
$(V \cdot \sim W) \vee (W \cdot \sim Y)$
$\therefore X \vee Z$

14. $[A \supset (B \vee C)] \supset [(D \cdot E) \equiv \sim F]$
$\sim[(D \cdot E) \equiv \sim F]$
$\therefore \sim[A \supset (B \vee C)]$

15. $\sim[G \supset (H \vee I)] \cdot \sim[(J \cdot K) \supset L]$
$\therefore \sim[G \supset (H \vee I)]$

II. Each of the following is a formal proof of validity for the indicated argument. State the 'justification' for each line that is not a premiss:

*1. 1. $(A \cdot B) \supset [A \supset (D \cdot E)]$
2. $(A \cdot B) \cdot C \quad / \therefore D \vee E$
3. $A \cdot B$
4. $A \supset (D \cdot E)$
5. A
6. $D \cdot E$
7. D
8. $D \vee E$

2. 1. $F \vee (G \vee H)$
2. $(G \supset I) \cdot (H \supset J)$
3. $(I \vee J) \supset (F \vee H)$
4. $\sim F \quad / \therefore H$
5. $G \vee H$
6. $I \vee J$
7. $F \vee H$
8. H

3. 1. $K \supset L$
2. $M \supset N$
3. $(O \supset N) \cdot (P \supset L)$
4. $(\sim N \vee \sim L) \cdot (\sim M \vee \sim O)$
$/ \therefore (\sim O \vee \sim P) \cdot (\sim M \vee \sim K)$
5. $(M \supset N) \cdot (K \supset L)$
6. $\sim N \vee \sim L$
7. $\sim M \vee \sim K$
8. $\sim O \vee \sim P$
9. $(\sim O \vee \sim P) \cdot (\sim M \vee \sim K)$

4. 1. $Q \supset (R \supset S)$
2. $(R \supset S) \supset T$
3. $(S \cdot U) \supset \sim V$
4. $\sim V \supset (R \equiv \sim W)$
5. $\sim T \vee \sim(R \equiv \sim W)$
$/ \therefore \sim Q \vee \sim(S \cdot U)$
6. $Q \supset T$
7. $(S \cdot U) \supset (R \equiv \sim W)$
8. $[Q \supset T] \cdot [(S \cdot U) \supset (R \equiv \sim W)]$
9. $\sim Q \vee \sim(S \cdot U)$

*5. 1. $(\sim X \vee \sim Y) \supset [A \supset (P \cdot \sim Q)]$
2. $(\sim X \cdot \sim R) \supset [(P \cdot \sim Q) \supset Z]$

3. $(\sim X \cdot \sim R) \cdot (\sim Z \vee A)$ $/ \therefore A \supset Z$
4. $\sim X \cdot \sim R$
5. $(P \cdot \sim Q) \supset Z$
6. $\sim X$
7. $\sim X \vee \sim Y$
8. $A \supset (P \cdot \sim Q)$
9. $A \supset Z$

6.
1. $A \supset B$
2. $C \supset D$
3. $\sim B \vee \sim D$
4. $\sim \sim A$
5. $(E \cdot F) \supset C$ $/ \therefore \sim (E \cdot F)$
6. $(A \supset B) \cdot (C \supset D)$
7. $\sim A \vee \sim C$
8. $\sim C$
9. $\sim (E \cdot F)$

7.
1. $(G \supset H) \supset (I \equiv J)$
2. $K \vee \sim (L \supset M)$
3. $(G \supset H) \vee \sim K$
4. $N \supset (L \supset M)$
5. $\sim (I \equiv J)$ $/ \therefore \sim N$
6. $\sim (G \supset H)$
7. $\sim K$
8. $\sim (L \supset M)$
9. $\sim N$

8.
1. $(O \supset \sim P) \cdot (\sim Q \supset R)$
2. $(S \supset T) \cdot (\sim U \supset \sim V)$
3. $(\sim P \supset S) \cdot (R \supset \sim U)$
4. $(T \vee \sim V) \supset (W \cdot X)$
5. $O \vee \sim Q$ $/ \therefore W \cdot X$
6. $\sim P \vee R$
7. $S \vee \sim U$
8. $T \vee \sim V$
9. $W \cdot X$

9.
1. $[(A \vee \sim B) \vee C] \supset [D \supset (E \equiv F)]$
2. $(A \vee \sim B) \supset [(F \equiv G) \supset H]$
3. $A \supset [(E \equiv F) \supset (F \equiv G)]$
4. A $/ \therefore D \supset H$
5. $A \vee \sim B$
6. $(A \vee \sim B) \vee C$
7. $D \supset (E \equiv F)$
8. $(E \equiv F) \supset (F \equiv G)$
9. $D \supset (F \equiv G)$
10. $(F \equiv G) \supset H$
11. $D \supset H$

***10.**
1. $H \supset (I \supset J)$
2. $K \supset (I \supset J)$
3. $(\sim H \cdot \sim K) \supset (\sim L \vee \sim M)$
4. $(\sim L \supset \sim N) \cdot (\sim M \supset \sim O)$
5. $(P \supset N) \cdot (Q \supset O)$
6. $\sim (I \supset J)$ $/ \therefore \sim P \vee \sim Q$
7. $\sim H$
8. $\sim K$
9. $\sim H \cdot \sim K$
10. $\sim L \vee \sim M$
11. $\sim N \vee \sim O$
12. $\sim P \vee \sim Q$

III. Construct a formal proof of validity for each of the following arguments:

***1.** $A \supset B$
$C \supset D$
$(\sim B \vee \sim D) \cdot (\sim A \vee \sim B)$
$\therefore \sim A \vee \sim C$

2. $E \supset (F \cdot \sim G)$
$(F \vee G) \supset H$
E
$\therefore H$

3. $J \supset K$
$J \vee (K \vee \sim L)$
$\sim K$
$\therefore \sim L \cdot \sim K$

4. $M \supset N$
$N \supset O$
$(M \supset O) \supset (N \supset P)$
$(M \supset P) \supset Q$
$\therefore Q$

***5.** $(R \supset \sim S) \cdot (T \supset \sim U)$
$(V \supset \sim W) \cdot (X \supset \sim Y)$
$(T \supset W) \cdot (U \supset S)$
$V \vee R$
$\therefore \sim T \vee \sim U$

6. $A \supset (B \cdot C)$
$\sim A \supset [(D \supset E) \cdot (F \supset G)]$
$(B \cdot C) \vee [(\sim A \supset D) \cdot (\sim A \supset F)]$
$\sim (B \cdot C) \cdot \sim (G \cdot D)$
$\therefore E \vee G$

7. $(\sim H \vee I) \supset (J \supset K)$
   $(\sim L \cdot \sim M) \supset (K \supset N)$
   $(H \supset L) \cdot (L \supset H)$
   $(\sim L \cdot \sim M) \cdot \sim O$
   $\therefore J \supset N$

8. $(P \supset Q) \cdot (R \supset S)$
   $(Q \supset T) \cdot (S \supset U)$
   $(\sim P \supset T) \cdot (\sim Q \supset S)$
   $\sim T$
   $\therefore \sim R \vee \sim Q$

9. $V \supset W$
   $X \supset Y$
   $Z \supset W$
   $X \supset A$
   $W \supset X$
   $[(V \supset Y) \cdot (Z \supset A)] \supset (V \vee Z)$
   $\therefore Y \vee A$

10. $(B \vee C) \supset (D \vee E)$
    $[(D \vee E) \vee F] \supset (G \vee H)$
    $(G \vee H) \supset \sim D$
    $E \supset \sim G$
    B
    $\therefore H$

IV. Construct a formal proof of validity for each of the following arguments, using the abbreviations suggested:

*1. If either algebra is required or geometry is required then all students will study mathematics. Algebra is required and trigonometry is required. Therefore all students will study mathematics. (*A*: Algebra is required. *G*: Geometry is required. *S*: All students will study mathematics. *T*: Trigonometry is required.)

2. Either Smith attended the meeting or Smith was not invited to the meeting. If the directors wanted Smith at the meeting then Smith was invited to the meeting. Smith did not attend the meeting. If the directors did not want Smith at the meeting and Smith was not invited to the meeting then Smith is on his way out of the company. Therefore Smith is on his way out of the company. (*A*: Smith attended the meeting. *I*: Smith was invited to the meeting. *D*: The directors wanted Smith at the meeting. *W*: Smith is on his way out of the company.)

3. If a scarcity of commodities develops then prices rise. If there is a change of administration then fiscal controls will not be continued. If the threat of inflation persists then fiscal controls will be continued. If there is overproduction then prices do not rise. Either there is overproduction or there is a change of administration. Therefore either a scarcity of commodities does not develop or the threat of inflation does not persist. (*S*: A scarcity of commodities develops. *P*: Prices rise. *C*: There is a change of administration. *F*: Fiscal controls will be continued. *I*: The threat of inflation persists. *O*: There is overproduction.)

4. If the investigation continues then new evidence is brought to light. If new evidence is brought to light then several leading citizens are implicated. If several leading citizens are implicated then the newspapers stop publicizing the case. If continuation of the investigation implies that the newspapers stop publicizing the case then the bringing to light of new evidence implies that the investigation continues. The investigation does not continue. Therefore new evidence is not brought to light. (*C*: The investigation continues. *N*: New evidence is brought to light. *I*: Several leading citizens are implicated. *S*: The newspapers stop publicizing the case.)

*5. If the king does not castle and the pawn advances then either the bishop is blocked or the rook is pinned. If the king does not castle then if the bishop is blocked then the game is a draw. Either the king castles or if the rook is pinned then the exchange is lost. The king does not castle and the pawn advances. Therefore either the game is a draw or the exchange is lost. (*K*: The king castles. *P*: The pawn advances. *B*: The bishop is blocked. *R*: The rook is pinned. *D*: The game is a draw. *E*: The exchange is lost.)

6. If Andrews is present then Brown is present, and if Brown is present then Cohen is not present. If Cohen is present then Davis is not present. If Brown is present then Emerson is present. If Davis is not present then Farley is present. Either Emerson is not present or Farley is not present. Therefore either Andrews is not present or Cohen is not present. (*A*: Andrews is present. *B*: Brown is present. *C*: Cohen is present. *D*: Davis is present. *E*: Emerson is present. *F*: Farley is present.)

7. If either George enrolls or Harry enrolls then Ira does not enroll. Either Ira enrolls or Harry enrolls. If either Harry enrolls or George does not enroll then Jim enrolls. George enrolls. Therefore either Jim enrolls or Harry does not enroll. (*G*: George enrolls. *H*: Harry enrolls. *I*: Ira enrolls. *J*: Jim enrolls.)

8. If Tom received the message then Tom took the plane, but if Tom did not take the plane then Tom missed the meeting. If Tom missed the meeting then Dave was elected to the board, but if Dave was elected to the board then Tom received the message. If either Tom did not miss the meeting or Tom did not receive the message then either Tom did not take the plane or Dave was not elected to the board. Tom did not miss the meeting. Therefore either Tom did not receive the message or Tom did not miss the meeting. (*R*: Tom received the message. *P*: Tom took the plane. *M*: Tom missed the meeting. *D*: Dave was elected to the board.)

9. If Dick was recently vaccinated then he has a fever. Either Dick was recently vaccinated or if pocks begin to appear then Dick must be quarantined. Either Dick has measles or if a rash develops then there are complications. If Dick has measles then he has a fever. If Dick was not recently vaccinated and Dick does not have measles then either a rash develops or pocks begin to appear. Dick does not have a fever. Therefore either there are complications or Dick must be quarantined. (*V*: Dick was recently vaccinated. *F*: Dick has a fever. *P*: Pocks begin to appear. *Q*: Dick must be quarantined. *M*: Dick has measles. *R*: A rash develops. *C*: There are complications.)

T E D

*10. Either taxes are increased or if expenditures rise then the debt ceiling is raised. If taxes are increased then the cost of collecting taxes increases. If a rise in expenditures implies that the government borrows more money then if the debt ceiling is raised then interest rates increase. If taxes are not increased and the cost of collecting taxes does not increase then if the debt ceiling is raised then the government borrows more money. The cost of collecting taxes does not increase. Either interest rates do not increase or the government does not borrow more money. Therefore either the debt ceiling is not raised or expenditures do not rise. (*T*: Taxes are increased. *E*: Expenditures rise. *D*: The debt ceiling is raised. *C*: The cost of collecting taxes increases. *G*: The government borrows more money. *I*: Interest rates increase.)

3.2 The Rule of Replacement

There are many valid truth-functional arguments that cannot be proved valid using only the nine Rules of Inference given thus far. For example, a formal proof of validity for the obviously valid argument

$$A \cdot B$$
$$\therefore B$$

requires additional Rules of Inference.

Now the only compound statements that concern us here are truth-functionally compound statements. Hence if any part of a compound statement is replaced by an expression which is logically equivalent to the part replaced, the truth value of the resulting statement is the same as that of the original statement. This is sometimes called the Rule of Replacement and sometimes the Principle of Extensionality.[1] We adopt the Rule of Replacement as an additional principle of inference. It permits us to infer from any statement the result of replacing all or part of that statement by any other statement logically equivalent to the part replaced. Thus using the Principle of Double Negation (D.N.), which asserts the logical equivalence of p and $\sim\sim p$, we can infer from $A \supset \sim\sim B$ any of the statements

$$A \supset B, \sim\sim A \supset \sim\sim B, A \supset \sim\sim\sim\sim B, \quad \text{or} \quad \sim\sim(A \supset \sim\sim B)$$

by the Rule of Replacement.

To make this new rule definite we list a number of logical equivalences with which it can be used. These equivalences constitute additional Rules of Inference that can be used in proving the validity of arguments. We number them consecutively after the first nine Rules already stated.

Rule of Replacement: Any of the following logically equivalent expressions can replace each other wherever they occur:

10. De Morgan's Theorems (De M.): $\sim(p \cdot q) \equiv (\sim p \vee \sim q)$.
    $\sim(p \vee q) \equiv (\sim p \cdot \sim q)$.
11. Commutation (Com.): $(p \vee q) \equiv (q \vee p)$.
    $(p \cdot q) \equiv (q \cdot p)$.
12. Association (Assoc.): $[p \vee (q \vee r)] \equiv [(p \vee q) \vee r]$.
    $[p \cdot (q \cdot r)] \equiv [(p \cdot q) \cdot r]$.
13. Distribution (Dist.): $[p \cdot (q \vee r)] \equiv [(p \cdot q) \vee (p \cdot r)]$.
    $[p \vee (q \cdot r)] \equiv [(p \vee q) \cdot (p \vee r)]$.

[1] It will be stated more formally, in an appropriate context, and demonstrated, in Chapter 7.

14. Double Negation (D.N.): $p \equiv \sim\sim p$.
15. Transposition (Trans.): $(p \supset q) \equiv (\sim q \supset \sim p)$.
16. Material Implication (Impl.): $(p \supset q) \equiv (\sim p \vee q)$.
17. Material Equivalence (Equiv.): $(p \equiv q) \equiv [(p \supset q) \cdot (q \supset p)]$.
$(p \equiv q) \equiv [(p \cdot q) \vee (\sim p \cdot \sim q)]$.
18. Exportation (Exp.): $[(p \cdot q) \supset r] \equiv [p \supset (q \supset r)]$.
19. Tautology (Taut.): $p \equiv (p \vee p)$.
$p \equiv (p \cdot p)$.

A formal proof of validity for the argument given on page 37 can now be written:

1. $A \cdot B$ $/ \therefore B$
2. $B \cdot A$ 1, Com.
3. B 2, Simp.

Some argument forms, although very elementary and perfectly valid, are not included in our list of nineteen Rules of Inference. Although the argument

$$A \cdot B$$
$$\therefore B$$

is obviously valid, its form

$$p \cdot q$$
$$\therefore q$$

is not included in our list. Hence B does not follow from $A \cdot B$ by any single elementary valid argument form *as defined by our list*. It can be deduced, however, using *two* elementary valid arguments as shown above. We could add the intuitively valid argument form

$$p \cdot q$$
$$\therefore q$$

to our list, of course, but if we expanded our list in this way we might end up with a list that was too long and therefore unmanageable.

