Algorithm 1 SURF (strong unification of reduced-space and full-space)

SURF unifies reduced and full space for a simplified, equality-constrained optimization setting.

loop
  Assemble A, b, $\tilde{M}^{-1}$
  Solve $\tilde{M}^{-1} A p = \tilde{M}^{-1} b$
  Compute α via a line search
  Update $\left[x_+, y_+, \psi_+, \lambda_+\right]^T = \left[x, y, \psi, \lambda\right]^T + \alpha p$
  Update y and ψ by inexactly solving $\mathcal{R}(x, y) = 0$ and $\mathcal{R}_y^T \psi = -\mathcal{F}_y^T - \mathcal{C}_y^T \lambda$
  Update the approximation to A
end loop

Note: p is the search direction and α is the step size, $Ap = b$ is the FS KKT system, and M^{-1} and $\tilde{M}^{-1}$ are exact and approximate preconditioners for A such that $M = M_1 M_2 M_3 M_4$ and

$$A = \begin{bmatrix} m_{xx} & m_{xy} & {\mathcal{R}_x}^T & {\mathcal{C}_x}^T \\ m_{yx} & m_{yy} & {\mathcal{R}_y}^T & {\mathcal{C}_y}^T \\ \mathcal{R}_x & \mathcal{R}_y & 0 & 0 \\ \mathcal{C}_x & \mathcal{C}_y & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} -m_x \\ -m_y \\ -\mathcal{R}(x, y) \\ -\mathcal{C}(x, y) \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 0 & \mathcal{I} & 0 \\ 0 & \mathcal{I} & 0 & 0 \\ \mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I} \end{bmatrix}, M_2 = \begin{bmatrix} \mathcal{R}_y & 0 & 0 & 0 \\ m_{yy} & \mathcal{R}_y^T & 0 & 0 \\ m_{xy} & \mathcal{R}_x^T & \mathcal{I} & 0 \\ \mathcal{C}_y & 0 & 0 & \mathcal{I} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \mathcal{I} & 0 & \mathcal{R}_y^{-1}\mathcal{R}_x & 0 \\ 0 & \mathcal{I} & \mathcal{R}_y^{-T} m_{yx} - \mathcal{R}_y^{-T} m_{yy} \mathcal{R}_y^{-1} \mathcal{R}_x & \mathcal{R}_y^{-T} \mathcal{C}_y^T \\ 0 & 0 & m_{xx} - m_{xy} \mathcal{R}_y^{-1} \mathcal{R}_x - \mathcal{R}_x^T \mathcal{R}_y^{-T} m_{yx} + \mathcal{R}_x^T \mathcal{R}_y^{-T} m_{yy} \mathcal{R}_y^{-1} \mathcal{R}_x & \mathcal{C}_x^T - \mathcal{R}_x^T \mathcal{R}_y^{-T} \mathcal{C}_y^T \\ 0 & 0 & \mathcal{C}_x - \mathcal{C}_y \mathcal{R}_y^{-1} \mathcal{R}_x & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & \mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{I} & 0 \\ \mathcal{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I} \end{bmatrix}$$