HOME-TAKEN EXAM 1A OF NONLINEAR ANALYSIS AND CONTROL IN MES-EMBEDDED CONTROL MODULE_

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By submitting this exam the authors certify that this is their original work, and they have cited all the referenced materials, in the forms of texts, models, and books properly.

Note:
Do NOT Change the Format of This File.
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Contents

DOCUMENT HISTORY

Date	Change Summary	Authors
2024.02.07	Created the document	Amin Mannani
2024.03.25	Added the space for Assignments 5 and 6	Amin Mannani

1 ASSIGNMENT 1

1.1 Problem statement

What to do	You rewite the assignment (or copy its picture) in this section. Furthermore 1. Explain what you learn from the assignment 2. Explain why this is important with reference to its applications.
Syllabus	The study material of this assignment as specified in the lesson.
Engineering indicators	The student shall identify what the problem is and why it is important.
Acceptance criteria	• Assignment is mentioned legibly, and its goal/importance is identified.
Size	Max 1 A4

Assignment 1.

Learning Goals

1. Knowling nonlinear systems specifics.
2. Analyzing the nonlinear systems behavior using simulation

Problem Statement

Construct a Simulink model for the Van Der Pol equation below, as given in Example 1.3 in [1], and analyze it through answering the following questions.

$$m\,\ddot{x}(t) + 2c(x^2(t) - 1)\dot{x}(t) + kx(t) = 0$$

1. Assume $m = 1\ [kg]$, $c = 0.1\ [\frac{Ns}{m^3}]$ and $k = 0.1\ [\frac{N}{m}]$.
   a. Simulate the system behavior starting from $x(0) = 0.1\ [m], \dot{x}(0) = 0\ \left[\frac{m}{s}\right]$.
   b. Show the signals $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ using Data Inspector.
   c. Why is the system behavior is so? Explain it using what you learned about the limit cycle.
2. Repeat the simulation but now starting from $x(0) = 3\ [m], \dot{x}(0) = -1\ \left[\frac{m}{s}\right]$.
   a. Show the signals $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ using Data Inspector.
   b. Compare the results for this case with those of the previous case using the differences and similarities of the two cases.

Note1:

1. Run the simulations long enough to see the behavior of the system at steady state.
2. Make sure the parameters have units in the Simulink model.

What was learnt from this assignment:

- Nonlinear Dynamics: The pendulum is inherently a nonlinear system due to the sine function in its equation of motion. This assignment helps us understand how nonlinearity can affect the behavior of a system.
- Importance of Damping: The presence of damping, characterized by the viscous friction term, has a significant effect on the behavior of the system, leading to the eventual ceasing of motion, as seen in the first plot.
- Stability Analysis: The assignment illustrates how stability can be analyzed by linearizing the system around equilibrium points. This teaches us about different types of equilibrium, such as stable, unstable, and potentially saddle points in more complex systems.

Importance and applications:

- Mechanical Systems: In mechanical systems like clock pendulums, cranes, or robotic arms, understanding the dynamics is crucial for accurate operation and preventing oscillatory instabilities that could lead to inefficiency or mechanical failure.
- Vibration Analysis: Damped oscillatory systems are a model for understanding vibrations in structures and vehicles. Controlling these oscillations is vital for the comfort and safety of buildings and transportation systems.
- Predictive Modeling: Modeling the behavior of physical systems under different conditions helps in predictive maintenance and the design of systems that can withstand various disturbances, which is essential in aerospace, automotive, and civil engineering.

1.2 Solution

What to do	Here you solve the assignment; you can type directly or attach pictures of your handwritten notes. 1. Write the formulas using the correct symbols legibly. 2. Respect the logical/mathematical way of writing and citation. 3. If relevant, refer to the names of the models and functions which you have in the corresponding MATLAB Project. 4. If requested in the assignment, provide the graphs.
Syllabus	The study material of this assignment as specified in the lesson.
Engineering indicators	The student should be able to show both the solution and the process of arrivinging at it.
Acceptance criteria	• Solution approach is clarified and sound. • The formulas, final results and simulation/analysis graphs are compared and justified against each other.
Size	Max 6 A4

The Simulink model for the Van Der Pol equation was made as can be seen below:

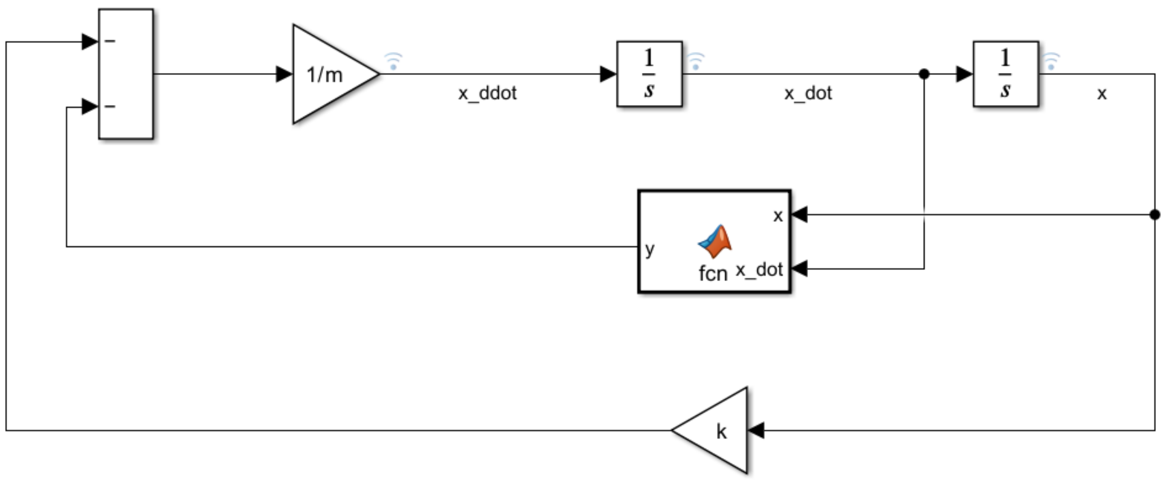


Figure 1 Simulnk model Van Der Pol Oscillator

1)

a) The system behavior was simulated starting from x(0)=0.1 [m], $\dot{x}(0) = 0$ [m/s] for 200 seconds

b) Below the signals from the data inspector can be seen:

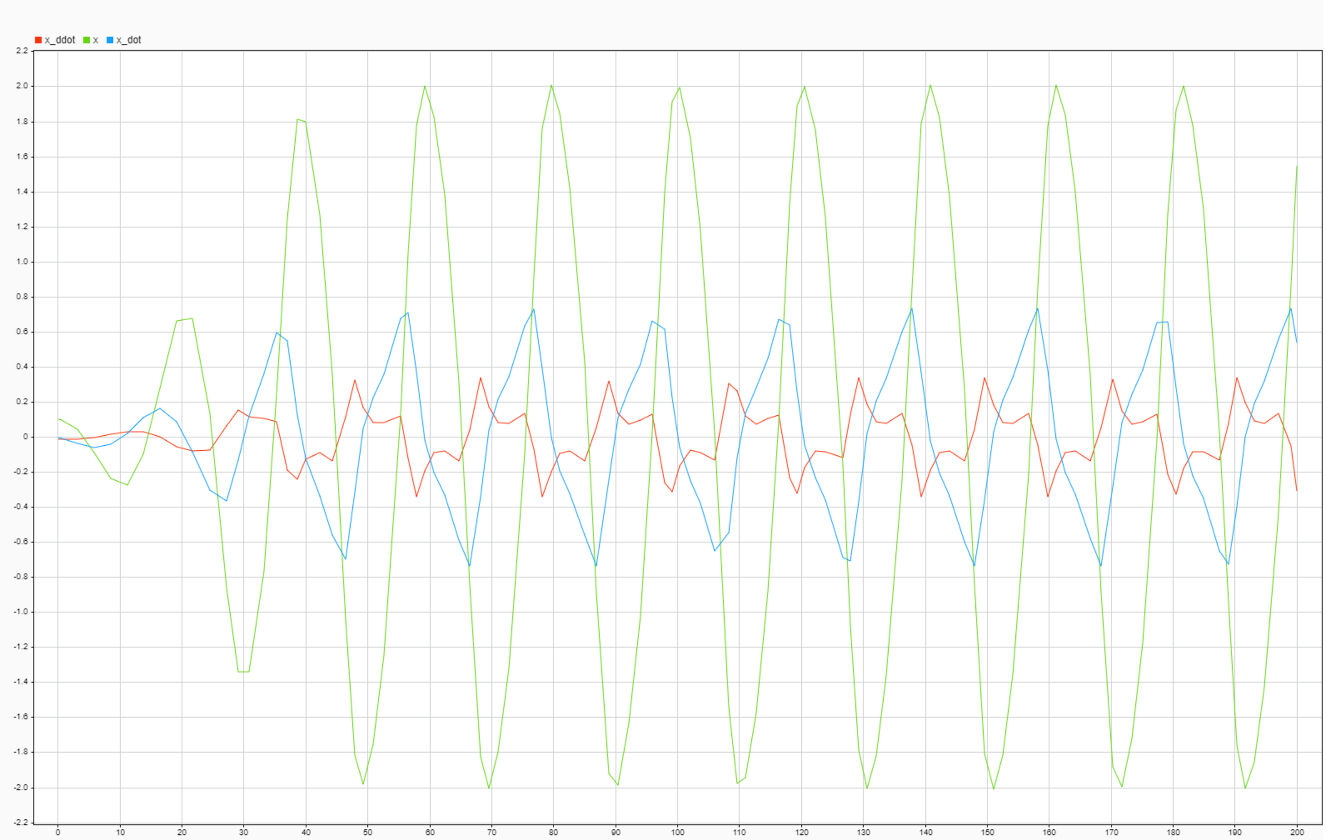


Figure 2 Signals for the first case

c) The plot shows that after some initial transients, the system's behavior settles into a repeating pattern. This is indicates stable oscillatory behaviour, which is a hallmark of the limit cycle in the Van Der Pol system. This behaviour is expected due to the non-linear damping term $2c(x^2(t) - 1)\dot{x}(t)$, which varies with amplitude of x(t). When |x(t)| is small $(x\hat{}2(t) < 1)$, the damping is negative, which means it acts as an energy input and amplifies the oscillations. When |x(t)| is large $(x\hat{}2(t) > 1)$, the damping is positive and suppresses the oscillations. This nonlinearity stabilizes the amplitude of the oscillations, leding to a constant amplitude in the limit cycle.

