Unit II (21MAB206T)

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21MAB206T- NUMERICAL METHODS AND ANALYSIS

Topics Discussed

- Finite Differences
  - Different types of Operators and their Relation
  - Difference of polynomial
  - Factorial Polynomials
- Interpolation
  - For equal Intervals:
    - Newton’s Forward Interpolation
    - Newton’s Backward Interpolation
  - For unequal Intervals:
    - Newton’s Divided Difference Interpolation
    - Lagrange’s Interpolation
  - Lagrange Formula for Inverse Interpolation

Operators

Introduction

Let $y = f(x)$ be a given function of x and let $y_0, y_1, ...y_n$ be the values of y corresponding to $x_0, x_1, ...x_n$ the values of x. Here the independent variable x is called argument and the corresponding dependent value y is called entry. The difference between any two consecutive values of x need not be same.

In general,

x	x_0	x_1	x_2	...	x_n
y	y_0	y_1	y_2	...	y_n

Here $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ is called the first difference of y and it is denoted by Δy. (i.e). $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, ..., $\Delta y_{n-1} = y_n - y_{n-1}$, here "$\Delta$" is called forward difference operator.

Types Of Operators

Suppose the argument $x_0, x_1, ...x_n$ are equally spaced, which means that $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, ..., x_n = x_0 + nh$, here h is called the interval length or length of an interval.

- Forward Difference Operator (Δ):

  $\Delta f(x) = f(x+h) - f(x)$

  (i.e). $\Delta y_0 = \Delta y_1 - \Delta y_0$

  similarly $\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta f(x+h) - \Delta f(x)$

  $= f(x+2h) - 2f(x+h) + f(x).$

- Backward Difference Operator (∇):

$\nabla f(x) = f(x) - f(x-h)$

(i.e). $\nabla y_1 = \nabla y_1 - \nabla y_0$

- Central Difference Operator (δ):

$\delta f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$

- Shifting Operator (E):

$Ef(x) = f(x+h), E^2 f(x) = f(x+2h), \ldots, E^n f(x) = f(x+nh)$

- Averaging Operator (μ):

$\mu f(x) = \frac{1}{2}[f(x + \frac{h}{2}) + f(x - \frac{h}{2})]$

- Difference Operator (D):

$Df(x) = \frac{d}{dx} f(x)$

- Unit Operator 1:

$1.f(x) = f(x)$

Properties of Operators

- The operators $\Delta, \nabla, \delta, E, \mu$ *and* D are all linear.

  Proof:

  $$\Delta(af(x)+bg(x)) = [af(x+h)+bg(x+h)] - [af(x)+bg(x)]$$
  $$= a[f(x+h)-f(x)] + b[g(x+h)-g(x)]$$
  $$= a\Delta f(x) + b\Delta g(x)$$

- The operator is distributive over addition.

  $$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^{n+m} f(x) = \Delta^n \Delta^m f(x)$$

Relation Between Operators

- Relation between Δ *and* E:

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= Ef(x) - 1.f(x)\\ &= (E-1)f(x)\\ \Delta &= E - 1.\end{aligned}$$

- Relation between ∇ *and* E:

$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h)\\ &= 1.f(x) - E^{-1}f(x)\\ &= (1-E^{-1})f(x)\\ \nabla &= 1 - E^{-1}.\end{aligned}$$

- Relation between δ *and* E:

$$\delta f(x) = f(x+\tfrac{h}{2}) - f(x-\tfrac{h}{2})$$
$$= E^{1/2}f(x) - E^{-1/2}f(x)$$
$$= (E^{1/2} - E^{-1/2})f(x)$$
$$\delta = E^{1/2} - E^{-1/2}.$$

- Relation between μ *and* E:

$$\mu f(x) = f(x+\tfrac{h}{2}) + f(x-\tfrac{h}{2})$$
$$= E^{1/2}f(x) + E^{-1/2}f(x)$$
$$= (E^{1/2} + E^{-1/2})f(x)$$
$$\mu = E^{1/2} + E^{-1/2}.$$

