UNIT IV
(21MAB206T)

ACADEMIC YEAR 2023-2024 (ODD SEMESTER)

NUMERICAL METHODS AND ANALYISIS

TOPICS

- ❖ Taylor Series Method,
- ❖Euler’s Method and it’s rate of convergence,
- ❖Improved Euler’s Method,
- ❖Modified Euler’s method,
- ❖Runge-Kutta Second-Order Method,
- ❖Runge-Kutta Fourth Order Method and their order of convergence.

Taylor Series Method

Solution of ODE with initial condition by Taylor series method

- To find the numerical solution of the equation
- $\frac{dy}{dx} = f(x, y)$
- Given the initial condition $y(x_0) = y_0$.
- First we expand $y(x)$ about the point $x = x_0$ in a Taylor's series in power of $(x - x_0)$. i.e.
- $y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$
- Where $y^{(r)}(x_0) = \left(\frac{d^r y}{dx^r}\right)_{x=x_0}$

- i.e.
- $y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots$

- So
- $y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$
- Where $h = x - x_0$ or $x_1 = x_0 + h$.

- We find $y_0', y_0'', \ldots$ by using our initial equation $\frac{dy}{dx} = f(x, y)$ and its derivatives at $x = x_0$.
- The series obtained in previous slide is an infinite series.
- Since h is small, we can truncate the infinite series for higher power of h.
- After obtaining y_1, we can calculate $y_1', y_1'', \ldots$ by using our initial equation $\frac{dy}{dx} = f(x, y)$ and its derivatives at $x = x_1$.

- Expanding $y(x)$, in a Taylor′s series about the point $x = x_1$, we get
- $y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$
- Proceeding in the same way, we get the
- $y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$
- Where $y_n^{(r)} = \left(\frac{d^r y}{dx^r}\right)_{(x_n, y_n)}$
- This infinite series is truncated at some term to have the numerical value calculated.

- If for calculation purpose, we retain the terms upto and including h^n and neglecting terms involving h^{n+1} and higher power of h, then the Taylor algorithm used is said to be of n^{th} order.
- The truncation error of Taylor method is $O(h^{n+1})$.
- By including more number of terms for calculation, the error can be reduced further.
- If h is small and the terms after n terms are neglected, the error is $\frac{h^n}{n!} f^n(\theta)$ where $x_0 < \theta < x_1$ if $x_1 - x_0 = h$.

Examples

- ***Example1***: Solve $\frac{dy}{dx} = x + y$, given $y(1) = 0$, and get $y(1.1), y(1.2)$ by Taylor series method. Compare the result with explicit solution.
- ***Solution***: Here, we take the step difference $h = 0.1$.
- So
- $x_0 = 1$
- $x_1 = x_0 + h = 1 + 0.1 = 1.1$
- $y_0 = y(x_0) = y(1) = 0$
- $y_0' = (y')_{(x_0,y_0)} = \left(\frac{dy}{dx}\right)_{(x_0,y_0)} = (x + y)_{(x_0,y_0)} = x_0 + y_0 = 1 + 0 = 1$

- $y_0'' = (y'')_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}y'\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(x+y)\right)_{(x_0,y_0)}$
- $= \left(1+\frac{dy}{dx}\right)_{(x_0,y_0)}$
- $= (1+x+y)_{(x_0,y_0)}$ (NOTE: In this step, we obtained the value of y'' which is $1+x+y$)
- $= 1 + x_0 + y_0$
- $= 1 + 1 + 0 = 2$

- $y_0''' = (y''')_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}y''\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(1+x+y)\right)_{(x_0,y_0)}$
- $= \left(1+\frac{dy}{dx}\right)_{(x_0,y_0)}$
- $= (1+x+y)_{(x_0,y_0)}$ (NOTE: In this step, we obtained the value of y''' which is $1+x+y$)
- $= 1 + x_0 + y_0$
- $= 1 + 1 + 0 = 2$

- $y_0^{(4)} = \left(y^{(4)}\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}y'''\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(1 + x + y)\right)_{(x_0,y_0)}$
- $= \left(1 + \frac{dy}{dx}\right)_{(x_0,y_0)}$
- $= (1 + x + y)_{(x_0,y_0)}$ (NOTE: In this step, we obtained the value of $y^{(4)}$ which is $1 + x + y$)
- $= 1 + x_0 + y_0$
- $= 1 + 1 + 0 = 2$

