Nonlinear control design using Lyapunov function for two-wheeled mobile robots

Conference Paper · January 2012

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Nonlinear Control Design Using Lyapunov Function for Two-Wheeled Mobile Robots

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***Abstract*— This paper presents a nonlinear feedback control framework for two-wheeled mobile robots. The approach uses a constructive Lyapunov function which allows the formulation of a control law with asymptotic stability of the equilibrium point of the system and a computable stability region. The dynamic equations are simplified through normalization and partial feedback linearization. The latter allows linearizing only the actuated coordinate. Description of the control law is complemented by the stability analysis of the closed loop dynamics of the system. The effectiveness of the method has been illustrated by its good performance and less control demand through simulations conducted for two control tasks: upright position stabilization and velocity tracking for a statically unstable two wheeled mobile robot.**

Index Terms— Lyapunov function, Nonlinear control, Stabilization, Two wheeled mobile robot

I. INTRODUCTION

Control of two-wheeled mobile robots (TWRs) attracted the attention of researchers in the last decade due to their inherent instability, nonlinear and complex dynamics. These robots have wide applications in transportation and industry [1-3]. A TWR consists of two drive wheels, joined by an axle, and a robot body called the intermediate body (IB). The statically unstable [4] configuration of TWR , shown in Figure 1, is more challenging as compared to a statically stable[5]. A TWR has been termed statically unstable if the center of mass of the body lies above the axle. The robot wheels are actuated by DC motors and the IB rotates freely. As there is no actuator dedicated directly to control the angular acceleration of vertical IB, the system is under actuated.The underlying control objective of this under actuated TWR, therefore, has been to drive the IB to upright position and to drive the wheels to rest or move at a desired velocity.

Concerning the stability problems of the IB, ensuing the stationary wheels, many control algorithms have been proposed in literature. In particular, state space, optimal and intelligent controllers have been implemented [4-12]. In Grasser et al. [6] dynamic equations were linearized around an operating point to design a controller made up of two decoupled state-space controllers. A pole placement controller was proposed by Nawawi et al. [7].But these controllers were non-robust with respect to un-modeled dynamic uncertainties. To avoid this problem a linear quadratic regulator (LQR) has been designed [8]. Since the presence of nonlinearities in the system limit the control authority and leads to limit cycles or even instability, nonlinear controllers such as Fuzzy adaptive and neural network controls [9, 10]have been designed to improve the stability control of TWRs. A nonlinear controller based on feedback linearization has been proposed in [11] but this lacks the robustness to parameter uncertainties. The robustness against modeling errors has been improved by [12] presenting a back-stepping with sliding mode controller. These nonlinear controllers improve the performance of the stability control but the issue of stability region is not addressed. In this paper, we propose a nonlinear controller designed for the required stability region that enhances the stability of TWRs. The proposed controller is later compared to a conventional optimal linear controller.

We developed a control law for a TWR which stabilizes the closed loop system asymptotically around the unstable upright position of the IB. The control law is devised using a Lyapunov function and control strategy presents computable stability region. The asymptotic stability of the closed loop system is analyzed using LaSalle invariance theorem. The Lyapunov function based (LFB) controllers have been used in many application such as ball on beam [13], turbo charged diesel engine [14] and transportation systems[15]. This paper contributes to the construction of a Lyapunov function and controller design for the statically unstable two-wheeled mobile robots, which does not exist in literature.

In Section II of this article, we present the dynamics model of a statically unstable TWR. The dynamic model is normalized to simplify the algebraic manipulation and then partially linearized to produce an equivalent feedback system in the affine form. Section III provides an insight to the construction of the Lyapunov function candidate, nonlinear controller design to stabilize upright position of the IB, corresponding stability and convergence analysis and stability region computation. In Section IV, we present the simulation results with a brief discussion for a TWR to verify the effectiveness of the proposed control algorithm.