The list of Rules of Inference contains several redundancies. For example, *Modus Tollens* could be dropped from our list without any real weakening of the machinery, for any step deduced by its use could be deduced using other Rules of the list instead. Thus in our first proof, on page 31, line 5, $\sim A$, which was deduced from lines 3 and 4, $A \supset C$ and $\sim C$, by *Modus Tollens*, could have been deduced without it, since $\sim C \supset \sim A$ follows from $A \supset C$ by Transposition, and $\sim A$ from $\sim C \supset \sim A$ and $\sim C$ by *Modus Ponens*. But

Modus Tollens is so common and intuitive a principle of inference that it has been included anyway, and others have also been included for convenience despite their logical redundancy.

The test of whether or not a given sequence of statements is a formal proof is *effective*. That is, direct observation will suffice to decide of every line beyond the premisses whether or not it actually does follow from preceding lines by one of the given Rules of Inference. No 'thinking' is required: neither thinking about what the statements mean, nor using logical intuition to check the validity of any line's deduction. Even where the 'justification' of a statement is not written beside it, there is a finite, mechanical procedure for deciding whether or not the deduction is legitimate. Each line is preceded by only a finite number of other lines, and only a finite number of Rules of Inference have been adopted. Although time consuming, it can be verified by inspection whether the line in question follows from any single preceding line or any pair of preceding lines by any one of the Rules of Inference listed. For example, in the foregoing proof, line 2, $B \cdot A$, is preceded only by line 1, $A \cdot B$. Its legitimacy can be decided by observing that although it does not follow from $A \cdot B$ by *Modus Ponens*, nor by *Modus Tollens*, nor by a Hypothetical Syllogism, and so on through number 10, when we come to number 11 we can *see*, simply by looking at their forms, that line 2 follows from line 1 by the principle of Commutation. Similarly, the legitimacy of any line can be decided by a finite number of observations, none of which involves anything more than comparing shapes or patterns. To preserve this effectiveness we lay down the rule that only one Rule of Inference should be applied at a time. The explanatory notation beside each statement is not, strictly speaking, part of the proof, but it is helpful and should always be included.

Although the test of whether or not a given sequence of statements is a formal proof is effective, *constructing* such a formal proof is *not* an effective procedure. In this respect the present method differs from the method of the preceding chapter. The use of truth tables is *completely* mechanical: given any argument of the general sort with which we are now concerned, its validity can always be tested by following the simple rules presented in Chapter 2. But in constructing a formal proof of validity on the basis of the nineteen Rules of Inference listed, it is necessary to *think* or 'figure out' where to begin and how to proceed. Although we have no effective or purely mechanical method of procedure, it is usually much easier to construct a formal proof of validity than to write out a truth table with perhaps dozens or hundreds or even thousands of rows.

There is an important difference between the first nine and the last ten Rules of Inference. The first nine can be applied only to whole lines of a proof. Thus A can be inferred from $A \cdot B$ by Simplification only if $A \cdot B$ constitutes a whole line. But neither A nor $A \supset C$ follows from $(A \cdot B) \supset C$ by Simplification or by any other Rule of Inference. A does not follow because A can be false while $(A \cdot B) \supset C$ is true. $A \supset C$ does not follow because if A is true and B and C are both false, $(A \cdot B) \supset C$ is true whereas $A \supset C$ is false. On the other hand, any of the last ten Rules of Inference can be applied either to whole lines or to parts of lines. Not only can the statement $A \supset (B \supset C)$ be inferred from the whole line $(A \cdot B) \supset C$ by Exportation, but from the line $[(A \cdot B) \supset C] \vee D$ we can infer $[A \supset (B \supset C)] \vee D$ by Exportation. The Rule of Replacement authorizes specified logically equivalent expressions to replace each other wherever they occur, even where they do not constitute whole lines of a proof. But the first nine Rules of Inference can be used only with whole lines of a proof serving as premisses.

In the absence of mechanical rules for the construction of formal proofs of validity, some rules of thumb or hints on procedure may be suggested. The first is simply to begin deducing conclusions from the premisses by the given Rules of Inference. As more and more of these subconclusions become available as premisses for further deductions, the greater is the likelihood of being able to see how to deduce the conclusion of the argument to be proved valid.

Another hint is to try to eliminate statements that occur in the premisses but not in the conclusion. Such elimination can proceed only in accordance with the Rules of Inference. But the Rules contain many techniques for eliminating statements. Simplification is one such rule: by it the right-hand conjunct of a conjunction can simply be dropped—provided that the conjunction is a whole line in the proof. And by Commutation the left-hand conjunct can be switched over to the right-hand side, from which it can be dropped by Simplification. The 'middle' term q can be eliminated by a Hypothetical Syllogism given two premisses or subconclusions of the patterns $p \supset q$ and $q \supset r$. Distribution is a useful rule for transforming a disjunction of the form $p \vee (q \cdot r)$ into the conjunction $(p \vee q) \cdot (p \vee r)$ whose right-hand conjunct $p \vee r$ can then be eliminated by Simplification. Another rule of thumb is to introduce by Addition a statement that occurs in the conclusion but not in the premisses. Another method is to work backward from the conclusion by looking for some statement or pair of statements from which it could be deduced by one of the Rules of Inference, and then trying to deduce those intermediate statements either from the premisses or from other intermediate statements, and so on, until you come to some which are derivable from the premisses. A judicious combination of these methods is often the best way to proceed. Practice, of course, is the best method of acquiring facility in using the method of deduction.

EXERCISES

I. For each of the following arguments state the Rule of Inference by which its conclusion follows from its premiss:

*1. $(\sim A \supset B) \cdot (C \vee \sim D)$
$\therefore (\sim A \supset B) \cdot (\sim D \vee C)$

2. $(\sim E \vee F) \cdot (G \vee \sim H)$
$\therefore (E \supset F) \cdot (G \vee \sim H)$

3. $(I \supset \sim J) \vee (\sim K \supset \sim L)$
$\therefore (I \supset \sim J) \vee (L \supset K)$

4. $M \supset \sim(N \vee \sim O)$
$\therefore M \supset (\sim N \cdot \sim\sim O)$

*5. $[P \supset (Q \vee R)] \vee [P \supset (Q \vee R)]$
$\therefore P \supset (Q \vee R)$

6. $[S \cdot (T \cdot U)] \supset (V \equiv \sim W)$
$\therefore [(S \cdot T) \cdot U] \supset (V \equiv \sim W)$

7. $[X \cdot (Y \cdot Z)] \supset (A \equiv \sim B)$
$\therefore X \supset [(Y \cdot Z) \supset (A \equiv \sim B)]$

8. $(C \cdot \sim D) \supset (E \equiv \sim F)$
$\therefore (C \cdot \sim D) \supset [(E \cdot \sim F) \vee (\sim E \cdot \sim\sim F)]$

9. $(G \vee H) \cdot (I \vee J)$
$\therefore [(G \vee H) \cdot I] \vee [(G \vee H) \cdot J]$

***10.** $(K \cdot L) \supset \{M \cdot [(N \cdot O) \cdot P]\}$
$\therefore (K \cdot L) \supset \{M \cdot [(O \cdot N) \cdot P]\}$

11. $\sim\{Q \vee \sim[(R \cdot \sim S) \cdot (T \vee \sim U)]\}$
$\therefore \sim\{Q \vee [\sim(R \cdot \sim S) \vee \sim(T \vee \sim U)]\}$

12. $\sim V \supset \{W \supset [\sim(X \cdot Y) \supset \sim Z]\}$
$\therefore \sim V \supset \{[W \cdot \sim(X \cdot Y)] \supset \sim Z\}$

13. $[A \vee (B \vee C)] \vee [(D \vee D) \vee E]$
$\therefore [A \vee (B \vee C)] \vee [D \vee (D \vee E)]$

14. $(F \supset G) \cdot \{[(G \supset H) \cdot (H \supset G)] \supset (H \supset I)\}$
$\therefore (F \supset G) \cdot \{(G \equiv H) \supset (H \supset I)\}$

***15.** $J \equiv \sim\{[(K \cdot \sim L) \vee \sim M] \cdot [(K \cdot \sim L) \vee \sim N]\}$
$\therefore J \equiv \sim\{(K \cdot \sim L) \vee (\sim M \cdot \sim N)\}$

16. $O \supset [(P \cdot \sim Q) \equiv (P \cdot \sim\sim R)]$
$\therefore O \supset [(P \cdot \sim Q) \equiv (\sim\sim P \cdot \sim\sim R)]$

17. $\sim S \equiv \{\sim\sim T \supset [\sim\sim\sim U \vee (\sim T \cdot S)]\}$
$\therefore \sim S \equiv \{\sim\sim\sim T \vee [\sim\sim\sim U \vee (\sim T \cdot S)]\}$

18. $V \supset \{(\sim W \supset \sim\sim X) \vee [(\sim Y \supset Z) \vee (\sim Z \supset \sim Y)]\}$
$\therefore V \supset \{(\sim X \supset W) \vee [(\sim Y \supset Z) \vee (\sim Z \supset \sim Y)]\}$

19. $(A \cdot \sim B) \supset [(C \cdot C) \supset (C \supset D)]$
$\therefore (A \cdot \sim B) \supset [C \supset (C \supset D)]$

20. $(E \cdot \sim F) \supset [G \supset (G \supset H)]$
$\therefore (E \cdot \sim F) \supset [(G \cdot G) \supset H)]$

II. Each of the following is a formal proof of validity for the indicated argument. State the 'justification' for each line which is not a premiss:

***1.**
1. $(A \vee B) \supset (C \cdot D)$
2. $\sim C \quad / \therefore \sim B$
3. $\sim C \vee \sim D$
4. $\sim(C \cdot D)$
5. $\sim(A \vee B)$
6. $\sim A \cdot \sim B$
7. $\sim B \cdot \sim A$
8. $\sim B$

2.
1. $(E \cdot F) \cdot G$
2. $(F \equiv G) \supset (H \vee I) \quad / \therefore I \vee H$
3. $E \cdot (F \cdot G)$
4. $(F \cdot G) \cdot E$
5. $F \cdot G$
6. $(F \cdot G) \vee (\sim F \cdot \sim G)$
7. $F \equiv G$
8. $H \vee I$
9. $I \vee H$

3.
1. $(J \cdot K) \supset L$
2. $(J \supset L) \supset M$
3. $\sim K \vee N \quad / \therefore K \supset (M \cdot N)$
4. $(K \cdot J) \supset L$
5. $K \supset (J \supset L)$
6. $K \supset M$
7. $\sim K \vee M$
8. $(\sim K \vee M) \cdot (\sim K \vee N)$
9. $\sim K \vee (M \cdot N)$
10. $K \supset (M \cdot N)$

4.
1. $(O \supset \sim P) \cdot (P \supset Q)$
2. $Q \supset O$
3. $\sim R \supset P \quad / \therefore R$
4. $\sim Q \vee O$
5. $O \vee \sim Q$
6. $(O \supset \sim P) \cdot (\sim Q \supset \sim P)$
7. $\sim P \vee \sim P$
8. $\sim P$
9. $\sim\sim R$
10. R

***5.**
1. $S \supset (T \supset U)$
2. $U \supset \sim U$
3. $(V \supset S) \cdot (W \supset T) \quad / \therefore V \supset \sim W$

4. $(S \cdot T) \supset U$
5. $\sim U \vee \sim U$
6. $\sim U$
7. $\sim(S \cdot T)$
8. $\sim S \vee \sim T$
9. $\sim V \vee \sim W$
10. $V \supset \sim W$

6. 1. $X \supset (Y \supset Z)$
2. $X \supset (A \supset B)$
3. $X \cdot (Y \vee A)$
4. $\sim Z \quad /\therefore B$
5. $(X \cdot Y) \supset Z$
6. $(X \cdot A) \supset B$
7. $(X \cdot Y) \vee (X \cdot A)$
8. $[(X \cdot Y) \supset Z] \cdot [(X \cdot A) \supset B]$
9. $Z \vee B$
10. B

7. 1. $C \supset (D \supset \sim C)$
2. $C \equiv D \quad /\therefore \sim C \cdot \sim D$
3. $C \supset (\sim\sim C \supset \sim D)$
4. $C \supset (C \supset \sim D)$
5. $(C \cdot C) \supset \sim D$
6. $C \supset \sim D$
7. $\sim C \vee \sim D$
8. $\sim(C \cdot D)$
9. $(C \cdot D) \vee (\sim C \cdot \sim D)$
10. $\sim C \cdot \sim D$

8. 1. $E \cdot (F \vee G)$
2. $(E \cdot G) \supset \sim(H \vee I)$
3. $(\sim H \vee \sim I) \supset \sim(E \cdot F)$
$/\therefore H \equiv I$
4. $(E \cdot G) \supset (\sim H \cdot \sim I)$
5. $\sim(H \cdot I) \supset \sim(E \cdot F)$
6. $(E \cdot F) \supset (H \cdot I)$
7. $[(E \cdot F) \supset (H \cdot I)] \cdot [(E \cdot G) \supset (\sim H \cdot \sim$
8. $(E \cdot F) \vee (E \cdot G)$
9. $(H \cdot I) \vee (\sim H \cdot \sim I)$
10. $H \equiv I$

9. 1. $J \vee (\sim K \vee J)$
2. $K \vee (\sim J \vee K) \quad /\therefore (J \cdot K) \vee (\sim J \cdot \sim$
3. $(\sim K \vee J) \vee J$
4. $\sim K \vee (J \vee J)$
5. $\sim K \vee J$
6. $K \supset J$
7. $(\sim J \vee K) \vee K$
8. $\sim J \vee (K \vee K)$
9. $\sim J \vee K$
10. $J \supset K$
11. $(J \supset K) \cdot (K \supset J)$
12. $J \equiv K$
13. $(J \cdot K) \vee (\sim J \cdot \sim K)$

10. 1. $(L \vee M) \vee (N \cdot O)$
2. $(\sim L \cdot O) \cdot \sim(\sim L \cdot M) \quad /\therefore \sim L \cdot N$
3. $\sim L \cdot [O \cdot \sim(\sim L \cdot M)]$
4. $\sim L$
5. $L \vee [M \vee (N \cdot O)]$
6. $M \vee (N \cdot O)$
7. $(M \vee N) \cdot (M \vee O)$
8. $M \vee N$
9. $\sim L \cdot (M \vee N)$
10. $(\sim L \cdot M) \vee (\sim L \cdot N)$
11. $\sim(\sim L \cdot M) \cdot (\sim L \cdot O)$
12. $\sim(\sim L \cdot M)$
13. $\sim L \cdot N$

III. Construct a formal proof of validity for each of the following arguments:

***1.** $\sim A$
$\therefore A \supset B$

2. C
$\therefore D \supset C$

3. $E \supset (F \supset G)$
$\therefore F \supset (E \supset G)$

4. $H \supset (I \cdot J)$
$\therefore H \supset I$

***5.** $K \supset L$
$\therefore K \supset (L \vee M)$

6. $N \supset O$
$\therefore (N \cdot P) \supset O$

7. $(Q \vee R) \supset S$
$\therefore Q \supset S$

8. $T \supset \sim(U \supset V)$
$\therefore T \supset U$

9. $W \supset (X \cdot \sim Y)$
$\therefore W \supset (Y \supset X)$

***10.** $A \supset \sim(B \supset C)$
$(D \cdot B) \supset C$
D
$\therefore \sim A$

11. $E \supset F$
$E \supset G$
$\therefore E \supset (F \cdot G)$

12. $H \supset (I \vee J)$
$\sim I$
$\therefore H \supset J$

13. $(K \vee L) \supset \sim(M \cdot N)$
$(\sim M \vee \sim N) \supset (O \equiv P)$
$(O \equiv P) \supset (Q \cdot R)$
$\therefore (L \vee K) \supset (R \cdot Q)$