- By Taylor series, we have
- $y_1 = y(x_1) = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{(4)} + \cdots$
- So,
- $y_1 = y(1.1) = 0 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2) + \frac{(0.1)^5}{5!}(2) + \cdots$
- Hence
- $y(1.1) = 0.1 + 0.01 + 0.00033 + 0.00000833 + 0.0000000166 + \cdots$
- So,
- $y(1.1) = 0.11033847$

- For finding $y_2 = y(x_2)$, we have
- $x_2 = x_1 + h = 1.1 + 0.1 = 1.2$
- $y_1 = 0.11033847$
- $y_1' = (y')_{(x_1,y_1)}$
- $= \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$
- $= (x + y)_{(x_1,y_1)}$
- $= x_1 + y_1$
- $= 1.1 + 0.11033847$
- $= 1.210333847$

- $y_1'' = (y'')_{(x_1,y_1)}$
- $= (1 + x + y)_{(x_1,y_1)}$
- $= 1 + x_1 + y_1$
- $= 1 + 1.1 + 0.11033847$
- $= 2.210333847$
- $y_1''' = (y''')_{(x_1,y_1)}$
- $= (1 + x + y)_{(x_1,y_1)}$
- $= 1 + x_1 + y_1$
- $= 1 + 1.1 + 0.11033847$
- $= 2.210333847$

- $y_1^{(4)} = \left(y^{(4)}\right)_{(x_1,y_1)}$
- $= (1 + x + y)_{(x_1,y_1)}$
- $= 1 + x_1 + y_1$
- $= 1 + 1.1 + 0.11033847$
- $= 2.210333847$
- $y_1^{(5)} = \left(y^{(5)}\right)_{(x_1,y_1)}$
- $= (1 + x + y)_{(x_1,y_1)}$
- $= 1 + x_1 + y_1$
- $= 1 + 1.1 + 0.11033847$
- $= 2.210333847$

- By Taylor series, we have
- $y_2 = y(x_2) = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{(4)} + \cdots$
- So,
- $y_2 = y(1.2) = 0.11033847 + \frac{(0.1)}{1!}(1.21033847) + \frac{(0.1)^2}{2!}(2.21033847) + \frac{(0.1)^3}{3!}(2.21033847) + \frac{(0.1)^4}{4!}(2.21033847) + \frac{(0.1)^5}{5!}(2.21033847) + \cdots$
- $y(1.2) = 0.11033847 + 0.121033847 + 2.21033847(0.005 + 0.0016666 + \cdots)$
- So,
- $y(1.2) = 0.24280160$

- The exact solution of $\frac{dy}{dx} = x + y$ is
- $y = -x - 1 - 2e^{x-1}$
- So,
- $y(1.1) = -1.1 - 1 + 2e^{0.1} = 0.11034$
- $y(1.2) = -1.2 - 1 + 2e^{0.2} = 0.2428$

- ***Example2***: Using Taylor series method, find, correct to four decimal places, the value of $y(0.1)$, given $\frac{dy}{dx} = x^2 + y^2$, given $y(0) = 1$.
- ***Solution***: Here, we take the step difference $h = 0.1$.
- So
- $x_0 = 0$
- $x_1 = x_0 + h = 0 + 0.1 = 0.1$
- $y_0 = y(x_0) = y(0) = 1$
- $y_0' = (y')_{(x_0, y_0)}$
- $= \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$
- $= (x^2 + y^2)_{(x_0, y_0)}$
- $= x_0^2 + y_0^2 = 0^2 + 1^2 = 0 + 1 = 1$

- $y_0'' = (y'')_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}y'\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(x^2 + y^2)\right)_{(x_0,y_0)}$
- $= \left(2x + 2y\frac{dy}{dx}\right)_{(x_0,y_0)}$
- $= (2x + 2yy')_{(x_0,y_0)}$ (NOTE: In this step, we obtained the value of y'' which is $2x + 2yy' = 2x + 2yx^2 + 2y^3$)
- $= 2x_0 + 2y_0y_0'$
- $= 2 * 0 + 2 * 1 * 1 = 2$

- $y_0''' = (y''')_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}y''\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(2x + 2yy')\right)_{(x_0,y_0)}$
- $= (2 + 2yy'' + 2(y')^2)_{(x_0,y_0)}$ (NOTE: In this step, we obtained the value of y''' which is $2 + 2yy'' + 2(y')^2$)
- $= 2 + 2y_0y_0'' + 2(y_0')^2$
- $= 2 + 2 * 1 * 2 + 2 * 1^2 = 8$