II. DYNAMICS MODEL

This section presents the modeling of dynamics of a two-wheeled mobile robot. The equations of motion are formulated, simplified and partially feedback linearized to

This work was supported by the Higher Education Commission (HEC) of Pakistan.

convert into a convenient form, suitable to design the nonlinear controller. The two-wheeled mobile robot is modeled as an inverted pendulum on two wheels.

A. Dynamic equations

In order to model the dynamics of a TWR, the motion of the robot is restricted in an x-z plane. Schematic diagram of a statically unstable TWR is shown in Figure 1. Both the IB and wheels are assumed to be rigid bodies. The mass of each wheel, *m,* is alleged to be located at the centre of the wheel having a radius r. The mass of the IB, M is located at a distance l from the axle joining two wheels. The mass moment of inertia for the IB and wheels (including gearbox and transmission) are denoted as I_p and I, respectively.

It is assumed that the wheels rotate at an angle $\emptyset$ without slipping and each wheel remains in contact with the ground at a single point. A differential drive is assumed such that motors apply total torque (T) on the wheels to navigate the robot a horizontal distance x along the x-plane. The IB rotates from the vertical, denoted by pitch angle, θ. The dynamics of the robot are modelled using Newton-Euler method. The resulting equations of motion using kinematic constraint of no-slip are given by (1) and (2).

$$(Ml\,cos\theta)\ddot{x} + \left(Ml^2 + I_p\right)\ddot{\theta} - Mgl\,sin\theta = 0 \quad (1)$$

$$r\,(M+m)\ddot{x} + (Mlr\cos\theta)\ddot{\theta} - (Mlr\sin\theta)\dot{\theta}^2 + I\ddot{\phi} = T \quad (2)$$

B. Model Simplification

Equations (1) and (2) are normalized to simplify the algebraic manipulation of them and to develop the affine form, following the approach used by [15]. The affine form of equations brings about the controller design easy. To achieve this normalization we introduce the following scaling transformations:

$$q = \frac{x}{l}, \quad u = \frac{T}{Mlr}, \quad \delta = \frac{m}{M}, \quad a = \frac{I_p}{Ml^2}, \quad \frac{d\tau}{dt} = \sqrt{\frac{g}{l}}$$

Substitution of the above scaling factors in (1) & (2) and algebraic manipulation gives the following simpler set of dynamic equations:

$$(\cos\theta)q'' + (1+a)\theta'' - \sin\theta = 0 \quad (3)$$

$$\left(1 + \delta + \frac{I}{r^2}\right)q'' + (\cos\theta)\theta'' - (\sin\theta)\theta'^2 = 0 \quad (4)$$

Due to normalization, the differentiation in (3) & (4) and all following equations are with respect to the dimensionless time τ defined as $\tau = t\sqrt{\frac{g}{l}}$.

C. Partial Feedback Linearization of the Model

The above simplified equations still have strong inertial coupling and the outputs (pitch and displacement) are collocated with the input u making input/output linearization difficult. The simplified system is therefore partially feedback linearized. Considering the inertia matrix to be symmetric and positive definite, we manipulate the equation (3) to solve for θ'' and substitute the result in equation (4).

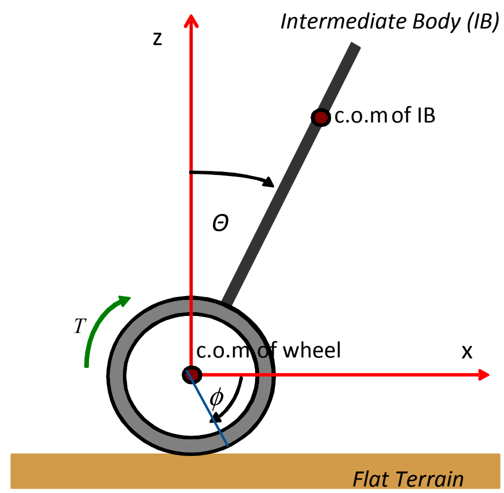

Figure 1: Schematic diagram of a two-wheeled mobile robot.

