UNIT III (21MAB206T)

ACADEMIC YEAR 2023-2024 (ODD SEMESTER)

NUMERICAL METHODS AND ANALYISIS

TOPICS

- Numerical Differenetiation
  - Numerical differentiation using Newton Forward difference formulae
  - Numerical differentiation using Newton backward difference formulae

- Numerical integration
  - Trapezoidal rule and their applications
  - Simpson’s one-third rule and their application
  - Simpson’s three-eighth rule and their application

NUMERICAL DIFFERENTIATION

Numerical differentiation using Newton forward difference formulae

- Suppose we have $(n+1)$ ordered pairs (x_i, y_i), for $i = 0,1,\ldots,n$.
- And we want to find the derivative of $y = f(x)$ passing through the $(n+1)$ points, at a point nearer to the starting value $x = x_0$.
- We use Newton forward difference interpolation formula to get the derivative.
- Newton's forward difference interpolation formula is
- $y(x) = y(x_0 + uh) = y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$
- Where $y(x)$ is a polynomial of degree n in x and $u = \frac{x-x_0}{h}$

- Differentiating $y(x)$ with respect to x, we get
- $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
- Also
- $\frac{du}{dx} = \frac{d\left(\frac{x-x_0}{h}\right)}{dx} = \frac{1}{h}$
- So
- $\frac{dy}{dx} = \frac{1}{h} \cdot \frac{dy}{du}$
- Again
- $\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots$
- Hence
- $\frac{dy}{dx} = \frac{1}{h}\{\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots\}$
- This is the value of $\frac{dy}{dx}$ at general x which may be anywhere in the interval.

- Some other higher order derivatives of $y(x)$ with respect to x are given as
- $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right\}$
- $\frac{d^3y}{dx^3} = \frac{1}{h^3}\{\Delta^3 y_0 + \frac{12u-18}{12}\Delta^4 y_0 + \cdots\}$

- In special case $x = x_0$, we have $u = 0$. So
- $\left[\frac{dy}{dx}\right]_{x=x_0} = \left[\frac{dy}{dx}\right]_{u=0} = \frac{1}{h}\{\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\}$
- This is the value of $\frac{dy}{dx}$ at general x which may be anywhere in the interval.
- Similarly,
- $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left\{\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right\}$
- $\left[\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\{\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\}$

Numerical differentiation using Newton backward difference formulae

- Suppose we have $(n+1)$ ordered pairs (x_i, y_i), for $i = 0,1,\dots,n$.
- And we want to find the derivative of $y = f(x)$ passing through the $(n+1)$ points, at a point nearer to the end value $x = x_n$.
- We use Newton backward difference interpolation formula to get the derivative.
- Newton's backward difference interpolation formula is
- $y(x) = y(x_n + vh) = y_v = y_n + \mathrm{v}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$
- Where $y(x)$ is a polynomial of degree n in x and v$= \frac{x-x_n}{h}$

- Differentiating $y(x)$ with respect to x, we get
- $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$
- Also
- $\frac{dv}{dx} = \frac{d\left(\frac{x-x_n}{h}\right)}{dx} = \frac{1}{h}$
- So
- $\frac{dy}{dx} = \frac{1}{h} \cdot \frac{dy}{dv}$
- Again
- $\frac{dy}{dv} = \nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots$
- Hence
- $\frac{dy}{dx} = \frac{1}{h}\{\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots\}$
- This is the value of $\frac{dy}{dx}$ at general x which may be anywhere in the interval.

- Some other higher order derivatives of $y(x)$ with respect to x are given as
- $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left\{\nabla^2 y_n + (v+1)\Delta^3 y_n + \frac{6v^2+18v+11}{12}\nabla^4 y_n + \cdots\right\}$
- $\frac{d^3y}{dx^3} = \frac{1}{h^3}\{\nabla^3 y_n + \frac{12v+18}{12}\nabla^4 y_n + \cdots\}$

- In special case $x = x_n$, we have v= 0. So
- $\left[\frac{dy}{dx}\right]_{x=x_n} = \left[\frac{dy}{dx}\right]_{v=0} = \frac{1}{h}\{\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\}$
- This is the value of $\frac{dy}{dx}$ at general x which may be anywhere in the interval.
- Similarly,
- $\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left\{\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right\}$
- $\left[\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}\{\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\}$

Examples

- Example 1:- Find the first two derivative of $(x)^{\frac{1}{3}}$ at $x = 50$ and $x = 56$ given the data below:

x	50	51	52	53	54	55	56
$y = (x)^{\frac{1}{3}}$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

- Solution:- Since we require $f'(x)$ at $x = 50$, we use Newton's forward difference formula and to get $f'(x)$ at $x = 56$, we use Newton's backward difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0
54	3.7798		-0.0003	
		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

- By the differentiation of $y(x)$ derived from Newton's forward interpolation formula, we have
- $\left[\frac{dy}{dx}\right]_{x=x_0} = \left[\frac{dy}{dx}\right]_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots\right]$
- $\Rightarrow \left[\frac{dy}{dx}\right]_{x=50} = \frac{1}{1}\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$
- $\Rightarrow \left[\frac{dy}{dx}\right]_{x=50} = 0.02455$
- Also,
- $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \left[\frac{d^2y}{dx^2}\right]_{u=0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 \ldots\right]$
- $\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=50} = 1[-0.0003] = -0.0003$