2) The simulation was repeated but now starting from x(0) = 3 [m], $\dot{x}(0) = -1[m/s]$

- Below the signals from the data inspector can be seen

Figure 3 Signals for the second case

b)

- The limit cycle reached by the system is the same in both cases, which is characteristic of nonlinear systems like the Van Der Pol oscillator where the limit cycle is an attractor.
- The time it takes for the system to settle into the limit cycle differs depending on the initial conditions.
- In the first case the transients die down more quickly as the initial condition is closer to the origin.
- In the second case the system starts further away from the origin, which results in a larger initial transient, and it takes longer time for the transients to die down.
- The amplitude of the limit cycle oscillations seems to be unaffected by the initial conditions, indicating that the system has a unique amplitude that is inherent to the parameters c and k.
- The frequency of the ocillations after transients also seem to be the same, reinforcing that the limit cycle has a specific frequency set by the systems parameters.
- In summary, while the transient behavior varies based on initial conditions, the long-term behavior of the Van der Pol oscillator converges to a stable limit cycle with a set amplitude and frequency determined by the system parameters, not the initial conditions. This reflects the fundamental nature of the Van der Pol equation, which describes a nonlinear system with a stable limit cycle attractor.

2 ASSIGNMENT 2

2.1 Problem statement

What to do	You rewite the assignment (or copy its picture) in this section. Furthermore 3. Explain what you learn from the assignment 4. Explain why this is important with reference to its applications.
Syllabus	The study material of this assignment as specified in the lesson.
Engineering indicators	The student shall identify what the problem is and why it is important.
Acceptance criteria	• Assignment is mentioned legibly, and its goal/importance is identified.
Size	Max 1 A4

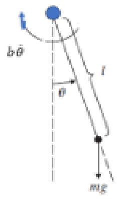
Assignment 2.

Learning Goals

1. Analyzing second-order nonlinear systems using phase plane analysis
2. Implementing nonlinear systems models and analyzing their behavior using simulation

Problem Statement

Consider the equation of a pendulum with a rod of length l $[m]$, a mass m $[kg]$ at the end of the rod, which swings in a plane around its hinge with angle $\theta(t)$ $[rad]$ measured from the vertical position. The rod experiences viscous fiction with coefficient b $[\frac{Ns}{rad}]$ at the hinge.



$$ml^2\frac{d^2\theta(t)}{dt^2} + mgl\sin\theta(t) + b\frac{d\theta(t)}{dt} = 0$$

Let $m = 0.2$ $[kg]$, $b = 0.1$ $[\frac{Ns}{rad}]$, $g = 9.8$ $[\frac{m}{s^2}]$ and $l = 0.7$ $[m]$.

1. Rewrite the model in the state space form by taking $\theta(t)$ and $\dot{\theta}(t)$ as the state variables.
2. Linearize the system around the equilibria $(0,0)$ and $(0,\pi)$.
3. Determine, based on the linearized models, the nature of the two equilibria.
4. Implement the model in Simulink. Using the following steps, show the behavior of the system around the two equilibria.
   a. Pick two initial conditions in the neighborhood of the two equilibria, respectively.
   b. Simulate the system for each of the above initial conditions.
   c. For each simulation, show the signals of the state variables in Data Inspector.
   d. For each simulation, use the Simulink block XY Graph (see Sinks folder in Simulink Library) to generate the phase plot of the system.
   e. Analyze the behavior which you see in the phase plots and Data Inspector for each case, and justify them with respect to your analysis in step 3 above. Make sure that you plot $\dot{\theta}(t)$ versus $\theta(t)$ in the XY Graph.

Note:

1. Run the simulations long enough to see the behavior of the system at steady state.
2. Make sure the parameters have units in the Simulink model.

1) Explain what you learn from the assignment

- Phase Plane Analysis: This technique is pivotal in understanding the qualitative behavior of second-order systems without solving the equations explicitly. It allows us to visualize the trajectory of a system in its state space, providing insights into the system's stability, convergence, and divergence.
- Linearization: It is a method used to approximate the behavior of a nonlinear system near an equilibrium point using a linear model. This simplification is crucial for analyzing system stability and designing controllers using linear control theory.
- Equilibrium Analysis: Determining the nature of the equilibrium points (stable, unstable, saddle points, etc.) helps in predicting the long-term behavior of the system under small perturbations around these points.

2) Why is this assignment importance and what are its applications?

- The pendulum is a fundamental system in physics and engineering, and understanding its behavior under various conditions is critical for the study and design of various mechanical and engineering systems, such as seismic vibration absorbers, clocks, and robotic arms.

2.2 Solution

What to do	Here you solve the assignment; you can type directly or attach pictures of your handwritten notes. 5. Write the formulas using the correct symbols legibly. 6. Respect the logical/mathematical way of writing and citation. 7. If relevant, refer to the names of the models and functions which you have in the corresponding MATLAB Project. 8. If requested in the assignment, provide the graphs.
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Size	Max 6 A4

2.2.1

Rewriting the model in the state space form:

Given the equation of motion of the pendulum:

$$ml^2\ddot{\theta}(t) + b\dot{\theta}(t) + mglsin\theta(t) = 0$$

where,

$\theta(t)$ is the angle of the pendulum,

m = 0.2 kg is the mass,

b = 0.1 Nms/rad is the damping coefficient,

g = 9.8 m/s2 is the acceleration due to gravity,

l = 0.7 m is the length of the pendulum,

Now, rewriting this in the statespace form,

$$x_1 = \theta(t),$$

$$x_2 = \dot{\theta}(t)$$

The state space equations become:

$$\dot{x_1} = \theta(t)$$

$$\ddot{x_2} = -\frac{b}{ml^2}x_2 - \frac{g}{l}sin(x_1)$$

2.2.2

Proceeding to linearize this system around the equilibria $(0,0)$ and $(\pi, 0)$:

Our conditions are:

$$x_1 = 0; x_1 = \pi$$

And

$$x_2 = 0; x_2 = 0$$

Therefore, $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

Now,

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{b}{ml^2} x_2 - \frac{g}{l} sin(x_1) \end{bmatrix}$$

The general Jacobian matrix will be,

$$\frac{Df}{Dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -14cos(x_1) & -1.02 \end{bmatrix}$$

Now, the jacobian matrices evaluated at the equilibrium points are as follows:

At $(0,0)$:

$$\frac{Df}{Dx}(\bar{x}) = \begin{bmatrix} 0 & 1 \\ -14.0 & -1.02 \end{bmatrix}$$

At $(\pi, 0)$:

$$\frac{Df}{Dx}(\bar{x}) = \begin{bmatrix} 0 & 1 \\ 14.0 & -1.02 \end{bmatrix}$$

2.2.3

Then, the eigenvalues for each equilibrium were calculated in matlab:

At $(0,0)$, the eigenvalues are complex with real negative parts: $-0.5100 \pm 3.7067i$. This indicates to a spiral sink or stable focus. This implies that the trajectories near this equilibrium will spiral inwards, indicating a stable but oscillatory behaviour due to damping.

At $(\pi, 0)$, the eigenvalues are real with one positive and one negative: 3.2663 and -4.2863. This indicates a saddle point, which is unstable. Trajectories near this point will diverge away along the direction of the positive eigenvalue and attract along the direction of the negative eigenvalue.

2.2.4

Implementing the model in Simulink:

a) The two equilibria we have are $(0,0)$ & $(\pi, 0)$, therefore the initial conditions picked in the neighbourhood of the two equilibria are,

- For the equilibrium at $\theta = 0$, which is the downward hanging position (downward equilibrium position) of the pendulum, the initial condition chosen was $\theta = 0.1\ rad$ & $\dot{\theta} = 0\ rad/s$, which indicates that the pendulum is slightly displaced from the vertical hanging position but initially stationary.
- For the equilibrium at $\theta = \pi$, which means the pendulum is in the upright position, the initial condtion chosen was $\theta = \pi + 0.1\ rad$ & $\dot{\theta} = 0\ rad/s$, which indicates that the pendulum is slightly displaced from the upright position.

b) The model was implemented in Simulink as you can see below:

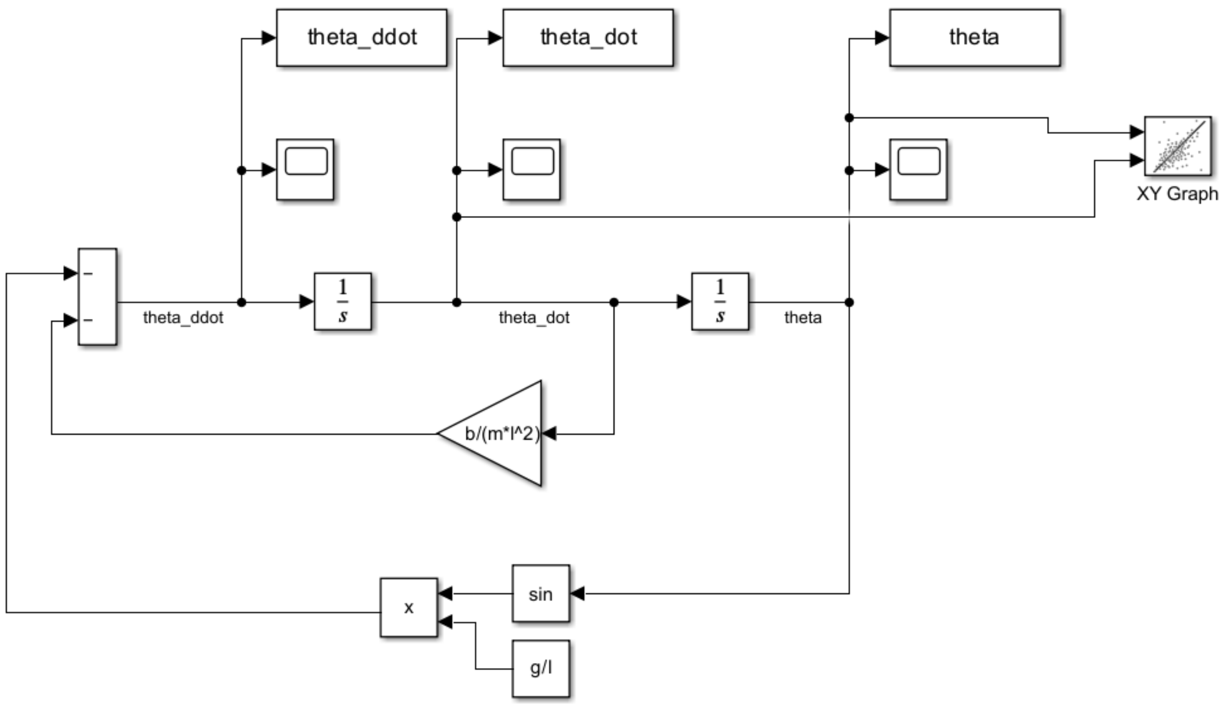

Figure 4: Simulink model for Assignment 2

c)

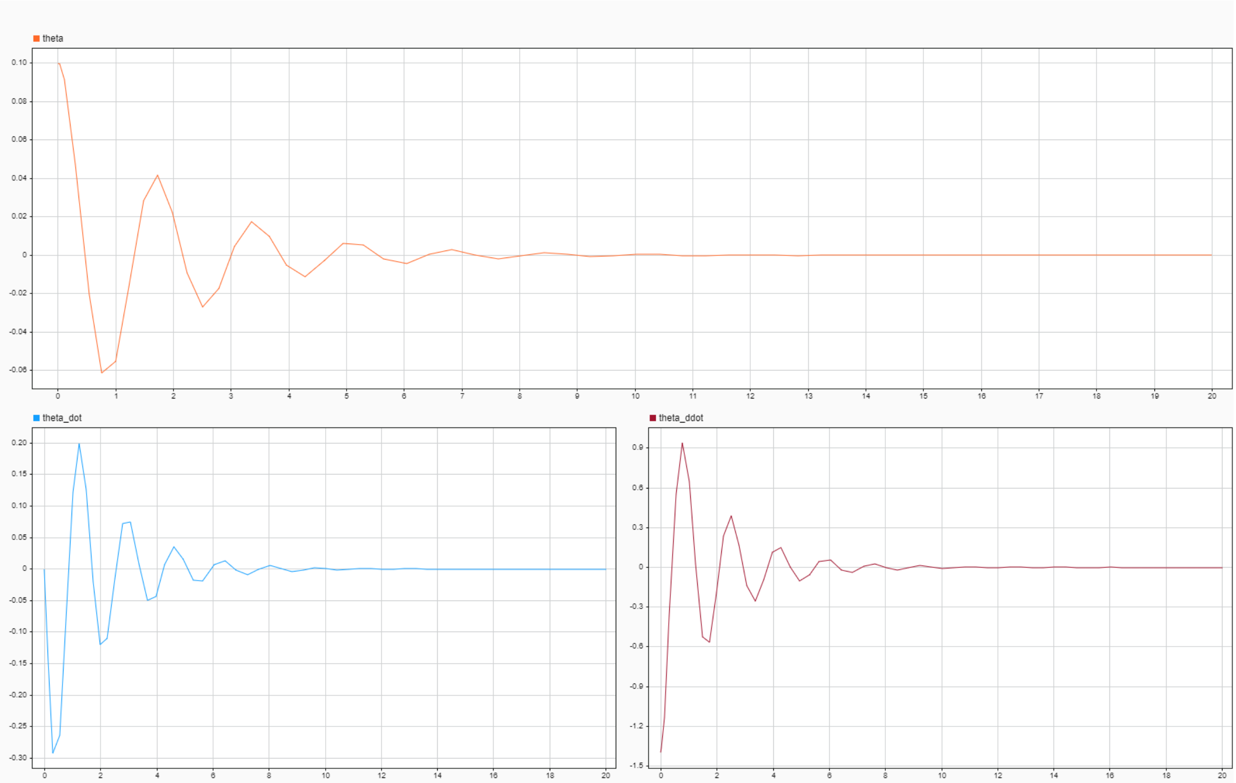

Figure 5: Signals for the first case

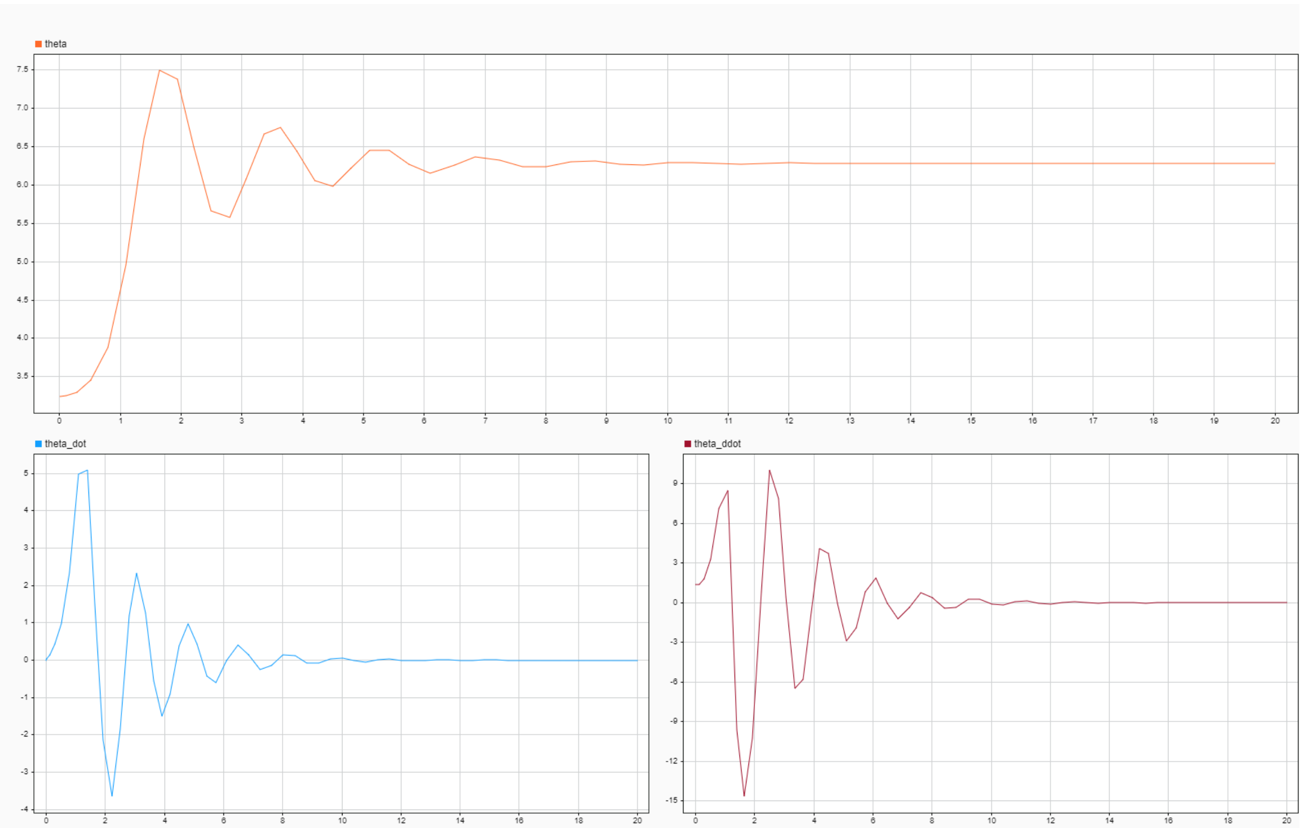

Figure 6: Signals for the second case

d)