Using some standard calculus we can write the new control variable, u, as (5):

$$u = \left[1 + \delta + b - \frac{\cos^2(\theta)}{1+a}\right]q'' + sin(\theta)cos(\theta) - \theta'^2 sin(\theta) \quad (5)$$

From (5) a partial feedback linearizing controller can be defined as (6)

$$u = \left[1 + \delta + b - \frac{\cos^2(\theta)}{1+a}\right]v + sin(\theta)cos(\theta) - \theta'^2 sin(\theta) \quad (6)$$

The control input of the robot, u, can be computed from (6) given the input , an additional control input yet to be defined. The feedback system equivalent to the normalized system of dynamic equations with new control input v is as following:

$$q'' = v \quad (7)$$

$$\theta'' = \frac{I.[sin(\theta) - v.cos(\theta)]}{1+a_1} \quad (8)$$

We see that the input/output system from v to q in (7) & (8) is linear and second order. The second equation therefore represents the internal dynamics. This set of equations can be evidently expressed now in the affine form as

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})v \quad (9)$$

with $\boldsymbol{x} = (q', \theta, \theta')^T$, the vector of state variables.

If $v = 0$ and $\theta \in [0, 2\pi]$, the system has two equilibrium points. One is a stable equilibrium point as $\boldsymbol{x} = (0,0,0)$ and the second is an unstable equilibrium point as $\boldsymbol{x} = (0,\pi,0)$. In the following section we will introduce a controller for this partially linearized system to stabilize the IB at its upright unstable equilibrium point.

III. CONTROLLER DESIGN

The control objective is to asymptotically stabilize the IB of the TWR to its upright position with velocity tracking. A Lyapunov method is applied to design the controller, assuming that the initial position of the IB is above the horizontal plane. In this section, a framework for designing a stabilizing controller is presented. The main idea of the Lyapunov method is to propose a scalar, positive definite energy or energy-like function $V(t)$for the system such that it has continuous partial derivatives. If its time derivative is provided to be at least negative semi definite, $V'(t) \leq 0$, the system is stable in the sense of Lyapunov in that the system states (energy) can be constrained for all future time to lie inside a ball in state space. This ball is directly related to the size of the initial states of the system. Based on the designed controller, the stability region and the asymptotic stability of the closed loop system, shown in Figure 2, are imparted before concluding this section.

A control Lyapunov function (CLF) for a nonlinear system of the form (9) is defined in [16] as a candidate Lyapunov function $V(\boldsymbol{x})$ with the property that for every fixed $x \neq 0$ there exists an admissible value u for the control such that $\nabla V(\boldsymbol{x}).f(\boldsymbol{x},u) < 0$. In other words CLF is simply a candidate Lyapunov function whose derivative can be made negative by the choice of control values.

The CLF is shaped by means of the technique of added integration, presented by Lozano [17]. First, we introduced a Lyapunov function that represents the energy of the IB of the two-wheeled mobile robots with certain constraint. The constraint bounds the states θ and θ'. This was used to develop the controller which stabilizes the pitch dynamics of the IB i.e. θ and θ'. Then we introduce an additional quadratic term for the translational part of the robot to formulate a new Lyapunov function.

We propose a Lyapunov function (10):

$$V_{c0}(\theta,\theta') = \tfrac{1}{2}(k_1 cos^2\theta - l)\theta'^2 + (1 - cos\theta)\frac{l}{1+a} \quad (10)$$

where, $\frac{l}{1+a}$ is a parametric constant of the system and has a positive value. A constraint function $(k_1 cos^2\theta - l)$ is introduced in the above Lyapunov function to bound the pitch $\theta$of the IB. The proposed $V_{c0}(\theta,\theta')$ is positive definite for:

(i) $k_1 > 1$, because $V_{c0}(\theta,\theta')$ is a locally convex function with a minimum at the origin if $k_1 > 1$. $k_1$is a constant that controls the boundary of initial pitch as well as performance of the controller.

(ii) $|\theta| < \tilde{\theta}$, such that the initial conditions (θ_0, θ'_0) with $|\theta_0| < \frac{\pi}{2}$ satisfies that $V_{c0}(\theta,\theta') < (1 - cos\tilde{\theta})$, where $\tilde{\theta} = cos^{-1}\left(\frac{l}{\sqrt{k_1}}\right)$.