- By the differentiation of $y(x)$ derived from Newton's backward interpolation formula, we have
- $\left[\frac{dy}{dx}\right]_{x=x_n} = \left[\frac{dy}{dx}\right]_{v=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$
- $\Rightarrow \left[\frac{dy}{dx}\right]_{x=56} = \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$
- $\Rightarrow \left[\frac{dy}{dx}\right]_{x=56} = 0.02275$
- Also,
- $\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \left[\frac{d^2y}{dx^2}\right]_{v=0} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n \ldots]$
- $\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=56} = 1[-0.0003] = -0.0003$

Examples

- Example 2:- The population f a certain town is given below. Find the rate of growth of the population in 1931,1941,1961 and 1971.

Year	x	**1931**	1941	**1951**	**1961**	**1971**
Population in thousands	y	40.62	60.80	79.95	103.56	132.65

- Solution:- We use the same difference table for backward and forward differences.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\boldsymbol{\Delta^4 y}$
1931	40.62	20.18			
1941	60.80	19.15	-1.03	5.49	
1951	79.95	23.61	4.46	1.02	-4.47
1961	103.56	29.09	5.48		
1971	132.65				

To get $f'(1931)$ and $f'(1941)$, we use the forward interpolation formula.

- By the differentiation of $y(x)$ derived from Newton's forward interpolation formula, we have
- $$\frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots\right\} \qquad (A)$$
- Here $x_0 = 1931$ and $x_1 = 1941$
- First we find the value of $f'(1931)$. So we use the following formula
- $$\left[\frac{dy}{dx}\right]_{x=x_0} = \left[\frac{dy}{dx}\right]_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots\right]$$
- So,
- $$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1931} = \frac{1}{10}\left[20.18 - \frac{1}{2}(-1.03) + \frac{1}{3}(5.49) - \frac{1}{4}(-4.47)\right]$$
- $$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1931} = 2.36425$$
- Also, if $x = 1941$, then $u = \frac{1941-1931}{10} = \frac{10}{10} = 1$.
- Putting $u = 1$ in equation (A), we get
- $$\left[\frac{dy}{dx}\right]_{x=1941} = \left[\frac{dy}{dx}\right]_{u=1} = \frac{1}{10}\left[20.18 + \frac{1}{2}(-1.03) - \frac{1}{6}(5.49) + \frac{1}{12}(-4.47)\right] = 1.83775.$$

To get $f'(1961)$ and $f'(1971)$, we use the backward interpolation formula.

- By the differentiation of $y(x)$ derived from Newton's backward interpolation formula, we have
- $$\frac{dy}{dx} = \frac{1}{h}\left\{\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \cdots\right\} \qquad (B)$$
- Here $x_n = 1971$ and $x_{n-1} = 1961$
- First we find the value of $f'(1931)$. So we use the following formula
- $$\left[\frac{dy}{dx}\right]_{x=x_n} = \left[\frac{dy}{dx}\right]_{v=0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$
- So,
- $$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1971} = \frac{1}{10}\left[29.09 + \frac{1}{2}(5.48) + \frac{1}{3}(1.02) + \frac{1}{4}(-4.47)\right]$$
- $$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1971} = 3.10525$$
- Also, if $x = 1961$, then $v = \frac{1961-1971}{10} = \frac{-10}{10} = -1.$
- Putting $v = 1$ in equation (B), we get
- $$\left[\frac{dy}{dx}\right]_{x=1961} = \left[\frac{dy}{dx}\right]_{v=1} = \frac{1}{10}\left[29.09 - \frac{1}{2}(5.48) - \frac{1}{6}(1.02) - \frac{1}{12}(-4.47)\right] = 2.65525.$$

Examples

- Example 3:- The table given below reveals the velocity v of a body at a time t specified. Find its acceleration at $t = 1.1$.

t	1.0	1.1	1.2	1.3	1.4
v	43.1	47.7	52.1	56.4	60.8

- Solution:-v is dependent on time t, i.e. $v = v(t)$. We require acceleration $= \frac{dv}{dt}$.
- Therefore, we have to find $v'(1.1)$.
- That is, it is a problem of numerical differentiation.

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1.0	43.1				
		4.6			
1.1	47.7		-0.2		
		4.4		0.1	
1.2	52.1		-0.1		0.1
		4.3		0.2	
1.3	56.4		0.1		
		4.4			
1.4	60.8				

As $\frac{dv}{dt}$ at $t = 1.1$ is require, (nearer to the beginning value), we use forward interpolation formula.