- Relation between D *and* Δ:

$$Df(x) = \frac{d}{dx}f(x)$$
$$D = \frac{1}{h}[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - ...]$$

Problems on Operators

- Prove that $E\nabla = \Delta = \nabla E$

Proof:

$$\begin{aligned}(E\nabla)f(x) &= E(\nabla f(x)) = E(f(x) - f(x-h)) \\ &= Ef(x) - Ef(x-h) \\ &= f(x+h) - f(x) = \Delta f(x)\end{aligned}$$

$\therefore E\nabla = \Delta$

similarly $(\nabla E)f(x) = \nabla(Ef(x)) = \nabla f(x+h)$

$$= f(x+h) - f(x) = \Delta f(x)$$

$\therefore \nabla E = \Delta$

Hence $E\nabla = \Delta = \nabla E$

- Prove that $\nabla\Delta = \Delta - \nabla = \delta^2$

Proof:

$$\begin{aligned}\nabla\Delta &= (1 - E^{-1})(E - 1)\\ &= E + E^{-1} - 2\\ &= (E^{1/2} - E^{-1/2})^2 = \delta^2\end{aligned}$$

$$\begin{aligned}\Delta - \nabla &= (E - 1) - (1 - E^{-1})\\ &= E + E^{-1} - 2 = \delta^2\end{aligned}$$

- Prove that $(1 - \nabla)(1 + \Delta) = 1$

Proof:

$$(1 - \nabla)(1 + \Delta) = E.E^{-1} = 1$$

- Prove that $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

Proof:

$$\frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(E - 1 + 1 - E^{-1}) = \frac{1}{2}(E - E^{-1}) = \mu\delta$$

Difference of a polynomial

Theorem. The nth differences (forward) of a polynomial of the nth degree are constants.

That is, if $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{x-2} + \cdots\cdots + a_n$

Then $\Delta^n f(x) = a_0 n\,!\, h^n$

where h is the interval of differencing.

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= a_0\,[(x+h)^n - x^n] + a_1\,[(x+h)^{n-1} - x^{n-1}] + \cdots\cdots + a_n\\ &= a_0\,[nhx^{n-1} + \cdots\cdots] + \cdots\cdots\\ &= a_0 nhx^{n-1} + \text{terms involving powers of } x \text{ less than } (n-1)\end{aligned}$$

That is, $\Delta f(x)$ = a polynomial of degree $(n - 1)$

$$\begin{aligned}\Delta^2 f(x) &= a_0\, nh\,[(x+h)^{n-1} - x^{n-1}] + \text{terms involving lesser degree}\\ &= a_0 n\,(n-1)\,h^2\,x^{n-2} + \text{terms involving degree less than } (n-2)\end{aligned}$$

i.e., second difference of a polynomial of degree n is a polynomial of degree x^{n-2}.

Proceeding like this

$$\begin{aligned}\Delta^n f(x) &= a_0\, n\,!\, h^n\, x^0\\ &= a_0\, n\,!\, h^n\end{aligned}$$

Note 1. The converse of the theorem is also true. That is, if the nth differences of a tabulated function are constants, then the function is a polynomial of degree n.

2. The $(n + 1)$th and higher differences of a polynomial of degree n are zeros.

- Find the missing term in the following data:

X:	1	2	3	4	5	6	7
y:	2	4	8	-	32	64	128

Solution:

There are 6 given values. Then we get polynomial of degree five to satisfies the given data.

(i.e). $\triangle^6 y_0 = 0.$

$(E-1)^6 y_0 = 0$

$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)y_0 = 0$

$y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$

$128 - 6(64) + 15(32) - 20y_3 + 15(8) - 6(4) + 2 = 0$

$20y_3 = 322$. Hence $y_3 = 16.1$.

Therefore Missing value is 16.1

Practice Questions

Prove the following:

- $1 + \mu^2\delta^2 = (1 + \frac{1}{2}\delta^2)^2$
- $E^{1/2} = \mu + \frac{1}{2}\delta$
- $E^{-1/2} = \mu - \frac{1}{2}\delta$
- $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$

- Estimate the production for 1964 and 1966 from the following data:

X:	1961	1962	1963	1964	1965	1966	1967
y:	200	220	260	-	350	-	430

Hint:There are 5 given values. Then we get polynomial of degree four to satisfies the given data.

