CONTROL OF AN INVERTED PENDULUM USING DIRECT MODEL REFERENCE ADAPTIVE CONTROL

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Abstract. This paper presents the design of a robust direct model reference adaptive control (DMRAC) algorithm with focus on the satisfaction of an almost strictly positive real (ASPR) condition over the maximal range of parameter variations of the plant, and its application to an inverted pendulum. To alleviate the ASPR condition, the pendulum is augmented with a feedforward compensator optimized for robustness. A DMRAC algorithm ensuring asymptotic model following is then integrated with the feedforward compensator design. These new experimental results validate the algorithm's use and applicability to the control of an inverted pendulum with a variety of parameter uncertainties.

Résumé. On presente la conception d'un algorithme robuste de reglage adaptif a' reference au modele directe (DMRAC) qui dois satisfaire la condition de realization positive presque partout (PPP) et son application a un pendule renverse. Pour alleger la condition PPP, le pendule est ameliore' par l'ajout d'un compensateur "feed-forward" qui est optimise' pour la robustess. Puis, un algorithme DMRAC qui assure que la modele soit suivi asymptotiquement est integre' dans la realisation du compensateur. Les nouvaux resultats experimentaux verifient l'usage et l'application de l'algorithme pour regler le pendule renverse' avec des incretitudes sur les valeurs des parametres.

Keywords. Adaptive Control, Inverted Pendulum, Model Reference Control

1. INTRODUCTION

Model reference adaptive methods may be classified according to three different approaches. First is the full state access method described by Landau (Landau, 1974), which assumes that the state variables are measurable. Second is the input - output method originating from Monopoli's augmented error signal concept (Monopoli, 1974). In this latter approach, adaptive observers are incorporated with the controller to estimate missing states. Third is the simple adaptive control approach originated by (Sobel *et al.*, 1979). This approach is an output feedback method which requires neither full state feedback nor adaptive observers. Other important properties of this class of algorithms include (1) their applicability to non-minimum phase systems, (2) the fact that the plant (physical system) order may be much higher than the order of the reference model, and (3) the applicability of this approach to multi-input multi-output (MIMO) systems. Its ease of implementation and inherent robustness properties make this simple adaptive control approach attractive.

The simple adaptive control approach to DMRAC of MIMO plants was first proposed by Sobel, Kaufman, and Mabius (Sobel *et al.*, 1979). This control structure uses a linear combination of feedforward model states and command inputs and feedback of the error between plant and model outputs. This class of algorithms requires neither full state access nor adaptive observers. Furthermore, asymptotic stability is guaranteed if the plant is ASPR; in other words, if there exists a feedback gain K_e

such that the resulting closed-loop transfer function is strictly positive real (SPR). This gain need not be physically realized during implementation.

Since most real systems are not ASPR, the algorithm was extended by (Kaufman *et al.*, 1994); (Kaufman and Neat, 1993) to the class of non-ASPR plants. It was shown that there exists a known dynamic output stabilizing feedback, with transfer matrix $H(s)^{-1}$, such that the plant in parallel with $H(s)$, the augmented plant, is ASPR. However, the error, which is bounded, is not the true difference between the plant and model outputs because the augmented plant output includes a contribution from the supplementary plant feedforward. However, it was argued in (Kaufman *et al.*, 1994) and (Kaufman and Neat, 1993) that if the supplementary feedforward can be made small, then the true output error can also be made small.

(Kaufman and Neat, 1993) suggested a further modification incorporating supplementary feedforward into the reference model in such a manner that asymptotic tracking of the augmented plant and model outputs implies asymptotic tracking of the original plant and model outputs. Such asymptotic tracking has indeed been observed in various applications, including the control of robotic dynamics (Kaufman *et al.*, 1994).

Although the modifications of (Kaufman and Neat, 1993) alleviated otherwise rather restrictive positive real constraint, thus greatly expanding the class of plants that can be adaptively controlled with asymptotically vanishing output error, their implementation required design of a feedforward compensator which satisfied the positive real constraint over the anticipated range of parameter variations.

Recently, (Iwai and Mizumoto, 1992) applied a robust feedforward compensation procedure to the direct adaptive algorithm described in (Bar-Kana, 1987) and (Bar-Kana and Kaufman, 1985). Because this feedforward path affects only the plant dynamics, and not that of the reference model, the error between plant and model outputs remained bounded, but not zero. They also showed that if the compensator high frequency gain is "sufficiently small", then the ASPR condition is thereby satisfied. However, they did not show exactly how to determine this suitably small gain. It is also important to note that their design applied only to minimum phase systems.