14. $S \supset T$
$S \vee T$
$\therefore T$

*15. $(\sim U \vee V) \cdot (U \vee W)$
$\sim X \supset \sim W$
$\therefore V \vee X$

16. $A \supset (B \supset C)$
$C \supset (D \cdot E)$
$\therefore A \supset (B \supset D)$

17. $E \supset F$
$G \supset F$
$\therefore (E \vee G) \supset F$

18. $[(H \cdot I) \supset J] \cdot [\sim K \supset (I \cdot \sim J)]$
$\therefore H \supset K$

19. $[L \cdot (M \vee N)] \supset (M \cdot N)$
$\therefore L \supset (M \supset N)$

20. $O \supset (P \supset Q)$
$P \supset (Q \supset R)$
$\therefore O \supset (P \supset R)$

21. $(S \supset T) \cdot (U \supset V)$
$W \supset (S \vee U)$
$\therefore W \supset (T \vee V)$

IV. Construct a formal proof of validity for each of the following arguments, in each case using the suggested notation.

*1. If I study I make good grades. If I do not study I enjoy myself. Therefore either I make good grades or I enjoy myself. (S, G, E)

2. If the supply of silver remains constant and the use of silver increases then the price of silver rises. If an increase in the use of silver implies that the price of silver rises then there will be a windfall for speculators. The supply of silver remains constant. Therefore there will be a windfall for speculators. (S, U, P, W)

3. Either Adams is elected chairman or both Brown and Clark are elected to the board. If either Adams is elected chairman or Brown is elected to the board then Davis will lodge a protest. Therefore either Clark is elected to the board or Davis will lodge a protest. (A, B, C, D)

4. If he uses good bait then if the fish are biting then he catches the legal limit. He uses good bait but he does not catch the legal limit. Therefore the fish are not biting. (G, B, C)

*5. Either the governor and the lieutenant-governor will both run for reelection, or the primary race will be wide open and the party will be torn by dissension. The governor will not run for reelection. Therefore the party will be torn by dissension. (G, L, W, T)

6. If the Dodgers win the pennant then they will win the Series. Therefore if the Dodgers win the pennant then if they continue to hit then they will win the Series. (P, S, H)

7. If he attracts the farm vote then he will carry the rural areas, and if he attracts the labor vote then he will carry the urban centers. If he carries both the urban centers and the rural areas then he is certain to be elected. He is not certain to be elected. Therefore either he does not attract the farm vote or he does not attract the labor vote. (F, R, L, U, C)

8. Either Argentina does not join the alliance or Brazil boycotts it, but if Argentina joins the alliance then Chile boycotts it. If Brazil boycotts the alliance then if Chile boycotts it then Ecuador will boycott it. Therefore if Argentina joins the alliance then Ecuador will boycott it. (*A*, *B*, *C*, *E*)
9. If Argentina joins the alliance then both Brazil and Chile will boycott it. If either Brazil or Chile boycotts the alliance then the alliance will be ineffective. Therefore if Argentina joins the alliance then the alliance will be ineffective. (*A*, *B*, *C*, *I*)

*10. Steve took either the bus or the train. If he took the bus or drove his own car then he arrived late and missed the first session. He did not arrive late. Therefore he took the train. (*B*, *T*, *C*, *L*, *M*)

11. If you enroll in the course and study hard then you will pass, but if you enroll in the course and do not study hard then you will not pass. Therefore if you enroll in the course then either you study hard and pass or you do not study hard and do not pass. (*E*, *S*, *P*)
12. If Argentina joins the alliance then either Brazil or Chile will boycott it. If Brazil boycotts the alliance then Chile will boycott it also. Therefore if Argentina joins the alliance then Chile will boycott it. (*A*, *B*, *C*)
13. If either Argentina or Brazil joins the alliance then both Chile and Ecuador will boycott it. Therefore if Argentina joins the alliance then Chile will boycott it. (*A*, *B*, *C*, *E*)
14. If prices fall or wages rise then both retail sales and advertising activities increase. If retail sales increase then jobbers make more money, but jobbers do not make more money. Therefore prices do not fall. (*P*, *W*, *R*, *A*, *J*)

*15. If I work then I earn money, but if I am idle then I enjoy myself. Either I work or I am idle. However, if I work then I do not enjoy myself, while if I am idle then I do not earn money. Therefore I enjoy myself if and only if I do not earn money. (*W*, *M*, *I*, *E*)

16. If he enters the primary then if he campaigns vigorously then he wins the nomination. If he wins the nomination and receives the support of the party regulars then he will be elected. If he takes the party platform seriously then he will receive the support of the party regulars but will not be elected. Therefore if he enters the primary then if he campaigns vigorously then he does not take the party platform seriously. (*P*, *C*, *N*, *R*, *E*, *T*)
17. Either the tariff is lowered, or imports continue to decrease and our own industries prosper. If the tariff is lowered then our own industries prosper. Therefore our own industries prosper. (*T*, *I*, *O*)
18. Either he has his old car repaired or he buys a new car. If he has his old car repaired then he will owe a lot of money to the garage. If he owes a lot of money to the garage then he will not soon be out of debt. If he buys a new car then he must borrow money from the bank, and if he must borrow money from the bank then he will not soon be out of debt. Either he will soon be out of debt or his creditors will force him into bankruptcy. Therefore his creditors will force him into bankruptcy. (*R*, *N*, *G*, *S*, *B*, *C*)
19. If he goes on a picnic then he wears sport clothes. If he wears sport clothes then he does not attend both the banquet and the dance. If he does not attend the banquet then he still has his ticket, but he does not still have his ticket. He does attend the dance. Therefore he does not go on a picnic. (*P*, *S*, *B*, *D*, *T*)

20. If he studies the sciences then he prepares to earn a good living, and if he studies the humanities then he prepares to live a good life. If he prepares to earn a good living or he prepares to live a good life then his college years are well spent. But his college years are not well spent. Therefore he does not study either the sciences or the humanities. (S, E, H, L, C)

*21. If you plant tulips then your garden will bloom early, and if you plant asters then your garden will bloom late. So if you plant either tulips or asters then your garden will bloom either early or late. (T, E, A, L)

22. If you plant tulips then your garden will bloom early, and if you plant asters then your garden will bloom late. So if you plant both tulips and asters then your garden will bloom both early and late. (T, E, A, L)

23. If we go to Europe then we tour Scandinavia. If we go to Europe then if we tour Scandinavia then we visit Norway. If we tour Scandinavia then if we visit Norway then we will take a trip on a fiord. Therefore if we go to Europe then we will take a trip on a fiord. (E, S, N, F)

24. If Argentina joins the alliance then either Brazil or Chile boycotts it. If Ecuador joins the alliance then either Chile or Peru boycotts it. Chile does not boycott it. Therefore if neither Brazil nor Peru boycotts it then neither Argentina nor Ecuador joins the alliance. (A, B, C, E, P)

25. If either Argentina or Brazil joins the alliance then if either Chile or Ecuador boycotts it then although Peru does not boycott it Venezuela boycotts it. If either Peru or Nicaragua does not boycott it then Uruguay will join the alliance. Therefore if Argentina joins the alliance then if Chile boycotts it then Uruguay will join the alliance. (A, B, C, E, P, V, N, U)

3.3 Proving Invalidity

We can establish the invalidity of an argument by using a truth table to show that the specific form of that argument is invalid. The truth table proves invalidity if it contains at least one row in which truth values are assigned to the statement variables in such a way that the premisses are made true and the conclusion false. If we can devise such a truth value assignment without constructing the entire truth table, we shall have a shorter method of proving invalidity.

Consider the invalid argument

> If the Senator votes against this bill then he is opposed to more severe penalties against tax evaders.
> If the Senator is a tax evader himself then he is opposed to more severe penalties against tax evaders.
> Therefore if the Senator votes against this bill he is a tax evader himself.

which may be symbolized as

$$V \supset O$$
$$H \supset O$$
$$\therefore V \supset H$$

Instead of constructing a truth table for the specific form of this argument, we can prove its invalidity by making an assignment of truth values to the component simple statements V, O, and H which will make the premisses true and the conclusion false. The conclusion is made false by assigning **T** to V and **F** to H; and both premisses are made true by assigning **T** to O. This method of proving invalidity is closely related to the truth table method. In effect, making the indicated truth value assignment amounts to describing one row of the relevant truth table—a row that suffices to establish the invalidity of the argument being tested. The relationship appears more clearly, perhaps, when the truth value assignments are written out horizontally, as

V	O	H	$V \supset O$	$H \supset O$	$V \supset H$
T	**T**	**F**	**T**	**T**	**F**

This new method of proving invalidity is shorter than writing out a complete truth table, and the amount of time and work saved is proportionally greater for more complicated arguments. In proving the invalidity of more extended arguments, a certain amount of trial and error may be needed to discover a truth value assignment which works. But even so, this method is quicker and easier than writing out the entire truth table. It is obvious that the present method will suffice to prove the invalidity of any argument which can be shown to be invalid by a truth table.

EXERCISES

Prove the invalidity of each of the following arguments by the method of assigning truth values:

*1. $A \supset B$
$C \supset D$
$B \vee C$
$\therefore A \vee D$

2. $E \supset (F \vee G)$
$G \supset (H \cdot I)$
$\sim H$
$\therefore E \supset I$

3. $J \supset (K \supset L)$
$K \supset (\sim L \supset M)$
$(L \vee M) \supset N$
$\therefore J \supset N$

4. $(O \vee P) \supset Q$
$Q \supset (P \vee R)$
$O \supset (\sim S \supset P)$
$(S \supset O) \supset \sim R$
$\therefore P \equiv Q$

*5. $T \equiv U$
$U \equiv (V \cdot W)$
$V \equiv (T \vee X)$
$T \vee X$
$\therefore T \cdot X$

6. $X \equiv (Y \supset Z)$
$Y \equiv (\sim X \cdot \sim Z)$
$Z \equiv (X \vee \sim Y)$
Y
$\therefore X \vee Z$

7. $(A \supset B) \cdot (C \supset D)$
$A \vee C$
$(B \vee D) \supset (E \cdot F)$
$E \supset (F \supset G)$
$G \supset (A \supset H)$
$\therefore H$

8. $I \vee (J \cdot K)$
$(I \vee J) \supset (L \equiv \sim M)$
$(L \supset \sim M) \supset (M \cdot \sim N)$
$(N \supset O) \cdot (O \supset M)$
$(J \supset K) \supset O$
$\therefore O$

9. $P \equiv (Q \equiv \sim R)$
$Q \supset (\sim R \vee \sim S)$
$[R \supset (Q \vee \sim T)] \cdot (P \supset Q)$
$[U \supset (S \cdot T)] \cdot (T \supset \sim V)$
$[(Q \cdot R) \supset \sim U] \cdot [U \supset (Q \vee R)]$
$(Q \vee V) \cdot \sim V$
$\therefore \sim U \cdot \sim V$

10. $W \equiv (X \vee Y)$
$X \equiv (Z \supset Y)$
$Y \equiv (Z \equiv \sim A)$
$Z \equiv (A \supset B)$
$A \equiv (B \equiv Z)$
$B \vee \sim W$
$\therefore W \equiv B$

3.4 Incompleteness of the Nineteen Rules[2]

The nineteen Rules of Inference presented thus far are *incomplete,* which is to say that there are truth functionally valid arguments whose validity cannot be proved using only those nineteen Rules. To discuss and establish this incompleteness it is useful to introduce the notion of a characteristic that is 'hereditary with respect to a set of rules of inference'. We offer this definition: a characteristic Φ is *hereditary with respect to a set of rules of inference* if and only if whenever Φ belongs to one or more statements it also belongs to every statement deduced from them by means of those Rules of Inference. For example, *truth* is a characteristic that is hereditary with respect to the nineteen Rules of Inference presented in the first two sections of the present chapter. As has already been remarked, any conclusion must be true if it can be deduced from true premisses by means of our nineteen Rules of Inference. Indeed, we should not want to use any rules of inference with respect to which truth was *not* hereditary.

Now to prove that a set of rules of inference is incomplete we must find a characteristic Φ and a valid argument α such that

(1) Φ is hereditary with respect to the set of rules of inference; and
(2) Φ belongs to the premisses of α but not to the conclusion of α.

The characteristic *truth* is hereditary with respect to any set of rules of inference in which we may be seriously interested, and therefore satisfies condition (1) above. But where α is a valid argument, it follows immediately from our definition of validity that *truth* can never satisfy condition (2) above. Hence to prove the incompleteness of our nineteen Rules we must find a characteristic other than *truth* which is hereditary with respect to our nineteen Rules, and can belong to the premisses but not to the conclusion of some valid argument α.

To obtain such a characteristic we introduce a three-element model in terms of which the symbols in our nineteen Rules can be interpreted. The three elements are the numbers **0**, **1**, and **2**, which play roles analogous to those of the truth values true (**T**) and false (**F**) introduced in Chapter 2. Every statement will have one of the three elements of the model assigned to it,

[2]The following proof of incompleteness was communicated to me by my friend Professor Leo Simons of the University of Cincinnati.

and it will be said to assume, take on, or have one of the three *values* **0**, **1**, or **2**. Just as in Chapter 2 the statement variables $p, q, r, \ldots$, were allowed to range over the two truth values **T** and **F**, so here we allow the statement variables $p, q, r, \ldots$ to range over the three values **0**, **1**, and **2**.

The five symbols '$\sim$', '$\cdot$', 'v', '$\supset$', and '$\equiv$' that occur in our nineteen Rules can be redefined for (or in terms of) our three-element model by the following three-valued tables:

p	$\sim p$
0	2
1	1
2	0

p	q	$p \cdot q$	$p \vee q$	$p \supset q$	$p \equiv q$
0	0	0	0	0	0
0	1	1	0	1	1
0	2	2	0	2	2
1	0	1	0	0	1
1	1	1	1	1	1
1	2	2	1	1	1
2	0	2	0	0	2
2	1	2	1	0	1
2	2	2	2	0	0

Alternative (but equivalent) analytical definitions can be given as follows, where 'min(x, y)' denotes the minimum of the numbers x and y, and 'max(x, y)' denotes the maximum of the numbers x and y.

$$\begin{aligned} \sim p &= 2 - p \\ p \cdot q &= \max(p, q) \\ p \vee q &= \min(p, q) \\ p \supset q &= \min(2 - p, q) \\ p \equiv q &= \max(\min(2 - p, q), \min(2 - q, p)) \end{aligned}$$

The desired characteristic Φ that is hereditary with respect to our nineteen Rules of Inference is the characteristic of having the value **0**. To prove that it is hereditary with respect to the nineteen Rules it will suffice to show that it is hereditary with respect to each of the nineteen Rules. This can be shown for each rule by means of a three-valued table. For example, that having the value **0** is hereditary with respect to *Modus Ponens*, can be seen by examining the table above that defines the value of '$p \supset q$' as a function of the values of 'p' and of 'q'. The two premisses 'p' and '$p \supset q$' have the value **0** only in the first row, and there the conclusion 'q' has the value **0** also. Examining the same table shows that having the value **0** is hereditary also for Simplification, Conjunction, and Addition. Filling in additional columns for '$\sim p$' and '$\sim q$' will show that having the value **0** is hereditary with respect to *Modus Tollens* and Disjunctive Syllogism also. That it is hereditary with respect to Hypothetical Syllogism can be shown by the following table:

p	q	r	$p \supset q$	$q \supset r$	$p \supset r$
0	0	0	0	0	0
0	0	1	0	1	1
0	0	2	0	2	2
0	1	0	1	0	0
0	1	1	1	1	1
0	1	2	1	1	2
0	2	0	2	0	0
0	2	1	2	0	1
0	2	2	2	0	2
1	0	0	0	0	0
1	0	1	0	1	1
1	0	2	0	2	1
1	1	0	1	0	0
1	1	1	1	1	1
1	1	2	1	1	1
1	2	0	1	0	0
1	2	1	1	0	1
1	2	2	1	0	1
2	0	0	0	0	0
2	0	1	0	1	0
2	0	2	0	2	0
2	1	0	0	0	0
2	1	1	0	1	0
2	1	2	0	1	0
2	2	0	0	0	0
2	2	1	0	0	0
2	2	2	0	0	0

Only in the first, tenth, nineteenth, twenty-second, twenty-fifth, twenty-sixth, and twenty-seventh rows do the two premisses '$p \supset q$' and '$q \supset r$' have the value **0**, and in each of them the conclusion '$p \supset r$' has the value **0** also. Even larger tables would be needed to show that having the value **0** is hereditary with respect to Constructive Dilemma and Destructive Dilemma, but they are easily constructed. (It is not absolutely necessary to construct them, however, because the alternative analytical definitions on page 48 can be used to show that having the value **0** is hereditary with respect to the Dilemmas, as on page 50.)