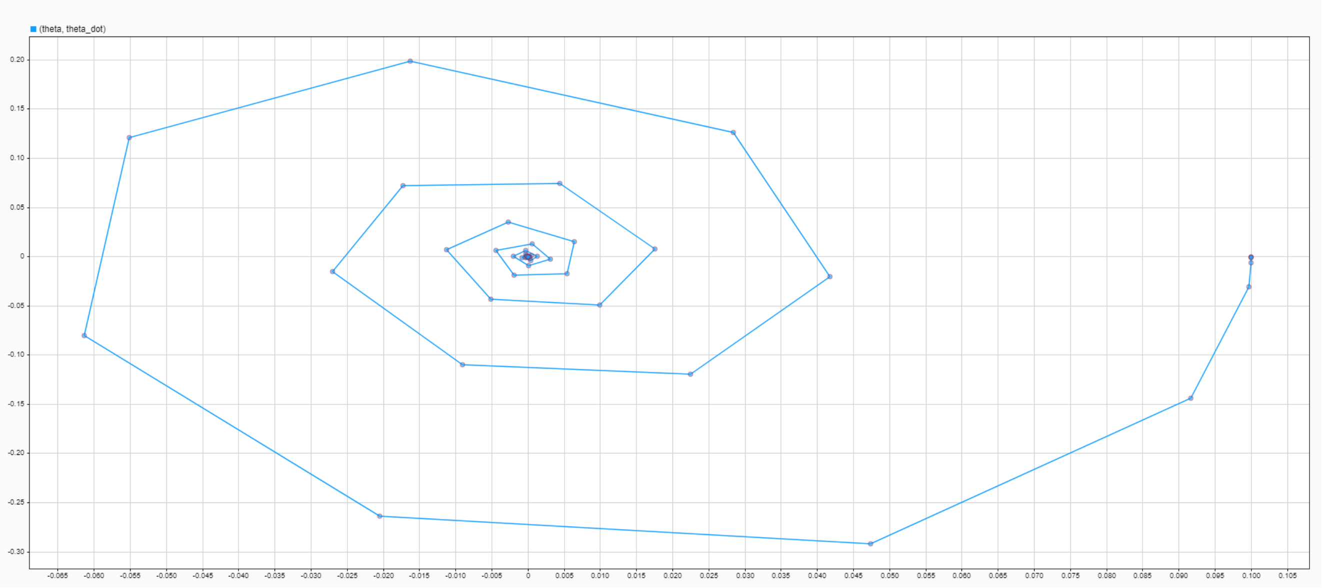

Figure 7: Phase plot for the first case

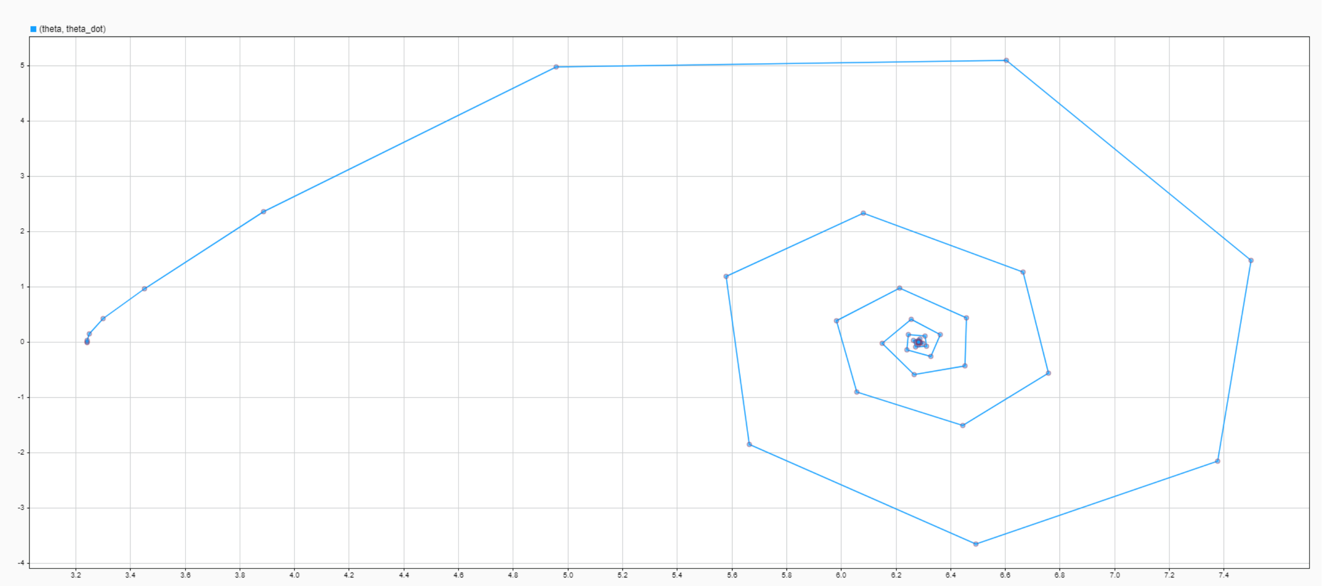

Figure 8: Phase plot for the second case

e) First case:

- The plot indicates that the pendulum starts with a small displacement and oscillates with diminishing amplitude, which suggests that the system is experiencing damping due to the viscous friction at the hinge. The angular velocity and acceleration fluctuate initially but gradually settle towards zero, consistent with the pendulum coming to rest due to the damping force.
- The first phase plot shows the phase trajectory for initial conditions near the stable equilibrium $\left(\theta = 0, \dot{\theta} = 0\right)$. The trajectory spirals into the center demonstrating the damping effect that brings the pendulum to rest at the downward hanging position. A stable equilibrium demonstrates such a behaviour, where small perturbations lead to motions that eventually decay back to equilibrium due to damping.

Second case:

- The angle oscillates without diminishing, which indicates a different behavior that is around the equilibrium point (π,0). The angular velocity spikes significantly before stabilizing, which suggests that the pendulum might have passed through the unstable equilibrium point where the nonlinear effects are more pronounced, and then it settled into a steady oscillatory behavior. A significant initial acceleration can also be observed, which then leads to a series of high-velocity oscillations before the system stabilizes, likely due to the non-linear dynamics of the pendulum and the effects of gravity and friction.
- The second phase plot shows the phase trajectory for initial conditions near the unstable equilibrium $\left(\theta = \pi + 1, \dot{\theta} = 0\right)$. Unlike the first case, the trajectory does not spiral into the center. Instead it shows more complex behaviour due to the pendulums motion slightly away from the upright position, indicating the instability at this equilibrium. Small perturbations lead to motion that diverge from this equilibrium point, as expected for an unstable equilibrium.

3 ASSIGNMENT 3

3.1 Problem statement

What to do	You rewite the assignment (or copy its picture) in this section. Furthermore 5. Explain what you learn from the assignment 6. Explain why this is important with reference to its applications.
Syllabus	The study material of this assignment as specified in the lesson.
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Acceptance criteria	• Assignment is mentioned legibly, and its goal/importance is identified.
Size	Max 1 A4

Assignment 3.

Learning Goals

1. Analyzing the stability of an equilibrium using the Lyapunov's Indirect method.
2. Verifying the asymptotic stability of an equilibrium using the Lyapunov's direct method and invariance theorems.
3. Comparing global and local stability cases.

Problem Statement

This assignment is based on Example 3.14 in the textbook [1].
Consider a nonlinear mass-spring-damper system with the following model

$$\ddot{x} + \dot{x}^3 + x - \sin^2(x) = 0$$

Where x is the position of the mass $m = 1\ [kg]$, the damper has the nonlinear model $\dot{x}^3$, and the spring satisfies the nonlinear equation $x - \sin^2(x)$.

1. Show that $(x, \dot{x}) = (0,0)$ is an equilibrium of the system.
2. Apply the Lyapunov's Linearization (or Indirect) method and state what you can conclude there.
3. Use the Lyapunov function $V(x) = \frac{1}{2}\dot{x}^2 + \int_0^x [y - \sin^2(y)]dy$ and local invariance theorem to show that $(0,0)$ is a locally asymptotically stable equilibrium.
   Hint: To prove the sign condition for the spring model, you may use the graph of the function $x - \sin(x)$ in MATLAB or an analytical argument.
4. Is $(0,0)$ globally asymptotically stable? Why?
5. Simulate the system for 100 seconds the two initial conditions $(x, \dot{x}) = (2, -0.2)$ and $(x, \dot{x}) = (0.2, -0.2)$ and show the graphs of $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ for both cases in Data Inspector. Plot the $x(t)$ graphs of both cases on the same diagram to simplify the comparison. Compare the two cases with each other, and comment on the speed of the convergence using the property of the damping $x - \sin(x)$.

What was learnt form this assignment.

- Equilibrium Analysis: The importance of identifying and characterizing equilibrium points in a system to understand its behavior at rest and the conditions under which it remains at rest.
- Lyapunov's Methods: The application of Lyapunov's indirect and direct methods provides a powerful framework for assessing stability. The indirect method involves linearizing the system around an equilibrium point and analyzing the eigenvalues, while the direct method involves constructing a Lyapunov function and examining its derivatives along the system's trajectories.
- Nonlinear Dynamics: Nonlinear terms, such as those in the damping and spring force of the mass-spring-damper system, introduce complex behavior that differs from linear systems. These nonlinearities can cause phenomena such as bifurcations, where small changes in system parameters can lead to drastic changes in behavior. Local vs. Global Stability: The distinction between local and global stability is crucial, especially for nonlinear systems where stability properties can drastically differ with changes in initial conditions or system parameters.

Importance of the assignment with applications:

Understanding the behavior of pendulums and similar nonlinear systems has significant practical applications:

- Seismic Vibration Absorbers: Nonlinearities are often introduced into vibration absorbers to improve their effectiveness across a wide range of frequencies. Understanding how nonlinear damping and restoring forces affect stability and performance is critical in designing absorbers that protect structures during earthquakes.
- Robotic Arms and Control Systems: The principles governing the motion of pendulums apply to robotic arms, which must move precisely and return to stable positions after completing tasks. Control strategies often rely on stability analyses to ensure the robots perform as expected in the presence of nonlinear dynamics.

3.2 Solution

What to do	Here you solve the assignment; you can type directly or attach pictures of your handwritten notes. 9. Write the formulas using the correct symbols legibly. 10. Respect the logical/mathematical way of writing and citation. 11. If relevant, refer to the names of the models and functions which you have in the corresponding MATLAB Project. 12. If requested in the assignment, provide the graphs.
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Size	Max 6 A4

1) To determine if $(x, \dot{x}) = (0,0)$ is an equilibrium point, we substitute $x = 0\ and\ \dot{x} = 0$ into the equation:

$$\ddot{x} + \dot{x}^3 + x - sin^2(x) = 0$$

Substituting $x = 0\ and\ \dot{x} = 0$,

$$\ddot{x} + 0^3 + 0 - sin^2(0) = \ddot{x}\ +\ 0 + 0 - 0 = 0$$

Thus, $(x, \dot{x}) = (0,0)$ satisfies the condition $\ddot{x} = 0$ when both the x and $\dot{x}$ are zero, confirming that it is indeed an equilibrium point.