In this paper, robust feedforward compensator design conditions are first developed for both minimum and non-minimum phase SISO plants in such a manner that the augmented plant satisfies the ASPR conditions over a wide range of plant parameter variations. Second, an easily implementable optimization procedure is developed to determine the feedforward compensator parameters necessary to satisfy the design conditions. Finally, a DMRAC algorithm is integrated with this robust feedforward compensator and applied to control an inverted pendulum with parametric variations.

In section 2, formulation of the DMRAC algorithm is discussed. Robust feedforward compensator design is formulated in section 3. Simulation results are given in Section 4. Finally, in section 5, results are discussed, and conclusions are drawn.

2. FORMULATION OF THE DMRAC ALGORITHM

The linear time invariant model reference adaptive control problem is considered for a plant described by the following state-space equations (Kaufman *et al.*, 1994)

$$\begin{aligned}\dot{x}_p(t) &= A_p x_p(t) + B_p u_p(t)\\ y_p(t) &= C_p x_p(t)\end{aligned}\qquad(1)$$

where $x_p(t)$ is the $(n \times 1)$ state vector, $u_p(t)$ is the $(m \times 1)$ control vector, $y_p(t)$ is the $(q \times 1)$ plant output vector, and A_p, B_p are matrices with appropriate dimensions. The range of the plant parameters is assumed to be bounded as defined by

$$\begin{aligned}\underline{a}_{ij} &\le a_p(i,j) \le \overline{a}_{ij}, i,j = 1,\cdots,n \qquad(2)\\ \underline{b}_{ij} &\le b_p(i,j) \le \overline{b}_{ij}, i,j = 1,\cdots,n\end{aligned}$$

where $a_p(i,j)$ is the $(i,j)^{th}$ element of A_p and $b_p(i,j)$ is the $(i,j)^{th}$ element of B_p.

The design objective is to find, without explicit knowledge of A_p and B_p, some control $u_p(t)$ such that the plant output vector $y_p(t)$ follows the output of the reference model

$$\begin{aligned}\dot{x}_m(t) &= A_m x_m(t) + B_m u_m(t)\\ y_m(t) &= C_m x_m(t)\end{aligned}\qquad(3)$$

The model incorporates desired plant behavior and in many cases

$$dim[x_p(t)] \gg dim[x_m(t)] \qquad(4)$$

The adaptive control algorithm being presented is based upon the command generator tracker concept (CGT) developed by (Broussard and O'Brien, 1979). In the CGT method it is assumed that there exists an ideal plant with ideal state and control trajectories, $x_p^*(t)$ and $u_p^*(t)$, respectively, which corresponds to perfect output tracking (i.e., when $y_p(t) = y_m(t)$ for $t \ge 0$). By definition, this ideal plant satisfies the same dynamics as the real plant, and the ideal plant output is identically equal to the model output. Thus,

$$\dot{x}_p^* = A_p x_p^* + B_p u_p^* \quad for\ all \quad t \geq 0 \tag{5}$$

$$y_p^* = y_m = C_p x_p^* = C_m x_m \tag{6}$$

Hence, when perfect tracking occurs, the real plant trajectories become the ideal plant trajectories, and the real plant output becomes the ideal plant output, which is defined to be the model output.

The ideal control law $u_p^*(t)$, generating perfect output tracking and the ideal state trajectories is assumed to be a linear combination of the model states and model input:

$$\begin{bmatrix} x_p^* \\ u_p^* \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} x_m(t) \\ u_m(t) \end{bmatrix} \tag{7}$$

where the S_{ij} submatrices satisfy the following conditions

$$\begin{aligned} S_{11} A_m &= A_p S_{11} + B_p S_{21} \\ S_{11} B_m &= A_p S_{12} + B_p S_{22} \\ C_m &= C_p S_{11} \\ 0 &= C_p S_{12} \end{aligned} \tag{8}$$

The adaptive control law based on this CGT approach is given as (Kaufman *et al.*, 1994)

$$u_p(t) = K_e(t) e_y(t) + K_x(t) x_m(t) + K_u(t) u_m(t) \tag{9}$$

where $e_y(t) = y_m(t) - y_p(t)$ is the output error and $K_e(t)$, $K_x(t)$, and $K_u(t)$ are adaptive gains, which can be concatenated into a single matrix

$$K(t) = [K_e(t) \quad K_x(t) \quad K_u(t)] \tag{10}$$

Defining a vector $r(t)$ as

$$r(t) = \begin{bmatrix} y_m(t) - y_p(t) \\ x_m(t) \\ u_m(t) \end{bmatrix} \tag{11}$$

