Let $\Gamma \leq G \leftarrow \text{Iso.}$ $\quad L = \{ b \in \mathbb{R}^2 \mid t_b \in \Gamma \}$

$\uparrow$ discrete $\quad \pi : \Gamma \to O_2 \quad \pi$ homomorphism

$t_a A \mapsto A \quad \bar{\Gamma} = \pi(\Gamma)$ point group

1) $L = \{0\}$

2) $L = a\mathbb{Z}$

3) $L = a\mathbb{Z} + b\mathbb{Z}$

Prop: L is preserved by $\bar{\Gamma}$

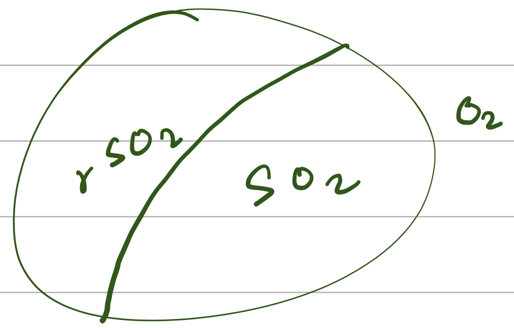
Theorem Crystallographic Restriction

1) If $L = \{0\}$, $\bar{\Gamma} = C_n, D_n$

2) If $L = a\mathbb{Z}$, $\bar{\Gamma} = C_1, \quad C_2 \simeq D_1, \quad D_2$

3) If $L = a\mathbb{Z} + b\mathbb{Z}$, $\bar{\Gamma} = C_n, D_n$ for $n = 1, 2, 3, 4, 6$

Proof: We observe that $\bar{\Gamma}$ is either cyclic or dihedral. We will reproduce the same proof as before. $\bar{\Gamma} \leq O_2$.



Case I: $\bar{\Gamma}$ contains only rotations

Choose θ_0 to be the smallest angle.

Claim: $\bar{\Gamma} = \langle \rho_{\theta_0} = \frac{2\pi}{n} \rangle$. Here $|\bar{\Gamma}| = n$.

θ arbitrary:

$$\theta = \lambda \theta_0 = (m + r_0)\theta_0 \quad \text{for } m \in \mathbb{Z}, \ r_0 \in [0,1).$$

$$\Rightarrow \theta = m\theta + r_0 \theta_0$$

$\Rightarrow \quad r_0\,\theta_0 = \theta - m\theta_0 \in \Gamma.$

$|r_0\,\theta_0| < |\theta_0| \Rightarrow r_0 = 0.$

Case II: Γ contains a reflection. We use part (1) on $\underbrace{\Gamma \cap SO_2}_{\text{rotation only}}$. So, $\Gamma \cap SO_2 = \langle \rho_{\frac{2\pi}{n}} \rangle$ where

$n = |\Gamma \cap SO_2|$

Now, $r_\ell \in \Gamma$ a reflection. Let us build D_n.

$D_n = \langle \rho_{\frac{2\pi}{n}}, r_\ell \rangle$

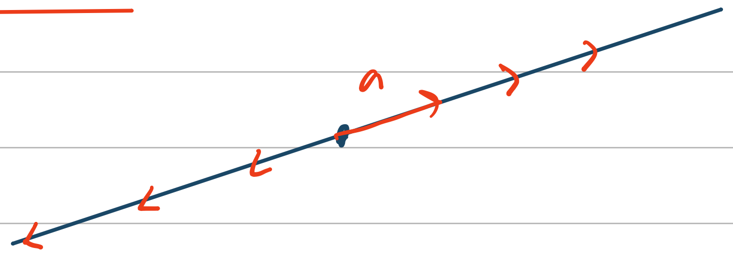
$(\rho_{2\pi/n})^n = 1, \quad r_\ell^2 = 1, \quad \rho_{2\pi/n} \cdot r_\ell = r_\ell \cdot \rho_{-2\pi/n}$

Claim: $D_n = \Gamma$ because $\underbrace{D_n}_{\text{order } 2n} \leq \underbrace{\Gamma}_{\text{order } 2n}.$

$\therefore D_n = \Gamma$

Let us describe the values of n for $L = a\mathbb{Z}$ and $L = a\mathbb{Z} + b\mathbb{Z}$.