When we construct three-valued tables to verify that having the value **0** is hereditary with respect to replacement of statements by their logical equivalents, we notice that although the biconditionals themselves need not have the value **0**, the expressions flanking the equivalence sign necessarily have the same value. For example, in the table appropriate to the first of De Morgan's Theorems,

p	q	$\sim p$	$\sim q$	$p \cdot q$	$\sim(p \cdot q)$	$\sim p \vee \sim q$	$\sim(p \cdot q) \equiv (\sim p \vee \sim q)$
0	0	2	2	0	2	2	0
0	1	2	1	1	1	1	1
0	2	2	0	2	0	0	0
1	0	1	2	1	1	1	1
1	1	1	1	1	1	1	1
1	2	1	0	2	0	0	0
2	0	0	2	2	0	0	0
2	1	0	1	2	0	0	0
2	2	0	0	2	0	0	0

the equivalent expressions '$\sim(p \cdot q)$' and '$\sim p \vee \sim q$' have the same value in every row even though the statement of their equivalence fails to have the value **0** in rows two, four, and five. It should be obvious, however, that having the value **0** is hereditary with respect to the replacement of all or part of any statement by any other statement equivalent to the part replaced.

Alternative proofs that having the value **0** is hereditary with respect to the nineteen Rules make use of our analytical definitions of the logical symbols. For example, that having the value **0** is hereditary with respect to Constructive Dilemma can be shown by the following argument. By assumption, '$(p \supset q) \cdot (r \supset s)$' and '$p \vee r$' both have the value **0**. Hence both '$p \supset q$' and '$r \supset s$' have the value **0**, so either $p = 2$ or $q = 0$ and either $r = 2$ or $s = 0$. Since '$p \vee r$' has the value **0**, either $p = 0$ or $r = 0$. If $p = 0$ then $p \neq 2$ whence $q = 0$, and if $r = 0$ then $r \neq 2$ whence $s = 0$, therefore either $q = 0$ or $s = 0$ whence '$q \vee s$' has the value **0**, which was to be shown.

Once it has been shown that the characteristic of having the value **0** is hereditary with respect to the nineteen Rules, to prove the incompleteness of those Rules one need only exhibit a valid argument whose premisses have the value **0** but whose conclusion does not have the value **0**. Such an argument is

$$A \supset B$$
$$\therefore A \supset (A \cdot B)$$

whose validity is easily established by a truth table. In case 'A' has the value **1** and 'B' has the value **0**, the premiss '$A \supset B$' has the value **0** but the conclusion '$A \supset (A \cdot B)$' has the value **1**. Therefore the nineteen Rules of Inference presented thus far are incomplete.

3.5 The Rule of Conditional Proof

Next we introduce a new rule to use in the method of deduction: the rule of Conditional Proof. In this section the new rule will be applied only to arguments whose conclusions are conditional statements. The new rule can best be explained and justified by reference to the principle of Exportation and the correspondence, noted in Chapter 2, between valid argument forms and tautologies.

To every argument there corresponds a conditional statement whose antecedent is the conjunction of the argument's premisses and whose consequent is the argument's conclusion. As has been remarked, an argument is valid if and only if its corresponding conditional is a tautology. If an argument has a conditional statement for its conclusion, which we may symbolize as $A \supset C$, then if we symbolize the conjunction of its premisses as P, the argument is valid if and only if the conditional

$$P \supset (A \supset C) \tag{1}$$

is a tautology. If we can deduce the conclusion $A \supset C$ from the premisses conjoined in P by a sequence of elementary valid arguments, we thereby prove the argument to be valid and the associated conditional (1) to be a tautology. By the principle of Exportation, (1) is logically equivalent to

$$(P \cdot A) \supset C \tag{2}$$

But (2) is the conditional associated with a somewhat different argument. This second argument has as its premisses all of the premisses of the first argument *plus* an additional premiss which is the antecedent of the conclusion of the first argument. And the conclusion of the second argument is the consequent of the conclusion of the first argument. Now if we deduce the conclusion of the second argument, C, from the premisses conjoined in $P \cdot A$ by a sequence of elementary valid arguments, we thereby prove that its associated conditional statement (2) is a tautology. But since (1) and (2) are logically equivalent, this fact proves that (1) is a tautology also, from which it follows that the original argument, with one less premiss and the conditional conclusion $A \supset C$, is valid also. Now the rule of Conditional Proof permits us to infer the validity of any argument

$$\begin{array}{l} P \\ \therefore A \supset C \end{array}$$

from a formal proof of validity for the argument

$$\begin{array}{l} P \\ A \\ \therefore C \end{array}$$

Given any argument whose conclusion is a conditional statement, a proof of its validity using the rule of Conditional Proof, that is, a conditional proof of its validity, is constructed by assuming the antecedent of its conclusion as an additional premiss and then deducing the consequent of its conclusion by a sequence of elementary valid arguments. Thus a conditional proof of validity for the argument

$$
\begin{array}{l}
(A \vee B) \supset (C \cdot D) \\
(D \vee E) \supset F \\
\therefore A \supset F
\end{array}
$$

may be written as

1.	$(A \vee B) \supset (C \cdot D)$	
2.	$(D \vee E) \supset F$	$/\therefore A \supset F$
3.	A	$/\therefore F$ (C.P.)
4.	$A \vee B$	3, Add.
5.	$C \cdot D$	1, 4, M.P.
6.	$D \cdot C$	5, Com.
7.	D	6, Simp.
8.	$D \vee E$	7, Add.
9.	F	2, 8, M.P.

Here the second slant line and three dot 'therefore' symbol, as well as the parenthesized 'C.P.', indicate that the rule of Conditional Proof is being used.

Since the rule of Conditional Proof can be used in dealing with any valid argument having a conditional statement as conclusion, it can be applied more than once in the course of the same deduction. Thus a conditional proof of validity for

$$
\begin{array}{l}
A \supset (B \supset C) \\
B \supset (C \supset D) \\
\therefore A \supset (B \supset D)
\end{array}
$$

will be a proof of validity for

$$
\begin{array}{l}
A \supset (B \supset C) \\
B \supset (C \supset D) \\
A \\
\therefore B \supset D
\end{array}
$$

and since the latter itself has a conditional conclusion, it can be given a conditional proof by proving the validity of

$$
\begin{array}{l}
A \supset (B \supset C) \\
B \supset (C \supset D) \\
A \\
B \\
\therefore D
\end{array}
$$

Each use of the conditional method should be signalled by an additional slant line and 'therefore' sign, in addition to the notation '(C.P.)'. The suggested proof would be written:

1. $A \supset (B \supset C)$
2. $B \supset (C \supset D)$ $/\therefore A \supset (B \supset D)$
3. A $/\therefore B \supset D$ (C.P.)
4. B $/\therefore D$ (C.P.)
5. $B \supset C$ 1, 3, M.P.
6. C 5, 4, M.P.
7. $C \supset D$ 2, 4, M.P.
8. D 7, 6, M.P.

The rule of Conditional Proof is a genuine addition to the proof apparatus of Sections 3.1 and 3.2. Not only does it permit the construction of *shorter* proofs of validity for arguments which could be proved valid by appealing to the original list of nineteen Rules of Inference alone, but it permits us to establish the validity of valid arguments whose validity could *not* be proved by reference to the original list alone. For example, it was shown in Section 3.4 that the obviously valid argument

$$A \supset B$$
$$\therefore A \supset (A \cdot B)$$

cannot be proved valid using only the original list of nineteen Rules of Inference. But it is easily proved valid by using, in addition, the rule of Conditional Proof. Its conditional proof of validity is

1. $A \supset B$ $/\therefore A \supset (A \cdot B)$
2. A $/\therefore A \cdot B$ (C.P.)
3. B 1, 2, M.P.
4. $A \cdot B$ 2, 3, Conj.

EXERCISES

Give conditional proofs of validity for Exercises *21, 22, 23, 24, and 25 on page 45.

3.6 The Rule of Indirect Proof

The method of *indirect proof*, often called the method of proof by *reductio ad absurdum*, is familiar to all who have studied elementary geometry. In deriving his theorems, Euclid often begins by assuming the opposite of what he wants to prove. If that assumption leads to a contradiction, or 'reduces to an absurdity', then that assumption must be false, and so its negation—the theorem to be proved—must be true.

An indirect proof of validity for a given argument is constructed by assuming, as an additional premiss, the negation of its conclusion, and then deriving an explicit contradiction from the augmented set of premisses. Thus an indirect proof of validity for the argument

$$A \supset (B \cdot C)$$
$$(B \vee D) \supset E$$
$$D \vee A$$
$$\therefore E$$

may be set down as follows:

1.	$A \supset (B \cdot C)$	
2.	$(B \vee D) \supset E$	
3.	$D \vee A$ $\quad /\therefore E$	
4.	$\sim E$	I.P. (Indirect Proof)
5.	$\sim(B \vee D)$	2, 4, M.T.
6.	$\sim B \cdot \sim D$	5, De M.
7.	$\sim D \cdot \sim B$	6, Com.
8.	$\sim D$	7, Simp.
9.	A	3, 8, D.S.
10.	$B \cdot C$	1, 9, M.P.
11.	B	10, Simp.
12.	$\sim B$	6, Simp.
13.	$B \cdot \sim B$	11, 12, Conj.

Here line 13 is an explicit contradiction, so the demonstration is complete, for the validity of the original argument follows by the rule of Indirect Proof.

It is easy to show that from a contradiction *any* conclusion can validly be deduced. In other words, any argument of the form

$$p$$
$$\sim p$$
$$\therefore q$$

is valid, no matter what statements are substituted for the variables p and q. Thus from lines 11 and 12 in the preceding proof, the conclusion E can be derived in just two more lines. Such a continuation would proceed:

14.	$B \vee E$	11, Add.
15.	E	14, 12, D.S.

Hence it is possible to regard an indirect proof of the validity of a given argument not as the deduction of its validity *from the fact that* a contradiction was obtained, but rather as deducing that argument's conclusion *from the* *contradiction itself*. Thus instead of viewing a *reductio ad absurdum* proof as proceeding *only up to* the contradiction, we can regard it as *going on through* the contradiction to the conclusion of the original argument. If we symbolize the conjunction of the premisses of an argument as P and its conclusion as C, then an indirect proof of the validity of

$$\begin{array}{l} P \\ \therefore C \end{array}$$

will be provided by a formal proof of validity for the argument

$$\begin{array}{l} P \\ \sim C \\ \therefore C \end{array}$$

What connection is there between the two arguments

$$\begin{array}{l} P \\ \therefore C \end{array} \quad \text{and} \quad \begin{array}{l} P \\ \sim C \\ \therefore C \end{array}$$

which makes proving the second valid suffice to establish the validity of the first? A formal proof of validity for the latter constitutes a conditional proof of validity for a third argument

$$\begin{array}{l} P \\ \therefore \sim C \supset C \end{array}$$

But the conclusion of this third argument is logically equivalent to the conclusion of the first. By the definition of material implication, $\sim C \supset C$ is logically equivalent to $\sim \sim C \vee C$, which is logically equivalent to $C \vee C$ by the principle of Double Negation. And $C \vee C$ and C are logically equivalent by the principle of Tautology. Since the first and third arguments have identical premisses and logically equivalent conclusions, any proof of validity for one is a proof of validity for the other also. A proof of validity for the second argument is both a conditional proof of the third and an indirect proof of the first. Thus we see that there is an intimate relationship between the conditional and the indirect methods of proof, that is, between the rule of Conditional Proof and the rule of Indirect Proof.

Adding the rule of Indirect Proof serves to strengthen our proof apparatus still further. Any argument whose conclusion is a tautology can be shown to be valid, regardless of what its premisses may be, by the method of truth tables. But if the tautologous conclusion of an argument is not a conditional statement, and the premisses are consistent with each other and quite irrelevant to that conclusion, then the argument cannot be proved valid by the method of deduction without use being made of the rule of Indirect Proof. Although the argument

$$A$$
$$\therefore B \vee (B \supset C)$$

cannot be proved valid by the means set forth in the preceding sections, its validity is easily established using the rule of Indirect Proof. One proof of its validity is this:

1.	A	$/\therefore B \vee (B \supset C)$
2.	$\sim[B \vee (B \supset C)]$	I.P.
3.	$\sim[B \vee (\sim B \vee C)]$	2, Impl.
4.	$\sim[(B \vee \sim B) \vee C]$	3, Assoc.
5.	$\sim(B \vee \sim B) \cdot \sim C$	4, De M.
6.	$\sim(B \vee \sim B)$	5, Simp.
7.	$\sim B \cdot \sim\sim B$	6, De M.

Our nineteen Rules of Inference plus the rules of Conditional and Indirect Proof provide us with a method of deduction that is complete. Any argument whose validity can be established by the use of truth tables can be proved valid by the method of deduction as set forth in Sections 3.1, 3.2, 3.5, and 3.6. This will not be proved, however, until the end of Chapter 7.

EXERCISES

For each of the following arguments construct both a formal proof of validity and an indirect proof, and compare their lengths:

1. $A \vee (B \cdot C)$
$A \supset C$
$\therefore C$

2. $(D \vee E) \supset (F \supset G)$
$(\sim G \vee H) \supset (D \cdot F)$
$\therefore G$

***3.** $(H \supset I) \cdot (J \supset K)$
$(I \vee K) \supset L$
$\sim L$
$\therefore \sim(H \vee J)$

4. $(M \vee N) \supset (O \cdot P)$
$(O \vee Q) \supset (\sim R \cdot S)$
$(R \vee T) \supset (M \cdot U)$
$\therefore \sim R$

***5.** $(V \supset \sim W) \cdot (X \supset Y)$
$(\sim W \supset Z) \cdot (Y \supset \sim A)$
$(Z \supset \sim B) \cdot (\sim A \supset C)$
$V \cdot X$
$\therefore \sim B \cdot C$

3.7 Proofs of Tautologies

The conditional and indirect methods of proof can be used not only to establish the validity of arguments, but also to prove that certain statements and statement forms are tautologies. Any conditional statement corresponds, in a sense, to an argument whose single premiss is the antecedent of the conditional, and whose conclusion is the conditional's consequent. The conditional is a tautology if and only if that argument is valid. Hence a conditional is proved tautologous by deducing its consequent from its antecedent by a sequence of elementary valid arguments. Thus the statement $(A \cdot B) \supset A$ is proved tautologous by the same sequence of lines which proves the validity of the argument

$$A \cdot B$$
$$\therefore A$$

It has already been noted that the conditional method can be used repeatedly in a single proof. Thus the conditional statement

$$(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]$$

is proved tautologous by the following:

1. $Q \supset R$ $/\therefore (P \supset Q) \supset (P \supset R)$ (C.P.)
2. $P \supset Q$ $/\therefore P \supset R$ (C.P.)
3. $P \supset R$ 2, 1, H.S.