the control $u_p(t)$ can be written in compact form as

$$u_p(t) = K(t) r(t) \tag{12}$$

Thus, $u_p(t)$ is composed of the feedback term

$$K_e(t) e_y(t) \tag{13}$$

together with the feedforward component

$$K_x(t) x_m(t) + K_u(t) u_m(t) \tag{14}$$

The adaptive gains are obtained as a combination of the following integral and proportional gains (Kaufman *et al.*, 1994)

$$\begin{aligned} K(t) &= K_p(t) + K_i(t) \\ K_p(t) &= [y_m(t) - y_p(t)] r^T(t) T_p, \; T_p \geq 0 \\ \dot{K}_i(t) &= [y_m(t) - y_p(t)] r^T(t) T_i, \; T_i > 0 \end{aligned} \tag{15}$$

Sufficiency for asymptotic tracking is achieved if:

(1) There exists a solution to the CGT problem, eq. (8).
(2) The plant is ASPR; that is, there exists a gain matrix K_e, not needed for implementation, such that the closed loop transfer function

$$G_c(s) = [I + G_p(s) K_e]^{-1} G_p(s) \tag{16}$$

is strictly positive real (SPR).

In general, the ASPR conditions are not satisfied by most real systems. (Bar-Kana and Kaufman, 1985) have remedied this situation by showing that a non-ASPR plant of the form $G_p(s) = C_p(sI - A_p)^{-1} B_p$ can be augmented with a feedforward compensator $H(s)$ so that the augmented plant transfer matrix

$$G_a(s) = G_p(s) + H(s) \tag{17}$$

is ASPR.

It can be shown (Kaufman *et al.*, 1994) that a SISO plant represented by the triple (A_p, B_p, C_p) is ASPR if the transfer function $G_p(s) = C_p(sI - A_p)^{-1} B_p$

(1) is minimum phase,
(2) has relative degree of unity or zero ($n - m = 1$ or $n - m = 0$),
(3) and has a positive leading coefficient.

It was shown in (Kaufman *et al.*, 1994) that the resulting adaptive controller will, in general, result in a model following error that is bounded, but not zero, in steady state. To improve upon this, a modification, incorporating the supplementary feedforward into the reference model output as well as to the plant output, has been developed (Kaufman and Neat, 1993).

3. ROBUST FEEDFORWARD COMPENSATOR DESIGN

In a recent paper by (Iwai and Mizumoto, 1992), existence of a feedforward compensator was shown for a minimum phase SISO plant, provided that the compensator high frequency gain is sufficiently small. However, it was not explicitly shown how one finds the necessary parameters such that the ASPR conditions are satisfied. Furthermore, their procedure is not applicable to non-minimum phase plants.

In the preceding section, the plant was described in state-space with uncertainties in the elements of the plant A_p and B_p matrices. In this section, the transfer function representation of a plant will be used instead, with uncertainties in the coefficients in its numerator and denominator.

Consider a non-ASPR plant of the form

$$G_p(s) = \frac{C_m s^m + C_{m-1}s^{m-1} + \cdots + C_0}{B_n s^n + B_{n-1}s^{n-1} + \cdots + B_0} \quad (18)$$

where the coefficients B_{n-j} and C_{m-j} can take any values within the given bounds

$$\underline{C}_{m-j} \le C_{m-j} \le \overline{C}_{m-j}, j = 0, 1, \cdots, m$$
$$\underline{B}_{n-j} \le B_{n-j} \le \overline{B}_{n-j}, j = 0, 1, \cdots, n \quad (19)$$

The following assumptions are made for the plant given above:

Assumption 1.

(1) Nominal plant parameters and their bounds are known
(2) The degree of the plant, n is known

Since the plant $G_p(s)$ is considered non-ASPR, a feedforward compensator $H(s)$ is to be augmented in parallel with the plant $G_p(s)$. Almost strictly positive realness of the augmented plant is described in the following lemma:

Lemma 1. (Bar-Kana, 1991), Let $G_p(s)$ be any transfer function with arbitrary relative degree ($n - m > 0$). $G_p(s)$ is not necessarily stable or of minimum phase. Let $H(s)^{-1}$ be any dynamic stabilizing controller as depicted in Figure 1.