1) $L = a\mathbb{Z}$:



Γ is either $C_n = \langle \rho_{2\pi/n} \rangle$ or $D = \langle \rho_{2\pi/n}, r_e \rangle$

Consider $\rho_{\frac{2\pi}{n}}$. Previously we showed the following.

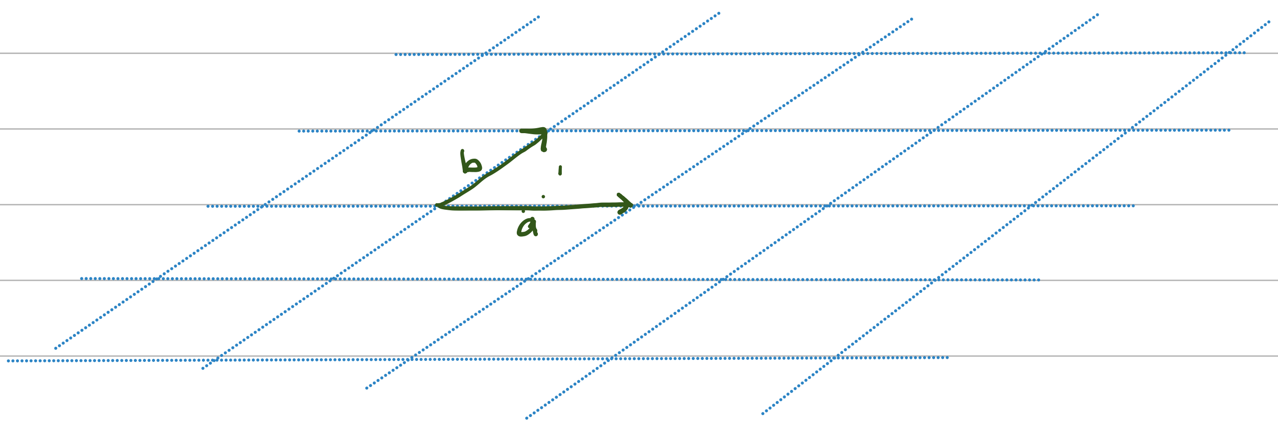
$$\forall \gamma \in \Gamma; \quad \gamma . L \subseteq L.$$

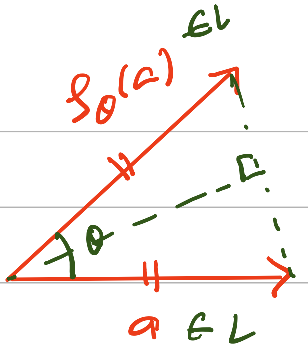
As $\rho_\theta = \rho_{\frac{2\pi}{n}}$ is a rotation fixing L, $\theta = \underset{2\pi}{\overset{\shortparallel}{0}}$ or π.

$$\theta = \frac{2\pi}{n} \longrightarrow n = 1 \text{ or } n = 2.$$

C_1, D_1 $\qquad$ C_2, D_2

(2) $L = a\mathbb{Z} + b\mathbb{Z}$.





since L is a group

L
$\vee$
$c := \rho_\theta(a) - a$

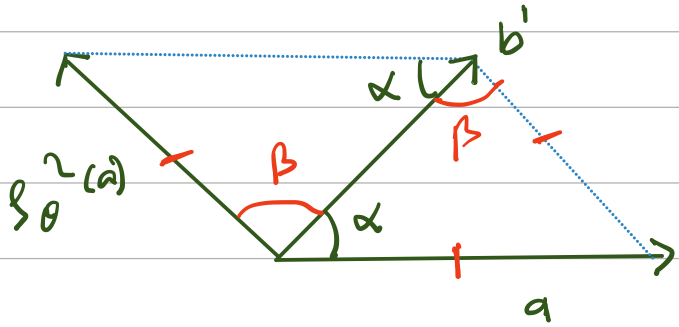
Since, a chosen to be shortest in construction of L, $|c| \geq |a|$. Need to find θ for which $|c| \geq |a|$.

$$|c| = 2|a|\cos\theta$$

So $|c| \geq |a|$ if $\cos\theta \leq 1/2 \Rightarrow \theta \geq 2\pi/6$

So, $n = 1,2,3,4,5,6$ ~~$\checkmark$~~

Now we show $n=5$ doesn't work.