2) To apply Lyapunov's linearization method, we need to linearize the system around the equilibrium point $(x, \dot{x}) = (0,0)$.

The systems equation is: $\ddot{x} = -\dot{x}^3 - x + sin^2(x)$

We need the Jacobian matrix of the system evaluated at $(x, \dot{x}) = (0,0)$

The Jacobian matrix J is computed from: $f(x, \dot{x}) = \dot{x}^3 + x - sin^2(x)$

The partial derivatives are:

$$\frac{\partial f}{\partial x} = 1 - 2sin(x)cos(x)$$

$$\frac{\partial f}{\partial \dot{x}} = 3\dot{x}^2$$

Evaluating these at $x = 0\ and\ \dot{x} = 0$, we get:

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 1,$$

$$\left.\frac{\partial f}{\partial \dot{x}}\right|_{(0,0)} = 0$$

Therefore, the linearized system of matrix around the equilibrium point is:

$$A\ =\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues of A can be obtained by solving the characteristic equation: $\lambda^2 + 1 = 0$, which gives $\lambda = \ \pm i$.

Sincethe eigenvalues are purely imaginary, the equilibrium at (0,0) is a center, indicating that the system is marginally stable in the linearized analysis.

3) For the given system we consider a Lyapunov function given in the problem statement:

$$V(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x \big(y - sin^2(y)\big)dy$$

Calculating the integral part

$$\int \big(y - sin^2(y)\big)dy = \frac{y^2}{2} - \int sin^2(y)dy$$

$$\int sin^2(y)dy = \frac{y}{2} - \frac{sin(2y)}{4}$$

Thus, the integral component becomes:

$$\int_0^x \big(y - sin^2(y)\big)dy = \frac{x^2}{2} - \left(\frac{x}{2} - \frac{sin(2x)}{4}\right) = \frac{x^2}{2} - \frac{x}{2} + \frac{sin(2x)}{4}$$

Hence, the Lyapunov function is:

$$V(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + \left(\frac{x^2}{2} - \frac{x}{2} + \frac{sin(2x)}{4}\right)$$

We know thayt for local asymptotic stability the Lyapunov function needs to be positive definite and its derivative along the trajectories of the system negative definite for local asymptotic stability:

$V(x,\dot{x})$ is clearly positive definite for all x, $\dot{x}$ near the origin, given the quadratic terms in x and $\dot{x}$.

The derivative of V with respect to time along the trajectories of the system is:

$$\dot{V}(x,\dot{x}) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial \ddot{x}}\ddot{x}$$

With $\ddot{x} = -\dot{x}^3 - x + sin^2(x)$ from the systems equation, and partial derivatives:

$$\frac{\partial V}{\partial x} = x - \frac{1}{2} + \frac{cos(2x)}{2},$$

$$\frac{\partial V}{\partial \dot{x}} = \dot{x},$$

$$\dot{V}(x,\dot{x}) = \left(x - \frac{1}{2} + \frac{cos(2x)}{2}\right)\dot{x} + \dot{x}\left(-\dot{x}^3 - x + sin^2(x)\right)$$

Simplifying,

$$\dot{V}(x,\dot{x}) = -\frac{1}{2}\dot{x} + \frac{cos(2x)}{2}\dot{x} + sin^2(x)\dot{x}$$

For small x and $\dot{x}$ near the equilibrium, the dominant term is $-\frac{1}{2}\dot{x}$, suggesting $\dot{V}$ could be negative or atleast non-positive near (0,0), validating local asymptotic stability.

cos(2x) can range from -1 to 1, and thus the term $\frac{cos(2x)}{2}\dot{x}$ can either reinforce or oppose the dominant negative term $-\frac{1}{2}\dot{x}$. The last term, $sin^2(x)\dot{x}$, is always non-negative and is zero when x is any muiltiple of pi.

4) From the linearization analysis, we know the system exhibits a center characteristic, indicating marginal stability in certain directions from the equilibrium. This typicslly negates global stability unless further restrictive conditions or global Lyapunov functions can be established.

5) Simulating the system in Simulink:
The following model was made in Simulink:

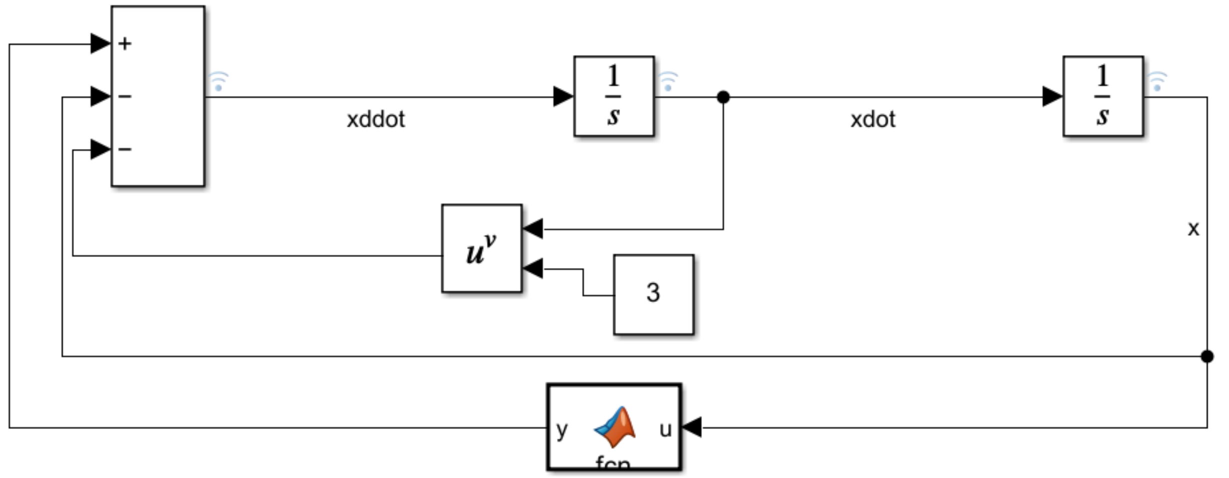

Figure 9 Simulink model Assignment 3

Below you can see the plots of both the cases:

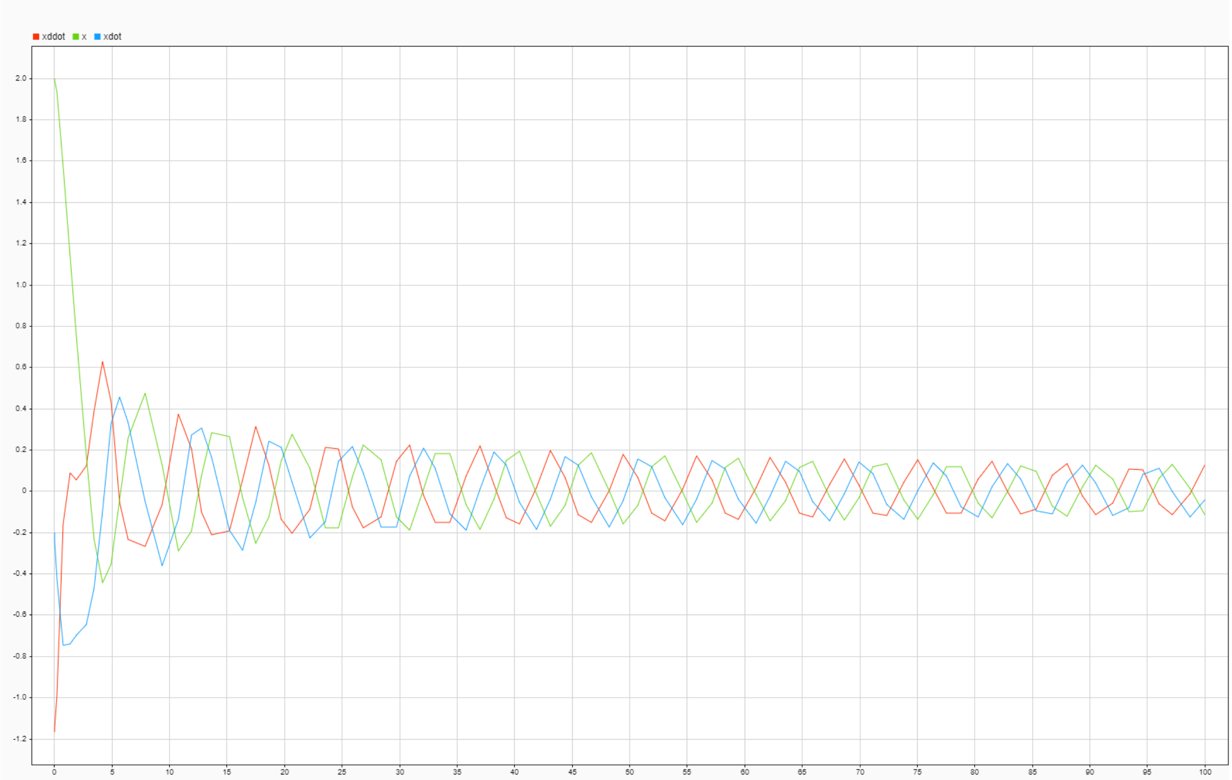

Figure 10 Assignment 3 Plot for Case 1

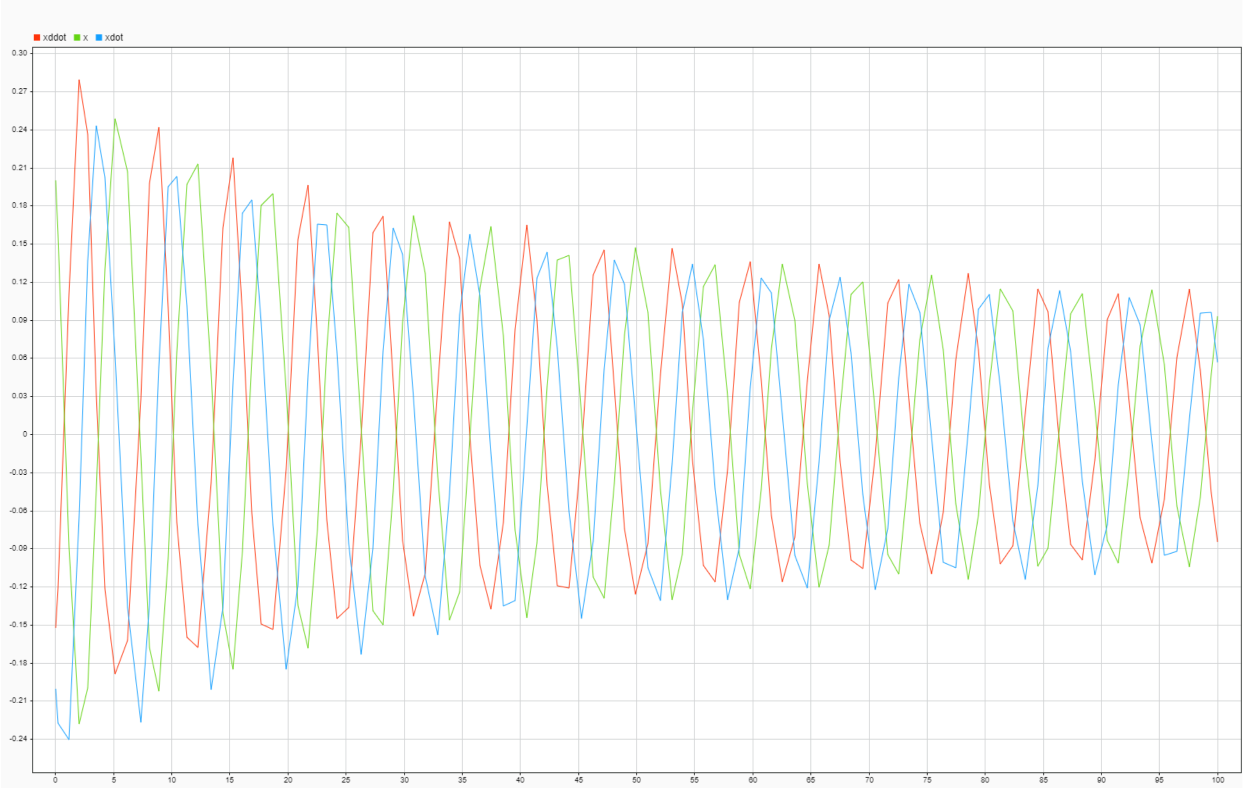

Figure 11 Assignment 3 Plot for Case 2

- It can be observed that the larger initial displacement in the first case results in a greater initial amplitude of oscillation.
- The nonlinearity in the damping term $\dot{x}^3$ plays a significant role. In the first case with a larger initial condition, we can see a rapid decrease in the amplitude of $\dot{x}$, indiciating strong damping effects due to the cubic relationship. As the velocity reduces, the rate of amplitude decrease lessens because the cubic term becomes smaller.
- The frequency of oscillation seems to be slightly higher in the second plot compared to the first. This might be attributed to the nonlinear equation $x - sin^2(x)$ which can alter the system's natural frequency depending on the amplitude of x.
- The convergence in the first case (larger initial condition) appears faster initially due to the stronger damping effect but then proceeds to oscillate with a somewhat consistent amplitude for a while before settling down. In contrast, the second case with a smaller initial condition exhibits a more gradual approach to equilibrium with consistent decay in amplitude. This suggests that the speed of convergence is initially high for larger displacements due to stronger damping but slows down as the system approaches the equilibrium.
- The system's energy dissipates over time due to the nonlinear damping term. Initially, when the speed is higher, energy dissipation is more rapid, as seen by the steep drop in the velocity and position amplitude.

4 ASSIGNMENT 4

4.1 Problem statement

What to do	You rewite the assignment (or copy its picture) in this section. Furthermore 7. Explain what you learn from the assignment 8. Explain why this is important with reference to its applications.
Syllabus	The study material of this assignment as specified in the lesson.
Engineering indicators	The student shall identify what the problem is and why it is important.
Acceptance criteria	• Assignment is mentioned legibly, and its goal/importance is identified.
Size	Max 1 A4

Assignment 4

Learning Goals

1. Analyzing the stability criteria for time-varying nonlinear systems
2. Deriving the conditions on the stability of linear time-varying systems

Problem Statement

1. Assume that $h(t)$ is a continuous and bounded function, and that $h(t) \geq 0$ for all $t \geq 0$. Use Theorem 4.1 in [1] and the Lyapunov function $V(\boldsymbol{x}, t) = [2 + h(t)]x_1^2 + 2x_1x_2 + x_2^2$ to show that if $\left[\dot{h}(t) - 2h(t)\right] \leq c < 0$ for some c, then the origin is a uniformly asymptotically stable equilibrium of the following nonlinear system.

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -h(t)x_1 - 2x_2 \end{cases}$$

2. Consider the model of a time-varying series RLC circuit with the resistance $R(t)$, inductance $L(t)$ and capacitance $C(t)$. Let $x_1(t)$ and $x_2(t)$ denote the capacitor voltage and inductor current, respectively.
   Use the result in equation (4.19) in [1] to find a sufficient condition on $R(t), L(t)$ and $C(t)$ to guarantee the exponential stability of the system.

$$\begin{cases} \dot{x}_1 = \dfrac{1}{C(t)}x_2 \\ \dot{x}_2 = -\dfrac{1}{L(t)}x_1 - \dfrac{R(t)}{L(t)}x_2 \end{cases}$$

What was learnt for this assignment:

- Understanding of Lyapunov Functions: The use of Lyapunov functions is a fundamental technique in the analysis of the stability of both linear and nonlinear systems. The assignment illustrates how to construct a Lyapunov function and calculate its derivative along the system trajectories to determine stability conditions.
- By examining the specific nonlinear system given, we gain insight into how non-constant parameters influence the system's stability.

Importance and applications:

- Control Systems: Stability analysis is paramount in designing controllers for systems like aircraft, vehicles, and even climate control systems in buildings. Ensuring stability means that the system will behave predictably and safely under various conditions.
- Electrical Circuits: In the design of electronic components such as amplifiers and oscillators, stability analysis prevents unwanted behaviors like oscillations and ensures that the circuits perform their intended functions reliably.

4.2 Solution

What to do	Here you solve the assignment; you can type directly or attach pictures of your handwritten notes. 13. Write the formulas using the correct symbols legibly. 14. Respect the logical/mathematical way of writing and citation. 15. If relevant, refer to the names of the models and functions which you have in the corresponding MATLAB Project. 16. If requested in the assignment, provide the graphs.
Syllabus	The study material of this assignment as specified in the lesson.
Engineering indicators	The student should be able to show both the solution and the process of arrivinging at it.
Acceptance criteria	• Solution approach is clarified and sound. • The formulas, final results and simulation/analysis graphs are compared and justified against each other.
Size	Max 6 A4

1) To show stability we compute the time derivative of the Lyapunov function:

Finding the partial derivatives of V with respect to x1 and x2, and the total derivative with respect to t:

$$\frac{\partial V}{\partial x_1} = 2[2 + h(t)]x_1 + 2x_2,$$

$$\frac{\partial V}{\partial x_2} = 2x_1 + 2x_2,$$

$$\frac{\partial V}{\partial t} = \dot{h}(t)x_1^2$$

Using the dynamics $\dot{x_1} = x_2 and \dot{x_2} = -h(t)x_1 - 2x_2$:

$$\dot{V}(x,t) = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + \frac{\partial V}{\partial t}$$

$$\dot{V}(x,t) = (4x_1 + 2h(t)x_1 + 2x_2)x_2 + (2x_1 + 2x_2)(-h(t)x_1 - 2x_2) + \dot{h}(t)x_1^2$$

$$\dot{V}(x,t) = (4x_1x_2 + 2h(t)x_1x_2 + 2x_2^2) - (2h(t)x_1x_2 + 2x_1x_2 + 4x_2^2) + \dot{h}(t)x_1^2$$

$$\dot{V}(x,t) = \dot{h}(t)x_1^2 - 2x_1x_2 - 2x_2^2$$

- From $[\dot{h}(t) - 2h(t)] \leq c < 0$, it can be observed that the contribution from $\dot{h}(t)x_1^2 to \dot{V}(x,t)$ needs to be negative or sufficiently small to guarantee $\dot{V}(x,t) \leq 0$ for al x1 and x2.
- For the term $\dot{h}(t)x_1^2$ to ensure $\dot{V}(x,t)$ is negative definite, $\dot{h}(t)$ must be dominated by -2h(t) such that the overall dynamics contribute negatively, thereby indicating decay of V(x,t) and ensuring stability of equilibrium at the origin.

2) Since the systems coefficients are time-varying, ensuring R(t), L(t) and C(t) are positive and R(t) is sufficiently large compared to the rates of change of L(t) and C(t).

$$R(t) > 2\sqrt{\frac{L(t)}{C(t)}}$$

The above condition ensures the damping in the the RLC circuit is dominant, leading to exponential stability.

5 ASSIGNMENT 5

5.1 Problem statement

What to do	You rewite the assignment (or copy its picture) in this section. Furthermore 9. Explain what you learn from the assignment 10. Explain why this is important with reference to its applications.
Syllabus	The study material of this assignment as specified in the lesson.
Engineering indicators	The student shall identify what the problem is and why it is important.
Acceptance criteria	• Assignment is mentioned legibly, and its goal/importance is identified.
Size	Max 1 A4

Assignment 5

Learning Goals

1. Knowing the concepts, limitations, challenges and advantages of nonlinear control design

Problem Statement

Answer the following questions using the information in Part II in [1]. Be concise.

