Unit I (21MAB206T)

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21MAB206T- NUMERICAL METHODS AND ANALYSIS

Topics Discussed

- Numerical Solution of Algebraic and Transcendental Equation
  - Bisection Method
  - Newton Raphson Method
  - Regula Falsi Method/Method of False Position
- Solving System of Simultaneous Linear Algebraic Equation
  - Direct Method
    - Gauss Elimination Method
    - Gauss Jordon Method
  - Iterative Method
    - Gauss Jacobi Method
    - Gauss Seidel Method

Numerical Solution of Algebraic and Transcendental Equation

INTRODUCTION

- In the field of Science and Engineering, the solution of equations of the form $f(x)=0$ occurs in many applications.
- If $f(x)$ is polynomial of degree two or three, exact formulae are available.
- But, if $f(x)$ is a transcendental function, the solution is not exact and we do not have formulae to get the solutions.
- When the coefficients are numerical values, we can adopt various numerical approximate methods to solve such algebraic and transcendental equations.

Bisection Method (or) Bolzano's Method (or) Interval halving Method

- Given an approximate value of a root of an equation, a better and close approximation to the root can be found by using an iterative process called Bisection Method.
- The iterative formula is given by

$$x_r = \frac{a+b}{2} \ , r = 0, 1, 2, \dots.$$

EXAMPLE

- 1. Find the positive root of $x = \cos x$ by using Bisection method and correct it to two decimal places.
- Solution:
- Let $f(x) = x - \cos x = 0$; ,

  $f(0) = -ve \quad \text{and} \quad f(1) = +ve$
- Therefore the root lies between 0 and 1.
- The Bisection method iterative formula is given by

$$x_r = \frac{a+b}{2} \;,\; r = 0, 1, 2, \ldots.$$

EXAMPLE

iteration	a $(-)$	b $(+)$	$x_r = \frac{a+b}{2}$	Sign of $f(x) = x - \cos x$
0	0	1	0.50	-ive
1	0.50	1	0.75	+ive
2	0.50	0.75	0.63	-ive
3	0.63	0.75	0.69	-ive
4	0.69	0.75	0.72	-ive
5	0.72	0.75	0.74	-ive
6	0.74	0.75	0.75	+ive
7	0.74	0.75	0.75	

$$\text{Therefore} \quad x = 0.75$$

NEWTON'S (or) NEWTON RAPHSON METHOD

- Given an approximate value of a root of an equation, a better and close approximation to the root can be found by using an iterative process called Newton Raphson Method.
- The iterative formula is given by

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} \qquad r = 0, 1, 2, \ldots.$$

- The criterion for convergence in this method is

$$|f(x).f''(x)| < [f'(x)]^2$$

- The convergence is quadratic and is of order 2.

NEWTON RAPHSON METHOD

- NOTE
- 1. The choice of x_0 is very important for convergence.
- 2. When $f'(x)$ is very large, the correct values of root can be found out with minimum number of iterations.
- 3.If $f(a)$ and $f(b)$ are of opposite signs, a root of $f(x)=0$ lies between a and b. This idea can be used to fix an approximate root.

EXAMPLE-1

- 1. Find the positive root of $f(x) = 3x - \cos x - 1 = 0$ by Newtion-Raphson method to five decimal places.
- Solution:
- Let $f(x) = 3x - \cos x - 1 = 0$; $f'(x) = 3 + \sin x$
  $f(0) = -2 = -ve$ and $f(1) = 2 - \cos 1 = +ve$
- Therefore the root lies between 0 and 1.
- The Newton-Raphson iterative formula is given by

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} \quad r = 0, 1, 2, ...$$

EXAMPLE-1

- Take $x_0 = 0.5$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.60852$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.60710$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.60710$$

- The better approximate root is 0.60710.

EXAMPLE-II

- Find an iterative formula to find $\sqrt{N}$ (where N is a positive number) .
- Solution:

$$x = \sqrt{N};\ \ x^2 - N = 0;\ f'(x) = 2x$$

- The iterative formula is given by

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)} \qquad r = 0, 1, 2, \ldots.$$

- Hence we have,

$$x_{r+1} = x_r - \frac{{x_r}^2 - N}{2x_r}$$

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{N}{x_r}\right)$$

Regula Falsi Method: (method of chords)

Regula Falsi method or the method of false position is a numerical method for solving an equation in one unknown. It is quite similar to bisection method algorithm and is one of the oldest approaches. It was developed because the bisection method converges at a fairly slow speed.