The time derivative of Lyapunov function is given as

$$V'_{c0} = \theta' cos\theta\big(k_1\beta(\theta,\theta') + v\gamma(\theta)\big) \quad (11)$$

where $\beta = \frac{cos\theta sin\theta}{1+a} - sin\theta\theta'^2 = sin\theta\left(\frac{cos\theta}{1+a} - \theta'^2\right)$,

$$\gamma = -\frac{k_1 cos^2\theta}{1+a} + \frac{l}{1+a}.$$

To control the robot velocity q' as well as pitch θ and pitch rateθ', a quadratic term is added to (10) such that the time derivative of the additional term has the same of structure as (11). The control Lyapunov function for $\boldsymbol{x} = (q',\theta,\theta')$ is, therefore, proposed as

$$V_{c1}(q',\theta,\theta') = k_d V_{c0}(\theta,\theta') + \tfrac{1}{2}W^2 \quad (12)$$

where W is the unknown auxiliary variable in the additional term. The auxiliary variable, a function of, q',θ,θ' with same structure as V'_{c0} satisfying

$$W'(\theta,\theta',q') = \theta'\frac{\partial W}{\partial\theta} + \sin\theta\frac{\partial W}{\partial\theta'} + \left(\frac{\partial W}{\partial q'} - \cos\theta\frac{\partial W}{\partial\theta'}\right)v$$

results into

$$W(\theta,\theta',q') = \tfrac{1}{1+a}(lq' + k_1\theta'\cos\theta). \quad (13)$$

Substituting (13) into (12), we get CLF as (14).

$$V_{c1} = k_d V_{c0}(\theta,\theta') + \tfrac{1}{2}\left[\tfrac{1}{1+a}(lq' + k_1\theta'\cos\theta)\right]^2 \quad (14)$$

Evidently, (13) is a positive definite function for all $|\theta| < \tilde{\theta} < \frac{\pi}{2}$ and $k_d > 0$. In the time derivative of CLF (14) substitution of (7), (8) & (11) and then replacement of $\beta(\theta,\theta')$, $\gamma(\theta)$ and $\dot{W}(\theta,\theta',\dot{x})$, wherever needed, leads to (15).

$$V'_{c1} = v\gamma(\theta)W' + k_1\beta(\theta,\theta')W' \quad (15)$$

where

$$\beta = \frac{cos\theta sin\theta}{1+a} - sin\theta\theta'^2 = sin\theta\left(\frac{cos\theta}{1+a} - \theta'^2\right),$$
$$\gamma = -\frac{k_1 cos^2\theta}{1+a} + \frac{l}{1+a}, \text{ and}$$
$$\dot{W} = \left(\frac{l\dot{x}}{(1+a)^2} + \frac{k_1\theta' \cos\theta}{(1+a)^2} + k_d\theta' cos\theta\right)$$

To craft the derivative of Lyapunov function to be semi negative definite $V'_{c1} \leq 0$, $k_1\beta(\theta,\theta') + v\gamma(\theta)$ should be equal to $-W'$ and the control law can be conveniently chosen as (16)

$$v_1 = \frac{-1}{\gamma(\theta)}\big(W'(q',\theta,\theta') + k_1\beta(\theta,\theta')\big) \quad (16)$$

The control law (16) makes the variables $q',\theta,$ to converge asymptotically to zero. This produces$V'_{c1} = -W'^2(q',\theta,\theta')$.

A. Stability Region

In (10), V_{c0} becomes locally convex function with a minimum at origin if $k_1 > 1$. This can be easily checked by plotting the corresponding level curves. If initial conditions $(\theta,\dot{\theta})$ with $|\theta_0| < \frac{\pi}{2}$ satisfies that $V_{c0} < 1 - \cos\tilde{\theta}$, then we have $|\theta| < \tilde{\theta}$ and a set defined by (17) is a compact set.