(i.e). $\triangle^5 y_0 = 0$.

Factorial Polynomial

A factorial polynomial $x^{(n)}$is defined as $x^{(n)} = x(x-h)(x-2h)\ldots(x-(n-1)h)$,where n is positive integer.

- Differences of $x^{(n)}$:

$\triangle x^{(n)} = (x+h)^{(n)} - x^{(n)}$

$= (x+h)x(x-h)\ldots(x-(n-2)h) - x(x-h)(x-2h)\ldots(x-(n-1)h)$

$= x(x-h)\ldots(x-(n-2)h)[(x+h)-(x-(n-1)h]$

$= n.hx^{(n-1)}$

Similarly $\triangle^2 x^{(n)} = \triangle x^{(n)}[n.hx^{(n-1)}]$

$= n.(n-1)h^2 x^{(n-2)}$

.

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$\triangle^r x^{(n)} = n.(n-1).(n-2)\ldots(n-r+1)h^r.x^{(n-r)}$,where r is positive integer and $r < n$.

- **Problem 1:**

  Express $x^4 + 3x^3 - 5x^2 + 6x - 7$ as a factorial polynomial and get their successive forward differences, taking h=1.

  Solution:

0	1	3	-5	6	-7
	0	0	0	0	0
1	1	3	-5	6	-7
	0	1	4	-1	
2	1	4	-1	5	
	0	2	12		
3	1	6	11		
	0	3			
	1	9			

Therefore, factorial polynimial is

$$f(x) = 1x^{(4)} + 9x^{(3)} + 11x^{(2)} + 5x^{(1)} - 7$$

$$\Delta f(x) = 4x^{(3)} + 27x^{(2)} + 22x^{(1)} + 5$$

$$\Delta^2 f(x) = 12x^{(2)} + 54x^{(1)} + 22$$

$$\Delta^3 f(x) = 24x^{(1)} + 54$$

$$\Delta^4 f(x) = 24$$

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$$\Delta^r f(x) = 0 \; if \; r > 4.$$

- **Problem 2:**

  Express $f(x) = x^3 - 3x^2 + 5x + 7$ as a factorial polynomial and get their successive forward differences, taking h=2.

  Solution:

$$
\begin{array}{c|cccc}
0 & 1 & -3 & 5 & 7 \\
 & 0 & 0 & 0 & 0 \\
\hline
2 & 1 & -3 & 5 & 7 \\
 & 0 & 2 & -2 & \\
\hline
4 & 1 & -1 & 3 & \\
 & 0 & 4 & & \\
\hline
 & 1 & 3 & &
\end{array}
$$

Therefore, factorial polynimial is

$$f(x) = 1x^{(3)} + 3x^{(2)} + 3x^{(1)} + 7$$

$$\Delta f(x) = 3x^{(2)}.2^1 + 6x^{(1)}.2^1 + 3$$

$$\Delta^2 f(x) = 6x^{(1)}.2^2 + 6.2^2$$

$$\Delta^3 f(x) = 6.2^3 = 48$$

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$$\Delta^r f(x) = 0 \; if \; r > 3.$$

Interpolation

Interpolation(for equal Interval)

Interpolation means the process of computing intermediate values of a function from a given set of tabular values of the function.

Suppose the following table represents a set of corresponding values of x and y.

x	:	x_0	x_1	x_2	x_3	$\cdots$	x_n
y	:	y_0	y_1	y_2	y_3	$\cdots$	y_n

We require the value of $y = y_i$ corresponding to a value $x = x_i$, where $x_0 < x_i < x_n$.

Extrapolation is used to denote the process of finding the values outside the interval (x_0, x_n).

Gregory-Newton forward interpolation formula or Newton’s forward interpolation formula (for equal intervals)

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, \ldots, y_n$ corresponding to the values $x_0, x_1, \ldots, x_n$ respectively of x.