For some complicated conditional statements, this method of proving them tautologous is shorter and easier than constructing truth tables.

There are many tautologies that are not conditional in form, and to these the preceding method cannot be applied. But any tautology can be established as tautologous by the indirect method. As applied to an *argument*, the indirect method of proving validity proceeds by adding the negation of its conclusion to the argument's premisses and then deducing a contradiction by a sequence of elementary valid arguments. As applied to a *statement*, the indirect method of proving it tautologous proceeds by taking its negation as premiss and then deducing an explicit contradiction by a sequence of elementary valid arguments. Thus the statement $B \vee \sim B$ is proved to be a tautology by the following:

1. $\sim(B \vee \sim B)$ $/\therefore B \vee \sim B$ (I.P.)
2. $\sim B \cdot \sim\sim B$ 1, De M.

To say that a statement is a tautology is to assert that its truth is unconditional, so it can be established without appealing to any other statements as premisses. Another, perhaps not too misleading, way of saying the same thing is to assert the validity of the 'argument' which has the statement in question as 'conclusion', but has no premisses at all. If the 'conclusion' *is* a tautology, then the method of deduction permits us to prove that the 'argument' is valid even though it has no premisses—using either the rule of Conditional Proof or the rule of Indirect Proof. Any tautology can be established by the method of deduction, although this will not be proved until the end of Chapter 7.

EXERCISES

I. Use the method of conditional proof to verify that the following are tautologies:

*1. $P \supset (Q \supset P)$
2. $[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$
3. $[P \supset (Q \supset R)] \supset [Q \supset (P \supset R)]$
4. $(P \supset Q) \supset (\sim Q \supset \sim P)$
*5. $\sim\sim P \supset P$
6. $P \supset \sim\sim P$
7. $(A \supset B) \supset [(B \supset C) \supset (A \supset C)]$
8. $[(A \supset B) \cdot (A \supset C)] \supset [A \supset (B \vee C)]$
9. $[(A \supset B) \cdot (A \supset C)] \supset [A \supset (B \cdot C)]$
*10. $(A \supset B) \supset [A \supset (A \cdot B)]$
11. $(A \supset B) \supset [(\sim A \supset B) \supset B]$
12. $(A \supset B) \supset [(A \cdot C) \supset (B \cdot C)]$
13. $[(A \supset B) \supset B] \supset (A \vee B)$
14. $(B \supset C) \supset [(A \vee B) \supset (C \vee A)]$
*15. $[A \supset (B \cdot C)] \supset \{[B \supset (D \cdot E)] \supset (A \supset D)\}$
16. $[(A \vee B) \supset C] \supset \{[(C \vee D) \supset E] \supset (A \supset E)\}$
17. $[(A \supset B) \supset A] \supset A$
18. $P \supset (P \cdot P)$
19. $(P \cdot Q) \supset P$
20. $(P \supset Q) \supset [\sim(Q \cdot R) \supset \sim(R \cdot P)]$

II. Use the method of indirect proof to verify that the following are tautologies:

*1. $(A \supset B) \vee (A \supset \sim B)$
2. $(A \supset B) \vee (\sim A \supset B)$
3. $(A \supset B) \vee (B \supset A)$
4. $(A \supset B) \vee (B \supset C)$
*5. $(A \supset B) \vee (\sim A \supset C)$
6. $A \vee (A \supset B)$
7. $P \equiv \sim\sim P$
8. $A \equiv [A \cdot (A \vee B)]$
9. $A \equiv [A \vee (A \cdot B)]$
10. $\sim[(A \supset \sim A) \cdot (\sim A \supset A)]$

3.8 The Strengthened Rule of Conditional Proof

In the preceding sections the method of Conditional Proof was applied only to arguments whose conclusions were conditional in form. But in the next chapter it will be convenient to use something like the method of Conditional Proof for arguments whose conclusions are not explicit conditional statements.

To accomplish this purpose we strengthen our rule of Conditional Proof and thereby give it wider applicability.

To formulate our strengthened rule of Conditional Proof it is useful to adopt a new method of writing out proofs that make use of the Conditional Method. As explained in Section 3.5, we used the method of Conditional Proof to establish the validity of an argument having a conditional as conclusion by adding the antecedent of that conditional to the argument's premisses as an assumption, and then deducing the conditional's consequent. The notation in Section 3.5 involved the use of an additional slant line and an extra *therefore* sign, as in proving the validity of the argument

$$A \supset B$$
$$\therefore A \supset (A \cdot B)$$

by the following four line proof:

1.	$A \supset B$	$/\therefore A \supset (A \cdot B)$	
2.	A	$/\therefore A \cdot B$	(C.P.)
3.	B	1, 2, M.P.	
4.	$A \cdot B$	2, 3, Conj.	

A Conditional Proof of validity for that same argument is set down in our new notation as the following sequence of five lines:

1.	$A \supset B$	$/\therefore A \supset (A \cdot B)$
2.	A	assumption
3.	B	1, 2, M.P.
4.	$A \cdot B$	2, 3, Conj.
5.	$A \supset (A \cdot B)$	2–4, C.P.

Here the fifth line is inferred not from any one or two of the preceding lines but from the *sequence* of lines 2, 3, 4, which constitutes a valid deduction of line 4 from lines 1 and 2. In line 5 we infer the validity of the argument

$$A \supset B$$
$$\therefore A \supset (A \cdot B)$$

from the demonstrated validity of the argument

$$A \supset B$$
$$A$$
$$\therefore A \cdot B$$

That inference is 'justified' by noting the sequence of lines to which appeal is made, and using the letters 'C.P.' to show that the principle of Conditional Proof is being used.

In the second of the preceding proofs, line 2, the assumption, has lines 3 and 4 dependent upon it. Line 5, however, is *not* dependent upon line 2, but only upon line 1. Line 5 is therefore *outside* or *beyond the scope* of the assumption made as line 2. When an assumption is made in a Conditional Proof of validity, its 'scope' is always *limited*, never extending all the way to the last line of the demonstration.

A notation is now introduced which is very helpful in keeping track of assumptions and their *scopes*. A bent arrow is used for this purpose, with its head pointing at the assumption from the left, its shaft bent down to run along all lines within the scope of the assumption, and then bent inward to mark the end of the scope of that assumption. The scope of the assumption in the preceding proof is indicated thus:

	1.	$A \supset B$	$/ \therefore A \supset (A \cdot B)$
→	2.	A	assumption
	3.	B	1, 2, M.P.
	4.	$A \cdot B$	2, 3, Conj.
	5.	$A \supset (A \cdot B)$	2–4, C.P.

It should be observed that *only* a line inferred by the principle of Conditional Proof ends the scope of an assumption, and that *every* use of the rule of Conditional Proof serves to end the scope of an assumption. When the scope of an assumption has been ended, the assumption is said to have been *discharged,* and no subsequent line can be justified by reference to it or to any line lying between it and the line inferred by the rule of Conditional Proof that discharges it. Only lines lying between an assumption of limited scope and the line that discharges it can be justified by reference to that assumption. After one assumption of limited scope has been discharged, another such assumption may be made and then discharged. Or a second assumption of limited scope may be written within the scope of a first. Scopes of different assumptions may follow each other, or one scope may be contained entirely within another.

If the scope of an assumption does *not* extend all the way to the end of a proof, then the final line of the proof does not *depend* on that assumption, but has been proved to follow from the original premisses alone. Hence we need not restrict ourselves to using as assumptions only the antecedents of conditional conclusions. *Any* proposition can be taken as an assumption of limited scope, for the final line that is the conclusion will always be beyond its scope and independent of it.

A more complex demonstration that involves making *two* assumptions is the following (incidentally, when our bent arrow notation is used, the word 'assumption' need not be written, since each assumption is sufficiently identified as such by the arrowhead on its left):

1.	$(A \vee B) \supset [(C \vee D) \supset E]$	$/ \therefore A \supset [(C \cdot D) \supset E]$
→ 2.	A	
3.	$A \vee B$	2, Add.
4.	$(C \vee D) \supset E$	1, 3, M.P.
→ 5.	$C \cdot D$	
6.	C	5, Simp.
7.	$C \vee D$	6, Add.
8.	E	4, 7, M.P.
9.	$(C \cdot D) \supset E$	5–8, C.P.
10.	$A \supset [(C \cdot D) \supset E]$	2–9, C.P.

In this proof, lines 2 through 9 lie within the scope of the first assumption, while lines 5, 6, 7, and 8 lie within the scope of the second assumption. From these examples it is clear that the scope of an assumption α in a proof consists of all lines α through φ where the line following φ is of the form $\alpha \supset \varphi$ and is inferred by C.P. from that sequence of lines. In the preceding proof, the second assumption lies within the scope of the first because it lies between the first assumption and line 10 which is inferred by C.P. from the sequence of lines 2 through 9.

When we use this new method of writing out a Conditional Proof of validity the scope of every original premiss extends all the way to the end of the proof. The original premisses may be supplemented by additional assumptions provided that the latter's scopes are limited and do not extend to the end of the proof. Each line of a formal proof of validity must be either a premiss, or an assumption of limited scope, or must follow validly from one or two preceding lines by a Rule of Inference, or must follow from a *sequence* of preceding lines by the principle of Conditional Proof.

It should be remarked that the strengthened principle of Conditional Proof includes the method of Indirect Proof as a special case. Since any assumption of limited scope may be made in a Conditional Proof of validity, we can take as our assumption the negation of the argument's conclusion. Once a contradiction is obtained, we can *continue on through* the contradiction to obtain the desired conclusion by Addition and the Disjunctive Syllogism. Once that is done, we can use the rule of Conditional Proof to end the scope of that assumption and obtain a conditional whose consequent is the argument's conclusion and whose antecedent is the negation of that conclusion. And from such a conditional the argument's conclusion will follow by Implication, Double Negation, and Tautology.

From now on, the strengthened Rule of Conditional Proof will be referred to simply as the Rule of Conditional Proof.

EXERCISES

Use the strengthened method of conditional proof to prove the validity of the following arguments:

*1. $A \supset B$
$B \supset [(C \supset {\sim}{\sim}C) \supset D]$
$\therefore A \supset D$

2. $(E \vee F) \supset G$
$H \supset (I \cdot J)$
$\therefore (E \supset G) \cdot (H \supset I)$

3. $(K \supset L) \cdot (M \supset N)$
$(L \vee N) \supset \{[O \supset (O \vee P)] \supset (K \cdot M)\}$
$\therefore K \equiv M$

4. $Q \vee (R \supset S)$
$[R \supset (R \cdot S)] \supset (T \vee U)$
$(T \supset Q) \cdot (U \supset V)$
$\therefore Q \vee V$

5. $[W \supset ({\sim}X \cdot {\sim}Y)] \cdot [Z \supset {\sim}(X \vee Y)]$
$({\sim}A \supset W) \cdot ({\sim}B \supset Z)$
$(A \supset X) \cdot (B \supset Y)$
$\therefore X \equiv Y$

6. $(C \vee D) \supset (E \supset F)$
$[E \supset (E \cdot F)] \supset G$
$G \supset [({\sim}H \vee {\sim}{\sim}H) \supset (C \cdot H)]$
$\therefore C \equiv G$

3.9 Shorter Truth Table Technique—*Reductio Ad Absurdum* Method

There is still another method of testing the validity of arguments, and of classifying statements as tautologous, contradictory, or contingent. In the preceding section it was pointed out that an argument is invalid if and only if it is possible to assign truth values to its component *simple* statements in such a way as to make all its premisses true and its conclusion false. It is impossible to make such truth value assignments in the case of a valid argument. Hence to prove the validity of an argument it suffices to prove that no such truth values can be assigned. We do so by showing that its premisses can be made true and its conclusion false only by assigning truth values *inconsistently*, so that some component statement is assigned both a **T** and an **F**. In other words, if the truth value **T** is assigned to each premiss of a valid argument and the truth value **F** is assigned to its conclusion, this will necessitate assigning *both* **T** and **F** to some component statement, which is, of course, a contradiction.

Thus to prove the validity of the argument

$$(A \vee B) \supset (C \cdot D)$$
$$(D \vee E) \supset F$$
$$\therefore A \supset F$$

we assign **T** to each premiss and **F** to the conclusion. Assigning **F** to the conclusion requires that **T** be assigned to A and **F** be assigned to F. Since **T** is assigned to A, the antecedent of the first premiss is true, and since the premiss has been assigned **T**, its consequent must be true also, so **T** must be assigned to both C and D. Since **T** is assigned to D, the antecedent of the second premiss is true, and since the premiss has been assigned **T**, its consequent must be true also, so **T** must be assigned to F. But we have already been forced to assign **F** to F to make the conclusion false. Hence the argument is invalid only if the statement F is both true and false, which is impossible.

This method of proving the validity of an argument is a version of the *reductio ad absurdum* technique, which uses truth value assignments rather than Rules of Inference.

It is easy to extend the use of this method to the classification of statements (and statement forms). Thus to certify that Peirce's Law $[(p \supset q) \supset p] \supset p$ is a tautology, we assign it the truth value **F**, which requires us to assign **T** to its antecedent $[(p \supset q) \supset p]$ and **F** to its consequent p. For the conditional $[(p \supset q) \supset p]$ to be true while *its* consequent p is false, its antecedent $(p \supset q)$ must be assigned the truth value **F** also. But for the conditional $p \supset q$ to be false, its antecedent p must be assigned **T** and its consequent q assigned **F**. However, we were previously forced to assign **F** to p, so assuming Peirce's Law false leads to a contradiction, which proves it a tautology.

If it *is* possible to assign truth values consistently to its components on the assumption that it is false, then the expression in question is not a tautology, but must be either contradictory or contingent. In such a case we attempt to assign truth values to make it true. If this attempt leads to a contradiction the expression cannot possibly be true and must be a contradiction. But if truth values can be assigned to make it true and (other) truth values assigned to make it false, then it is neither a tautology nor a contradiction, but is contingent.

The *reductio ad absurdum* method of assigning truth values is by far the quickest and easiest method of testing arguments and classifying statements. It is, however, more readily applied in some cases than in others. If **F** is assigned to a disjunction, **F** must be assigned to both disjuncts, and where **T** is assigned to a conjunction, **T** must be assigned to both conjuncts. Here the sequence of assignments is forced. But where **T** is assigned to a disjunction or **F** to a conjunction, that assignment by itself does not determine *which* disjunct is true or *which* conjunct is false. Here we should have to experiment and make various 'trial assignments', which will tend to diminish the advantage of the method for such cases. Despite these complications, however, in the vast majority of cases the *reductio ad absurdum* method is superior to any other method known.

EXERCISES

1. Use the *reductio ad absurdum* method of assigning truth values to decide the validity or invalidity of the arguments and argument forms in the Exercises on pages 23–25.
2. Use the *reductio ad absurdum* method of assigning truth values to establish that the statements in Exercises I and II on page 58 are tautologies.
3. Use the *reductio ad absurdum* method of assigning truth values to classify the statement forms in Exercise I on page 29 as tautologous, contradictory, or contingent.

4

Propositional Functions and Quantifiers

4.1 Singular Propositions and General Propositions

The logical techniques developed thus far apply only to arguments whose validity depends upon the way in which simple statements are truth-functionally combined into compound statements. Those techniques cannot be applied to such arguments as the following:

All humans are mortal.
Socrates is human.
Therefore Socrates is mortal.

The validity of such an argument depends upon the inner logical structure of the noncompound statements it contains. To appraise such arguments we must develop methods for analyzing noncompound statements and symbolizing their inner structures.

The second premiss of the preceding argument is a *singular proposition;* it asserts that the individual Socrates has the attribute of being human. We call 'Socrates' the *subject* term and 'human' the *predicate* term. Any (affirmative) singular proposition asserts that the individual referred to by its subject term has the attribute designated by its predicate term. We regard as individuals not only persons, but any *things,* such as animals, cities, nations, planets, or stars, of which attributes can significantly be predicated. Attributes can be designated not only by adjectives, but by nouns or even verbs: thus 'Helen is a gossip' and 'Helen gossips' have the same meaning, which can also be expressed as 'Helen is gossipy'.