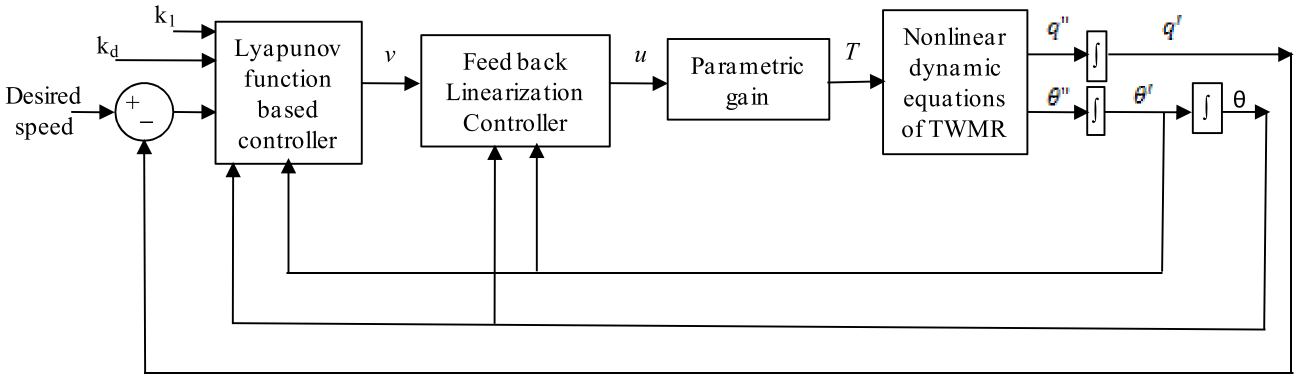

Figure 2: Block diagram of the closed-loop system.

$$\Omega_0 = \left\{(\theta, \theta') \, \epsilon \, R^2 : \; V_{c0} < 1 - \cos\tilde{\theta}\right\} \quad (17)$$

This compact set represents the stability region of the system with two controlled states, θ, θ'. Similarly, we can define the stability region of the system with three controlled states q', θ, θ'. Equation (16) suggests the system has no singularity when $|\theta| < \tilde{\theta} < \frac{\pi}{2}$ .To avoid singularity at $|\theta| = \pm\tilde{\theta}$, $|\theta_0| < \frac{\pi}{2}$ and $V_{c1} < K = k_d(1 - cos\theta)$should belong to the neighborhood of the origin. In other words states are bounded with $|\theta| < \tilde{\theta}$ if $|\theta(t)| < \tilde{\theta}$ and $V_{c1} < K$. $V_{c1} < K$, is an outcome from the fact that V_{c1} is a non- increasing function as $V'_{c1} = -W'^2(q', \theta, \theta')$. This also defines stability region,Ω_1, for the proposed closed loop system of the two-wheeled robot with three states (θ, θ', q').

$$\Omega_1 = \left\{(\theta, \theta', q') \, , |\theta| < \tilde{\theta} : V_{c1} < K\right\} \quad (18)$$

We conclude that the unstable equilibrium point of a TWR is stable with proposed controller in closed loop, in the sense of Lyapunov as $V_{c1}(\boldsymbol{x})$is a positive definite function forall $\boldsymbol{x}\epsilon\Omega_1$ and $\dot{V_{c1}}(\boldsymbol{x})$ is negative semi definite for $\boldsymbol{x}$.

IV. SIMULATIONS

In order to illustrate the effectiveness and limitations of the proposed control law, we carried out experiments in simulation using *Simulink* and MATLAB software. The simulation tests were conducted for the evaluation of velocity tracking and transient performance of the closed-loop system. Performance of the proposed controller is compared with a linear quadratic regulator (LQR) design for reference tracking, referred to as the baseline (BL) controller. The controllers are tuned keeping one of the performance metrics constant to allow fair comparisons. The performance was measured as transient response, integrated squared error of pitch, speed tracking and torque demand to accomplish the task. Both the controllers were designed with an assumption that all the states are measurable and available for feedback. The physical parameters of the system, used for all experiments, are given in the Table 1. These parameters represent a real time two-wheeled robotic platform system. The initial conditions of states were maintained at same values for both controllers.