- $y_0^{(4)} = \left(y^{(4)}\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx} y'''\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(2 + 2yy'' + 2(y')^2)\right)_{(x_0,y_0)}$
- $= (2yy''' + 2y'y'' + 4y'y'')_{(x_0,y_0)}$
- $= (2yy''' + 6y'y'')_{(x_0,y_0)}$ (NOTE: In this step, we obtained the value of $y^{(4)}$ which is $2yy''' + 6y'y''$)
- $= 2y_0y_0''' + 6y_0'y_0''$
- $= 2 * 1 * 8 + 6 * 1 * 2 = 28$

- By Taylor series, we have
- $y_1 = y(x_1) = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{(4)} + \cdots$
- So,
- $y_1 = y(0.1) = 1 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(28) + \cdots$
- Hence
- $y(0.1) = 1 + 0.1 + 0.01 + 0.00133333 + 0.000116666$
- So,
- $y(0.1) = 1.11144999$
- $y(0.1) \approx 1.11145$

- ***Example3***: Using Taylor series method, compute $y(0.2)$ and $y(0.4)$, correct to 4 decimal places given $\frac{dy}{dx} = 1 - 2xy$, given $y(0) = 0$.
- ***Solution***: Here, we take the step difference $h = 0.2$.
- So
- $x_0 = 0$
- $x_1 = x_0 + h = 0 + 0.2 = 0.2$
- $y_0 = y(x_0) = y(0)$ =0
- $y_0' = (y')_{(x_0, y_0)}$
- $= \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$
- $= (1 - 2xy)_{(x_0, y_0)}$
- $= 1 - 2x_0y_0 = 1 - 2 * 0 * 0 = 1$

- $y_0'' = (y'')_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}y'\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(1-2xy)\right)_{(x_0,y_0)}$
- $= (-2xy' - 2y)_{(x_0,y_0)}$ (NOTE: In this step, we obtained the value of y'' which is $-2xy' - 2y$)
- $= -2x_0y_0' - 2y_0$
- $= -2 * 0 * 1 - 2 * 0 = 0$

- $y_0''' = (y''')_{(x_0,y_0)}$
- $= \left(\frac{d}{dx} y''\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(-2xy' - 2y)\right)_{(x_0,y_0)}$
- $= (-2xy'' - 2y' - 2y')_{(x_0,y_0)}$
- $= (-2xy'' - 4y')_{(x_0,y_0)}$(NOTE: In this step, we obtained the value of y''' which is$-2xy'' - 4y'$)
- $= -2x_0y_0'' - 4y_0'$
- $= -2 * 0 * 0 - 4 * 1 = -4$

- $y_0^{(4)} = \left(y^{(4)}\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx} y'''\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(-2xy'' - 4y')\right)_{(x_0,y_0)}$
- $= (-2xy''' - 2y'' - 4y'')_{(x_0,y_0)}$
- $= (-2xy''' - 6y'')_{(x_0,y_0)}$(NOTE: In this step, we obtained the value of $y^{(4)}$ which is $-2xy''' - 6y''$)
- $= -2x_0y_0''' - 6y_0''$
- $= -2 * 0 * (-4) - 4 * 0 = 0$

- $y_0^{(5)} = \left(y^{(5)}\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx} y^{(4)}\right)_{(x_0,y_0)}$
- $= \left(\frac{d}{dx}(-2xy''' - 6y'')\right)_{(x_0,y_0)}$
- $= \left(-2xy^{(4)} - 2y''' - 6y'''\right)_{(x_0,y_0)}$
- $= \left(-2xy^{(4)} - 8y'''\right)_{(x_0,y_0)}$(NOTE: In this step, we obtained the value of $y^{(5)}$ which is$-2xy^{(4)} - 8y'''$)
- $= -2x_0y_0^{(4)} - 8y_0'''$
- $= -2 * 0 * 0 - 8 * (-4) = 32$

- By Taylor series, we have
- $y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(4)} + \cdots$
- So,
- $y_1 = y(0.2) = 0 + \frac{(0.2)}{1!}(1) + \frac{(0.2)^2}{2!}(0) + \frac{(0.2)^3}{3!}(-4) + \frac{(0.2)^4}{4!}(0) + \frac{(0.2)^5}{5!}(32) + \cdots$
- Hence
- $y(0.2) = 0.2 - 0.00533333 + 0.000085333$
- So,
- $y(0.2) = 0.194752003$