In symbolizing singular propositions we use the small letters '*a*' through '*w*' to denote individuals, ordinarily using the first letter of an individual's name to denote that individual. Because these symbols denote individuals we call them 'individual constants'. To designate attributes we use capital letters, being guided by the same principle in their selection. Thus in the context of the preceding argument we denote Socrates by the small letter '*s*' and symbolize the attributes *human* and *mortal* by the capital letters '*H*' and '*M*'.

To express a singular proposition in our symbolism we write the symbol for its predicate term to the left of the symbol for its subject term. Thus we symbolize 'Socrates is human' as '*Hs*' and 'Socrates is mortal' as '*Ms*'.[1]

Examining the symbolic formulations of singular propositions having the same predicate term, we observe them to have a common pattern. The symbolic formulations of the singular propositions 'Aristotle is human', 'Boston is human', 'California is human', 'Descartes is human', . . . , which are '*Ha*', '*Hb*', '*Hc*', '*Hd*', . . . , each consists of the attribute symbol '*H*' followed by an individual constant. We use the expression '*Hx*' to symbolize the pattern common to all singular propositions asserting individuals to have the attribute *human*. The small letter '*x*'—called an 'individual variable'—is a mere *place marker* that serves to indicate where an individual constant can be written to produce a singular proposition. The singular propositions '*Ha*', '*Hb*', '*Hc*', '*Hd*', . . . are either true or false; but '*Hx*' is neither true nor false, not being a proposition. Such expressions as '*Hx*' are called 'propositional functions'. These are defined to be expressions which contain individual variables and become propositions when their individual variables are replaced by individual constants.[2] Any singular proposition can be regarded as a *substitution instance* of the propositional function from which it results by the substitution of an individual constant for the individual variable in the propositional function. The process of obtaining a proposition from a propositional function by substituting a constant for a variable is called 'instantiation'. The negative singular propositions 'Aristotle is not human' and 'Boston is not human', symbolized as '$\sim Ha$' and '$\sim Hb$', result by *instantiation* from the propositional function '$\sim Hx$', of which they are substitution instances. Thus we see that symbols other than attribute symbols and individual variables can occur in propositional functions.

General propositions such as 'Everything is mortal' and 'Something is mortal' differ from singular propositions in not containing the names of any individuals. However, they also can be regarded as resulting from propositional functions, not by instantiation, but by the process called 'generalization' or 'quantification'. The first example, 'Everything is mortal', can alternatively be expressed as

> Given any individual thing whatever, it is mortal.

Here the relative pronoun 'it' refers back to the word 'thing' which precedes it in the statement. We can use the individual variable '*x*' in place of the pronoun 'it' and its antecedent to paraphrase the first general proposition as

> Given any *x*, *x* is mortal.

[1] Some logicians enclose the individual constant in parentheses, symbolizing 'Socrates is human' as '*H*(*s*)', but we shall not follow that practice here.

[2] Some writers have defined 'propositional functions' to be the *meanings* of such expressions; but here we define them to be the expressions themselves.

Then we can use the notation already introduced to rewrite it as

Given any x, Mx.

The phrase 'Given any x' is called a 'universal quantifier', and is symbolized as '(x)'. Using this new symbol we can completely symbolize our first general proposition as

$$(x)Mx$$

We can similarly paraphrase the second general proposition, 'Something is mortal', successively as

There is at least one thing which is mortal.
There is at least one thing such that it is mortal.
There is at least one x such that x is mortal.

and as

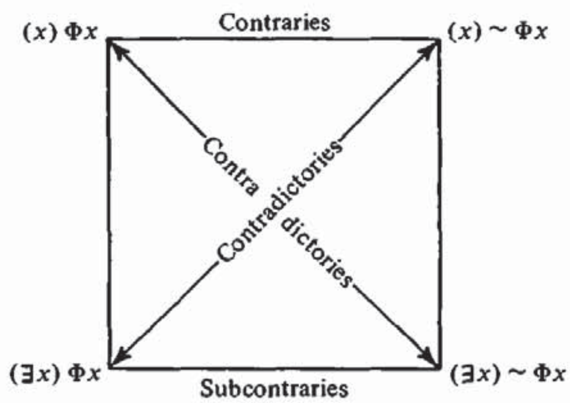
There is at least one x such that Mx.

The phrase 'There is at least one x such that' is called an 'existential quantifier' and is symbolized as '$(\exists x)$'. Using the new symbol we can completely symbolize our second general proposition as

$$(\exists x)Mx$$

A general proposition is formed from a propositional function by placing either a universal or an existential quantifier before it. It is obvious that the universal quantification of a propositional function is true if and only if all of its substitution instances are true, and that the existential quantification of a propositional function is true if and only if it has at least one true substitution instance. If we grant that there is at least one individual then every propositional function has at least one substitution instance (true or false). Under this assumption, if the universal quantification of a propositional function is true, then its existential quantification must be true also.

A further relationship between universal and existential quantification can be shown by considering two additional general propositions, 'Something is not mortal' and 'Nothing is mortal', which are the respective negations of the first two general propositions considered. 'Something is not mortal' is symbolized as '$(\exists x)\sim Mx$' and 'Nothing is mortal' is symbolized as '$(x)\sim Mx$'. These show that the negation of the universal (existential) quantification of a propositional function is logically equivalent to the existential (universal) quantification of the new propositional function which results from placing a negation symbol in front of the first propositional function. Where we use the Greek letter *phi* to represent any attribute symbol whatever, the general connections between universal and existential quantification can be described in terms of the following square array:



Assuming the existence of at least one individual: we can say that the two top propositions are *contraries,* i.e., they might both be false but cannot both be true; the two bottom propositions are *subcontraries,* i.e., they can both be true but cannot both be false; propositions which are at opposite ends of the diagonals are *contradictories,* of which one must be true and the other false; and finally, on each side, the truth of the lower proposition is implied by the truth of the proposition which is directly above it.

Traditional logic emphasized four types of subject-predicate propositions illustrated by the following:

All humans are mortal.
No humans are mortal.
Some humans are mortal.
Some humans are not mortal.

These were classified as 'universal affirmative', 'universal negative', 'particular affirmative', and 'particular negative', respectively, and their types abbreviated as '*A*', '*E*', '*I*', '*O*', again respectively. (The letters names have been presumed to come from the Latin '*AffIrmo*' and '*nEgO*', meaning *I affirm* and *I deny*.) These four special forms of subject-predicate propositions are easily symbolized by means of propositional functions and quantifiers.[3] The first of them, the *A* proposition, can successively be paraphrased as

Given any individual thing whatever, if it is human then it is mortal.
Given any x, if x is human then x is mortal.
Given any x, x is human $\supset$ x is mortal.

and finally symbolized as

$$(x)[Hx \supset Mx]$$

[3]An alternative method of symbolizing them is presented in Appendix B.

Our symbolic formulation of the *A* proposition is the universal quantification of the complex propositional function '$Hx \supset Mx$', which has as its substitution instances not singular propositions but conditionals whose antecedents and consequents are singular propositions having the same subject terms. Among the substitution instances of the propositional function '$Hx \supset Mx$' are the conditionals '$Ha \supset Ma$', '$Hb \supset Mb$', '$Hc \supset Mc$', and so on. In symbolizing an *A* proposition the square brackets serve as punctuation marks to indicate that the universal quantifier '(x)' *applies to* or *has within its scope* the whole of the complex propositional function '$Hx \supset Mx$'. The notion of the *scope of a quantifier* is very important, for differences in scope correspond to differences in meaning. The expression '$(x)[Hx \supset Mx]$' is a proposition which asserts that all substitution instances of the propositional function '$Hx \supset Mx$' are true. On the other hand, the expression '$(x)Hx \supset Mx$' is a propositional function whose substitution instances are '$(x)Hx \supset Ma$', '$(x)Hx \supset Mb$', '$(x)Hx \supset Mc$', etc.[4]

The *E* proposition 'No humans are mortal' may similarly be paraphrased successively as

Given any individual thing whatever, if it is human then it is not mortal.
Given any x, if x is human then x is not mortal.
Given any x, x is human $\supset$ x is not mortal.

and then symbolized as

$$(x)[Hx \supset \sim Mx]$$

Similarly, the *I* proposition 'Some humans are mortal', may be paraphrased as

There is at least one thing which is human and mortal.
There is at least one thing such that it is human and it is mortal.
There is at least one x such that x is human and x is mortal.
There is at least one x such that x is human $\cdot$ x is mortal.

and completely symbolized as

$$(\exists x)[Hx \cdot Mx]$$

Finally, the *O* proposition 'Some humans are not mortal', becomes

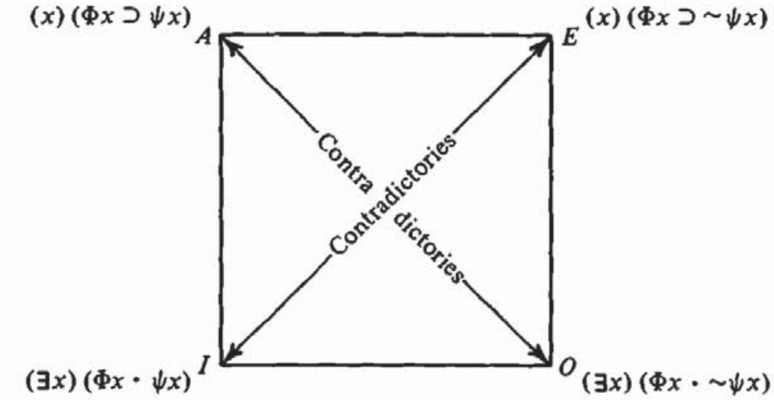
There is at least one thing which is human but not mortal.
There is at least one thing such that it is human and it is not mortal.
There is at least one x such that x is human and x is not mortal.

[4] We have the same symbolic convention for quantifiers (both universal and existential) that we established for negation on page 11: a quantifier applies to or has for its scope the smallest component that the punctuation permits.

and then symbolized as the existential quantification of a complex function

$$(\exists x)[Hx \cdot \sim Mx]$$

Where the Greek letters *phi* and *psi* are used to represent any attribute symbols whatever, the four general subject-predicate propositions of traditional logic may be represented in a square array as



Of these, the ***A*** and the ***O*** are contradictories, and the ***E*** and the ***I*** are contradictories also. But none of the other relationships discussed in connection with the square array on page 67 hold for the traditional ***A***, ***E***, ***I***, and ***O*** propositions, even where we assume that there is at least one individual in the universe. Where 'Φx' is a propositional function that has no true substitution instances, then regardless of what attribute is symbolized by 'Ψ', the propositional functions '$\Phi x \supset \Psi x$' and '$\Phi x \supset \sim\Psi x$' have only true substitution instances, for all their substitution instances are conditional statements with false antecedents. In such cases the **A** and **E** propositions that are the universal quantifications of these complex propositional functions are true, so ***A*** and ***E*** propositions are not contraries. Again, where 'Φx' is a propositional function that has no true substitution instances, then regardless of what 'Ψx' might be, the propositional functions '$\Phi x \cdot \Psi x$' and '$\Phi x \cdot \sim\Psi x$' have only false substitution instances, for their substitution instances are conjunctions whose first conjuncts are false. In such cases the ***I*** and ***O*** propositions that are the existential quantifications of these complex propositional functions are false, so ***I*** and ***O*** propositions are not subcontraries. In all such cases, then, since the ***A*** and ***E*** propositions are true and the ***I*** and ***O*** propositions are false, the truth of a universal does *not* imply the truth of the corresponding particular; no implication relation holds between them.

If we make the assumption that there is at least one individual, then '$(x)[\Phi x \supset \Psi x]$' does imply '$(\exists x)[\Phi x \supset \Psi x]$'. But the latter is *not* an ***I*** proposition. An ***I*** proposition of the form 'Some Φ's are Ψ's' is symbolized as '$(\exists x)[\Phi x \cdot \Psi x]$', which asserts that there is at least one thing having both the attribute Φ and the attribute Ψ. But the proposition '$(\exists x)[\Phi x \supset \Psi x]$' asserts only that there is at least one object which either has the attribute Ψ or does not have the attribute Φ, which is a very different and much weaker assertion.

The four traditional subject-predicate forms **A**, **E**, **I**, and **O** are not the only forms of general propositions. There are others that involve the quantification of more complicated propositional functions. Thus the general proposition 'All members are either parents or teachers', which does *not* mean the same as 'All members are parents or all members are teachers', is symbolized as '$(x)[Mx \supset (Px \vee Tx)]$'. And the general proposition 'Some Senators are either disloyal or misguided', is symbolized as '$(\exists x)[Sx \cdot (Dx \vee Mx)]$'. It should be observed that such a proposition as 'Apples and bananas are nourishing' can be symbolized either as the conjunction of two **A** propositions, '$\{(x)[Ax \supset Nx]\} \cdot \{(x)[Bx \supset Nx]\}$', or as a single noncompound general proposition, '$(x)[(Ax \vee Bx) \supset Nx]$'. But it should *not* be symbolized as '$(x)[(Ax \cdot Bx) \supset Nx]$', for to say that apples and bananas are nourishing is to say that anything is nourishing which is *either* an apple *or* a banana, *not* to say that anything is nourishing which is *both* an apple and a banana (whatever that might be). It must be emphasized that there are no mechanical rules for translating statements from English into our logical notation. In every case one must *understand the meaning* of the English sentence, and then *re-express* that meaning in terms of propositional functions and quantifiers.

EXERCISES

I. Translate each of the following into the logical notation of propositional functions and quantifiers, in each case using the abbreviations suggested, and having each formula begin with a quantifier, *not* with a negation symbol:

***1.** Snakes are reptiles. (Sx: x is a snake. Rx: x is a reptile.)
2. Snakes are not all poisonous. (Sx: x is a snake. Px: x is poisonous.)
3. Children are present. (Cx: x is a child. Px: x is present.)
4. Executives all have secretaries. (Ex: x is an executive. Sx: x has a secretary.)
***5.** Only executives have secretaries. (Ex: x is an executive. Sx: x has a secretary.)
6. Only property owners may vote in special municipal elections. (Px: x is a property owner. Vx: x may vote in special municipal elections.)
7. Employees may use only the service elevator. (Ux: x is an elevator that employees may use. Sx: x is a service elevator.)
8. Only employees may use the service elevator. (Ex: x is an employee. Ux: x may use the service elevator.)
9. All that glitters is not gold. (Gx: x glitters. Ax: x is gold.)
***10.** None but the brave deserve the fair. (Bx: x is brave. Dx: x deserves the fair.)
11. Not every visitor stayed for dinner. (Vx: x is a visitor. Sx: x stayed for dinner.)
12. Not any visitor stayed for dinner. (Vx: x is a visitor. Sx: x stayed for dinner.)
13. Nothing in the house escaped destruction. (Hx: x was in the house. Ex: x escaped destruction.)
14. Some students are both intelligent and hard workers. (Sx: x is a student. Ix: x is intelligent. Hx: x is a hard worker.)
***15.** No coat is waterproof unless it has been specially treated. (Cx: x is a coat. Wx: x is waterproof. Sx: x has been specially treated.)
16. Some medicines are dangerous only if taken in excessive amounts. (Mx: x is a medicine. Dx: x is dangerous. Ex: x is taken in excessive amounts.)