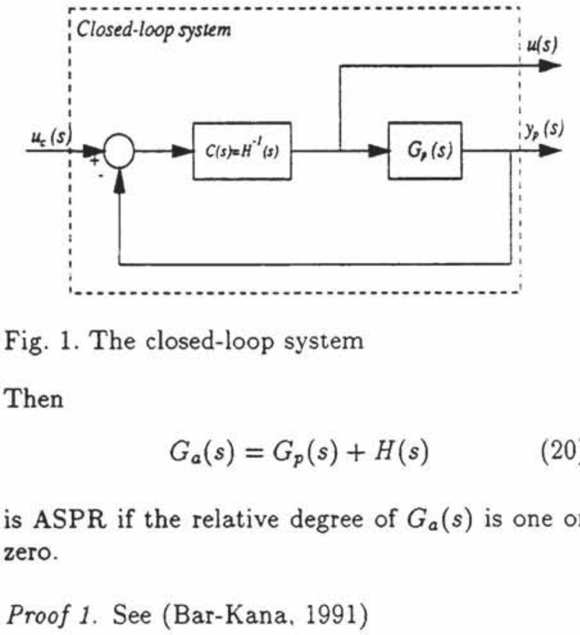

Fig. 1. The closed-loop system

Then

$$G_a(s) = G_p(s) + H(s) \quad (20)$$

is ASPR if the relative degree of $G_a(s)$ is one or zero.

Proof 1. See (Bar-Kana, 1991)

At this point, it is desired that some knowledge on the order of a feedforward compensator be known, so that the plant $G_p(s)$ can be stabilized. As discussed in (Ozcelik, 1996), the order of a feedforward compensator is chosen to be equal to the order of the plant. Using this result in conjuction with above Lemma 1, a feedforward compensator for an n^{th} order plant can be constructed as

$$H(s) = \frac{f_{n-1}s^{n-1} + f_{n-2}s^{n-2} + \cdots + f_0}{s^n + h_{n-1}s^{n-1} + \cdots + h_0} \quad (21)$$

Noted that this feedforward compensator will always satisfy the relative degree condition of the Lemma 1. The denominator of $H(s)$ might be predetermined in a manner such that the compensator time constant is faster than that of the reference model, because it is desired that the transients from the feedforward compensator should be much faster than the transients from the reference model.

In view of the Lemma 1, the augmented plant $G_a(s) = G(s) + H(s)$ will be ASPR if $H(s)^{-1}$ stabilizes the closed loop. In other words, the augmented plant will be ASPR if the characteristic polynomial of the closed-loop is stable for all possible variations of plant parameters. The characteristic polynomial of the closed loop may be given as

$$P(s) = a_k s^k + a_{k-1}s^{k-1} + \cdots + a_0 \quad (22)$$

$$a_k = f(C_i, B_{n-j}, f_{n-j}) \quad \begin{cases} k = 0, \cdots, 2n-1 \\ i = 0, \cdots, m \\ j = 1, \cdots, n \end{cases} \quad (23)$$

To determine the necessary coefficients of the feedforward compensator which guarantees the stability of the closed-loop characteristic equation for all possible variations of plant parameters, we propose the following design procedure:

(1) Obtain conditions for stability of the characteristic polynomial, $P(s)$ from the first column of Routh-Hurwitz table.
(2) Feedforward compensator parameters are then determined by the following optimization procedure

$$\underset{f_i}{\text{minimize}} \left\{ \sum f_i^2 \right\}$$

$$\text{subject to:} \quad \min_{(C_i, B_j)}(g_k) > 0 \quad k = 1, 2, \cdots, 2n-1$$

where g_k are parameter constraints obtained from first column of the Routh-Hurwitz table. Robust feedforward compensators designed using the above procedure will certainly satisfy all the conditions of Lemma 1; therefore, the augmented plant will be ASPR for the allowed variations in the plant parameters.

4. SIMULATION RESULTS

The feedforward compensator designed using the proposed method was implemented with the DMRAC algorithm given in (Kaufman *et al.*, 1994).

The resulting robust adaptive control algorithm was applied to control the inverted pendulum.

The linearized equations of motion of this system are

$$\ddot{\theta} = \frac{6}{(4M+m)l}\left[(M+m)g\theta - F\right] \quad (24)$$

$$\ddot{z} = \frac{1}{4M+m}\left[4F - 3mg\theta\right] \quad (25)$$

where M, m, l, θ, z, F, and g are cart mass, pendulum mass, pendulum length, pendulum angle, cart position, control force, and gravity, respectively. The transfer function representation of the system to be used for feedforward compensator

$$G_p(s) = \frac{(4l-600)s^2 - 6g}{(-4Ml - ml)s^4 + 6g(M+m)gs^2} \quad (26)$$

The control objective is to bring the pendulum to a vertical position from a given initial angle while driving the cart to the desired location. In our experimental setup, two states; the cart position and the pendulum angle are available for measurements. Thus, we used the output for the plant consisting of these two states.