The order of convergence is 1.618.

The iterative formula for this method is given below:

$$x_n = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Example 1: Find an approximate root of $x\log_{10} x - 1.2 = 0$ by False position method

Solution:

Let $f(x) = x\log_{10} x - 1.2$

f(1) = -1.2 (-ve)

f(2) = -0.5979 (-ve)

f(3) = 0.2313 (+ve)

Therefore a root lies between 2 and 3, a=2 and b=3

$$x_1 = \frac{2\times f(3) - 3\times f(2)}{f(3) - f(2)} = 2.72101$$

$$f(x_1) = -0.01710$$

Therefore a root lies between 2.72101 and 3, a=2.72101 and b=3

$$x_2 = \frac{2.72101\times f(3) - 3\times f(2.72101)}{f(3) - f(2.72101)} = \frac{0.68084}{0.24846} = 2.74021$$

$$f(x_2) = 2.74021\times \log(2.74021) - 1.2 = -0.000389$$

Therefore a root lies between 2.74021 and 3, a=2.74021 and b=3

$$x_3 = \frac{2.74021 \times f(3) - 3 \times f(2.74021)}{f(3) - f(2.74021)} = \frac{0.63514}{0.23175} = 2.74062$$

$$f(x_3) = 2.74062 \times \log(2.74062) - 1.2 = 0.000119$$

Therefore a root lies between 2.74021 and 2.74062, a=2.74021 and b=2.74062

$$x_4 = \frac{2.74021 \times f(2.74062) - 3 \times f(2.74062)}{f(3) - f(2.74062)} = \frac{0.001395}{0.000509} = 2.7405$$

Hence the root is 2.7405.

Example 2: Solve for a positive root of x – cos x = 0 by Regula Falsi method

Solution:

Let f(x) = x-cosx

f(0)=-1 (-ve)

f(1)=0.4596 (+ve)

Therefore a root lies between 0 and 1. Let a=0 and b=1

$$x_n = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_1 = \frac{0 \times f(1) - 1 \times f(0)}{f(1) - f(0)} = \frac{1}{1.45968} = 0.68507$$

$$f(x_1) = 0.68507 - \cos(0.68507) = -0.0892$$

Therefore a root lies between 0.68507 and 1, a=0.68507 and b=1

$$x_2 = \frac{0.68507 \times f(1) - 1 \times f(0.68507)}{f(1) - f(0.68507)} = \frac{0.40422}{0.54899} = 0.73629$$

$$f(x_2) = 0.73629 - \cos(0.73629) = -0.00466$$

Continuing the same process for $x_4\ and\ x_5$ values

The root is 0.7391 correct to 4 decimal places.

Solving System of Simultaneous Linear Algebraic Equation

Solution of Simultaneous Linear Algebraic Equations

Gauss - Elimination Method (Direct Method)

This is a method based on the elimination of the unknowns by combining equations such that the n equations in n unknowns are reduced to an equivalent upper triangular system which could be solved by back substitution method.

Gauss -Jordan Method (Direct Method)

This is the modification of the above method. Here the co-efficient matrix 'A' of the system is brought to a diagonal matrix or unit matrix by making the matrix 'A' not only upper triangular but also lower triangular by making all the elements above the leading diagonal of 'A' also zeroes. We get the values of x, y, z immediately without using the back substitution method.

Gauss Jacobi method: (Iterative method)

It is an iterative method for obtaining the solutions of a strictly diagonally dominant system of linear equations. The process is then iterated until it converges.

Consider the system of equations,

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

The diagonally dominant condition is as follows:

$$\begin{aligned} |a_1| &\succ |b_1| + |c_1| \\ |b_2| &\succ |a_2| + |c_2| \\ |c_3| &\succ |a_3| + |b_3| \end{aligned}$$

Note: The initial values are taken to be x = y = z = 0

Gauss Seidel method: (Iterative method)

This is also an iterative method used for solving linear equations. The diagonally dominant condition has to be checked. Here the initial values are y = z = 0.