17. All fruits and vegetables are wholesome and nourishing. (Fx: x is a fruit. Vx: x is a vegetable. Wx: x is wholesome. Nx: x is nourishing.)
18. Everything enjoyable is either immoral, illegal, or fattening. (Ex: x is enjoyable. Mx: x is moral. Lx: x is legal. Fx: x is fattening.)
19. A professor is a good lecturer if and only if he is both well informed and entertaining. (Px: x is a professor. Gx: x is a good lecturer. Wx: x is well informed. Ex: x is entertaining.)
*20. Only policemen and firemen are both indispensable and underpaid. (Px: x is a policeman. Fx: x is a fireman. Ix: x is indispensable. Ux: x is underpaid.)
21. Not every actor is talented who is famous. (Ax: x is an actor. Tx: x is talented. Fx: x is famous.)
22. Any girl is attractive if she is neat and well groomed. (Gx: x is a girl. Ax: x is attractive. Nx: x is neat. Wx: x is well groomed.)
23. It is not true that every watch will keep good time if and only if it is wound regularly and not abused. (Wx: x is a watch. Kx: x keeps good time. Rx: x is wound regularly. Ax: x is abused.)
24. Not every person who talks a great deal has a great deal to say. (Px: x is a person. Tx: x talks a great deal. Hx: x has a great deal to say.)
*25. No automobile that is over ten years old will be repaired if it is severely damaged. (Ax: x is an automobile. Ox: x is over ten years old. Rx: x will be repaired. Dx: x is severely damaged.)

In symbolizing the following, use the abbreviations: Hx: x is a horse. Gx: x is gentle. Tx: x has been well trained.

26. Some horses are gentle and have been well trained.
27. Some horses are gentle only if they have been well trained.
28. Some horses are gentle if they have been well trained.
29. Any horse is gentle that has been well trained.
*30. Any horse that is gentle has been well trained.
31. No horse is gentle unless it has been well trained.
32. Any horse is gentle if it has been well trained.
33. Any horse has been well trained if it is gentle.
34. Any horse is gentle if and only if it has been well trained.
35. Gentle horses have all been well trained.

II. Symbolize the following, using propositional functions and quantifiers:

*1. Blessed is he that considereth the poor. (Psalm 41:1)
2. He that hath knowledge spareth his words. (Proverbs 17:27)
3. Whoso findeth a wife findeth a good thing. (Proverbs 18:22)
4. He that maketh haste to be rich shall not be innocent. (Proverbs 28:20)
*5. They shall sit every man under his vine and under his fig-tree. (Micah 4:4)
6. He that increaseth knowledge increaseth sorrow. (Ecclesiastes 1:18)
7. Nothing is secret which shall not be made manifest. (Luke 8:17)
8. Whom The Lord loveth He chasteneth. (Hebrews 12:6)
9. There is a lion in the way; a lion is in the streets. (Proverbs 26:13)
10. He that hateth dissembleth with his lips, and layeth up deceit within him. (Proverbs 26:24)

4.2 Proving Validity: Preliminary Quantification Rules

To construct formal proofs of validity for arguments symbolized by means of quantifiers and propositional functions we must augment our list of Rules of Inference. We shall add four rules governing quantification, offering an oversimplified preliminary statement of them in this section, and giving a more adequate formulation in Section 4.5.

1. Universal Instantiation (Preliminary Version). Because the universal quantification of a propositional function is true if and only if all substitution instances of that propositional function are true, we can add to our list of Rules of Inference the principle that any substitution instance of a propositional function can validly be inferred from its universal quantification. We can express this rule symbolically as

$$\frac{(x)\Phi x}{\therefore \Phi \nu} \text{ (where } \nu \text{ is any individual symbol)}$$

Since this rule permits substitution instances to be inferred from universal quantifications, we refer to it as the 'principle of Universal Instantiation', and abbreviate it as '**UI**'.[5] The addition of **UI** permits us to give a formal proof of validity for the argument: 'All humans are mortal; Socrates is human; therefore Socrates is mortal'.

1. $(x)[Hx \supset Mx]$
2. Hs $\quad /\therefore Ms$
3. $Hs \supset Ms$ $\quad$ 1, **UI**
4. Ms $\quad$ 3, 2, M.P.

2. Universal Generalization (Preliminary Version). We can explain our next rule by analogy with fairly standard mathematical practice. A geometer may begin a proof by saying, 'Let *ABC* be any arbitrarily selected triangle'. Then he may go on to prove that the triangle *ABC* has some specified attribute, and concludes that *all* triangles have that attribute. Now what justifies his final conclusion? Why does it follow from triangle *ABC*'s having a specified attribute that *all* triangles do? The answer is that if no assumption other than its triangularity is made about *ABC*, then the expression '*ABC*' can be taken as denoting any triangle you please. And if the argument has established that *any* triangle must have the attribute in question, then it follows that *all* triangles do. We now introduce a notation analogous to that of the geometer in his reference to 'any arbitrarily selected triangle'. The hitherto unused small

[5]This rule and the three which follow are variants of rules for 'natural deduction' which were devised independently by Gerhard Gentzen and Stanislaw Jaśkowski in 1934.

letter 'y' will be used to denote *any arbitrarily selected individual.* In this usage the expression 'Φy' is a substitution instance of the propositional function 'Φx', and it asserts that *any arbitrarily selected individual* has the property Φ. Clearly 'Φy' follows validly from '$(x)\Phi x$' by **UI**, since what is true of all individuals is true of any arbitrarily selected individual. The inference is equally valid in the other direction, since what is true of *any arbitrarily selected individual* must be true of *all* individuals. We augment our list of Rules of Inference further by adding the principle that the universal quantification of a propositional function can validly be inferred from its substitution instance with respect to the symbol 'y'. Since this rule permits the inference of general propositions that are universal quantifications, we refer to it as the 'principle of Universal Generalization', and abbreviate it as '**UG**'. Our symbolic expression for this second quantification rule is

$$\frac{\Phi y}{\therefore (x)\Phi x} \text{ (where '}y\text{' denotes any arbitrarily selected individual)}$$

We can use the new notation and the additional rule **UG** to construct a formal proof of validity for the argument: 'No mortals are perfect; all humans are mortal; therefore no humans are perfect'.

1. $(x)[Mx \supset \sim Px]$
2. $(x)[Hx \supset Mx]$ $\quad /\therefore (x)[Hx \supset \sim Px]$
3. $Hy \supset My$ $\quad$ 2, **UI**
4. $My \supset \sim Py$ $\quad$ 1, **UI**
5. $Hy \supset \sim Py$ $\quad$ 3, 4, H.S.
6. $(x)[Hx \supset \sim Px]$ $\quad$ 5, **UG**

3. Existential Generalization (Preliminary Version). Because the existential quantification of a propositional function is true if and only if that propositional function has at least one true substitution instance, we can add to our list of Rules of Inference the principle that the existential quantification of a propositional function can validly be inferred from any substitution instance of that propositional function. This rule permits the inference of general propositions which are existentially quantified, so we call it the 'principle of Existential Generalization' and abbreviate it as '**EG**'. Its symbolic formulation is

$$\frac{\Phi \nu}{\therefore (\exists x)\Phi x} \text{ (where } \nu \text{ is any individual symbol)}$$

4. Existential Instantiation (Preliminary Version). One further quantification rule is required. The existential quantification of a propositional function asserts that there exists at least one individual the substitution of whose name for the variable 'x' in that propositional function will yield a true substitution instance of it. Of course we may not know anything else about that individual. But we can take any individual constant other than 'y', say, 'w', which has had no prior occurrence in that context and use it to denote the individual, or one of the individuals, whose existence has been asserted by the existential quantification. Knowing that there is such an individual, and having agreed to denote it by 'w', we can infer from the existential quantification of a propositional function the substitution instance of that propositional function with respect to the individual symbol 'w'. We add as our final quantification rule the principle that from the existential quantification of a propositional function we may validly infer the truth of its substitution instance with respect to an individual constant which has no prior occurrence in that context. The new rule may be written as

$\dfrac{(\exists x)\Phi x}{\therefore \Phi\nu}$ (where ν is an individual constant, other than 'y', which has no prior occurrence in the context)

It is referred to as the 'principle of Existential Instantiation' and abbreviated as '**EI**'.

We make use of the last two quantification rules in constructing a formal proof of validity for the argument: 'All dogs are carnivorous; some animals are dogs; therefore some animals are carnivorous'.

1.	$(x)[Dx \supset Cx]$	
2.	$(\exists x)[Ax \cdot Dx]$	$/\therefore (\exists x)[Ax \cdot Cx]$
3.	$Aw \cdot Dw$	2, **EI**
4.	$Dw \supset Cw$	1, **UI**
5.	$Dw \cdot Aw$	3, Com.
6.	Dw	5, Simp.
7.	Cw	4, 6, M.P.
8.	Aw	3, Simp.
9.	$Aw \cdot Cw$	8, 7, Conj.
10.	$(\exists x)[Ax \cdot Cx]$	9, **EG**

We can show the need for the indicated restriction on the use of **EI** by considering the obviously invalid argument: 'Some cats are animals; some dogs are animals; therefore some cats are dogs'. If we ignored the restriction on **EI** that the substitution instance inferred by it can contain only an individual constant which had no prior occurrence in the context, we might be led to construct the following 'proof':

1.	$(\exists x)[Cx \cdot Ax]$	
2.	$(\exists x)[Dx \cdot Ax]$	$/\therefore (\exists x)[Cx \cdot Dx]$
3.	$Cw \cdot Aw$	1, **EI**
4.	$Dw \cdot Aw$	2, **EI** (wrong)

5.	Cw	3, Simp.
6.	Dw	4, Simp.
7.	$Cw \cdot Dw$	5, 6, Conj.
8.	$(\exists x)[Cx \cdot Dx]$	7, **EG**

The mistake here occurs at line 4. The second premiss assures us that there is at least one thing that is both a dog and an animal. But we are not free to use the symbol 'w' to denote that thing because 'w' has already been used to denote one of the things asserted by the first premiss to be both a cat and an animal. To avoid errors of this sort we must obey the indicated restriction in using **EI**. It should be clear that whenever we use both **EI** and **UI** in a proof to instantiate with respect to the same individual constant, we must use **EI** first. (It has been suggested that we could avoid the necessity of using **EI** before **UI** by changing the restriction on **EI** to read 'where ν is an individual constant, other than 'y', which was not introduced into the context by any previous use of **EI**.' But even apart from its apparent circularity, that formulation would not prevent the construction of an erroneous 'formal proof of validity' for such an invalid argument as 'Some men are handsome. Socrates is a man. Therefore Socrates is handsome.')

Like the first nine Rules of Inference presented in Section 3.1, the four quantification rules **UI, UG, EG,** and **EI** can be applied only to whole lines in a proof.

Any assumption of limited scope may be made in a Conditional Proof of validity, and in particular we are free to make an assumption of the form 'Φy'. Thus the argument 'All freshmen and sophomores are invited and will be welcome; therefore all freshmen are invited' is proved valid by the following Conditional Proof:

1.	$(x)[(Fx \vee Sx) \supset (Ix \cdot Wx)]$	$/ \therefore (x)[Fx \supset Ix]$
→ 2.	Fy	
3.	$(Fy \vee Sy) \supset (Iy \cdot Wy)$	1, **UI**
4.	$Fy \vee Sy$	2, Add.
5.	$Iy \cdot Wy$	3, 4, M.P.
6.	Iy	5, Simp.
7.	$Fy \supset Iy$	2–6, C.P.
8.	$(x)[Fx \supset Ix]$	7, **UG**

More than one assumption of limited scope can be made in proving the validity of arguments involving quantifiers, as in the following Conditional Proof:

1.	$(x)[(Ax \vee Bx) \supset (Cx \cdot Dx)]$	
2.	$(x)\{(Cx \vee Ex) \supset [(Fx \vee Gx) \supset Hx\}$	$/ \therefore (x)[Ax \supset (Fx \supset Hx)]$
3.	$(Ay \vee By) \supset (Cy \cdot Dy)$	1, **UI**
4.	$(Cy \vee Ey) \supset [(Fy \vee Gy) \supset Hy]$	2, **UI**

5.	Ay	
6.	$Ay \vee By$	5, Add.
7.	$Cy \cdot Dy$	3, 6, M.P.
8.	Cy	7, Simp.
9.	$Cy \vee Ey$	8, Add.
10.	$(Fy \vee Gy) \supset Hy$	4, 9, M.P.
11.	Fy	
12.	$Fy \vee Gy$	11, Add.
13.	Hy	10, 12, M.P.
14.	$Fy \supset Hy$	11–13, C.P.
15.	$Ay \supset (Fy \supset Hy)$	5–14, C.P.
16.	$(x)[Ax \supset (Fx \supset Hx)]$	15, UG

EXERCISES

I. Construct formal proofs of validity for the following arguments, using the rule of Conditional Proof wherever you wish:

*1. $(x)[Ax \supset Bx]$
$\sim Bt$
$\therefore \sim At$

2. $(x)[Cx \supset Dx]$
$(x)[Ex \supset \sim Dx]$
$\therefore (x)[Ex \supset \sim Cx]$

3. $(x)[Fx \supset \sim Gx]$
$(\exists x)[Hx \cdot Gx]$
$\therefore (\exists x)[Hx \cdot \sim Fx]$

4. $(x)[Ix \supset Jx]$
$(\exists x)[Ix \cdot \sim Jx]$
$\therefore (x)[Jx \supset Ix]$

*5. $(x)[Kx \supset Lx]$
$(x)[(Kx \cdot Lx) \supset Mx]$
$\therefore (x)[Kx \supset Mx]$

6. $(x)[Nx \supset Ox]$
$(x)[Px \supset Ox]$
$\therefore (x)[(Nx \vee Px) \supset Ox]$

7. $(x)[Qx \supset Rx]$
$(\exists x)[Qx \vee Rx]$
$\therefore (\exists x)Rx$

8. $(x)[Sx \supset (Tx \supset Ux)]$
$(x)[Ux \supset (Vx \cdot Wx)]$
$\therefore (x)[Sx \supset (Tx \supset Vx)]$

9. $(x)[(Xx \vee Yx) \supset (Zx \cdot Ax)]$
$(x)[(Zx \vee Ax) \supset (Xx \cdot Yx)]$
$\therefore (x)[Xx \equiv Zx]$

10. $(x)[(Bx \supset Cx) \cdot (Dx \supset Ex)]$
$(x)[(Cx \vee Dx) \supset \{[Fx \supset (Gx \supset Fx)] \supset (Bx \cdot Dx)\}]$
$\therefore (x)[Bx \equiv Dx]$

II. Construct formal proofs of validity for the following arguments, using the rule of Conditional Proof wherever you wish:

*1. All athletes are brawny. Charles is not brawny. Therefore Charles is not an athlete. (Ax, Bx, c)

2. No contractors are dependable. Some contractors are engineers. Therefore some engineers are not dependable. (Cx, Dx, Ex)

3. All fiddlers are gay. Some hunters are not gay. Therefore some hunters are not fiddlers. (Fx, Gx, Hx)

4. No judges are idiots. Kanter is an idiot. Therefore Kanter is not a judge. (Jx, Ix, k)

*5. All liars are mendacious. Some liars are newspapermen. Therefore some newspapermen are mendacious. (Lx, Mx, Nx)

6. No osteopaths are pediatricians. Some quacks are pediatricians. Therefore some quacks are not osteopaths. (Ox, Px, Qx)

7. Only salesmen are retailers. Not all retailers are travelers. Therefore some salesmen are not travelers. (Sx, Rx, Tx)

8. There are no uniforms that are not washable. There are no washable velvets. Therefore there are no velvet uniforms. (Ux, Wx, Vx)

9. Only authoritarians are bureaucrats. Authoritarians are all churlish. Therefore any bureaucrat is churlish. (Ax, Bx, Cx)

*10. Dates are edible. Only food is edible. Food is good. Therefore all dates are good. (Dx, Ex, Fx, Gx)

11. All dancers are graceful. Mary is a student. Mary is a dancer. Therefore some students are graceful. (Dx, Gx, Sx, m)

12. Tigers are fierce and dangerous. Some tigers are beautiful. Therefore some dangerous things are beautiful. (Tx, Fx, Dx, Bx)

13. Bananas and grapes are fruits. Fruits and vegetables are nourishing. Therefore bananas are nourishing. (Bx, Gx, Fx, Vx, Nx)