- For finding $y_2 = y(x_2)$, we have
- $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$
- $y_1 = 0.194752003$
- $y_1' = (y')_{(x_1,y_1)}$
- $= (1 - 2xy)_{(x_1,y_1)}$
- $= 1 - 2x_1y_1$
- $= 1 - 2(0.2)(0.194752003)$
- $= 0.9220992$

- $y_1'' = (y'')_{(x_1,y_1)}$
- $= (-2xy' - 2y)_{(x_1,y_1)}$
- $= -2x_1y_1' - 2y_1$
- $= -2$[(0.2)(0.9220992)+0.194752003]
- $= -0.758343686$
- $y_1''' = (y''')_{(x_1,y_1)}$
- $= (-2xy'' - 4y')_{(x_1,y_1)}$
- $= -2x_1y_1'' - 4y_1'$
- $= -2[(0.2)(-0.758343686) + 2(0.9220992)]$
- $= -3.38505933$

- $y_1^{(4)} = \left(y^{(4)}\right)_{(x_1,y_1)}$
- $= (-2xy''' - 6y'')_{(x_1,y_1)}$
- $= -2x_1y_1''' - 6y_1''$
- $= -2$[(0.2)(-3.38505933)+3(-0.758343686)]
- $=$ 5.90408585

- By Taylor series, we have
- $y_2 = y(x_2) = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{(4)} + \cdots$
- So,
- $y_1 = y(0.4) = 0.194752003 + \frac{(0.2)}{1!}(1) + \frac{(0.2)^2}{2!}(0) + \frac{(0.2)^3}{3!}(-4) + \frac{(0.2)^4}{4!}(0) + \cdots$
- Hence
- $y(0.4) = 0.2 + (0.2)(0.9220992) + \frac{(0.2)^2}{2!}(-0.758343686) + \frac{(0.2)^3}{3!}(-3.38505933) + \frac{(0.2)^4}{4!}(5.90408585)$
- So,
- $y(0.4) = 0.359883723$

Euler’s Method

Solution of ODE with initial condition by Euler's method

- To find the numerical solution of the equation
- $\frac{dy}{dx} = f(x, y)$
- Given the initial condition $y(x_0) = y_0$.
- Euler's algorithm is given by
- $y_{n+1} = y_n + hf(x_n, y_n); n = 0,1,2,3, \ldots$
- In other words,
- $y(x + h) = y(x) + hf(x, y)$
- It is of order h^2

Examples

- ***Example1***: Given $y' = -y$ and $y(0) = 1$, determine the values of y at $x = (0.01)(0.01)(0.04)$ by Euler method.
- **Note:** The notation $(0.01)(0.01)(0.04)$ means starting point is 0.01 and step difference is 0.01 and final point is 0.04.
- ***Solution***: Here, we have the step difference $h = 0.01$.
- So
- $x_0 = 0, x_1 = 0.01, x_2 = 0.02, x_3 = 0.03, x_4 = 0.04$
- $y_0 = y(x_0) = y(0) = 1$
- We have to find y_1, y_2, y_3, y_4.

- By Euler algorithm,
- $y_{n+1} = y_n + hf(x_n, y_n)$
- So,
- $y_1 = y_0 + hf(x_0, y_0)$
- $= 1 + (0.01)(-y_0)$
- $= 1 + (0.01)(-1) = 1 - 0.01 = 0.99$
- $y_2 = y_1 + hf(x_1, y_1)$
- $= 0.99 + (0.01)(-y_1)$
- $= 0.99 + (0.01)(-0.99) = 0.9801$

- $y_3 = y_2 + hf(x_2, y_2)$
- $= 0.9801 + (0.01)(-y_2)$
- $= 0.9801 + (0.01)(-0.9801) = 0.9703$
- $y_4 = y_3 + hf(x_3, y_3)$
- $= 0.9703 + (0.01)(-y_3)$
- $= 0.9703 + (0.01)(-0.9703\ ) = 0.9606$

- ***Example2***: Using Euler's method, solve numerically the equation, $y' = x + y$ and $y(0) = 1$, for $x = (0.0)(0.2)(1.0)$.
- **Note:** The notation $(0.0)(0.2)(1.0)$ means starting point is 0.0 and step difference is 0.2 and final point is 1.0.
- ***Solution***: Here, we have the step difference $h = 0.2$.
- So
- $x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8,\ x_5 = 1.0$
- $y_0 = y(x_0) = y(0) = 1$
- We have to find y_1, y_2, y_3, y_4, y_5.