Suppose that the values of x viz. $x_0, x_1, \ldots, x_n$ are equidistant. That is, $x_i - x_{i-1} = h$, for $i = 1, 2, \ldots, n$.

We can prove the formula using symbolic operator methods.

$$
\begin{aligned}
P_n(x) &= P_n(x_0 + uh) = E^u P_n(x_0) = E^u y_0 \\
&= (1 + \Delta)^u y_0
\end{aligned}
$$

$$
= [1 + {}^uC_1\Delta + {}^uC_2\Delta^2 + {}^uC_3\Delta^3 + \cdots + {}^uC_r\Delta^r + \cdots + {}^uC_n\Delta^n + \cdots]y_0
$$

$$= y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \cdots + \frac{u^{(r)}}{r!}\Delta^r y_0 + \cdots + \frac{u^{(n)}}{n!}\Delta^n y_0 + \cdots$$

where $u = \dfrac{x - x_0}{h}$

If $y(x)$ is a polynomial of nth degree $\Delta^{n+1} y_0, \ldots$ are zero. Hence

$$\boxed{P_n(x) = P_n(x_0 + uh) = y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \cdots + \frac{u^{(n)}}{n!}\Delta^n y_0}$$

Note:

- The first two terms will give the linear interpolation and the first three terms will give a parabolic interpolation and so on.
- Since this formula involves forward differences of y_0, we call it Newton's forward interpolation formula. Since this involves the forward differences of y_0, this is used to interpolate the values of y nearer to the beginning value of the table.
- This is applicable only if the interval of differencing h is constant.

Gregory-Newton Backward interpolation formula (for equal intervals)

Newton's forward interpolation formula cannot be used for interpolating a value of y nearer to the end of the table of values. For this purpose, we get another backward interpolation formula.

Suppose $y = f(x)$ takes the values $y_0, y_1, \ldots, y_n$ corresponding to the values $x_0, x_1, \ldots, x_n$ of x.

Let $x_i - x_{i-1} = h$ for $i = 1, 2, \ldots, n$.(equal intervals)

$$\therefore \qquad x_i = x_0 + ih, i = 0, 1, \ldots, n.$$

We can prove the formula using symbolic operator methods.

$$
\begin{aligned}
P_n(x) &= P_n(x_n + vh) = E^v P_n(x_n) \\
&= (1-\nabla)^{-v} y_n \qquad \textit{since} \qquad E = (1-\nabla)^{-1} \\
&= \left[1 + v\nabla + \frac{v(v+1)}{2!}\nabla^2 + \frac{v(v+1)(v+2)}{3!}\nabla^3 + \cdots\right] y_n
\end{aligned}
$$

$$
\boxed{P_n(x) = P_n(x_n + vh) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots}
$$

where $v = \dfrac{x - x_n}{h}$

Note:

- Since the formula involves the backward difference operator, it is named as backward interpolation formula
- This is used to interpolate the values of y nearer to the end of a set tabular values. This may also be used to extrapolate closure to the right of y_n.

Examples:

Example:1 Find the values of y at $x = 21$ and $x = 28$ from the following data

x	:	20	23	26	29
y	:	0.3420	0.3907	0.4384	0.4848

Solution: Since $x = 21$ is nearer to the beginning of the table, we use Newton's forward formula. Here, h =constant=3.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.3420			
		0.0487		
23	0.3907		−0.0010	
		0.0477		−0.0003
26	0.4384		−0.0013	
		0.0464		
29	0.4848			

The topmost diagonal gives the forward differences of y_0 while the lowermost diagonal gives the backward differences of y_n.

There are only 4 data given. Hence the collocation polynomial will be of degree 3.