Controller Performance:

The results, simulated for the evaluation of velocity tracking and transient performance of the closed loop system with the proposed controller are shown in Fig. 3. These results clearly indicate that all the states converge to the origin with a smooth speed tracking. The transient behaviour of the closed loop system portrays a good performance of the controller. All the states converge to steady state condition with an acceptable overshoot, settling within 10 seconds and no steady state error. A few tests were conducted with variation of mass, inertia and the height of centre of mass of the IB. The proposed control strategy proved robust for large mass range (50-100 kg), inertia range (0.05-1.35 kg-m^2) and for high centre of mass positions (0.1-1 m).

Controller Comparison:

The objective of the three comparative tests designed was to evaluate if the proposed control algorithm has a benefit over linear control. As the tests are conducted in different conditions, they give a confidence that the proposed algorithm may be used for variety of test conditions and may describe the limitations of the proposed controller.

The first test is conducted to compare the controller output and transient response such that the two closed loop systems have the same settling time. In this experiment, the controller parameters (gains) are tuned such that both the systems' response has the same settling time. The objective was to compare the transient and steady state behaviour as well as the control demand by both the controllers to accomplish the task. The initial conditions of all the states are same for both the closed loop systems i.e. all initiated from zero except pitch with 85 degrees, a pitch angle close to the boundary of initial pitch calculated using $\tilde{\theta} = cos^{-1}\left(\frac{l}{\sqrt{k_1}}\right)$.

TABLE I
PHYSICAL PARAMETERS OF THE TWR

Parameter	Value	Parameter	Value
M (kg)	43.0	I_p (kg-m^2)	1.30
m (kg)	2.80	l (m)	0.19
I (kg-m^2)	0.09	r (m)	0.20

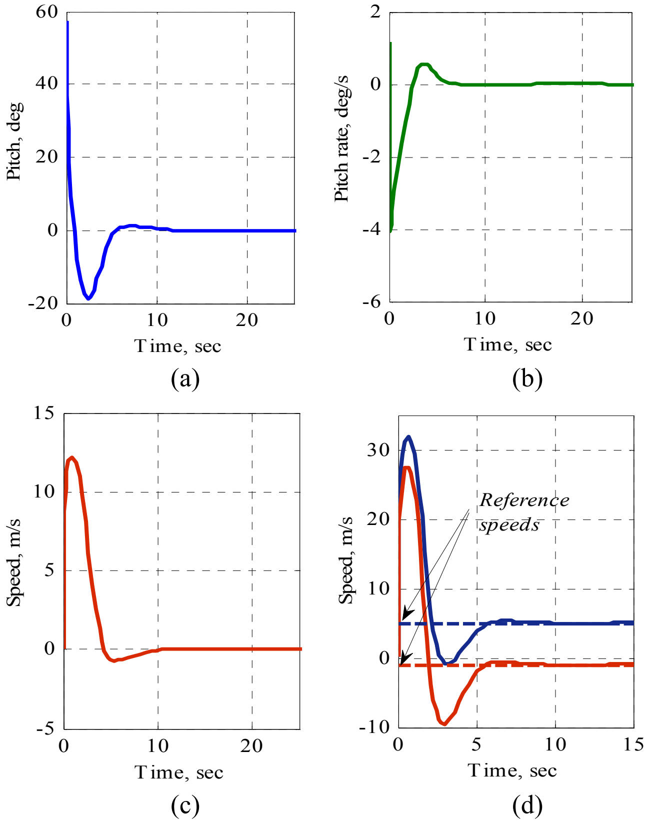

Figure 3: Transient performance and speed tracking of a LFB controller.