- By Euler algorithm,
- $y_{n+1} = y_n + hf(x_n, y_n)$
- So,
- $y_1 = y_0 + hf(x_0, y_0)$
- $= 1 + (0.2)(x_0 + y_0)$
- $= 1 + (0.2)(0 + 1) = 1.2$
- $y_2 = y_1 + hf(x_1, y_1)$
- $= 1.2 + (0.2)(x_1 + y_1)$
- $= 1.2 + (0.2)(0.2 + 1.2) = 1.48$

- $y_3 = y_2 + hf(x_2, y_2)$
- $= 1.48 + (0.2)(x_2 + y_2)$
- $= 1.48 + (0.2)(0.4 + 1.48) = 1.856$
- $y_4 = y_3 + hf(x_3, y_3)$
- $= 1.856 + (0.2)(x_3 + y_3)$
- $= 1.856 + (0.2)(0.6 + 1.856) = 2.3472$
- $y_5 = y_4 + hf(x_4, y_4)$
- $= 2.3472 + (0.2)(x_4 + y_4)$
- $= 2.3472 + (0.2)(0.8 + 2.3472\ ) = 2.94664$

Improved Euler’s Method

Solution of ODE with initial condition by Improved Euler's method

- To find the numerical solution of the equation
- $\frac{dy}{dx} = f(x, y)$
- Given the initial condition $y(x_0) = y_0$.
- Improved Euler's algorithm is given by
- $y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right]; n = 0,1,2,3, \ldots$

Examples

- ***Example1***: Solve $y' = y + e^x$ and $y(0) = 0$, for $x = 0.2, 0.4$ by Improved Euler method.
- ***Solution***: Here, we have the step difference $h = 0.2$.
- So
- $x_0 = 0, x_1 = 0.2, x_2 = 0.4,$
- $y_0 = y(x_0) = y(0) = 0$
- We have to find y_1, y_2.

- By Improved Euler algorithm,
- $y_{n+1} = y_n + \frac{1}{2}h\,[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))]$
- So,
- $y_1 = y_0 + \frac{1}{2}h\,[f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0))]$
- $y_1 = y_0 + \frac{1}{2}h\,[y_0 + e^{x_0} + f(x_1, y_0 + hf(x_0, y_0))]$
- $y_1 = y_0 + \frac{1}{2}(0.2)\,[y_0 + e^{x_0} + y_0 + hf(x_0, y_0) + e^{x_1}]$
- $y_1 = y_0 + \frac{1}{2}(0.2)\,[y_0 + e^{x_0} + y_0 + h(y_0 + e^{x_0}) + e^{x_1}]$
- $= 0 + (0.1)[0 + 1 + 0 + 0.2(0 + 1) + e^{0.2}]$
- $= (0.1)[1 + 0.2 + 1.2214]$=0.24214

- $y_2 = y_1 + \frac{1}{2}h\,[f(x_1, y_1) + f(x_1 + h, y_1 + hf(x_1, y_1))]$
- $y_2 = y_1 + \frac{1}{2}h\,[y_1 + e^{x_1} + f(x_2, y_1 + hf(x_1, y_1))]$
- $y_2 = y_1 + \frac{1}{2}(0.2)\,[y_1 + e^{x_1} + y_1 + hf(x_1, y_1) + e^{x_2}]$
- $y_2 = y_1 + \frac{1}{2}(0.2)\,[y_1 + e^{x_1} + y_1 + h(y_1 + e^{x_1}) + e^{x_2}]$
- $= 0.24214 + (0.1)[0.24124 + e^{0.2} + 0.24124 + (0.2)(0.24124 +$

Modified Euler’s Method

Solution of ODE with initial condition by Modified Euler's method

- To find the numerical solution of the equation
- $\frac{dy}{dx} = f(x, y)$
- Given the initial condition $y(x_0) = y_0$.
- Modified Euler's algorithm is given by
- $y_{n+1} = y_n + h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right]; n = 0,1,2,3, \dots$
- Or
- $y(x + h) = y(x) + h\left[f(x + \frac{1}{2}h, y + \frac{1}{2}hf(x, y))\right]$

Examples

- ***Example1***: Compute y at $x = 0.25$ by Modified Euler method given $y' = 2xy$ and $y(0) = 1$.
- ***Solution***: Here, we have the step difference $h = 0.25$.
- So
- $x_0 = 0$, $x_1 = 0.25$.
- $y_0 = y(x_0) = y(0) = 1$
- We have to find y_1.