By Newton's forward interpolation formula,

$$y(x) \approx P_3(x) = y_0 + \frac{u^{(1)}}{1!}\Delta y_0 + \frac{u^{(2)}}{2!}\Delta^2 y_0 + \frac{u^{(3)}}{3!}\Delta^3 y_0 + \cdots \text{ where}$$

$$u = \frac{x - x_0}{h} = \frac{21 - 20}{3} = 0.3333$$

$$y(21) \approx P_3(21) = 0.3420 + (0.3333)(0.0487) +$$
$$\frac{(0.3333)(-0.6666)}{2}(-0.001) + \frac{(0.3333)(-0.6666)(-1.6666)}{6}(-0.0003)$$

$$\therefore y(21) \approx 0.3583.$$

Since $x = 28$ is nearer to end value, we use Newton's backward interpolation formula.

$$y(x) \approx P_3(x) = P_3(x_n + vh) = y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{6}\nabla^3 y_n + \cdots$$

$$y(28) \approx P_3(28) = P_3\left[29 + \left(-\frac{1}{3}\right)3\right] \text{ where } v = \frac{x - x_n}{h} = \frac{28-29}{3} = -\frac{1}{3}$$

$$= 0.4848 + \left(-\frac{1}{3}\right)(0.0464) + \frac{\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)}{2}(-0.0013) + \frac{\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{6}(-0.0003) + \cdots$$

=0.4848-0.015465+0.0001444+0.0000185

$y(28) \approx 0.4695.$

Example:2 The population of a town is as follows.

Year	x	:	1941	1951	1961	1971	1981	1991
Population in lakhs	y	:	20	24	29	36	46	51

Estimate the population increase during the period 1946 to 1976.

Solution: Since six data are given, $P(x)$ is of degree 5. To find y at $x = 1946$ use forward interpolation and to find y at $x = 1976$, use backward interpolation formula. Here, h =constant=10.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1941	20					
		4				
			1			
1951	24			1		
		5				
			2			
					0	
1961	29			1		
		7				
			3			−9
					−9	
1971	36			−8		
		10				
			−5			
1981	46					
		5				
1991	51					

$$u = \frac{x - x_0}{h} = \frac{1946 - 1941}{10} = \frac{1}{2}$$

$$\begin{aligned}
y(1946) &\approx P_5(1946) = P_5\left[1941 + \frac{1}{2}(10)\right] \\
&= y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \cdots \\
&= 20 + \frac{1}{2}(4) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(1) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(1) \\
&+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{24}(0) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{120}(-9) \\
&= 20 + 2 - 0.125 + 0.0625 - 0.24609 \\
&= 21.69
\end{aligned}$$

$$
\begin{aligned}
y(1976) &\approx P_5(1976) = P_5\left[1991 - \frac{3}{2}(10)\right] \because v = \frac{x - x_n}{h} = \frac{1976 - 1991}{10} = -\frac{3}{2} \\
&= y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{6}\nabla^3 y_n + \cdots \\
&= 51 - \frac{3}{2}(5) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}{2}(-5) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}{6}(-8) \\
&+ \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{24}(-9) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{120}(-9) \\
&= 51 - 7.5 - 1.875 - 0.5 - 0.2109375 - 0.10546875 \\
&= 40.8085938
\end{aligned}
$$

Therefore, increase in population during the period

$$
= 40.809 - 21.69 = 19.119 \text{lakhs.}
$$

Example:3 From the data given below, find the number of students whose weight is between 60 and 70.

Weight in lbs	:	0 – 40	40 – 60	60 – 80	80 – 100	100 – 120
No. of students	:	250	120	100	70	50

Solution:

x weight	y (No. of students)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		−20		
		100		−10	
Below 80	470		−30		20
		70		10	
Below 100	540		−20		
		50			
Below 120	590				

To calculate the number of students whose weight is less than 70.
We will use forward difference formula

$$u = \frac{x - x_0}{h} = \frac{70 - 40}{20} = \frac{3}{2}$$

$$\begin{aligned}
y(70) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \cdots \\
&= 250 + \frac{3}{2}(120) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2}(-20) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{6}(-10) \\
&+ \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{24}(20) \\
&= 250 + 180 - 7.5 + 0.625 + 0.46875 \\
&= 423.59 \\
&\approx 424
\end{aligned}$$

Number of students whose weights is between 60 and 70

$$y(70) - y(60) = 424 - 370 = 54.$$

Practice Problems:

1. From the following table of half-yearly premium for policies maturing at different ages,estimate the premium for policies maturing at age 46 and 63.