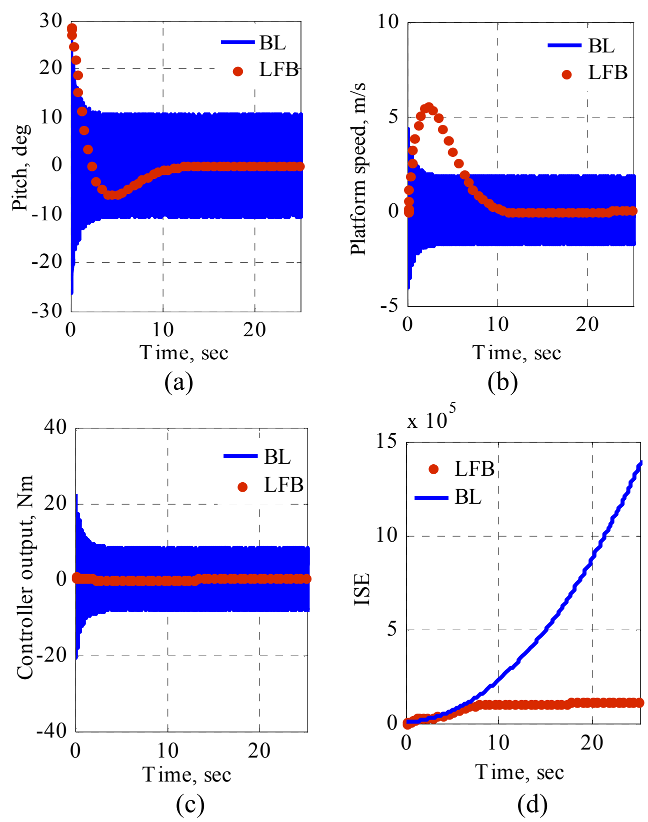

Figure 4: Comparison of transient performance and control input for same closed loop polynomials.

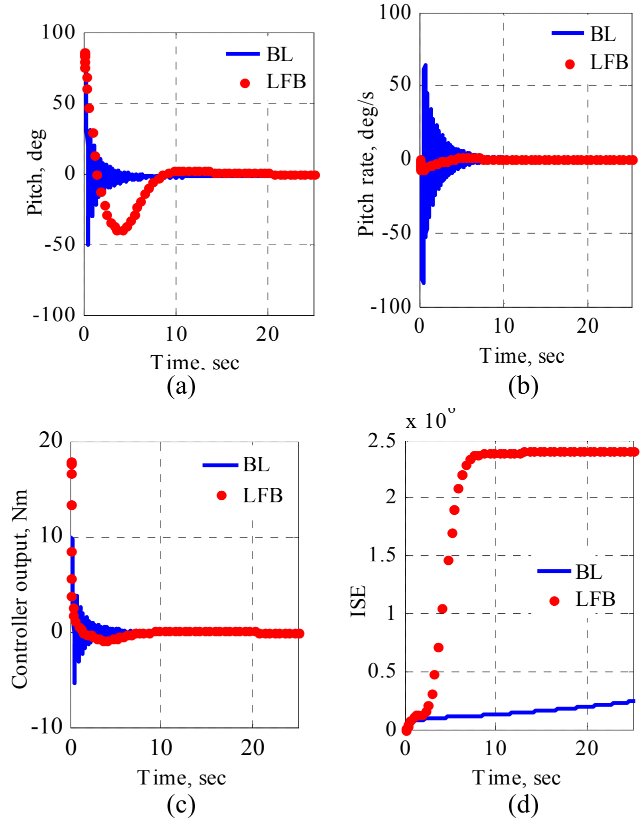

Figure 5: Comparison of transient performance and control input for same settling time.