Note: The convergence in Gauss Seidel method is very fast (roughly 2 times) compared to that of Gauss Jacobi method.

Example :3 Solve the system of equations by (i) Gauss elimination method (ii) Gauss Jordan method.

x+2y+z=3, 2x+3y+3z=10, 3x-y+2z=13.

Solution: (By Gauss elimination method)

The given system is equivalent to

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}$$

$$AX \ = \ B$$

$$(A,B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

Now , We will make the matrix A upper triangular.

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 3 & -7 & -1 & 4 \end{array}\right) \quad R_2 = R_2 - 2R_1, R_3 = R_3 - 3R_1$$

Now take $b_{22} = -1$ as the pivot and make b_{32} as zero.

$$(A,B) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) \quad R_3 = R_3 - 7R_2$$

x+2y+z = 3

-y+z = 4

-8z = -24

By back substitution, we get x = 2, y = -1, z = 3.

Solution: (By Gauss Jordan method)

$$(A,B) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right)$$

$$(A,B) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) \quad R_1 = 2R_2 + R_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right) \quad R_3 = R_3\left(\frac{1}{8}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right) \quad R_{13}(3), R_{23}(1)$$

x = 2, y = -1, z = 3

Example :4 Solve the system of equations by Gauss elimination method

2x+3y-z = 5, 4x+4y-3z = 3, 2x-3y+2z = 2.

Solution:

The given system is equivalent to

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$A \quad X \;=\; B$$

$$(A,B)=\left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right) \quad R_2 = R_2 - 2R_1, R_3 = R_3 - R_1$$

$$(A,B)\sim\left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{array}\right) \quad R_3 = R_3 - 3R_2$$

2x+3y-z = 5

-2y-z = -7

6z = 18

By back substitution, we get x = 1, y = 2, z = 3

Example :5 Solve the system of equations by Gauss Jordan method

10x+y+z = 12, 2x+10y+z = 13, x+y+5z = 7

Solution:

Since the coefficient of x in the last equation is unity, we rewrite the equations interchanging the first and the last. Hence the augmented matrix is

$$(A,B)=\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right) \quad R_2 = R_2 - 2R_1, R_3 = R_3 - 10R_1$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & -9 & -49 & -58 \end{array}\right) \quad R_2 = R_2\left(\frac{1}{8}\right)$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & -\frac{473}{8} & -\frac{473}{8} \end{array}\right) \quad R_3 = R_3 + 9R_2$$

$$\sim\left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_3 = R_3\left(-\frac{8}{473}\right)$$

$$\sim\left(\begin{array}{ccc|c} 1 & 0 & \frac{49}{8} & \frac{57}{8} \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_1 = R_1 - R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_2 = R_2 + \left(\frac{9}{8}\right)R_3, R_1 = R_1 + \left(-\frac{49}{8}\right)R_3$$

Hence the solution is x = 1, y = 1, z = 1

Example :5 Solve the following system by Gauss- Jacobi and Gauss- Seidel method

10x-5y-2z = 3, 4x-10y+3z = -3, x+6y+10z = -3.

Solution:

Since the diagonal elements are dominant in the coefficient matrix

$$\begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$$

Since $|10| \succ |-5| + |-2|$,

$|-10| \succ |4| + |3|$,

$|10| \succ |1| + |6|$

Gauss- Jacobi method :

Solving for x, y, z we have

$$x = \frac{1}{10}\left(3 + 5y + 2z\right) \ldots\ldots\ldots\ldots(1)$$

$$y = \frac{1}{10}\left(3 + 4x + 3z\right) \ldots\ldots\ldots\ldots(2)$$

$$z = \frac{1}{10}\left(-3 - x - 6y\right) \ldots\ldots\ldots\ldots(3)$$

First iteration : Let the initial values be (0,0,0)