Age (x)	:	45	50	55	60	65
Premium (y)	:	114.84	96.16	83.32	74.48	68.48

2. The following data are taken from the steam table.

Temp. degree/cel.:	140	150	160	170	180
Pressure kgf/cm^2:	3.685	4.854	6.302	8.076	10.225

Find the pressure at temperature $t = 142^\circ$ and $t = 175^\circ$

3. Find a polynomial of degree two which takes the values

x	:	0	1	2	3	4	5	6	7
y	:	1	2	4	7	11	16	22	29

Interpolation with Unequal Intervals

Divided differences

Let the function $y = f(x)$ assume the values $f(x_0), f(x_1), \ldots, f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$ respectively where the intervals $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$ need not be equal.

Definitions

The **first divided differences** of $f(x)$ for the arguments x_0, x_1 is defined as $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$. It is denoted by $f(x_0, x_1)$ or $[x_0, x_1]$ or $\underset{x_1}{\Delta} f(x_0)$.

$$f(x_0, x_1) = [x_0, x_1] = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad (1)$$

The **second divided differences** of $f(x)$ for the three arguments x_0, x_1, x_2 is defined as

$$f(x_0, x_1, x_2) = \underset{x_1, x_2}{\Delta^2} f(x_0) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} \qquad (2)$$

The **third divided differences** of $f(x)$ for the four arguments x_0, x_1, x_2, x_3 as

$$f(x_0, x_1, x_2, x_3) = \underset{x_1, x_2, x_3}{\Delta^3} f(x_0) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \qquad (3)$$

Equations (1), (2), (3) refer to divided differences of order one, two, and three respectively.

Example:1 Find the divided differences of $f(x) = x^3 + x + 2$ for the arguments 1,3,6,11.

Solution: The divided difference table is

x	y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	4			
		$\frac{32-4}{3-1} = 14$		
3	32		$\frac{64-14}{6-1} = 10$	
		$\frac{224-32}{6-3} = 64$		$\frac{20-10}{11-1} = 1$
6	224		$\frac{224-64}{11-3} = 20$	
		$\frac{1344-224}{11-6} = 224$		
11	1344			

Newton's interpolation formula for unequal intervals (or Newton's divided difference formula)

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f(x_0, x_1, \ldots, x_n)$$

is called Newton's divided difference interpolation formula for unequal intervals.

Example 2: From the following table find $f(x)$ and hence $f(6)$ using Newton's interpolation formula

x:	1	2	7	8
f(x):	1	5	5	4

Solution:

Here, intervals are not equal.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		$\frac{5-1}{2-1}=4$		
2	5		$\frac{0-4}{7-1}=-\frac{2}{3}$	
		$\frac{5-5}{7-2}=0$		$\frac{-\frac{1}{6}+\frac{2}{3}}{8-1}=\frac{1}{14}$
7	5		$\frac{-1-0}{8-2}=-\frac{1}{6}$	
		$\frac{4-5}{8-7}=-1$		
8	4			

By Newton's divided difference formula,

$$\begin{aligned}
f(x) &= f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+\cdots \\
&= 1+(x-1)4+(x-1)(x-2)\left(-\frac{2}{3}\right)+(x-1)(x-2)(x-7)\left(\frac{1}{14}\right) \\
&= \frac{1}{42}(3x^3-58x^2+321x-224) \\
f(6) &= \frac{1}{42}[3\times 216-36\times 58+1926-224] \\
&= 6.23809524.
\end{aligned}$$

Practice Problems:

1. Form the divided difference table for the following data

x :	-2	0	3	5	7	8
y=f(x):	-792	108	-72	48	-144	-252

2. Using Newton's divided difference formula, find the values of $f(2)$, $f(8)$ and f(15) given the following table:

x :	4	5	7	10	11	13
f(x):	48	100	294	900	1210	2028

3. Find the equation $y = f(x)$ of least degree and passing through the points (-1,-21),(1,15),(2,12),(3,3). Find also y at $x = 0$.