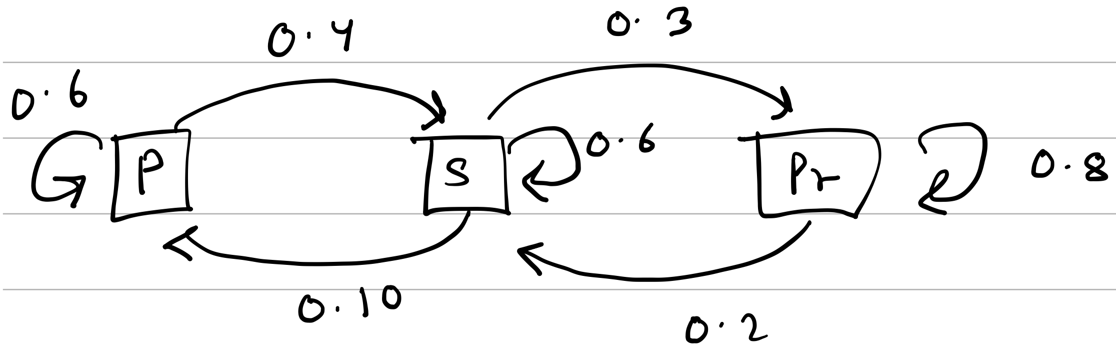


$b' = \rho_\theta^2(a) + a.$

Isosceles $\alpha = \beta.$

$$|b'| = 2|a|\cos(\pi/5) < |a|.$$

So $|b'| < |a|$ contradiction



a.

	P	S	Pr
P	0.6	0.4	0
S	0.1	0.6	0.3
Pr	0	0.2	0.8

b. limiting dist $\Leftrightarrow$ stationary distribution

i.e. $\pi P = \pi$

Let $\pi = \{x, y, 1-x-y\}$ then

$$\begin{bmatrix} x & y & 1-x-y \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} x & y & 1-x-y \end{bmatrix}$$

$\Leftrightarrow \quad 0.6x + 0.1y = x \quad \Rightarrow \quad 4x = y$

. .

$0.3y + 0.8 - 0.8x - 0.8y = 1 - x - y$

$\Rightarrow \quad 0.5y + 0.2x = 0.2 \quad \Rightarrow \quad 5y + 2x = 2$

$\Rightarrow \quad 20x + 2x = 2$

$\Rightarrow \quad x = 1/11$

$\Rightarrow \quad y = 4/11 \;, \quad 1-x-y = 6/11$

So limiting dist is $\left[\ \frac{1}{11} \quad \frac{4}{11} \quad \frac{6}{11}\ \right]$

2. $E(X_{n+1} \mid F_n) = X_n$ — (2)

$E(|X_n|) < \infty$ — (1)

(1) $E\left(r^{-n} e^{X_1 + \dots + X_n} \right)$

$= E\left(r^{-n} e^{X_1} \cdot e^{X_2} \cdots e^{X_n} \right)$

$= r^{-n} E(e^{X_1}) E(e^{X_2}) \cdots E(e^{X_n})$

$= r^{-n} \cdot r \cdot r \cdots r = 1$

(2) $E(M_{n+1} \mid F_n)$

$= E\left(r^{-n-1} e^{X_1 + \dots + X_n + X_{n+1}} \mid F_n \right)$

$= E\left(r^{-n-1} \cdot e^{X_1 + \dots + X_n} \cdot e^{X_{n+1}} \mid F_n \right)$