- By Modified Euler algorithm,
- $y_{n+1} = y_n + h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right]$
- So,
- $y_1 = y_0 + h\left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0)\right)\right]$
- $y_1 = y_0 + h\left[2(x_0 + \frac{1}{2}h)(y_0 + \frac{1}{2}hf(x_0, y_0))\right]$
- $y_1 = y_0 + (0.25)\left[2(x_0 + \frac{1}{2}h)(y_0 + \frac{1}{2}h(2x_0y_0))\right]$
- $y_1 = 1 + (0.25)\left[2(0 + \frac{1}{2}(0.25))(1 + \frac{1}{2}(0.25)(2(0)(1))\right]$
- $y_1 = 1 + (0.25)\left[2(\frac{1}{2}(0.25))(1)\right]$
- $y_1 = 1 + (0.25)\,[0.25\,]$
- $y_1 = 1 + 0.0625 = 1.0625$

- ***Example2***: Solve the equation $y' = 1 - y$ given $y(0) = 0$ using Modified Euler method at $x = 0.1, 0.2$ and 0.3.
- ***Solution***: Here, we have the step difference $h = 0.1$.
- So
- $x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$.
- $y_0 = y(x_0) = y(0) = 0$
- We have to find y_1, y_2, y_3.

- By Modified Euler algorithm,
- $y_{n+1} = y_n + h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right]$
- So,
- $y_1 = y_0 + h\left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0)\right)\right]$
- $y_1 = y_0 + h\left[1 - y_0 - \frac{1}{2}hf(x_0, y_0)\right]$
- $y_1 = y_0 + h\left[1 - y_0 - \frac{1}{2}h(1 - y_0)\right]$
- $y_1 = 0 + (0.1)\left[1 - 0 + \frac{1}{2}(0.1)(1 - 0))\right]$
- $y_1 = (0.1)\,[1 - 0.05]$
- $y_1 = (0.1)\,[0.95]$
- $y_1 = 0.095$

- $y_2 = y_1 + h\ \left[f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}hf(x_1, y_1)\right)\right]$
- $y_2 = y_1 + h\ \left[1 - y_1 - \frac{1}{2}hf(x_1, y_1)\right]$
- $y_2 = y_1 + h\ \left[1 - y_1 - \frac{1}{2}h(1 - y_1)\right]$
- $y_2 = 0.095 + (0.1)\ \left[1 - (0.095) + \frac{1}{2}(0.1)(1 - 0.095))\ \right]$
- $y_2 = 0.095 + (0.1)\ [1 - 0.14025]$
- $y_2 = 0.18098$

- $y_3 = y_2 + h\left[f\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}hf(x_2, y_2)\right)\right]$
- $y_3 = y_2 + h\left[1 - y_2 - \frac{1}{2}hf(x_2, y_2)\right]$
- $y_3 = y_1 + h\left[1 - y_2 - \frac{1}{2}h(1 - y_2)\right]$
- $y_3 = 0.18098 + (0.1)\left[1 - (0.18098) + \frac{1}{2}(0.1)(1 - 0.18098))\right]$
- $y_3 = 0.18098 + (0.1)[1 - 0.22193]$
- $y_3 = 0.258787$

Second order Runge-Kutta Method

Solution of ODE with initial condition by Second order Runge-Kutta method

- To find the numerical solution of the equation
- $\frac{dy}{dx} = f(x, y)$
- Given the initial condition $y(x_0) = y_0$.
- Second order Runge-Kutta method is given by
- $y_{n+1} = y_n + k_2; n = 0,1,2,3, \ldots$
- Where
- $k_2 = h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)\right]$
- Where
- $k_1 = hf(x_n, y_n)$

Solution of ODE with initial condition by Second order Runge-Kutta method

- So for finding y_{n+1}, we need to find k_1 and k_2 first.
- Put the value of k_1 and k_2 in $k_2 = h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)\right]$ and $y_{n+1} = y_n + k_2; n = 0,1,2,3, \ldots$, we get
- $y_{n+1} = y_n + h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right)\right]; n = 0,1,2,3, \ldots$
- We observe that the value of y_{n+1} is same as in Modified Euler's method.
- So, the Runge-Kutta method of second order is nothing but the Modified Euler's method.
- The order of convergence for Second order Runge-Kutta method is 2.