1. What are the two purposes of using a reference model in tracking problem?
2. Using an example, show why perfect tracking is impossible for a non-minimum-phase system.
3. What is the difference between a stable linear and stable nonlinear system in reacting to persistent disturbances?
4. What are the roles of feedback and feedforward in a (nonlinear) controller?
5. What is the idea behind Gain Scheduling technique?

5.2 Solution

What to do	Here you solve the assignment; you can type directly or attach pictures of your handwritten notes. 17. Write the formulas using the correct symbols legibly. 18. Respect the logical/mathematical way of writing and citation. 19. If relevant, refer to the names of the models and functions which you have in the corresponding MATLAB Project. 20. If requested in the assignment, provide the graphs.
Syllabus	The study material of this assignment as specified in the lesson.
Engineering indicators	The student should be able to show both the solution and the process of arrivinging at it.
Acceptance criteria	• Solution approach is clarified and sound. • The formulas, final results and simulation/analysis graphs are compared and justified against each other.
Size	Max 6 A4

1) Purposes of Using a Reference Model in Tracking Problems

Error Evaluation: A reference model specifies the desired trajectory or output for the system, which allows for precise measurement and evaluation of the tracking error i.e.the difference between the actual system output and the desired output.

Stability and Performance Assurance: The reference model serves as a template forcontroller design. By ensuring that the systems response aligns with the reference model, the controller can be designed to achieve both stability and optimal performance. Ths alignment helps in adjusting the control inputs to minimize the tracking error while ensuring that the system remains stable under different operating conditions.

2) Perfect Tracking in Non-minimum-phase Systems

Example: Consider a non-minimum-phase system like an inverted pendulum on a cart where the goal is to control the position of the cart such that the pendulum remains upright. In this case, moving the cart too quickly to a desired position could cause the pendulum to fall over, demonstrating an intrinsic limitation: the system's initial response to control inputs moves in the opposite direction of what is eventually needed (negative initial response).

Explanation: Non-minimum-phase systems have right half plane (RHP) zeroes. These RHP zeroes cause an initial response opposite to the desired direction when a control input is applied, making perfect tracking difficult. This phenomenon occurs because the system needs to counteract the initial adverse response before achieving the desired output, leading to a compromise in transient performance.

3) Difference Between Stable Linear and Nonlinear Systems in Reacting to Persistent Disturbances

Linear Systems:For stable linear systems, the response to persistent disturbances is predictable and can be analyzed using superposition. The system will return to its equilibrium state after the disturbance is removed, provided it remains within the systems stability margins.

Nonlinear Systems: Stable nonlinear systems can exhibit more complex behaviours in response to persistent disturbances. These systems may experience bifurcations, chaos, or saturation, depending on the magnitude and state of the disturbance. Unlike linear systems, the response is not simply additive and small disturbances can sometimes lead to significant changes in system behaviour.

4) Roles of Feedback and Feedforward in Nonlinear Controllers

Feedback: The primary role of feedback in a nonlinear controller is to correct the system's behavior by minimizing the error between the output and the reference signal. It adjusts the control action based on the observed state of the system, helping to stabilize and optimize performance under varying conditions and disturbances.

Feedforward: Feedforward control anticipates the needed control actions based on the desired trajectory and system dynamics, independent of the current system state. It helps improve the responsiveness and performance of the system by addressing predictable changes and disturbances before they occur.

5) Idea Behind Gain Scheduling Technique

Concept: Gain scheduling involves varying the controller parameters (gains) based on a predefined schedule that corresponds to different operating conditions or system states. This technique allows the controller to adjust its behaviour to accommodate changes in the systems dynamics.

Purpose: The main purpose of gain scheduling is to handle nonliniearities in the system by approximating it with a series of linear time invariant (LTI) models at different operating points. By adjusting the controller gains according to the current operating point, the system can maintain optimal performance across a wide range of conditions, ensuring stability and improving control accuracy.

6 ASSIGNMENT 6

6.1 Problem statement

What to do	You rewite the assignment (or copy its picture) in this section. Furthermore 11. Explain what you learn from the assignment 12. Explain why this is important with reference to its applications.
Syllabus	The study material of this assignment as specified in the lesson.
Engineering indicators	The student shall identify what the problem is and why it is important.
Acceptance criteria	• Assignment is mentioned legibly, and its goal/importance is identified.
Size	Max 1 A4

Assignment 6

Learning Goals

1. Design of a nonlinear controller using Feedback Linearization technique
2. Verification of the controlled system using simulation.

Problem Statement

Consider the system in Example 6.3 in [1].

$$\begin{cases} \dot{x_1} = x_2^3 + u \\ \dot{x_2} = u \end{cases}, \qquad y = x_1$$

The control objective is to track the reference $y_d(t) = sin\, t + \cos t$.

Design a tracking controller using feedback linearization technique.

1. Show that the tracking error dynamics is as in equation (6.29). Show that it is stable.
2. Show that the internal dynamics is as in equation (6.30).
3. Show that de zero dynamics is as in equation (6.45). Prove, using the Lyapunov function suggested there, that the $x_2 = 0$ is an asymptotically stable equilibrium of the zero dynamics.
4. Simulate the controlled system from a nonzero initial point and show the graphs of the system output tracking the reference, and the graph of the internal dynamics being asymptotically stable.

6.2 Solution

What to do	Here you solve the assignment; you can type directly or attach pictures of your handwritten notes. 21. Write the formulas using the correct symbols legibly. 22. Respect the logical/mathematical way of writing and citation. 23. If relevant, refer to the names of the models and functions which you have in the corresponding MATLAB Project. 24. If requested in the assignment, provide the graphs.
Syllabus	The study material of this assignment as specified in the lesson.
Engineering indicators	The student should be able to show both the solution and the process of arrivinging at it.
Acceptance criteria	• Solution approach is clarified and sound. • The formulas, final results and simulation/analysis graphs are compared and justified against each other.
Size	Max 6 A4

1)

Given the system dynamics:

$$\begin{matrix} \dot{x_1} = x_2^3 + u \\ \dot{x_2} = u \end{matrix}, y = x_1$$

And the control objective is to track the reference trajectory $y_d(t) = sin(t) + cos(t)$

with the derivative: $\dot{y_d} = -cos(t) + sin(t)$

$$\dot{y} = \dot{x_1} = x_2^3 + u$$

$$u = -x_2^3 - e + \dot{y_d} \text{ with } e = y - y_d$$

$$\dot{y} = x_2^3 - x_2^3 - e + \dot{y_d} \rightarrow \dot{y} - \dot{y_d} = -e \rightarrow \dot{e} + e = 0$$

To show the stability of the error dynamics, we solve the differential equation for e(t):

$$\dot{e} + e = 0$$

This is a first-order linear homogenous differential equation. The solution is given by :

$$e(t) = e(0)e^{-t}$$

Since e(t) decays exponentially to zero as $t \rightarrow \infty$, the error dynamics are asymptotically stable. Thus, y will track y_d asymptotically.

2)

The internal dynamics of the system are:

$$\dot{x_2} = -x_2^3 - e + \dot{y_d} \rightarrow \dot{x_2} + x_2^3 = \dot{y_d} - e$$

This is a non-autonomous nonlinear system. Since e and $\dot{y_d}$ are bounded: $|e - \dot{y_d}| \leq D, D > 0$

Observe:

$$\begin{cases} x_2 > D^{1/3} \rightarrow \dot{x_2} = -x_2^3 + \dot{y_d} - e < 0 \\ x_2 < -D^{1/3} \rightarrow \dot{x_2} = -x_2^3 + \dot{y_d} - e > 0 \end{cases} \rightarrow |x_2| < d^{1/3} \rightarrow bounded$$

Input output linearization can be used by setting $y_d = 0$, so $\dot{y_d} = 0$:

$$\begin{cases} x_2 > D^{1/3} \rightarrow \dot{x_2} = -x_2^3 + 0 - x_1 < 0 \\ x_2 < -D^{1/3} \rightarrow \dot{x_2} = -x_2^3 + 0 - x_1 > 0 \end{cases} \rightarrow |x_2| < d^{1/3} \rightarrow bounded$$

3)

The zero dynamics are defined when the output $y = x_1 = 0$. Thus:

$$\begin{cases} \dot{x_1} = x_2^3 + u \\ \dot{x_2} = u \\ 0 = x_1 \end{cases}$$

When $y_d = y = e = 0$, the control input u becomes:

$$u = -x_2^3$$

Thus, the zero dynamics are:

$$\dot{x_2} + x_2^3 = 0$$

To analyze the stability of this differential equation, we use a Lyapunov function:

$$V(x_2) = \frac{1}{2}x_2^2$$

Lyapunov function derivative:

$$\dot{V}(x_2) = \frac{d}{dt}\left(\frac{1}{2}x_2^2\right) = x_2\dot{x_2} = x_2(-x_2^3) = -x_2^4$$

Since $\dot{V}(x_2) = -x_2^4$ is negative definite for all $x_2 \neq 0$ and $\dot{V}(x_2) = 0$ only when $x_2 = 0$, the equilibrium point $x_2 = 0$ is locally asymptotically stable.

4)

The controlled system was simulated form a non-zero initial point.