- By the differentiation of v(t) derived from Newton's forward interpolation formula, we have
- $\frac{dv}{dt} = \frac{1}{h}\left\{\Delta v_0 + \frac{2u-1}{2}\Delta^2 v_0 + \frac{3u^2-6u+2}{6}\Delta^3 v_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 v_0 + \cdots\right\}$ $\quad (A)$
- Here $u = \frac{t-t_0}{h} = \frac{1.1-1}{0.1} = 1.$
- $\left[\frac{dv}{dt}\right]_{t=1.1} = \left[\frac{dv}{dt}\right]_{u=1} = \frac{1}{0.1}\left[4.6 + \frac{1}{2}(-0.2) - \frac{1}{6}(0.1) + \frac{1}{12}(0.1)\right] = 44.917.$

NUMERICAL   INTEGRATION

- In this section we will be using
- Trapezodial rule
- Simpson’s one – third rule
- Simpson’s three-eighth rule

DEFINITION :

The term Numerical integration is the numerical evaluation of a definite integral

$$A=\int_{a}^{b} f(x)dx$$

where 'a' and 'b' are given constants and f(x) is a function given analytically by a formula or emprically by a table of values

Trapezoidal Rule

Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

Note: The trapezoidal rule is the simplest of the formulae for numerical integration, but it is also the least accurate. The accuracy of the result can be improved by decreasing the interval h.

The Error in the trapezoidal rule is of the order h^2.

Simpson’s One-third Rule

Simpson's $\frac{1}{3}$ rule:

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\left[\begin{array}{l}(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + \\ 2(y_2 + y_4 + \ldots)\end{array}\right]$$

$$= \frac{h}{3}\left[\begin{array}{l}(y_0 + y_n) + 4(\text{sum of odd ordinates}) \\ +2(\text{sum of even ordinates})\end{array}\right]$$

Note: The interval of integration must be divided into an even number of subintervals of width h

The Error in the Simpson's $\frac{1}{3}$ rule is of the order h^4.

Simpson's Three-eighth Rule

Simpson's $\dfrac{3}{8}th$ rule:

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\left[\begin{array}{l}(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \ldots) + \\ 2(y_3 + y_6 + y_9 \ldots)\end{array}\right]$$

Note: This rule can be applied only if the number of subinterval is a multiple of 3.

Problems

Problem 01:

Compute the value of $\int_{4}^{5.2} \log_e x\,dx$ using Trapezoidal and Simpson's rule.

Solution:

Divide the interval of integration into 6 equal parts with h=0.2.

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log_e x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Trapezoidal rule:

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

$$= \frac{0.2}{2}[(y_0 + y_6) + 2(y_1 + y_2 + \dots + y_5)] = 0.18276$$

Simpson's $\frac{1}{3}$rd rule is applicable an n=6, even no.

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\begin{bmatrix}(y_0+y_n)+4(y_1+y_3+\ldots)+ \\ 2(y_2+y_4+\ldots)\end{bmatrix}$$

$$= \frac{h}{3}\begin{bmatrix}(y_0+y_n)+4(\text{sum of odd ordinates}) \\ +2(\text{sum of even ordinates})\end{bmatrix}$$

$$= \frac{0.2}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= 1.8278$$

Simpson's $\frac{3}{8}th$ rule is applicable as n=6, a multiple of 3.

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{3h}{8}\begin{bmatrix}(y_0+y_n)+3(y_1+y_2+y_4+y_5\ldots)+ \\ 2(y_3+y_6+y_9\ldots)\end{bmatrix}$$

$$= \frac{3(0.2)}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)] = 1.8278$$

Problem 02 :

A river is 80 feet wide. The depth 'd' in fet at a distance x feet from one bank is given by the following table. Find the area of cross section of the river.

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Solution:

Since n=8 we can apply either Trapezoidal rule os Simpson's1/3rd rule, but not Simpson's 3/8th rule.

h=10. Using Simpson's1/3rd rule

$$\text{Area}=\int_0^{80} y\,dx$$

$$=\frac{10}{3}[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$

$$=710 \text{ sq.feet}$$

Problem 03:

Evaluate $I = \int_0^6 \frac{dx}{1+x}$ using (i) Trapezodial rule (ii) Simpson's $\frac{1}{3}$ Rule (iii) Simpson's $\frac{3}{8}$ Rule with h = 1 also check by direct integration.

Solution :

Given h = 1

x :	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$:	1	0.5	0.333	0.25	0.2	0.166	0.142

(i) Trapezodial rule

$$\int_0^1 \frac{dx}{1+x} = \frac{1}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.142) + 2(0.5+0.333+0.25+0.2+0.166)\right]$$

$$= 2.0214$$

(ii) Simpson's $\frac{1}{3}$ rule :

$$\int_0^6 \frac{dx}{1+x} = \frac{1}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{1}{3}\left[1 + 0.142 + 4(0.5 + 0.25 + 0.166) + 2(0.33 + 0.2)\right]$$

$$= 1.9587$$

(iii) Simpson's $\frac{3}{8}$ rule :

$$\int_0^6 \frac{dx}{1+x} = \frac{3}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$= \frac{1}{3}\left[1 + 0.142 + 4(0.5 + 0.333 + 0.2 + 0.166) + 2(0.25)\right]$$

$$= 1.966$$

By actual integration

$$\int_0^6 \frac{dx}{1+x} = [\log_e (1+x)]_0^6$$

$$= \log_e 7$$

$$=1.9459$$

THE END