Lagrange's interpolation formula for unequal intervals

Let $y = f(x)$ be a function such that $f(x)$ takes the values $y_0, y_1, \ldots, y_n$ corresponding to $x_0, x_1, \ldots, x_n$.

$$\begin{aligned}
y = f(x) \quad &= \quad \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}.y_0 \\
&+ \quad \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}.y_1 \\
&+ \quad \cdots\cdots\cdots\cdots \\
&+ \quad \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}.y_n
\end{aligned}$$

This equation is called Lagrange's interpolation formula for unequal intervals.

NOTE: Lagrange's interpolation formula can also be used when the values of x are equally spaced.

Example:1 Using Lagrange's formula of interpolation find $y(9.5)$ given

x:	7	8	9	10
y:	3	1	1	9

Solution: By Lagrange's formula,

$$
\begin{aligned}
y = f(x) \quad &= \quad \frac{(x-8)(x-9)(x-10)}{(7-8)(7-9)(7-10)} \times 3 \\
&+ \quad \frac{(x-7)(x-9)(x-10)}{(8-7)(8-9)(8-10)} \times 1 \\
&+ \quad \frac{(x-7)(x-8)(x-10)}{(9-7)(9-8)(9-10)} \times 1 \\
&+ \quad \frac{(x-7)(x-8)(x-9)}{(10-7)(10-8)(10-9)} \times 9
\end{aligned}
$$

$$
\begin{aligned}
f(9.5) &= \frac{(1.5)(0.5)(-0.5)}{(-1)(-2)(-3)} \times 3 \\
&+ \frac{(2.5)(0.5)(-0.5)}{(1)(-1)(-2)} \times 1 \\
&+ \frac{(2.5)(1.5)(-0.5)}{(2)(1)(-1)} \times 1 \\
&+ \frac{(2.5)(1.5)(0.5)}{(3)(2)(1)} \times 9 \\
&= 0.1875 - 0.3125 + 0.9375 + 2.8125 \\
&= 3.625
\end{aligned}
$$

Example:2 Find the parabola of the form $y = ax^2 + bx + c$ passing through the points $(0,0), (1,1)$, and $(2,20)$.

Solution:

By Lagrange's interpolation formula,

$$\begin{aligned} y = f(x) &= \frac{(x-1)(x-2)}{(0-1)(0-2)} \times 0 \\ &+ \frac{(x-0)(x-2)}{(1-0)(1-2)} \times 1 \\ &+ \frac{(x-0)(x-1)}{(2-0)(2-1)} \times 20 \\ &= 0 - x(x-2) + 10x(x-1) \\ y &= 9x^2 - 8x. \end{aligned}$$

Inverse Interpolation

For a given set of table values, we interpolate the value of x for a given value of y. In such case, we will take y as independent variable and x as dependent variable. Replacing $'y'$ by $'x'$ and $'x'$ by $'y'$ in Lagrange's interpolation formula, we get Inverse Lagrange's interpolation formula.

$$
\begin{aligned}
x &= \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}.x_0 \\
&+ \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}.x_1 \\
&+ \cdots \\
&+ \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}.x_n
\end{aligned}
$$

This formula is called formula of inverse interpolation.

Example:1 Find the age corresponding to the annuity value 13.6 given the table

Age (x) :	30	35	40	45	50
Annuity value (y):	15.9	14.9	14.1	13.3	12.5

Solution: By using inverse interpolation formula,

$$
\begin{aligned}
x &= \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)}\times 30 \\
&+ \frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)}\times 35 \\
&+ \frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)}\times 40 \\
&+ \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)}\times 45 \\
&+ \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)}\times 50 \\
\therefore x(y=13.6) &= 43.
\end{aligned}
$$

Practice Problems:

1. Use Lagrange's formula to fit a polynomial to the data

x	:	-1	0	2	3
y	:	-8	3	1	12

and hence find $y(x = 1)$.

2. From the data given below, find the value of x when $y = 13.5$

x	:	93.0	96.2	100.0	104.2	108.7
f(x)	:	11.38	12.80	14.70	17.07	19.91