Below you can see the simulation model:

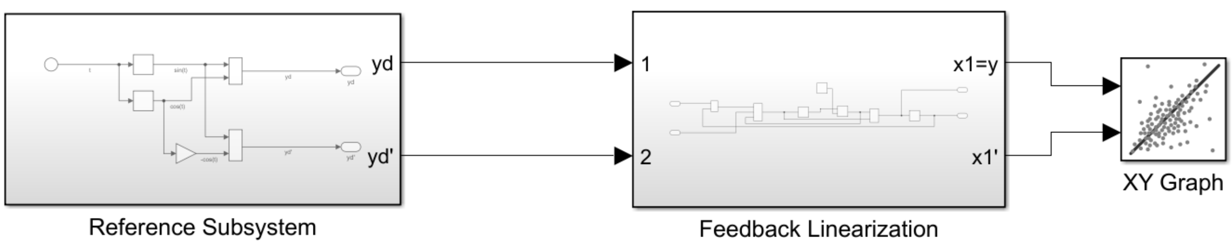

Figure 12 Feedback linearization top layer

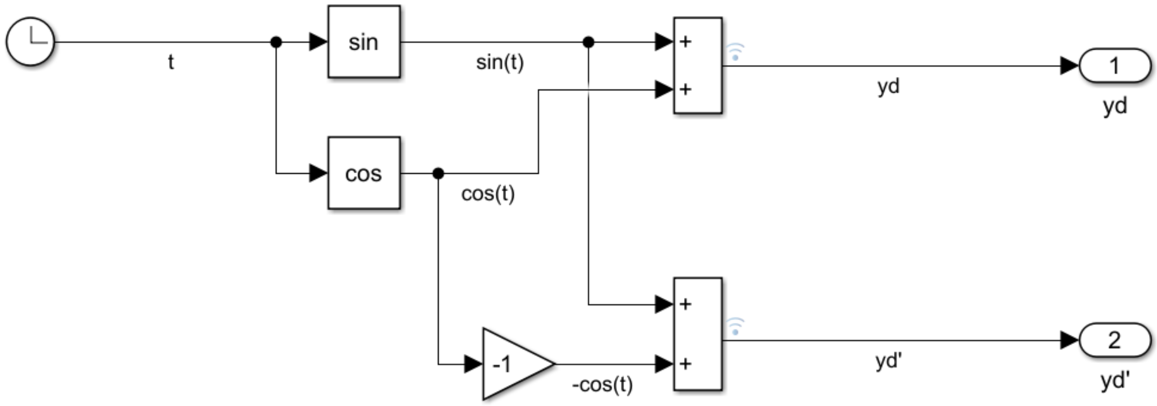

Figure 13 Reference subsystem

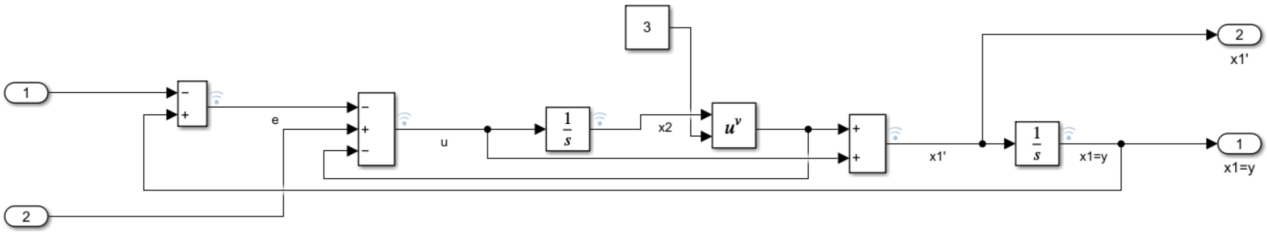

Figure 14 Feedback linearization subsystem

Below you can see the plots of the system output tracking the reference. The simulation time was set to 20sec. At the beginning of the simulation, the initial conditions ensure that the system output does not exactly match the reference signals. After approximately one period of the sine wave, the system output closely follows the reference signal, demonstrating the effectiveness of the feedback linearization control in achieving the desired tracking performance.

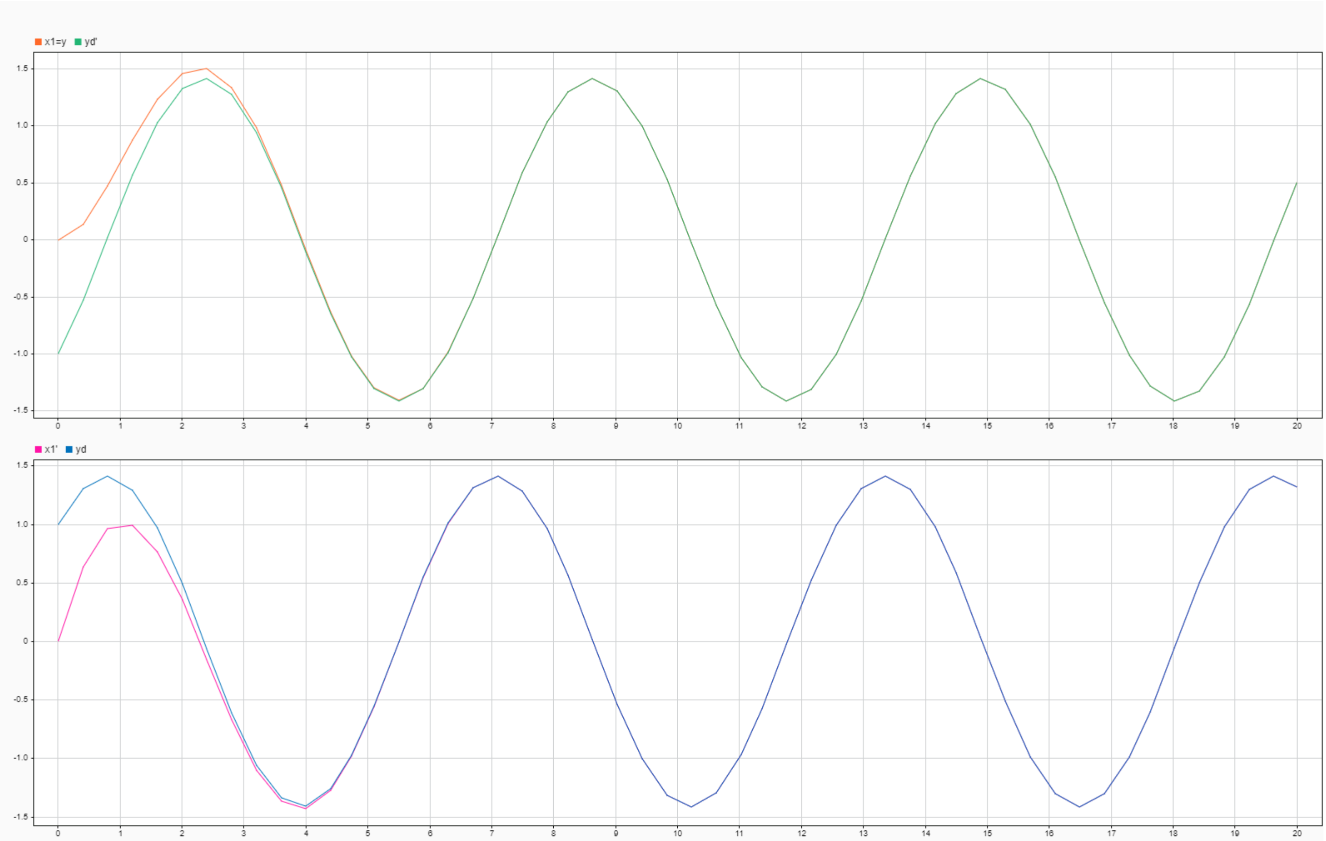

Figure 15 Reference tracking

The figure below shows that the internal dynamics of the system is asymptotically stable.

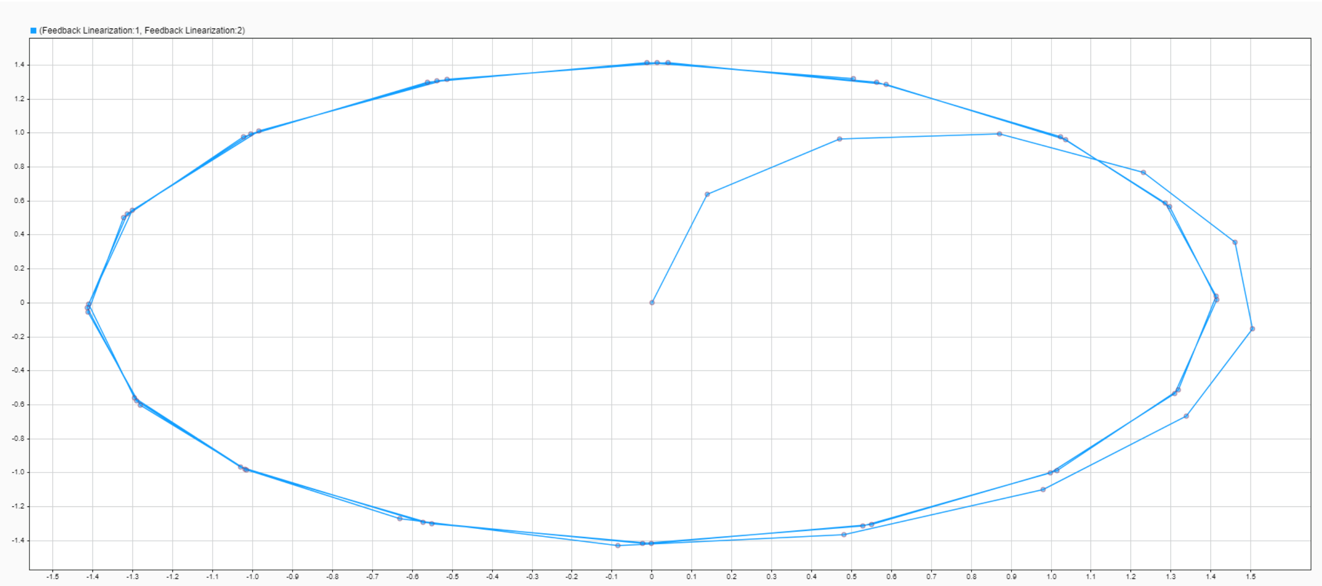

Figure 16 Asymptotic stability

7 REFERENCES

[1] The Mathworks, Inc, "Control Algorithm Modeling Guidelines Using MATLAB, Simulink, and Stateflow - Version 5.0," March 2020. [Online]. Available: nl.mathworks.com/help/simulink/mab-modeling-guidelines.html. [Accessed 01 2021].