Examples

- ***Example1***: Obtain the values of y at $x = 0.1, 0.2$ using second order R.K. method for the differential equation $y' = -y$ given $y(0) = 1$.
- ***Solution***: Here, we have the step difference $h = 0.1$.
- So
- $x_0 = 0, x_1 = 0.1,\ x_2 = 0.2$.
- $y_0 = y(x_0) = y(0) = 1$
- We have to find y_1, y_2.

- For finding y_1 using second order R.K method ,
- we first find k_1 and k_2 as
- $y_1 = y_0 + k_2$ where $k_2 = h\left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)\right]$ where $k_1 = hf(x_0, y_0)$
- So
- $k_1 = hf(x_0, y_0) = h(-y_0) = (0.1)(-1) = -0.1$
- $k_2 = h\left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)\right]$
- $k_2 = hf\left(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(-0.1)\right)$
- $k_2 = hf(0.05, 0.95)$
- $k_2 = (0.1)(-0.95) = -0.095$

- So,
- $y_1 = y(0.1) = y_0 + k_2$
- $y_1 = 1 - 0.095$
- $y_1 = 0.905$
- For finding y_2 using second order R.K method ,
- we again find k_1 and k_2 as
- $y_2 = y_1 + k_2$ where $k_2 = h\left[f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)\right]$
- where $k_1 = hf(x_1, y_1)$

- So
- $k_1 = hf(x_1, y_1) = h(-y_1) = (0.1)(-0.905) = -0.0905$
- $k_2 = h\left[f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)\right]$
- $k_2 = hf\left(0.1 + \frac{1}{2}(0.1), 0.905 + \frac{1}{2}(-0.0905)\right)$
- $k_2 = hf(0.15, 0.85975)$
- $k_2 = (0.1)(-0.85975) = -0.085975$
- Hence
- $y_2 = y_1 + k_2 = 0.905 - 0.085975 = 0.819025$

Fourth order Runge-Kutta Method

Solution of ODE with initial condition by Fourth order Runge-Kutta method

- To find the numerical solution of the equation
- $\frac{dy}{dx} = f(x, y)$
- Given the initial condition $y(x_0) = y_0$.
- Fourth order Runge-Kutta method is given by
- $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); n = 0,1,2,3, \ldots$
- Where
- $k_1 = hf(x_n, y_n)$
- $k_2 = h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)\right]$
- $k_3 = h\left[f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)\right]$
- $k_4 = h\left[f(x_n + h, y_n + k_3)\right]$
- The order of convergence for Fourth order Runge-Kutta method is 4.

Examples

- ***Example1***: Apply the fourth order Runge-Kutta method to find $y(0.2)$ given that $y' = x + y$ given $y(0) = 1$.
- ***Solution***: Here, we take the step difference $h = 0.1$.
- So
- $x_0 = 0, x_1 = 0.1,\ x_2 = 0.2$.
- $y_0 = y(x_0) = y(0) = 1$
- We have to find y_1, y_2.

- For finding y_1 using fourth order R.K method ,
- we first find k_1, k_2, k_3, k_4 as
- $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- So
- $k_1 = hf(x_0, y_0) = h(x_0 + y_0) = (0.1)(0 + 1) = 0.1$
- $k_2 = h\left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)\right]$
- $k_2 = hf\left(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(0.1)\right)$
- $k_2 = hf(0.05, 1.05)$
- $k_2 = (0.1)(0.05 + 1.05) = 0.11$

- $k_3 = h \left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)\right]$
- $k_3 = hf\left(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(0.11)\right)$
- $k_3 = hf(0.05, 1.055)$
- $k_3 = (0.1)(0.05 + 1.055) = 0.1105$
- $k_4 = h\,[f(x_0 + h, y_0 + k_3)]$
- $k_4 = hf\big(0 + (0.1), 1 + (0.1105)\big)$
- $k_4 = hf(0.1, 1.11055)$
- $k_4 = (0.1)(0.1 + 1.11055) = 0.12105$

- So,
- $y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- $y_1 = 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105)$
- $y_1 = 0.110341667 \approx 0.110342$
- For finding y_2 using fourth order R.K method ,
- we again find k_1, k_2, k_3, k_4 as
- $y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