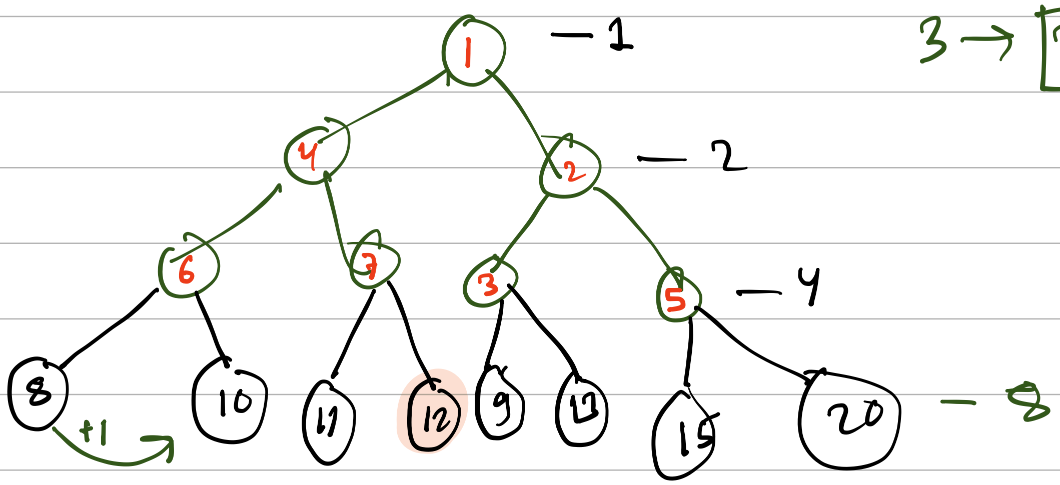
$= \frac{r^{-n}}{r} e^{X_1 + \dots + X_n} E(e^{X_{n+1}} \mid F_n)$

$= \frac{M_n}{r} \cdot E(e^{X_{n+1}})$

$= \frac{M_n}{r} \cdot r = M_n$

Binary tree (Complete)

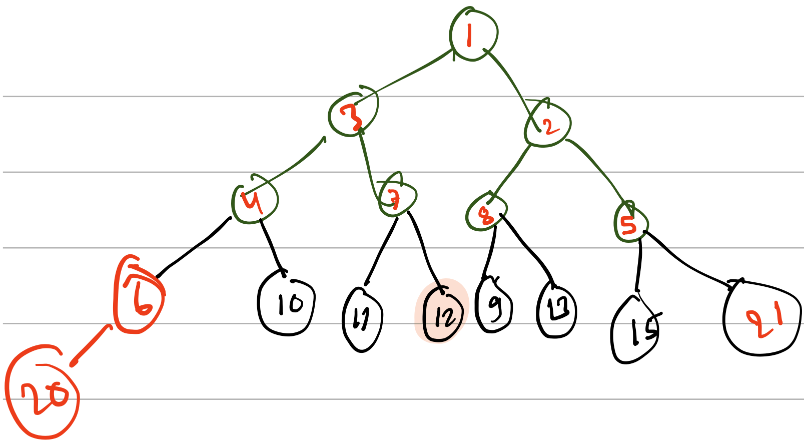
left
3 → $\boxed{3 \times 2} + 1$



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	1	4	2	6	7	3	5	8	10	11	12	9	13	15	20				

Index of left child = Index of parent × 2.
Index of right child = Index of left child + 1.
= 2 × Index of parent + 1

To find parent index,
divide by 2 and take
integer part



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	1	4	2	6	7	8	5	20	10	11	12	9	13	15	21	3			

3 < 20 → replace 3 and 20.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	1	4	2	6	7	8	5	3	10	11	12	9	13	15	21	20			

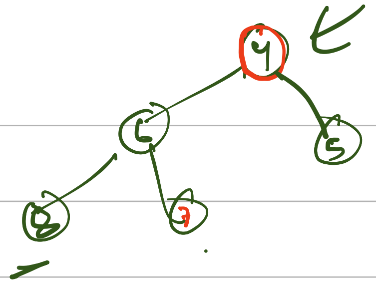
3 < 6 → swap 3 and 6

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	1	4	2	3	7	8	5	6	10	11	12	9	13	15	21	20			

3 < 4

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	1	3	2	4	7	8	5	6	10	11	12	9	13	15	21	20			

$\boxed{3 > 1 \rightarrow \text{stop}}$



root has the minimum

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
	4	6	5	8	7														

1 2 4 5 6 7 8

$\boxed{1 \quad 2 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8}$