As seen from (26), the plant has negative high frequency gain. Therefore, we chose the reference model with negative high frequency gain as follows

$$G_m(s) = \frac{y_m}{u_m} = -16\frac{s+1}{s^2+8s+16} \quad (27)$$

with u_m set to zero.

In order to achieve the control objective in a reasonable amount of time, we set the reference model settling time to about 1.25 sec.

Using the result discussed in (Ozcelik, 1996), a feedforward compensator was formed for the above plant as

$$H(s) = \frac{f_3s^3 + f_2s^2 + f_1s + f_0}{(s+5)^4} \quad (28)$$

After the characteristic polynomial was constructed, an optimization based design procedure was applied using MATLAB's *constr* function and the following feedforward compensator

$$H(s) = \frac{0.00028s^3 + 76.55s^2 + 1496.6s - 1139.6}{(s+5)^4} \quad (29)$$

for the variations in the parameters of the inverted pendulum given in Table 1. For evaluation, the cases from Table 2 were considered. In all cases, the weights were $T_i = [1e3,\ 1e2,\ 1e2,\ 1e2]$ and $T_p = [2.5e4,\ 1.5e4,\ 1.5e4,\ 1.5e2]$. Initial conditions were set to 0.08 rad. and zero for the pendulum angle and the cart position, respectively.

Table 1. Plant parameter values

Parameter	Nominal	Range
$m[kg]$	0.1	0.05 to 1
$l[m]$	0.635	0.5 to 2
$M[kg]$	1	-

Table 2. Cases considered for simulation

Param.	Case1	Case2	Case3	Case4	Case5
m	0.1	0.05	1	0.05	1
l	0.635	0.5	2	2	0.5

Plant and model responses along with pendulum angles and cart positions for the above cases are shown in Figures 2-4. Reasonable model following is observed for each case.

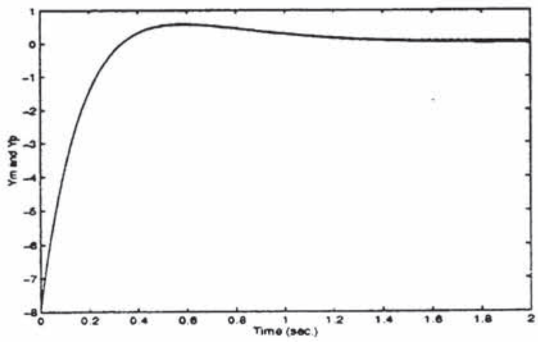

Fig. 2. Plant and model responses for the cases given in Table 2

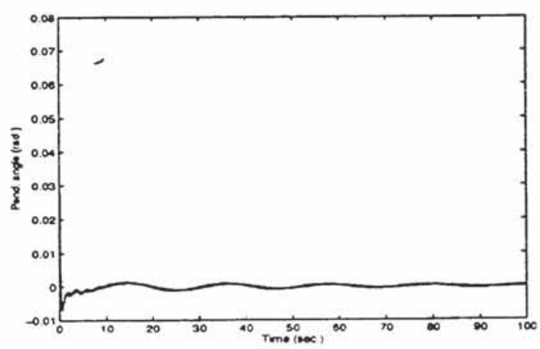

Fig. 3. Pendulum angle for the cases given in Table 2

5. DISCUSSION AND CONCLUSION

An easily implementable design procedure has been presented for determining the DMRAC feedforward compensator needed for satisfying certain positive real conditions. This development enables the augmented plant to satisfy the ASPR condition over a wide range of plant parameter variations. The proposed design procedures were then implemented with the DMRAC algorithm, and applied to the control of an inverted pendulum on a cart. Simulation results demonstrate the viability of the DMRAC algorithm designed using this new method.

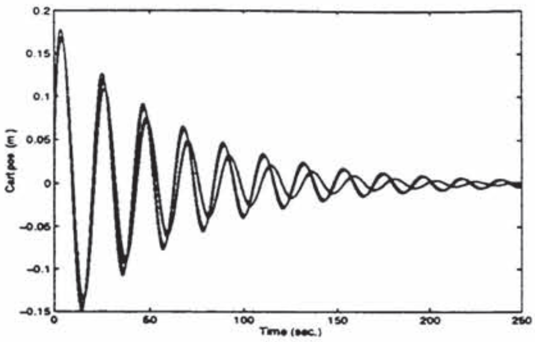

Fig. 4. Cart position for the cases given in Table 2

6. ACKNOWLEDGEMENT

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