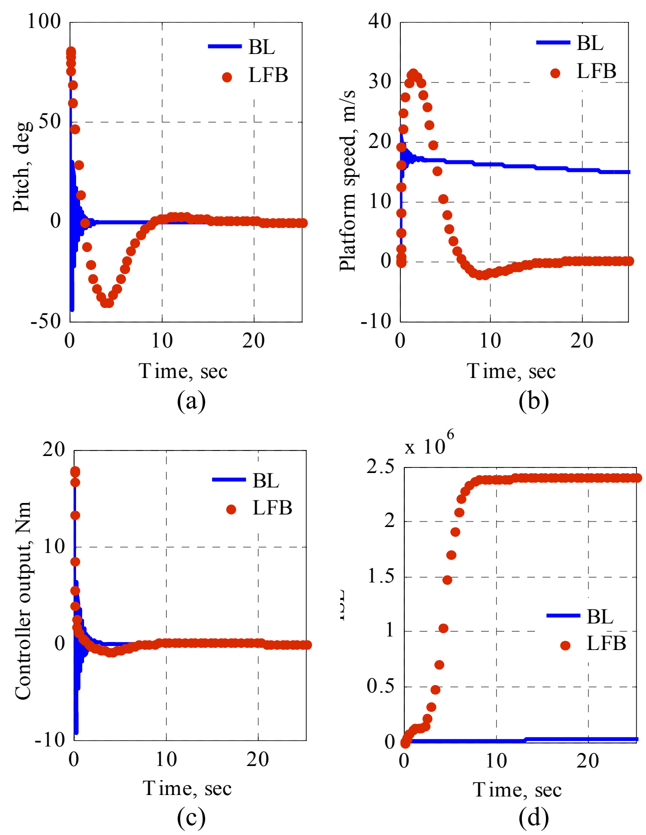

Figure 6: Comparison of transient performance and control input for same initial control demand

Fig.4 shows a comparison of the transient behaviour, pitch, pitch rate, integrated squared error of pitch and control demand of the robot motion in the horizontal x- direction. Maximum settling time is aimed within 10 seconds in this case and results are presented for comparison for both closed loop systems.

The closed loop systems response, for both the controllers with same settling time, suggests that a LFB controller is better in sense of smooth transition of the states in comparison to the BL controller. This produces higher integrated squared error of pitch but demands less control input to the system.

The second experiment is designed to compare the performance and control demand of TWR systems having the same closed loop polynomial but different controllers. In this experiment, the controller parameters are selected such that they produce same closed loop system polynomial. The objective was to see if both the controllers perform equally well close to the equilibrium states then how they perform at the initial conditions far from the equilibrium point. The parameters of the proposed control law are tuned and chosen such that the system shows same performance as in previous section. The control parameters of the BL controller were selected such that the characteristic polynomial of the closed loop system matches the characteristic polynomial computed for the closed loop system with proposed controller. The initial conditions were set at the origin except for the pitch of the IB. This is fixed as 28 degrees. The results shown in Fig.5 compare the transient behaviour of the IB position, integrated squared error, control demand and the speed of the robot motion in longitudinal direction for two closed loop systems. When the system is made equivalent closed loop system the LFB controller shows good results while the BL controller could not stabilize the system and caused large oscillations.

Lastly, the transient performance and control demand are evaluated and compared with a condition that both the controllers produce the same initial torque at the same initial conditions. In this experiment we aimed to observe the response of the closed-loop systems if both produce the same control demand initially. For this purpose the controller gains were selected such that they produce same initial control input demand with both the controllers, for same initial conditions of the states. All the states were kept at zero initial condition except pitch. The pitch had an initial value of 86 degrees. The results shown in Fig.6 compare the transient behaviour of the IB position, rate of the IB pitch, integrated squared error and the control demand for the robot motion in longitudinal direction for two closed loop systems. The results show although the LFB controller settles slowly it drops down the control demand over transition very smoothly and gradually as compared to the oscillated behaviour of the linear controller.

V. Conclusion

We proposed a nonlinear Lyapunov function based controller for stability control of two-wheeled mobile robots and compared its performance with a linear controller, in different scenarios. The conclusions from the presented results are following.

1. The proposed controller tracked the velocity and stabilized the IB position from far initial position within 10 seconds and with very low power demand.
2. The linear controller, as compare to the proposed controller, has a very high control demand that limits its practical use.
3. Velocity tracking performance of the linear controller is not to the desired standard.

Simulations with realistic data show that the proposed approach is effective. Further work is required to make the controller design robust with respect to parameter-uncertainties.

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