14. A communist is either a fool or a knave. Fools are naive. Not all communists are naive. Therefore some communists are knaves. (Cx, Fx, Kx, Nx)

*15. All butlers and valets are both obsequious and dignified. Therefore all butlers are dignified. (Bx, Vx, Ox, Dx)

16. All houses built of brick are warm and cozy. All houses in Englewood are built of brick. Therefore all houses in Englewood are warm. (Hx, Bx, Wx, Cx, Ex)

17. All professors are learned. All learned professors are savants. Therefore all professors are learned savants. (Px, Lx, Sx)

18. All diplomats are statesmen. Some diplomats are eloquent. All eloquent statesmen are orators. Therefore some diplomats are orators. (Dx, Sx, Ex, Ox)

19. Doctors and lawyers are college graduates. Any altruist is an idealist. Some lawyers are not idealists. Some doctors are altruists. Therefore some college graduates are idealists. (Dx, Lx, Cx, Ax, Ix)

*20. Bees and wasps sting if they are either angry or frightened. Therefore any bee stings if it is angry. (Bx, Wx, Sx, Ax, Fx)

21. Any author is successful if and only if he is well read. All authors are intellectuals. Some authors are successful but not well read. Therefore all intellectuals are authors. (Ax, Sx, Wx, Ix)

22. Every passenger is either in first class or in tourist class. Each passenger is in tourist class if and only if he is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore some passengers are in tourist class. (Px, Fx, Tx, Wx)

23. All members are both officers and gentlemen. All officers are fighters. Only a pacifist is either a gentleman or not a fighter. No pacifists are gentlemen if they are fighters. Some members are fighters if and only if they are officers. Therefore not all members are fighters. (Mx, Ox, Gx, Fx, Px)

24. Wolfhounds and terriers are hunting dogs. Hunting dogs and lap dogs are domesticated animals. Domesticated animals are gentle and useful. Some wolfhounds are neither gentle nor small. Therefore some terriers are small but not gentle. (Wx, Tx, Hx, Lx, Dx, Gx, Ux, Sx)

25. No man who is a candidate will be defeated if he is a good campaigner. Any man who runs for office is a candidate. Any candidate who is not defeated will be elected. Every man who is elected is a good campaigner. Therefore any man who runs for office will be elected if and only if he is a good campaigner. (Mx, Cx, Dx, Gx, Rx, Ex)

4.3 Proving Invalidity

In the preceding chapter we proved the invalidity of invalid arguments containing truth-functional compound statements by assigning truth values to their component simple statements in such a way as to make their premisses true and their conclusions false. We use a closely related method to prove the invalidity of invalid arguments involving quantifiers. The method of proving invalidity about to be described is connected with our basic assumption that there is at least one individual.

The assumption that there is at least one individual could be satisfied in infinitely many different ways: if there is exactly one individual, or if there are exactly two individuals, or if there are exactly three individuals, or etc. For any such case there is a strict logical equivalence between noncompound general propositions and truth-functional compounds of singular propositions. If there is exactly one individual, say, a, then

$$[(x)\Phi x] \equiv \Phi a \quad \text{and} \quad [(\exists x)\Phi x] \equiv \Phi a$$

If there are exactly two individuals, say a and b, then

$$[(x)\Phi x] \equiv [\Phi a \cdot \Phi b] \quad \text{and} \quad [(\exists x)\Phi x] \equiv [\Phi a \vee \Phi b]$$

And for any number k, if there are exactly k individuals, say $a, b, c, \ldots, k$, then

$$[(x)\Phi x] \equiv [\Phi a \cdot \Phi b \cdot \Phi c \cdot \ldots \cdot \Phi k]$$

and

$$[(\exists x)\Phi x] \equiv [\Phi a \vee \Phi b \vee \Phi c \vee \ldots \vee \Phi k]$$

The truth of these biconditionals is an immediate consequence of our definitions of the universal and existential quantifiers. No use is made here of the four quantification rules presented in the preceding section. So for any possible nonempty universe or model containing any finite number of individuals, every general proposition is logically equivalent to some truth-functional compound of singular propositions. Hence for any such model every argument involving quantifiers is logically equivalent to some argument containing only singular propositions and truth-functional compounds of them.

An argument involving quantifiers is valid if and only if it is valid no matter how many individuals there are, so long as there is at least one. So an argument involving quantifiers is valid if and only if for every possible nonempty universe or model it is logically equivalent to a truth-functional argument which is valid. Hence we can prove the invalidity of a given argument by displaying or describing a model for which the given argument is logically equivalent to an *invalid* truth-functional argument. We can accomplish this purpose by translating the given argument involving quantifiers into a logically equivalent argument involving only singular propositions and truth-functional compounds of them, and then using the method of assigning truth values to prove the latter invalid. For example, given the argument

All whales are heavy.
All elephants are heavy.
Therefore all whales are elephants.

we first symbolize it as

$$(x)[Wx \supset Hx]$$
$$(x)[Ex \supset Hx]$$
$$\therefore (x)[Wx \supset Ex]$$

In the case of a model containing exactly one individual, say, a, the given argument is logically equivalent to

$$Wa \supset Ha$$
$$Ea \supset Ha$$
$$\therefore Wa \supset Ea$$

which is proved invalid by assigning the truth value **T** to 'Wa' and 'Ha' and **F** to 'Ea'. (This assignment of truth values is a shorthand way of describing the model in question as one which contains only the one individual a which is W (a whale) and H (heavy) but not E (an elephant).)[6] Hence the original argument is not valid for a model containing exactly one individual, and is therefore *invalid*.

It must be emphasized that in proving the invalidity of arguments involving quantifiers *no use is made of our quantification rules*. For a model containing just the one individual a we do not *infer* the statement '$Wa \supset Ha$' from the statement '$(x)[Wx \supset Hx]$' by **UI**; those two statements are logically equivalent for that model because in it '$Wa \supset Ha$' is the *only* substitution instance of the propositional function '$Wx \supset Hx$'.

[6]Here we assume that the simple propositional functions 'Ax', 'Bx', 'Cx', ... are neither necessary, that is, logically true of all individuals (for example, x is identical with itself), nor impossible, that is, logically false of all individuals (for example, x is different from itself). We also assume that the only logical relations among the simple propositional functions are those asserted or logically implied by the premisses of the argument being proved invalid. The point of these restrictions is to permit the arbitrary assignment of truth values to substitution instances of these simple propositional functions without inconsistency—for our model-descriptions must of course be consistent.

It *can* happen that an invalid argument involving quantifiers is logically equivalent, for any model containing exactly one individual, to a valid truth-functional argument, although it will be logically equivalent, for any model containing more than one individual, to an invalid truth-functional argument. For example, consider the argument

All whales are heavy.
Some elephants are heavy.
Therefore all whales are elephants.

which is symbolized as

$$\begin{array}{l}(x)[Wx \supset Hx]\\ (\exists x)[Ex \cdot Hx]\\ \therefore (x)[Wx \supset Ex]\end{array}$$

For a model containing just the one individual a this argument is logically equivalent to

$$\begin{array}{l}Wa \supset Ha\\ Ea \cdot Ha\\ \therefore Wa \supset Ea\end{array}$$

which is a valid argument. But for a model consisting of the two individuals a and b the given argument is logically equivalent to

$$\begin{array}{l}(Wa \supset Ha) \cdot (Wb \supset Hb)\\ (Ea \cdot Ha) \vee (Eb \cdot Hb)\\ \therefore (Wa \supset Ea) \cdot (Wb \supset Eb)\end{array}$$

which is proved invalid by assigning the truth value **T** to 'Wa', 'Wb', 'Ha', 'Hb', 'Eb', and the truth value **F** to 'Ea'. Hence the original argument is invalid, because there is a model for which it is logically equivalent to an invalid truth-functional argument.

Another illustration is

Some dogs are pointers.
Some dogs are spaniels.
Therefore some pointers are spaniels.

which we symbolize as

$$\begin{array}{l}(\exists x)[Dx \cdot Px]\\ (\exists x)[Dx \cdot Sx]\\ \therefore (\exists x)[Px \cdot Sx]\end{array}$$

For a model containing just the one individual a it is logically equivalent to

$$\begin{array}{l} Da \cdot Pa \\ Da \cdot Sa \\ \therefore Pa \cdot Sa \end{array}$$

which is valid. But for a model consisting of the two individuals a and b it is equivalent to

$$\begin{array}{l} (Da \cdot Pa) \vee (Db \cdot Pb) \\ (Da \cdot Sa) \vee (Db \cdot Sb) \\ \therefore (Pa \cdot Sa) \vee (Pb \cdot Sb) \end{array}$$

which is proved invalid by assigning the truth value **T** to '*Da*', '*Db*', '*Pa*', '*Sb*', and the truth value **F** to '*Pb*' and '*Sa*'. Here, too, the original argument is invalid, because there is a model for which it is logically equivalent to an invalid truth-functional argument.

An invalid argument involving quantifiers may be valid for any model containing fewer than k individuals, even though it must be invalid for every model containing k or more individuals. Hence in using this method to prove the invalidity of an argument involving quantifiers it may be necessary to consider larger and larger models. The question naturally arises. how large a model must we consider in trying to prove the invalidity of an argument of this type? A theoretically satisfactory answer to this question has been found. If an argument contains n different predicate symbols then if it is valid for a model containing 2^n individuals then it is valid for every model, or universally valid.[7] This result holds only for propositional functions of one variable, and is not true of the relational predicates discussed in Chapter 5. Although theoretically satisfactory, this solution is not of much practical help. If we were to go straight to the theoretically crucial case for deciding the validity or invalidity of any of the arguments already considered in this section we should have to consider models containing eight individuals. And for some of the following exercises the theoretically crucial case would be a model containing $2^8 = 256$ individuals. In fact, however, none of the following exercises requires consideration of models containing more than three individuals to prove their invalidity.

EXERCISES

I. Prove that each of the following arguments is invalid:

*1. $(\exists x)[Ax \cdot Bx]$	2. $(x)[Cx \supset \sim Dx]$
Ac	$\sim Cj$
$\therefore Bc$	$\therefore Dj$

[7] See Paul Bernays and Moses Schönfinkel, 'Zum Entscheidungsproblem der mathematischen Logik,' *Mathematische Annalen*, vol. 99 (1928), and Wilhelm Ackermann, *Solvable Cases of the Decision Problem*, Amsterdam, 1954, chap. IV.

3. $(x)[Ex \supset Fx]$
   $(x)[Gx \supset Fx]$
   $\therefore (x)[Ex \supset Gx]$

4. $(x)[Hx \supset \sim Ix]$
   $(\exists x)[Jx \cdot \sim Ix]$
   $\therefore (x)[Hx \supset Jx]$

*5. $(\exists x)[Kx \cdot Lx]$
   $(\exists x)[\sim Kx \cdot \sim Lx]$
   $\therefore (\exists x)[Lx \cdot \sim Kx]$

6. $(x)[Mx \supset (Nx \cdot Ox)]$
   $(\exists x)[Px \cdot Nx]$
   $(\exists x)[Px \cdot \sim Ox]$
   $\therefore (x)[Mx \supset \sim Px]$

7. $(x)[Qx \supset (Rx \cdot Sx)]$
   $(\exists x)[Tx \cdot Rx]$
   $(\exists x)[Tx \cdot \sim Sx]$
   $\therefore (x)[Qx \supset Tx]$

8. $(x)[Ux \supset (Vx \supset Wx)]$
   $(x)[Vx \supset (Ux \supset \sim Wx)]$
   $(\exists x)[Ux \cdot Wx]$
   $\therefore (\exists x)[Ux \cdot Vx]$

9. $(\exists x)[Xx \cdot Yx]$
   $(x)[Xx \supset Zx]$
   $(\exists x)[Zx \cdot \sim Xx]$
   $\therefore (\exists x)[Zx \cdot \sim Yx]$

10. $(x)[Ax \supset Bx]$
   $(\exists x)[Cx \cdot Bx]$
   $(\exists x)[Cx \cdot \sim Bx]$
   $\therefore (x)[Ax \supset Cx]$

II. Prove that each of the following arguments is invalid:

*1. All astronauts are brave. Jim is brave. Therefore Jim is an astronaut.
2. No cowboys are dudes. Bill is not a dude. Therefore Bill is a cowboy.
3. All evergreens are fragrant. Some gum trees are not fragrant. Therefore some evergreens are not gum trees.
4. All heathens are idolaters. No heathen is joyful. Therefore no idolater is joyful.
*5. No kittens are large. Some mammals are large. Therefore no kittens are mammals.
6. All novelists are observant. Some poets are not observant. Therefore no novelists are poets.
7. All statesmen are intelligent. Some politicians are intelligent. Not all politicians are intelligent. Therefore no statesmen are politicians.
8. All statesmen are intelligent. Some politicians are intelligent. Not all politicians are intelligent. Therefore all statesmen are politicians.
9. All statesmen are politicians. Some statesmen are intelligent. Some politicians are not statesmen. Therefore some politicians are not intelligent.
10. Horses and cows are mammals. Some animals are mammals. Some animals are not mammals. Therefore all horses are animals.

III. Prove the validity or prove the invalidity of each of the following arguments:

*1. All aviators are brave. Jones is brave. Therefore Jones is an aviator.
2. All collegians are debonair. Smith is a collegian. Therefore Smith is debonair.
3. No educators are fools. All gamblers are fools. Therefore no educators are gamblers.
4. No historians are illiterates. All illiterates are underprivileged. Therefore no historians are underprivileged.
*5. Only citizens are voters. Not all residents are citizens. Therefore some residents are not voters.

6. Only citizens are voters. Not all citizens are residents. Therefore some voters are not residents.
7. Automobiles and wagons are vehicles. Some automobiles are Fords. Some automobiles are trucks. All trucks are vehicles. Therefore some Fords are trucks.
8. Automobiles and wagons are vehicles. Some automobiles are Fords. Some automobiles are trucks. All vehicles are trucks. Therefore some Fords are trucks.
9. Automobiles and wagons are vehicles. Some automobiles are Fords. Some automobiles are trucks. Some wagons are not vehicles. Therefore some Fords are trucks.

*10. All tenors are either overweight or effeminate. No overweight tenor is effeminate. Some tenors are effeminate. Therefore some tenors are overweight.

11. All tenors are either overweight or effeminate. No overweight tenor is effeminate. Some tenors are effeminate. Therefore some tenors are not overweight.
12. No applicant will be either hired or considered who is either untrained or inexperienced. Some applicants are inexperienced beginners. All applicants who are women will be disappointed if they are not hired. Every applicant is a woman. Some women will be hired. Therefore some applicants will be disappointed.
13. No candidate is either elected or appointed who is either a liberal or a radical. Some candidates are wealthy liberals. All candidates who are politicians are disappointed if they are not elected. Every candidate is a politician. Some politicians are elected. Therefore some candidates are not disappointed.
14. Abbots and bishops are churchmen. No churchmen are either dowdy or elegant. Some bishops are elegant and fastidious. Some abbots are not fastidious. Therefore some abbots are dowdy.
15. Abbots and bishops are churchmen. No churchmen are both dowdy and elegant. Some bishops are elegant and fastidious. Some abbots are not fastidious. Therefore some abbots are not dowdy.

4.4 Multiply General Propositions

Thus far we have limited our attention to general propositions containing only a single quantifier. A general proposition that contains exactly one quantifier is said to be *singly* general. We turn next to *multiply* general propositions, which contain two or more quantifiers. In our use of the term, any compound statement whose components are general propositions is to be counted as a multiply general proposition. For example, the conditional 'If all dogs are carnivorous then some animals are carnivorous', symbolized as '$(x)[Dx \supset Cx] \supset (\exists x)[Ax \cdot Cx]$', is a multiply general proposition. Other multiply general propositions are more complex and require a more complicated notation. To develop the new notation we must turn again to the notion of a propositional function.