Using the initial values in (1),(2),(3), we get

$$x^{(1)} = \frac{1}{10}\left(3 + 5(0) + 2(0)\right) = 0.3$$

$$y^{(1)} = \frac{1}{10}\left(3 + 4(0) + 3(0)\right) = 0.3$$

$$z^{(1)} = \frac{1}{10}\left(-3 - 0 - 6(0)\right) = -0.3$$

Second iteration : Using the values of $x^{(1)}, y^{(1)}, z^{(1)}$ in (1),(2),(3), we get

$$x^{(2)} = \frac{1}{10}\left(3 + 5(0.3) + 2(-0.3)\right) = 0.39$$

$$y^{(2)} = \frac{1}{10}\left(3 + 4(0.3) + 3(-0.3)\right) = 0.33$$

$$z^{(2)} = \frac{1}{10}\left(-3 - (0.3) - 6(0.3)\right) = -0.51$$

Iteration	x	y	z
1	0.3	0.3	-0.3
2	0.39	0.33	-0.51
3	0.363	0.303	-0.537
4	0.3441	0.2841	-0.5181
5	0.33843	0.2822	-0.50487
6	0.34012	0.28391	-0.50316
7	0.34132	0.28510	-0.50435
8	0.34167	0.28522	-0.50519
9	0.34157	0.28511	-0.50530

Gauss- Seidel method :

First iteration : Initial values y = 0, z = 0

$$x^{(1)} = \frac{1}{10}\left(3+5(0)+2(0)\right) = 0.3$$

$$y^{(1)} = \frac{1}{10}\left(3+4(0.3)+3(0)\right) = 0.42$$

$$z^{(1)} = \frac{1}{10}\left(-3-(0.3)-6(0.42)\right) = -0.582$$

Second iteration : Using the values of $x^{(1)}, y^{(1)}, z^{(1)}$ in (1),(2),(3), we get

$$x^{(2)} = \frac{1}{10}\left(3+5(0.42)+2(-0.582)\right) = 0.3936$$

$$y^{(2)} = \frac{1}{10}\left(3+4(0.3936)+3(-0.582)\right) = 0.28284$$

$$z^{(2)} = \frac{1}{10}\left(-3-(0.3936)-6(0.28284)\right) = -0.50906$$

Iteration	x	y	z
1	0.3	0.42	-0.582
2	0.3936	0.28284	-0.50906
3	0.33960	0.28312	-0.50383
4	0.34079	0.28516	-0.50517
5	0.34155	0.28506	-0.50519
6	0.34149	0.28503	-0.50517
7	0.34148	0.28504	-0.50517

The values correct to 3 decimal places are

$$x = 0.342,\ y = 0.285 \text{ and } z = -0.505$$

Example :6 Solve the following system by Gauss- Jacobi and Gauss- Seidel method Correct to three decimal places. $x+y+54z = 110$, $27x+6y-z = 85$, $6x+15y+2z = 72$.

Solution:

As the coefficient matrix is not diagonally dominant as it is, we rearrange the equations.

$$27x+6y-z = 85$$

$$6x+15y+2z = 72$$

$$x+y+54z = 110$$

Solving for x, y, z, we get

$$x = \frac{1}{27}(85 - 6y + z) \text{............(1)}$$

$$y = \frac{1}{15}(72 - 6x - 2z) \text{............(2)}$$

$$z = \frac{1}{54}(110 - x - y) \text{............(3)}$$

Starting with the initial value x = 0, y = 0, z = 0 and using (1),(2),(3) and repeating the process we get the values of x, y, z as the tabulated by both methods. (Gauss- Jacobi and Gauss- Seidel method)

	Gauss- Jacobi method			Gauss- Seidel method		
Iteration	**x**	**y**	**z**	**x**	**y**	**z**
1	3.14815	4.8	2.03704	3.14815	3.54074	1.91317
2	2.15693	3.26913	1.88985	2.43218	3.57204	1.92585
3	2.49167	3.68525	1.93655	2.42569	3.57294	1.92595
4	2.40093	3.54513	1.92265	2.42549	3.57301	1.92595
5	2.43155	3.58327	1.92692	2.42548	3.57301	1.92595
6	2.42323	3.57046	1.92565	2.42548	3.57301	1.92595
7	2.42603	3.57395	1.92604			
8	2.42527	3.57278	1.92593			

Hence the values are $x = 2.425$, $y = 3.573$ and $z = 1.926$

Thank You