- So
- $k_1 = hf(x_1, y_1) = h(x_1 + y_1) = (0.1)(0.1 + 1.110342) = 0.1210342$
- $k_2 = h\left[f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)\right]$
- $k_2 = hf\left(0.1 + \frac{1}{2}(0.1), 1.110342 + \frac{1}{2}(0.1210342)\right)$
- $k_2 = hf(0.15, 1.170859)$
- $k_2 = (0.1)(0.15 + 1.170859) = 0.1320859$

- $k_3 = h\left[f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)\right]$
- $k_3 = hf\left(0.1 + \frac{1}{2}(0.1), 1.110432 + \frac{1}{2}(0.1320859)\right)$
- $k_3 = hf(0.15, 1.1763848)$
- $k_3 = (0.1)(0.15 + 1.1763848) = 0.13263848$
- $k_4 = h\,[f(x_1 + h, y_1 + k_3)]$
- $k_4 = hf\big(0.1 + (0.1), 1.110432 + (0.13263848)\big)$
- $k_4 = hf(0.2, 1.24298048)$
- $k_4 = (0.1)(0.2 + 1.24298048) = 0.144298048$

- So,
- $y_2 = y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- $y_2 = 1.110342 + \frac{1}{6}(0.794781008)$
- $y_2 = 1.2428055 \approx 1.2428$

Examples

- ***Example2***: Compute $y(0.3)$ given $y' + y + xy^2 = 0$, $y(0) = 1$ by taking $h = 0.1$ using fourth order Runge-Kutta method.
- ***Solution***: Here, we take the step difference $h = 0.1$.
- So
- $x_0 = 0, x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3.$
- $y_0 = y(x_0) = y(0) = 1$
- We have to find y_1, y_2, y_3.
- Also
- $y' = -(xy^2 + y) = f(x, y)$

- For finding y_1 using fourth order R.K method ,
- we first find k_1, k_2, k_3, k_4 as
- $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- So
- $k_1 = hf(x_0, y_0) = h\{-(x_0 y_0^2 + y_0)\} = -0.1$
- $k_2 = h\ \left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)\right]$
- $k_2 = hf\left(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(-0.1)\right)$
- $k_2 = hf(0.05, 0.95)$
- $k_2 = -(0.1)\big((0.05)(0.95)^2 + 0.95\big) = -0.0995$

- $k_3 = h\left[f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)\right]$
- $k_3 = hf\left(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(-0.0995)\right)$
- $k_3 = hf(0.05,0.95025)$
- $k_3 = -(0.1)\left((0.05)(0.95025)^2 + 0.95025\right) = -0.09953987 \approx -0.0995$
- $k_4 = h\left[f(x_0 + h, y_0 + k_3)\right]$
- $k_4 = hf\left(0 + (0.1), 1 + (-0.0995)\right)$
- $k_4 = hf(0.1, 0.9005)$
- $k_4 = -(0.1)\left((0.1)(0.9005)^2 + 0.9005\right) = -0.0982$

- So,
- $y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- $y_1 = 0.9006$
- For finding y_2 using fourth order R.K method ,
- we again find k_1, k_2, k_3, k_4 as
- $y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

- So
- $k_1 = hf(x_1, y_1)$
- $= hf(0.1, 0.9006$
- $= -0.0982$
- $k_2 = h\left[f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)\right]$
- $k_2 = hf(0.15, 0.8515)$
- $k_2 = -0.0960$

- $k_3 = h\left[f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)\right]$
- $k_3 = hf(0.15, 0.8526)$
- $k_3 = -0.0962$
- $k_4 = h\,[f(x_1 + h, y_1 + k_3)]$
- $k_4 = hf(0.2, 0.8044)$
- $k_4 = -0.0934$

- So,
- $y_2 = y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- $y_2 = 0.8046$
- Again for finding y_3 using fourth order R.K method ,
- we again find k_1, k_2, k_3, k_4 as
- $y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- Here,
- $k_1 = hf(x_2, y_2)$
- $k_2 = h\left[f\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right)\right]$
- $k_3 = h\left[f\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right)\right]$
- $k_4 = h\,[f(x_2 + h, y_2 + k_3)]$

- Here,
- $k_1 = -0.0934$
- $k_2 = -0.0902$
- $k_3 = -0.0904$
- $k_4 = -0.0867$
- So,
- $y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- $y_3 = y(0.3) = 0.7144$